Boundary-value problems for higher-order elliptic equations in non-smooth domains

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Abstract. This paper presents a survey of recent results, methods, and open problems in the theory of higher order elliptic boundary value problems on Lipschitz and more general non-smooth domains. The main topics include the maximum principle and pointwise estimates on solutions in arbitrary domains, analogues of the Wiener test governing continuity of solutions and their derivatives at a boundary point, and well-posedness of boundary value problems in domains with Lipschitz boundaries.

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1. Introduction

The last three decades have witnessed a surge of activity on boundary value problems on Lipschitz domains. The Dirichlet, Neumann, and regularity problems for the Laplacian are now well-understood for data in L^p, Sobolev, and Besov spaces. More generally, well-posedness in L^p has been established for divergence form elliptic equations with non-smooth coefficients $-\operatorname{div} A\nabla$ and, at least in the context of real symmetric matrices, the optimal conditions on A needed for solvability of the Dirichlet problem in L^p are known. We direct the interested reader to Kenig's 1994 CBMS book [Ken94] for an excellent review of these matters and to [KKPT00, KR09, Rul07, AAA$^+$11, DPP07, DR10, AAH08, AAM10, AA11, AR11, HKMP12] for recent results.

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Unfortunately, this beautiful and powerful theory has mostly been restricted to the case of second-order operators. Higher-order elliptic boundary problems, while having abundant applications in physics and engineering, have mostly been out of reach of the methods devised to study the second order case. The present survey is devoted to major recent results in this subject, new techniques, and principal open problems.

The prototypical example of a higher-order elliptic operator is the bilaplacian $\Delta^2 = \Delta(\Delta)$ or, more generally, the polyharmonic operator Δ^m, $m \geq 2$. The biharmonic problem in a domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary data consists, roughly speaking, of finding a function u such that for given f, g, h,

$$\Delta^2 u = h \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f, \quad \partial_\nu u\big|_{\partial\Omega} = g,$$

subject to the appropriate estimates on u in terms of the data. To make it precise, as usual, one needs to properly interpret restriction of solution to the boundary $u\big|_{\partial\Omega}$ and its normal derivative $\partial_\nu u\big|_{\partial\Omega}$, as well as specify the desired estimates. The biharmonic equation arises in numerous problems of structural engineering. It models the displacements of a thin plate clamped near its boundary, the stresses in an elastic body, the stream function in creeping flow of a viscous incompressible fluid, to mention just a few applications (see, e.g., [Mel03]).

The primary goal of this survey is to address the biharmonic problem and more general higher order partial differential equations in domains with non-smooth boundaries, specifically, in the class of Lipschitz domains. However, the analysis of such delicate questions as well-posedness in Lipschitz domains requires preliminary understanding of fundamental properties of the solutions, such as boundedness, continuity, and regularity near a boundary point. For the Laplacian, these properties of solutions in general domains are described by the maximum principle and by the 1924 Wiener criterion; for the bilaplacian, they turn out to be highly nontrivial and partially open to date. For the purposes of the introduction, let us mention just a few highlights and outline the paper.

In Section 3, we discuss the maximum principle for higher-order elliptic equations. Loosely, one expects that for a solution u to the equation $Lu = 0$ in Ω, where L is a differential operator of order $2m$, there holds

$$\max_{|\alpha| \leq m-1} \|\partial^\alpha u\|_{L^\infty(\Omega)} \leq C \max_{|\beta| \leq m-1} \|\partial^\beta u\|_{L^\infty(\partial\Omega)},$$

with the usual convention that the zeroth-order derivative of u is simply u itself. For the Laplacian ($m = 1$), this formula is a slightly weakened formulation of the maximum principle. In striking contrast with the case of harmonic functions, the maximum principle for an elliptic operator of order $2m \geq 4$ may fail, even in a Lipschitz domain. To be precise, in general, the derivatives of order $(m-1)$ of a solution to an elliptic equation of order $2m$ need not be bounded. We discuss relevant counterexamples, known positive results, as well as a more general question of pointwise bounds on solutions and their derivatives in arbitrary domains, e.g., whether u (rather than $\nabla^{m-1} u$) is necessarily bounded in Ω.

