**Inverse functions**

Let $f(x)$ be a one-to-one function, so $f(x) = f(y)$ means that $x = y$. For example, let $f(x) = x^3$. Suppose that $f^{-1}(x)$ is the inverse of $f(x)$. If $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$. You should know these three inverse formulas:

$$\begin{aligned} f(f^{-1}(x)) &= x \text{ for all } x \text{ in the range of } f(x), \\ f^{-1}(f(x)) &= x \text{ for all } x \text{ in the domain of } f(x), \\ \text{If } x = f^{-1}(y),\ &\text{then } f(x) = y. \\ \text{If } f(x) = y,\ &\text{then } x = f^{-1}(y). \end{aligned}$$

You should also know that

$$f^{-1}(x) \text{ does } \textbf{NOT} \text{ mean } \frac{1}{f(x)}.$$

It is also true (although you don't need to memorize this) that the range of $f^{-1}(x)$ equals the domain of $f(x)$, and the domain of $f^{-1}(x)$ equals the range of $f(x)$. Furthermore,

$$\begin{aligned} f^{-1}(x) &\text{ does not exist if } x \text{ is not in the range of } f(x), \\ f(f^{-1}(x)) &\text{ does not exist if } x \text{ is not in the range of } f(x), \\ f^{-1}(f(x)) &\text{ does not exist if } x \text{ is not in the domain of } f(x). \end{aligned}$$

---

Now, let $f(x)$ be a function which is not one-to-one, such as $x^2$ or $\sin x$. Let $f^{-1}(x)$ be *an* inverse to $f(x)$. Again, you should know the four inverse formulas

$$\begin{aligned} f(f^{-1}(x)) &= x \text{ for all } x \text{ in the range of } f(x), \\ f^{-1}(f(x)) &= x \text{ for all } x \text{ in the range of } f^{-1}(x), \text{ but } \textit{not} \text{ for all } x \text{ in the domain of } f(x), \\ \text{If } x = f^{-1}(y),\ &\text{then } f(x) = y. \\ \text{If } f(x) = y,\ &\text{and } x \text{ is in the range of } f^{-1}, \text{ then } x = f^{-1}(y). \end{aligned}$$

For example,

$$\begin{aligned} (\sqrt{x})^2 &= x \text{ for all } x \geq 0, \quad (\sqrt{x})^2 \text{ does not exist if } x < 0, \\ \sqrt{x^2} &= |x|, \text{ which equals } x \text{ if and only if } x \geq 0. \\ \text{If } x = \sqrt{y},\ &\text{then } x^2 = y. \\ \text{If } x^2 = y,\ &\text{then } x = \pm\sqrt{y}, \text{ so } x = \sqrt{y} \text{ provided } x \geq 0. \end{aligned}$$

You should also recall that such functions have more than one inverse. For example, if $f(x) = x^2$, then $g(x) = \sqrt{x}$ and $h(x) = -\sqrt{x}$ are both inverses of $f(x)$.

It is also true (although you don't need to memorize this) that the domain of the inverse $f^{-1}(x)$ is equal to the range of $f(x)$; however, it is no longer true that the range of $f^{-1}(x)$ is equal to the domain of $f(x)$.

---

We can summarize important properties of $\ln x$ and the inverse trigonometric functions using these facts.

$$\begin{aligned} &e^{\ln b} = b \text{ if } b > 0, && \sin(\arcsin(x)) = x \text{ if } -1 \leq x \leq 1, \\ &\ln e^b = b \text{ for all } b, && \arcsin(\sin(x)) = x \text{ if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \\ &e^x = y \text{ if and only if } x = \ln(y), && \text{if } x = \arcsin(y) \text{ then } \sin(x) = y, \\ & && \text{if } \sin(x) = y \text{ and } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \text{ then } x = \arcsin(y), \\ &\ln b \text{ does not exist if } b \leq 0, && \arcsin(x) \text{ does not exist if } |x| > 1. \end{aligned}$$

Similar facts hold for arctan, arccos and so on.

By the end of your first calculus course, you should be able to compute the derivative of an inverse function using implicit differentiation.

**Exponentials and Logarithms**

You should recall some basic properties of exponentials:

$$a^{b+c} = a^b a^c, \quad a^{b-c} = \frac{a^b}{a^c}, \quad (a^b)^c = a^{bc}, \quad (ab)^c = a^c b^c, \quad \left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}, \quad a^{1/n} = \sqrt[n]{a},$$
$$a^{-b} = \frac{1}{a^b}, \quad a^{-1} = \frac{1}{a}, \quad \text{if } a > 0 \text{ then } a^0 = 1 \text{ and } a^x > 0 \text{ for all } x.$$

Let $a$ be any number with $0 < a$ and $a \neq 1$. You should remember the definition of $\log_a$ and the relationships between an exponential and a logarithm:

$$a^{\log_a b} = b \text{ for all } b > 0, \quad \log_a a^b = b \text{ for all } b, \quad a^x = y \text{ if and only if } x = \log_a(y).$$

In particular,

$$e^{\ln b} = b \text{ for all } b > 0, \quad \ln e^b = b \text{ for all } b, \quad e^x = y \text{ if and only if } x = \ln(y).$$

You should know these properties of logarithms:

$$\log_a b^c = c \log_a b, \quad \log_a(bc) = \log_a b + \log_a c, \quad \log_a \left(\frac{b}{c}\right) = \log_a b - \log_a c, \quad \log_a \left(\frac{1}{b}\right) = -\log_a b,$$
$$\log_a a = 1, \quad \log_a 1 = 0, \quad \frac{\log_a b}{\log_a c} = \log_c b, \quad \log_a x \text{ does not exist if } x \leq 0.$$

You should be able to use these properties to solve equations. For example, if you know that $\ln w = -\ln t + c$, you should be able to deduce that $w = e^{-\ln t + c} = e^c/t$.

By the end of your first calculus course, you should know that

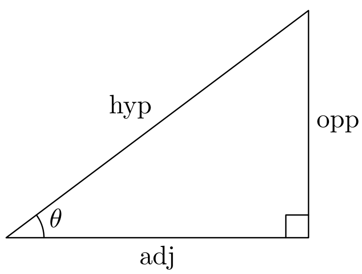
$$\begin{aligned} \frac{d}{dx} e^x &= e^x, & \frac{d}{dx} \ln|x| &= \frac{1}{x}, \\ \frac{d}{dx} a^x &= \frac{d}{dx} e^{x \ln a} = a^x \ln a, & \frac{d}{dx} \log_a |x| &= \frac{d}{dx} \frac{\ln|x|}{\ln a} = \frac{1}{x \ln a}. \end{aligned}$$

**Trigonometry (precalculus)**

You need to understand radians. Recall that

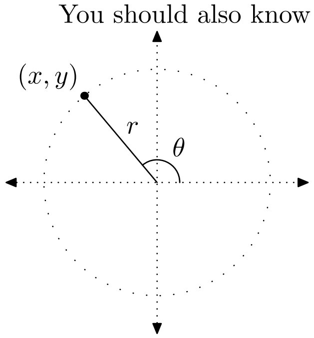
$$\begin{aligned}\frac{\pi}{2} \text{ radians} &= 90^\circ = \text{a right angle,}\\ \pi \text{ radians} &= 180^\circ = \text{a straight line,}\\ 2\pi \text{ radians} &= 360^\circ = \text{a full circle.}\end{aligned}$$

You should know the definitions of the trigonometric functions in terms of a right triangle:




$$\begin{aligned}\sin(\theta) &= \frac{\text{opp}}{\text{hyp}} & \sec(\theta) &= \frac{\text{hyp}}{\text{adj}} = \frac{1}{\cos\theta}\\ \cos(\theta) &= \frac{\text{adj}}{\text{hyp}} & \csc(\theta) &= \frac{\text{hyp}}{\text{opp}} = \frac{1}{\sin\theta}\\ \tan(\theta) &= \frac{\text{opp}}{\text{adj}} = \frac{\sin\theta}{\cos\theta} & \cot(\theta) &= \frac{\text{adj}}{\text{opp}} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}\end{aligned}$$

You should also know the definitions of the trigonometric functions in terms of the unit circle:




$$\begin{aligned}\sin(\theta) &= \frac{y}{r} & \sec(\theta) &= \frac{r}{x} = \frac{1}{\cos\theta}\\ \cos(\theta) &= \frac{x}{r} & \csc(\theta) &= \frac{r}{y} = \frac{1}{\sin\theta}\\ \tan(\theta) &= \frac{y}{x} = \frac{\sin\theta}{\cos\theta} & \cot(\theta) &= \frac{x}{y} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}\end{aligned}$$

(You need these to define trig functions of negative angles and angles greater than $\pi/2$.)

In particular, you should be able to find the values of the trig functions evaluated at multiples of $\frac{\pi}{2}$:

$$\begin{aligned}&\sin 0 = 0, && \sin\left(\frac{\pi}{2}\right) = 1, && \sin\pi = 0, && \sin\left(\frac{3\pi}{2}\right) = -1,\\ &\cos 0 = 1, && \cos\left(\frac{\pi}{2}\right) = 0, && \cos\pi = -1, && \cos\left(\frac{3\pi}{2}\right) = 0.\end{aligned}$$

You should be able to sketch the graphs of all six trig functions. These graphs are provided later in this document.