Section 4 is devoted to continuity of solutions to higher-order equations and their derivatives near the boundary of the domain. Specifically, if for some operator the aforementioned boundedness of the $(m-1)$-st derivatives holds, one next would need to identify conditions assuring their continuity near the boundary. For instance, in the particular case of the bilaplacian, the gradient of a solution is bounded in an arbitrary three-dimensional domain, and one would like to study the continuity of the gradient near a boundary point. As is well known, for second-order equations, necessary and sufficient conditions for continuity of the solutions have been provided by the celebrated Wiener criterion. Analogues of the Wiener test for higher order PDEs are known only for some operators, and in a restricted range of dimensions. We shall discuss these results, testing conditions, and the associated capacities, as well as similarities and differences with their second-order antecedents.

Finally, Sections 5 and 6 are devoted to boundary-value problems in Lipschitz domains. The simplest example is the Dirichlet problem for the bilaplacian,

$$\Delta^2 u = 0 \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = f \in W_1^p(\partial\Omega), \quad \partial_\nu u\big|_{\partial\Omega} = g \in L^p(\partial\Omega), \tag{1.1}$$

in which case the expected sharp estimate on the solution is

$$\|N(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|\nabla_\tau f\|_{L^p(\partial\Omega)} + C\|g\|_{L^p(\partial\Omega)}, \tag{1.2}$$

where N denotes the non-tangential maximal function and $W_1^p(\partial\Omega)$ is the Sobolev space of functions with one tangential derivative in L^p (cf. Section 2 for precise definitions). In Sections 5.1–5.6 we discuss (1.1) and (1.2), and more general higher-order homogeneous Dirichlet and regularity boundary value problems with constant coefficients, with boundary data in L^p. Section 5.7 describes the specific case of convex domains. The Neumann problem for the bilaplacian is addressed in Section 5.8. In Section 5.9, we discuss inhomogeneous boundary value problems with data in Besov and Sobolev spaces, which, in a sense, are intermediate between those with Dirichlet and regularity data. Finally, in Section 6, we discuss boundary-value problems with variable coefficients.

The sharp range of p, such that the aforementioned biharmonic (or any other higher order) Dirichlet problem with data in L^p is well-posed in Lipschitz domains, is not yet known in high dimensions. However, over the recent years numerous advances have been made in this direction and some new methods have emerged. For instance, *several* different layer potential constructions have proven to be useful (again, note the difference with the second order case when the relevant layer potentials are essentially uniquely defined by the boundary problem), as well as recently discovered equivalence of well-posedness to certain reverse Hölder estimates on the non-tangential maximal function. It is interesting to point out that the main local estimates which played a role in recent well-posedness results actually come from the techniques developed in connection with the Wiener test discussed above. Thus, in the higher order case the two issues are intimately intertwined; this was one of the reasons for the particular choice of topics in the present survey.

The Neumann problem and the variable coefficient case are even more puzzling. The details will be presented in the body of the paper. Here, let us just point out that in both cases even the proper statement of the "natural" boundary problem presents a challenge. For instance, in the higher order case the choice of Neumann data is not unique. Depending on peculiarities of the Neumann operator, one can be led to well-posed and ill-posed problems even for the bilaplacian, and more general operators give rise to new issues related to the coercivity of the underlying form. However, despite the aforementioned challenges, the first well-posedness results have recently been obtained and will be discussed below.

To conclude this introduction, we refer the reader to the excellent expository paper [Maz99b] by Vladimir Maz'ya on the topic of the Wiener criterion and pointwise estimates. This paper largely inspired the corresponding sections of the present manuscript, and its exposition of the related historical material extends and complements that in Sections 3 and 4. Our main goal here, however, was to discuss the most recent achievements (some of which appeared after the aforementioned survey was written) and their role in the well-posedness results on Lipschitz domains which constitute the main topic of the present review. We also would like to mention that this paper does not touch upon the methods and results of the part of elliptic theory studying the behavior of solutions in the domains with isolated singularities, conical points, cuspidal points, etc. Here, we have intentionally concentrated on the case of Lipschitz domains, which can display accumulating singularities—a feature drastically affecting both the available techniques and the actual properties of solutions.