You should know the following basic identities:

$$\begin{aligned}\sin(-x) &= -\sin x, & \sin(x + 2\pi) &= \sin x,\\ \cos(-x) &= \cos x, & \cos(x + 2\pi) &= \cos x.\end{aligned}$$

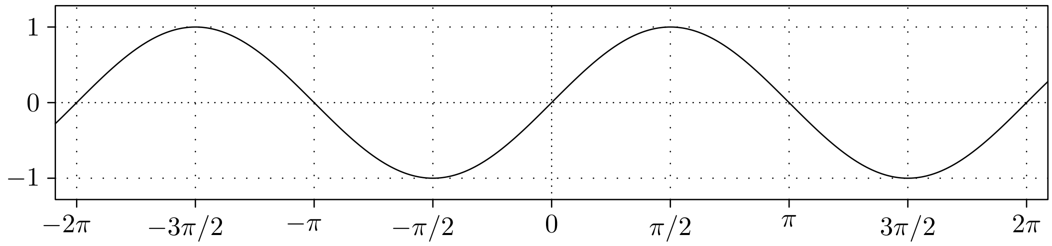
You should be able to combine all of the above, to deduce (for example) that

$$\csc\left(\frac{3\pi}{2}\right) = -1 \quad \text{or} \quad \tan(x + 2\pi) = \tan x.$$

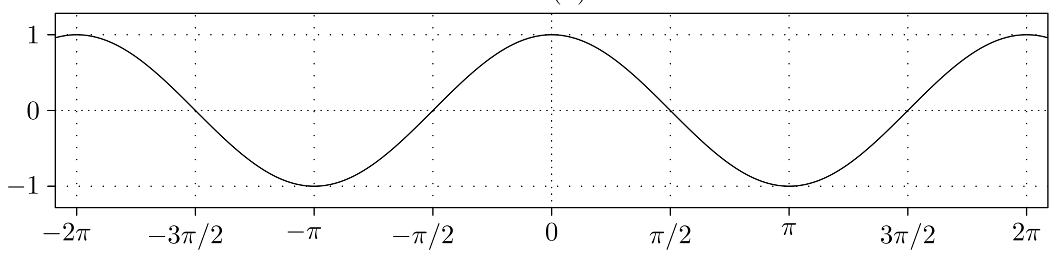
You should know the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ and should be able to deduce that

$$\tan^2 x + 1 = \sec^2 x, \quad \cot^2 x + 1 = \csc^2 x.$$

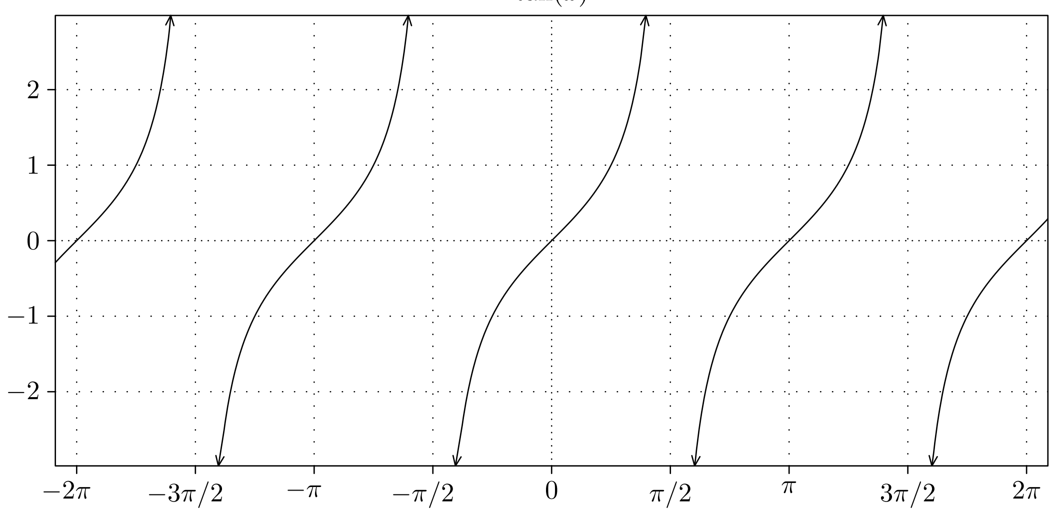
$\sin(x)$

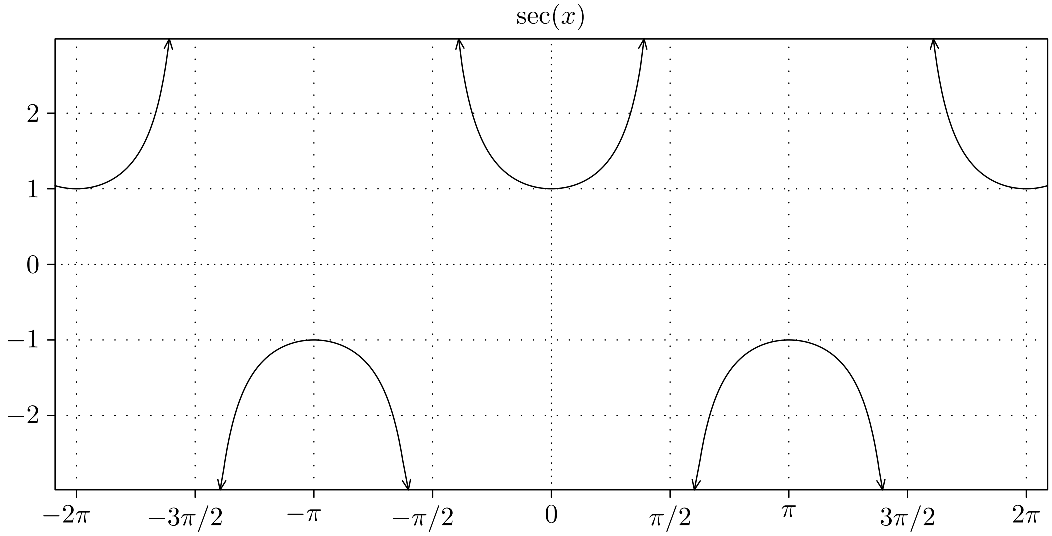


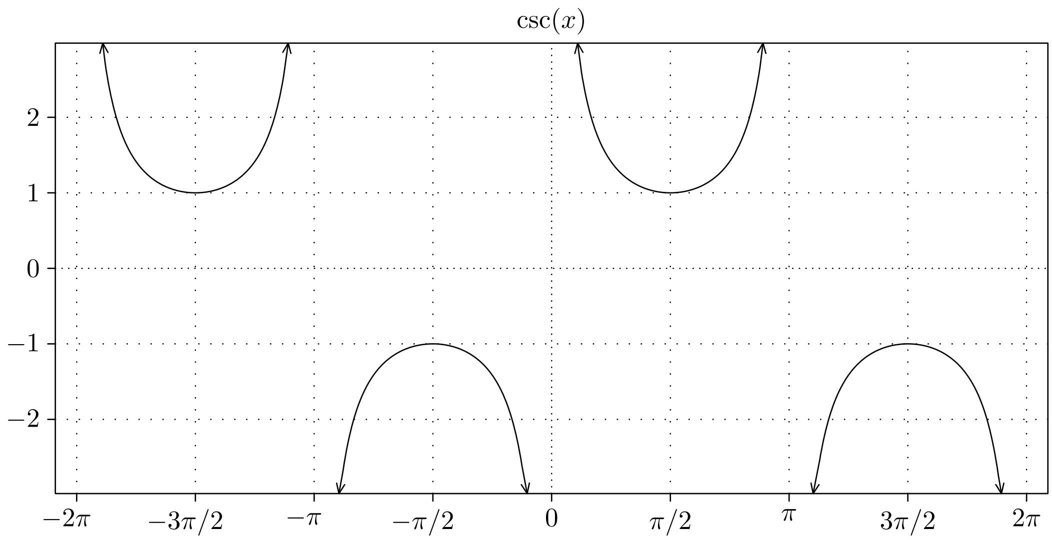

$\cos(x)$

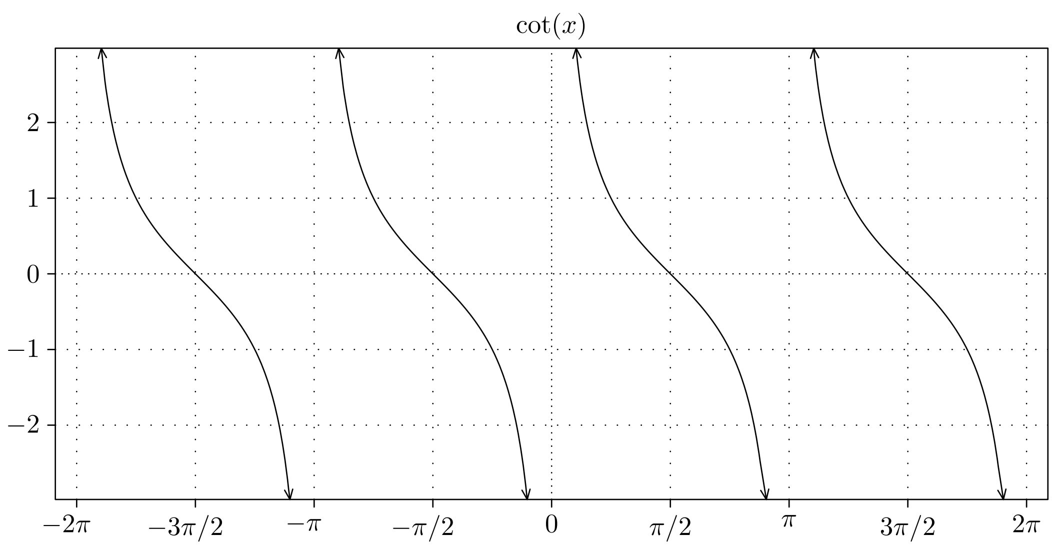



$\tan(x)$




sec($x$)
2
1
0
−1
−2
−2π
−3π/2
−π
−π/2
0
π/2
π
3π/2
2π


csc($x$)
2
1
0
−1
−2
−2π
−3π/2
−π
−π/2
0
π/2
π
3π/2
2π


cot($x$)
2
1
0
−1
−2
−2π
−3π/2
−π
−π/2
0
π/2
π
3π/2
2π

**Inverse trigonometric functions (precalculus)**

You should be able to sketch the graphs of the inverse trigonometric functions. These are provided on the next page. Notice the ways in which they are related to the graphs of the trigonometric functions.