2. Definitions

As we pointed out in the introduction, the prototypical higher-order elliptic equation is the biharmonic equation $\Delta^2 u = 0$, or, more generally, the polyharmonic equation $\Delta^m u = 0$ for some integer $m \geq 2$. It naturally arises in numerous applications in physics and in engineering, and in mathematics it is a basic model for a higher-order partial differential equation. These operators may be generalized to constant-coefficient differential operators of order $2m$, or to variable-coefficient operators in either divergence or nondivergence form.

Let us discuss the details. To start, a general *constant coefficient* elliptic operator is defined as follows.

Definition 2.1. Let L be an operator acting on functions $u : \mathbb{R}^n \mapsto \mathbb{C}^\ell$. Suppose that we may write

$$(Lu)_j = \sum_{k=1}^{\ell} \sum_{|\alpha|=|\beta|=m} \partial^\alpha a^{jk}_{\alpha\beta} \partial^\beta u_k \tag{2.2}$$

for some coefficients $a^{jk}_{\alpha\beta}$ defined for all $1 \leq j, k \leq \ell$ and all multiindices α, β of length n with $|\alpha| = |\beta| = m$. Then we say that L is a *differential operator of order* $2m$.

Suppose the coefficients $a^{jk}_{\alpha\beta}$ are constant and satisfy the Legendre-Hadamard ellipticity condition

$$\operatorname{Re}\sum_{j,k=1}^{\ell}\sum_{|\alpha|=|\beta|=m} a^{jk}_{\alpha\beta}\xi^\alpha\xi^\beta\zeta_j\bar\zeta_k \geq \lambda|\xi|^{2m}|\zeta|^2 \tag{2.3}$$

for all $\xi\in\mathbb{R}^n$ and all $\zeta\in\mathbb{C}^\ell$, where $\lambda>0$ is a real constant. Then we say that L is an *elliptic operator of order* $2m$.

If $\ell=1$ we say that L is a *scalar operator* and refer to the equation $Lu=0$ as an elliptic equation; if $\ell>1$ we refer to $Lu=0$ as an elliptic system. If $a^{jk}=a^{kj}$, then we say the operator L is *symmetric*. If $a^{jk}_{\alpha\beta}$ is real for all α, β, j, and k, we say that L has real coefficients.

Here if α is a multiindex of length n, then $\partial^\alpha=\partial^{\alpha_1}_{x_1}\dots\partial^{\alpha_n}_{x_n}$.

Now let us discuss the case of variable coefficients. A divergence-form higher-order elliptic operator is given by

$$(Lu)_j(X)=\sum_{k=1}^{\ell}\sum_{|\alpha|=|\beta|=m}\partial^\alpha(a^{jk}_{\alpha\beta}(X)\partial^\beta u_k(X)). \tag{2.4}$$

This form affords a notion of weak solution; we say that $Lu=h$ weakly if

$$\sum_{j=1}^{\ell}\int_\Omega \varphi_j\,h_j=\sum_{|\alpha|=|\beta|=m}\sum_{j,k=1}^{\ell}(-1)^m\int_\Omega \partial^\alpha\varphi_j\,a^{jk}_{\alpha\beta}\,\partial^\beta u_k \tag{2.5}$$

for any function $\varphi:\Omega\mapsto\mathbb{C}^\ell$ smooth and compactly supported.

If the coefficients $a^{jk}_{\alpha\beta}:\mathbb{R}^n\to\mathbb{C}$ are sufficiently smooth, we may rewrite (2.4) in nondivergence form

$$(Lu)_j(X)=\sum_{k=1}^{\ell}\sum_{|\alpha|\leq 2m}a^{jk}_\alpha(X)\partial^\alpha u_k(X). \tag{2.6}$$

This form is particularly convenient when we allow equations with lower-order terms (note their appearance in (2.6)).

A simple criterion for ellipticity of the operators L of (2.6) is the condition that (2.3) holds with $a^{jk}_{\alpha\beta}$ replaced by $a^{jk}_\alpha(X)$ for any $X\in\mathbb{R}^n$, that is, that

$$\operatorname{Re}\sum_{j,k=1}^{\ell}\sum_{|\alpha|=2m}a^{jk}_\alpha(X)\xi^\alpha\zeta_j\bar\zeta_k\geq\lambda|\xi|^{2m}|\zeta|^2 \tag{2.7}$$

for any fixed $X\in\mathbb{R}^n$ and for all $\xi\in\mathbb{R}^n$, $\zeta\in\mathbb{C}^2$. This means in particular that ellipticity is only a property of the highest-order terms of (2.6); the value of a^{jk}_α, for $|\alpha|<m$, is not considered.

For divergence-form operators, some known results use a weaker notion of ellipticity, namely that $\langle\varphi,L\varphi\rangle\geq\lambda\|\nabla^m\varphi\|^2_{L^2}$ for all smooth compactly supported functions φ; this notion is written out in full in Formula (6.5) below.

Finally, let us mention that throughout we let C and ε denote positive constants whose value may change from line to line. We let $\fint$ denote the average integral, that is, $\fint_E f\,d\mu = \frac{1}{\mu(E)}\int_E f\,d\mu$. The only measures we will consider are the Lebesgue measure dX (on $\mathbb{R}^n$ or on domains in $\mathbb{R}^n$) or the surface measure $d\sigma$ (on the boundaries of domains).

3. The maximum principle and pointwise estimates on solutions

The maximum principle for harmonic functions is one of the fundamental results in the theory of elliptic equations. It holds in arbitrary domains and guarantees that every solution to the Dirichlet problem for the Laplace equation, with bounded data, is bounded. Moreover, it remains valid for all second-order divergence-form elliptic equations with real coefficients.

In the case of equations of higher order, the maximum principle has been established only in relatively nice domains. It was proven to hold for operators with smooth coefficients in smooth domains of dimension two in [Mir48] and [Mir58], and of arbitrary dimension in [Agm60]. In the early 1990s, it was extended to three-dimensional domains diffeomorphic to a polyhedron ([KMR01, MR91]) or having a Lipschitz boundary ([PV93, PV95b]). However, in general domains, no direct analog of the maximum principle exists (see Problem 4.3, p. 275, in Nečas's book [Neč67]). The increase of the order leads to the failure of the methods which work for second order equations, and the properties of the solutions themselves become more involved.

To be more specific, the following theorem was proved by Agmon.

Theorem 3.1 ([Agm60, Theorem 1])**.** *Let $m\geq 1$ be an integer. Suppose that Ω is domain with C^{2m} boundary. Let*

$$L=\sum_{|\alpha|\leq 2m} a_\alpha(X)\partial^\alpha$$

be a scalar operator of order $2m$, where $a_\alpha\in C^{|\alpha|}(\overline{\Omega})$. Suppose that L is elliptic in the sense of (2.7)*. Suppose further that solutions to the Dirichlet problem for L are unique.*

Then, for every $u\in C^{m-1}(\overline{\Omega})\cap C^{2m}(\Omega)$ that satisfies $Lu=0$ in Ω, we have

$$\max_{|\alpha|\leq m-1}\|\partial^\alpha u\|_{L^\infty(\Omega)}\leq C\max_{|\beta|\leq m-1}\|\partial^\beta u\|_{L^\infty(\partial\Omega)}. \tag{3.2}$$

We remark that the requirement that the Dirichlet problem have unique solutions is not automatically satisfied for elliptic equations with lower-order terms; for example, if λ is an eigenvalue of the Laplacian then solutions to the Dirichlet problem for $\Delta u-\lambda u$ are not unique.