You should know the ranges of the trig trigonometric. Again, you can probably figure these out if you know what the graph looks like. Notice that the range of a trigonometric function is equal to the domain of the corresponding inverse trigonometric function.

$$\begin{aligned} &\sin \text{ and } \cos \text{ have range } [-1,1], && \arcsin \text{ and } \arccos \text{ have domain } [-1,1], \\ &\sec \text{ and } \csc \text{ have range } (-\infty,-1]\cup[1,\infty), && \operatorname{arcsec} \text{ and } \operatorname{arccsc} \text{ have domain } (-\infty,-1]\cup[1,\infty), \\ &\tan \text{ and } \cot \text{ have range } (-\infty,\infty), && \arctan \text{ and } \operatorname{arccot} \text{ have domain } (-\infty,\infty). \end{aligned}$$

You should know the definitions and basic identities for the inverse trig functions:

$$\begin{aligned} \sin(\arcsin x) &= x \text{ or does not exist}, \\ \arcsin(\sin x) &= x \text{ if } 0 < x < \pi/2, \\ \text{If } 0 < x < \pi/2, \text{ then } \sin(x) &= y \text{ if and only if } x = \arcsin(y), \end{aligned}$$

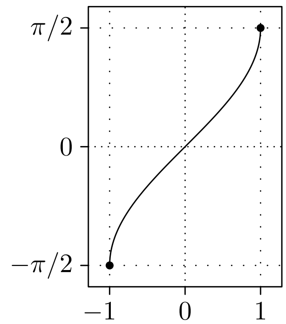
(These identities hold with sin replaced by any of the six trig functions. They are also true for some values of $x \geq \pi/2$ or $x \leq 0$, depending on the trig function.)

You need to be able to go from equations in terms of trig functions to equations in terms of inverse trig functions. For example, if I tell you that $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, you need to be able to deduce that $\arctan\frac{\sqrt{3}}{2} = \frac{\pi}{6}$.

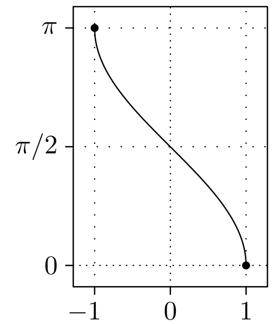
You should be able to deduce the following identities and values:

$$\begin{array}{llll} \arcsin(-x) = -\arcsin x, & \arctan(-x) = -\arctan x, & \arcsin 1 = \dfrac{\pi}{2}, & \arcsin 0 = 0, \\ \arcsin(-1) = -\dfrac{\pi}{2}, & \arctan 0 = 0, & \operatorname{arcsec} 1 = 0, & \operatorname{arcsec}(-1) = \pi. \end{array}$$