Equation (3.2) is called the *Agmon-Miranda maximum principle.* In [Šul75], Šul'ce generalized this to systems of the form (2.6), elliptic in the sense of (2.7), that satisfy a positivity condition (strong enough to imply Agmon's requirement that solutions to the Dirichlet problem be unique).

Thus the Agmon-Miranda maximum principle holds for sufficiently smooth operators and domains. Moreover, for some operators, the maximum principle is valid even in domains with Lipschitz boundary, provided the dimension is small enough. We postpone a more detailed discussion of the Lipschitz case to Section 5.6; here we simply state the main results. In [PV93] and [PV95b], Pipher and Verchota showed that the maximum principle holds for the biharmonic operator Δ^2, and more generally for the polyharmonic operator Δ^m, in bounded Lipschitz domains in $\mathbb{R}^2$ or $\mathbb{R}^3$. In [Ver96, Section 8], Verchota extended this to symmetric, strongly elliptic systems with real constant coefficients in three-dimensional Lipschitz domains.

For Laplace's equation and more general second order elliptic operators, the maximum principle continues to hold in *arbitrary* bounded domains. In contrast, the maximum principle for higher-order operators in rough domains generally *fails*.

In [MNP83], Maz'ya, Nazarov and Plamenevskii studied the Dirichlet problem (with zero boundary data) for constant-coefficient elliptic systems in cones. Counterexamples to (3.2) for systems of order $2m$ in dimension $n \geq 2m+1$ immediately follow from their results. (See [MNP83, Formulas (1.3), (1.18) and (1.28)].) Furthermore, Pipher and Verchota constructed counterexamples to (3.2) for the biharmonic operator Δ^2 in dimension $n = 4$ in [PV92, Section 10], and for the polyharmonic equation $\Delta^m u = 0$ in dimension n, $4 \leq n < 2m+1$, in [PV95b, Theorem 2.1]. Independently Maz'ya and Rossmann showed that (3.2) fails in the exterior of a sufficiently thin cone in dimension n, $n \geq 4$, where L is any constant-coefficient elliptic scalar operator of order $2m \geq 4$ (without lower-order terms). See [MR92, Theorem 8 and Remark 3].

Moreover, with the exception of [MR92, Theorem 8], the aforementioned counterexamples actually provide a stronger negative result than simply the failure of the maximum principle: they show that the left-hand side of (3.2) may be infinite even if the data of the elliptic problem is as nice as possible, that is, smooth and compactly supported.

The counterexamples, however, pertain to high dimensions and do not indicate, e.g., the behavior of the derivatives of order $(m-1)$ of a solution to an elliptic equation of order $2m$ in the lower-dimensional case.

Recently in [MM09b], the second author of the present paper together with Maz'ya have considered this question for the inhomogeneous Dirichlet problem

$$\Delta^2 u = h \text{ in } \Omega, \quad u \in \mathring{W}_2^2(\Omega). \tag{3.3}$$

Here Ω is a bounded domain in $\mathbb{R}^3$ or $\mathbb{R}^2$, the Sobolev space $\mathring{W}_2^2(\Omega)$ is a completion of $C_0^\infty(\Omega)$ in the norm $\|u\|_{\mathring{W}_2^2(\Omega)} = \|\nabla^2 u\|_{L^2(\Omega)}$, and h is a reasonably nice function (e.g., $C_0^\infty(\Omega)$). We remark that if Ω is an arbitrary domain, defining $\nabla u\big|_{\partial\Omega}$ is a delicate matter, and so considering the Dirichlet problem with homogeneous boundary data is somewhat more appropriate.

Motivated by (3.2), the authors showed that if u solves (3.3), then $\nabla u \in L^\infty(\Omega)$, under no restrictions on Ω other than its dimension. Moreover, they proved the following bounds on the Green's function.

Let Ω be an arbitrary bounded domain in $\mathbb{R}^3$ and let G be the Green's function for the biharmonic equation. Then

$$|\nabla_X \nabla_Y G(X,Y)| \leq C|X-Y|^{-1}, \qquad X, Y \in \Omega, \tag{3.4}$$

$$|\nabla_X G(X,Y)| \leq C \quad \text{and} \quad |\nabla_Y G(X,Y)| \leq C, \qquad X, Y \in \Omega, \tag{3.5}$$

where C is an absolute constant.