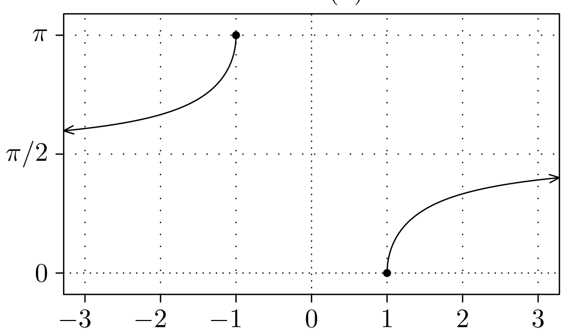
arcsin$(x)$



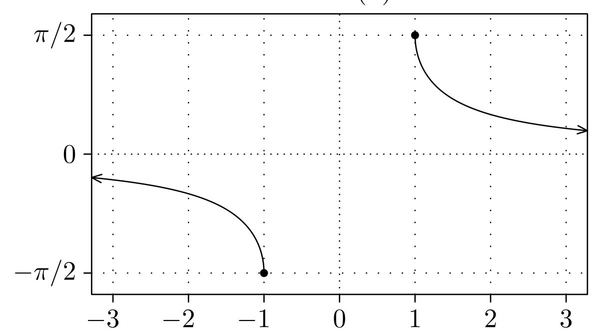

arccos$(x)$



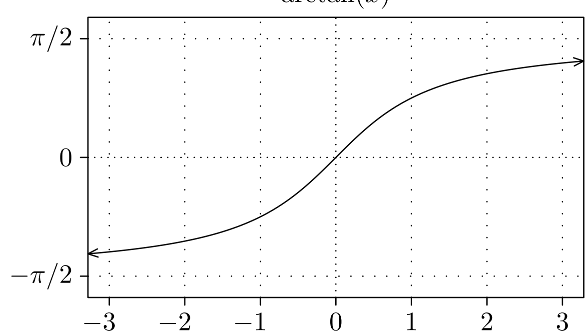

arcsec$(x)$



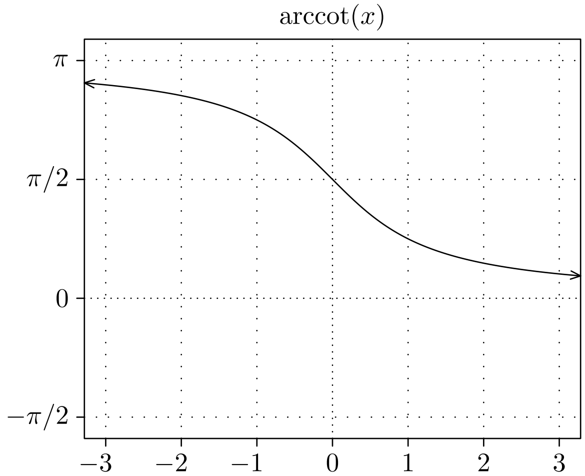

arccsc$(x)$




arctan$(x)$

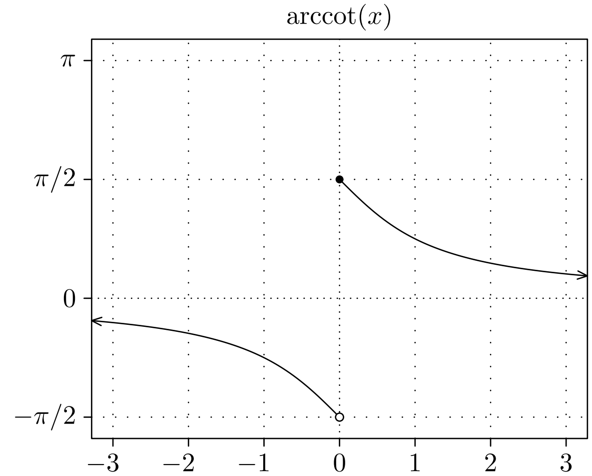



arccot$(x)$




or

arccot$(x)$



**Trigonometry (calculus)**

By the end of your first calculus course, you should know (or be able to compute from the quotient rule) the derivatives of all six trigonometric functions:

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\sec x = \sec x \tan x,$$
$$\frac{d}{dx}\cos x = -\sin x, \qquad \frac{d}{dx}\cot x = -\csc^2 x, \qquad \frac{d}{dx}\csc x = -\csc x \cot x.$$

(Notice that the "co" functions are the ones with minus signs in their derivatives.)

You should know the integrals of $\sin x$ and $\cos x$:

$$\int \sin x\,dx = -\cos x + C, \qquad \int \cos x\,dx = \sin x + C$$

You should be able to compute the derivatives of the inverse trig functions using implicit differentiation:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\operatorname{arccot} x = -\frac{1}{1+x^2}, \qquad \frac{d}{dx}\operatorname{arccsc} x = -\frac{1}{|x|\sqrt{x^2-1}}.$$

However, you need not memorize these derivatives.

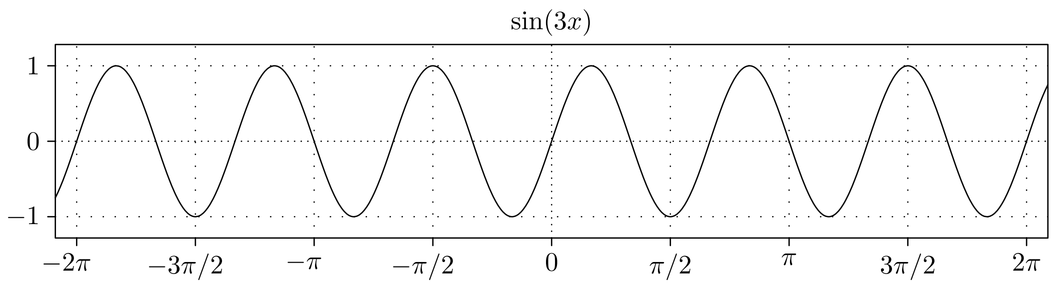
The following information might also be useful (although you don't need to memorize it):

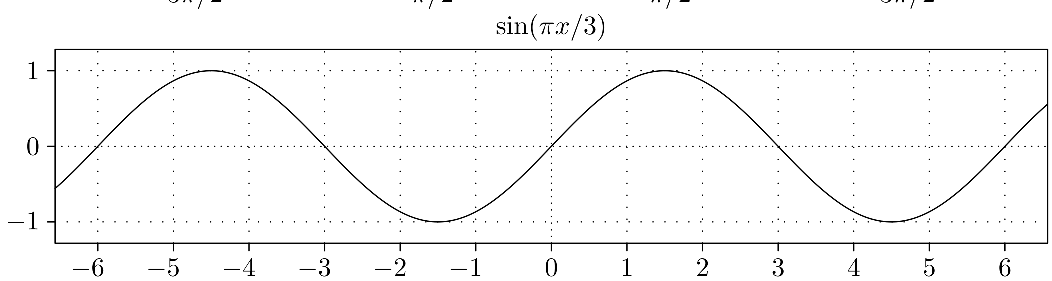
$$\int \tan x\,dx = \ln|\sec x| + C = -\ln|\cos x| + C, \qquad \int \cot x\,dx = \ln|\sin x| + C = -\ln|\csc x| + C,$$
$$\int \sec x\,dx = \ln|\sec x + \tan x| + C, \qquad \int \csc x\,dx = -\ln|\csc x + \cot x| + C.$$

**Trigonometry (Fourier analysis)**

At the start of a course on Fourier analysis, you should know the information outlined above, and also should know the following.