The boundedness of the gradient of a solution to the biharmonic equation in a three-dimensional domain is a sharp property in the sense that the function u satisfying (3.3) generally does not exhibit more regularity. For example, let Ω be the three-dimensional punctured unit ball $B(0,1)\setminus\{0\}$, where $B(X,r) = \{Y \in \mathbb{R}^3 : |X-Y| < r\}$, and consider a function $\eta \in C_0^\infty(B(0,1/2))$ such that $\eta = 1$ on $B(0,1/4)$. Let

$$u(X) := \eta(X)|X|, \qquad X \in B_1 \setminus \{0\}. \tag{3.6}$$

Obviously, $u \in \mathring{W}_2^2(\Omega)$ and $\Delta^2 u \in C_0^\infty(\Omega)$. While ∇u is bounded, it is not continuous at the origin. Therefore, the *continuity* of the gradient *does not hold* in general and must depend on some delicate properties of the domain. These questions will be addressed in Section 4 in the framework of the Wiener criterion.

In the absence of boundedness of the gradient ∇u of a harmonic function, or the higher-order derivatives $\nabla^{m-1} u$ of a solution to a higher-order equation, we may instead consider boundedness of a solution itself. Let

$$\Delta^m u = h \text{ in } \Omega, \quad u \in \mathring{W}_m^2(\Omega), \tag{3.7}$$

and $h \in C_0^\infty(\Omega)$. Observe that if $\Omega \subset \mathbb{R}^n$ for $n \leq 2m-1$, then every $u \in \mathring{W}_m^2(\Omega)$ is Hölder continuous on $\overline{\Omega}$ and so must necessarily be bounded.

In [Maz99b, Section 10], Maz'ya showed that the Green's function $G_m(X,Y)$ for Δ^m in an arbitrary bounded domain $\Omega \subset \mathbb{R}^n$ satisfies

$$|G_m(X,Y)| \leq C(2m) \log \frac{C \operatorname{diam}\Omega}{\min(|X-Y|, \operatorname{dist}(Y, \partial\Omega))} \tag{3.8}$$

in dimension $n = 2m$, and satisfies

$$|G_m(X,Y)| \leq \frac{C(n)}{|X-Y|^{n-2m}} \tag{3.9}$$

if $n = 2m+1$ or $n = 2m+2$. If $m = 2$, then (3.9) also holds in dimension $n = 7 = 2m+3$ (cf. [Maz79]). Whether (3.9) holds in dimension $n \geq 8$ (for $m = 2$) or $n \geq 2m+3$ (for $m > 2$) is an open problem; see [Maz99b, Problem 2].

If (3.9) holds, then solutions to (3.7) satisfy

$$\|u\|_{L^\infty(\Omega)} \leq C(m,n,p) \operatorname{diam}(\Omega)^{2m-n/p} \|h\|_{L^p(\partial\Omega)}$$

provided $p > n/2m$ (see, e.g., [Maz99b, Section 2]).

Thus, if $\Omega \subset \mathbb{R}^n$ is bounded for $n \leq 2m+2$, $n \neq 2m$, and if u satisfies (3.7) for a reasonably nice function h, then $u \in L^\infty(\Omega)$. This result also holds if $\Omega \subset \mathbb{R}^7$ and $m = 2$.