Here are the graphs of $y = \sin(3x)$ and $y = \sin \pi x/3$:




You should be able to sketch the graph of $y = \sin(\omega x)$ or $y = \cos(\omega x)$ for any positive real number $\omega$.

You should know derivatives and integrals of the trig functions:

$$\frac{d}{dx}\sin(\omega x) = \omega\cos(\omega x), \qquad \frac{d}{dx}\cos(\omega x) = -\omega\sin(\omega x),$$
$$\int \sin(\omega x)\,dx = -\frac{1}{\omega}\cos(\omega x) + C, \qquad \int \cos(\omega x)\,dx = \frac{1}{\omega}\sin(\omega x) + C.$$

You should know that $\sin(\omega x)$ is odd and $\cos(\omega x)$ is even, that is,

$$\sin(-\omega x) = -\sin(\omega x), \qquad \cos(-\omega x) = \cos(\omega x).$$

You should know that $\sin x$ and $\cos x$ have period $2\pi$, and that $\sin(\omega x)$ and $\cos(\omega x)$ have period $2\pi/\omega$. That is,

$$\sin(x + 2\pi) = \sin(x), \quad \cos(x + 2\pi) = \cos(x), \quad \sin(\omega(x + 2\pi/\omega)) = \sin(\omega x), \quad \cos(\omega(x + 2\pi/\omega)) = \cos(\omega x).$$

You should know that

$$\sin(n\pi) = 0, \quad \cos\frac{(2n+1)\pi}{2} = 0, \quad \text{for any integer } n.$$

It may help to know that if $n$ is an integer, then $\sin(n\pi) = 0$, $\sin((2n+1)\pi/2) = (-1)^n$, $\cos(n\pi) = (-1)^n$, $\cos((2n+1)\pi/2) = 0$.

**Extra trigonometry**

The following information might also be useful (although you don't need to memorize it):

$$\sin(x+\pi) = -\sin x, \qquad \cos\left(\frac{\pi}{2} - x\right) = \sin x,$$
$$\cos(x+\pi) = -\cos(x), \qquad \sin\left(\frac{\pi}{2} - x\right) = \cos x.$$

The sum-angle, double-angle and half-angle identities:

$$\begin{aligned}
\sin(x+y) &= \sin(x)\cos(y) + \sin(y)\cos(x),\\
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y),\\
\tan(x+y) &= \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)},\\
\sin(2x) &= 2\sin(x)\cos(x),\\
\cos(2x) &= \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x),\\
\sin\left(\frac{x}{2}\right) &= \pm\sqrt{\frac{1-\cos(x)}{2}},\\
\cos\left(\frac{x}{2}\right) &= \pm\sqrt{\frac{1+\cos(x)}{2}}.
\end{aligned}$$

Product formulas:

$$\begin{aligned}
\sin x \sin y &= \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y),\\
\cos x \cos y &= \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y),\\
\sin x \cos y &= \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y).
\end{aligned}$$

Euler's formula states that

$$e^{ix} = \cos(x) + i\sin(x)$$

where $i = \sqrt{-1}$.

You can derive the sum-angle identities for sine and cosine from Euler's formula

$$\begin{aligned}
\cos(x+y) + i\sin(x+y) = e^{i(x+y)} = e^{ix}e^{iy} &= (\cos(x) + i\sin(x))(\cos(y) + i\sin(y))\\
&= \cos(x)\cos(y) - \sin(x)\sin(y) + i\cos(x)\sin(y) + i\sin(x)\cos(y)
\end{aligned}$$

by equating real and imaginary parts.

**Extra inverse trigonometry**

The following information might also be useful (although you don't need to memorize it):
The ranges of the inverse trig functions are given by:

$$\begin{aligned}
&\text{The range of arcsin is } [-\pi/2, \pi/2],\\
&\text{The range of arccos is } [0, \pi],\\
&\text{The range of arctan is } (-\pi/2, \pi/2),\\
&\text{The range of arcsec is } [0, \pi/2) \cup (\pi/2, \pi],\\
&\text{The range of arccsc is } [-\pi/2, 0) \cup (0, \pi/2].
\end{aligned}$$

It is sometimes convenient to define $\operatorname{arccot}(x)$ to have range $(0, \pi)$, and sometimes convenient to define it to have range $(-\pi/2, 0) \cup (0, \pi/2]$.

If $x > 0$, then

$$\begin{aligned}
\arcsin x + \arccos x &= \frac{\pi}{2}, & \operatorname{arcsec} x &= \arccos(1/x),\\
\operatorname{arcsec} x + \operatorname{arccsc} x &= \frac{\pi}{2}, & \operatorname{arccsc} x &= \arcsin(1/x),\\
\arctan x + \operatorname{arccot} x &= \frac{\pi}{2}, & \operatorname{arccot} x &= \arctan(1/x)
\end{aligned}$$

whenever both inverse trig functions are defined. (The first four equations are also true for $x < 0$; $\arctan(x) + \operatorname{arccot}(x) = \frac{\pi}{2}$ for all $x$ if the range of arccot is taken to be $(0, \pi)$, and $\operatorname{arccot} x = \arctan(1/x)$ for all $x$ if the range of arccot is taken to be $(-\pi/2, 0) \cup (0, \pi/2]$.)

Some limits:

$$\lim_{x\to\infty} \arctan x = \frac{\pi}{2}, \quad \lim_{x\to-\infty} \arctan x = -\frac{\pi}{2}, \quad \lim_{x\to-\infty} \operatorname{arcsec} x = \lim_{x\to\infty} \operatorname{arcsec} x = \frac{\pi}{2}.$$

We can state the exact region of validity of the formulas defining the inverse trigonometric functions:

$$\begin{aligned}
\sin(\arcsin x) &= x \text{ if } -1 \le x \le 1, & \sin(\arcsin(x)) &\text{ does not exist if } x > 1 \text{ or } x < -1,\\
\cos(\arccos x) &= x \text{ if } -1 \le x \le 1, & \cos(\arccos(x)) &\text{ does not exist if } x > 1 \text{ or } x < -1,\\
\tan(\arctan x) &= x \text{ if } -\infty < x < \infty, & &\\
\cot(\operatorname{arccot} x) &= x \text{ if } -\infty < x < \infty, & &\\
\sec(\operatorname{arcsec} x) &= x \text{ if } x \ge 1 \text{ or } x \le -1, & \sec(\operatorname{arcsec} x) &\text{ does not exist if } -1 < x < 1,\\
\csc(\operatorname{arccsc} x) &= x \text{ if } x \ge 1 \text{ or } x \le -1, & \csc(\operatorname{arccsc} x) &\text{ does not exist if } -1 < x < 1.
\end{aligned}$$

The following statements are true. (Compare to the statement "if $x^2 = y^2$ then $x = \pm y$.")

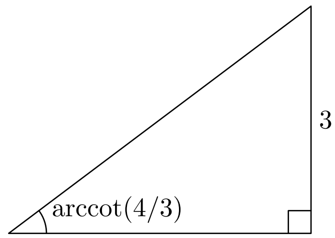
$$\begin{aligned}
&\text{If } \sin x = \sin z \text{ then } x = z + 2n\pi \text{ or } x = -z + (2n+1)\pi \text{ for some integer } n.\\
&\text{If } \cos x = \cos z \text{ then } x = \pm z + 2n\pi \text{ for some integer } n.\\
&\text{If } \tan x = \tan z \text{ then } x = z + n\pi \text{ for some integer } n.\\
&\text{If } \cot x = \cot z \text{ then } x = z + n\pi \text{ for some integer } n.\\
&\text{If } \sec x = \sec z \text{ then } x = \pm z + 2n\pi \text{ for some integer } n.\\
&\text{If } \csc x = \csc z \text{ then } x = z + 2n\pi \text{ or } x = -z + (2n+1)\pi \text{ for some integer } n.
\end{aligned}$$

Thus, if (for example) $\tan x = y$, then $x = \arctan y + n\pi$ for some integer $n$. Because the formula for tan and cot is so much simpler than the formulas for sin, cos, sec and csc, if you are solving a trigonometric equation and it is important to find *all* solutions, it is often helpful to try to rewrite the equation in terms of tangents and cotangents.

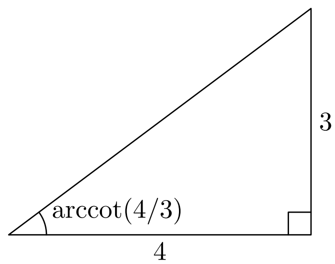
It's also good to be able to compute any trig function of any inverse trig function. For example,

$$\tan(\operatorname{arcsec} x) = \sqrt{x^2 - 1}, \quad \cos\left(\operatorname{arccot}\frac{4}{3}\right) = \frac{4}{5}.$$

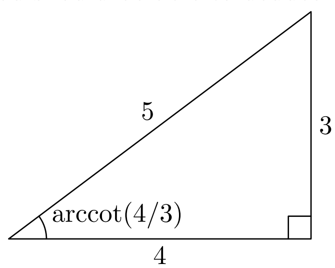
Here's how: Draw a right triangle and let one angle be $\operatorname{arccot}\frac{4}{3}$. Pick a side and let its length be any convenient number:




By knowing $\cot\left(\operatorname{arccot}\frac{4}{3}\right)$, you should be able to deduce the length of a second side:




From the Pythagorean theorem, you should be able to deduce the length of the third side:




From there $\cos\left(\operatorname{arccot}\frac{4}{3}\right) = \frac{4}{5}$ may be easily found.