As in the case of the Green function estimates, if $\Omega \subset \mathbb{R}^n$ is bounded and $n \geq 2m+3$, or if $m = 2$ and $n \geq 8$, then the question of whether solutions u to (3.7) are bounded is open. In particular, it is not known whether solutions u to

$$\Delta^2 u = h \text{ in } \Omega, \quad u \in \mathring{W}_2^2(\Omega)$$

are bounded if $\Omega \subset \mathbb{R}^n$ for $n \geq 8$. However, there exists another fourth-order operator whose solutions are *not* bounded in higher-dimensional domains. In [MN86], Maz'ya and Nazarov showed that if $n \geq 8$ and if $a > 0$ is large enough, then there exists an open cone $K \subset \mathbb{R}^n$ and a function $h \in C_0^\infty(\overline{K} \setminus \{0\})$ such that the solution u to

$$\Delta^2 u + a\partial_n^4 u = h \text{ in } K, \quad u \in \mathring{W}_2^2(K) \tag{3.10}$$

is unbounded near the origin.

To conclude our discussion of Green's functions, we mention two results from [MM11]; these results are restricted to relatively well-behaved domains. In [MM11], D. Mitrea and I. Mitrea showed that, if Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, and G denotes the Green's function for the bilaplacian Δ^2, then the estimates

$$\nabla^2 G(X, \cdot\,) \in L^3(\Omega), \quad \operatorname{dist}(\,\cdot\,, \partial\Omega)^{-\alpha} \nabla G(X, \cdot\,) \in L^{3/\alpha,\infty}$$

hold, uniformly in $X \in \Omega$, for all $0 < \alpha \leq 1$.

Moreover, they considered more general elliptic systems. Suppose that L is an arbitrary elliptic operator of order $2m$ with constant coefficients, as defined by Definition 2.1, and that G denotes the Green's function for L. Suppose that $\Omega \subset \mathbb{R}^n$, for $n > m$, is a Lipschitz domain, and that the unit outward normal ν to Ω lies in the Sarason space $VMO(\partial\Omega)$ of functions of vanishing mean oscillations on $\partial\Omega$. Then the estimates

$$\begin{aligned} \nabla^m G(X, \cdot\,) &\in L^{\frac{n}{n-m},\infty}(\Omega), \\ \operatorname{dist}(\,\cdot\,, \partial\Omega)^{-\alpha} \nabla^{m-1} G(X, \cdot\,) &\in L^{\frac{n}{n-m-1+\alpha},\infty}(\Omega) \end{aligned} \tag{3.11}$$

hold, uniformly in $X \in \Omega$, for any $0 \leq \alpha \leq 1$.

4. The Wiener test

In this section, we discuss conditions that ensure that solutions (or appropriate gradients of solutions) are continuous up to the boundary. These conditions parallel the famous result of Wiener, who in 1924 formulated a criterion that ensured continuity of *harmonic* functions at boundary points [Wie24]. Wiener's criterion has been extended to a variety of second-order elliptic and parabolic equations ([LSW63, FJK82, FGL89, DMM86, MZ97, AH96, TW02, Lab02, EG82]; see also the review papers [Maz97, Ada97]). However, as with the maximum principle, extending this criterion to higher-order elliptic equations is a subtle matter, and many open questions remain.

We begin by stating the classical Wiener criterion for the Laplacian. If $\Omega\subset\mathbb{R}^n$ is a domain and $Q\in\partial\Omega$, then Q is called *regular* for the Laplacian if every solution u to

$$\Delta u = h \text{ in } \Omega, \quad u\in \mathring{W}_1^2(\Omega)$$

for $h\in C_0^\infty(\Omega)$ satisfies $\lim_{X\to Q} u(X)=0$. According to Wiener's theorem [Wie24], the boundary point $Q\in\partial\Omega$ is regular if and only if the equation

$$\int_0^1 \operatorname{cap}_2(\overline{B(Q,s)}\setminus\Omega)s^{1-n}\,ds=\infty \tag{4.1}$$

holds, where

$$\operatorname{cap}_2(K)=\inf\Big\{\|u\|^2_{L^2(\mathbb{R}^n)}+\|\nabla u\|^2_{L^2(\mathbb{R}^n)} : u\in C_0^\infty(\mathbb{R}^n),\ u\geq 1 \text{ on } K\Big\}.$$

For example, suppose Ω satisfies the exterior cone condition at Q. That is, suppose there is some open cone K with vertex at Q and some $\varepsilon>0$ such that $K\cap B(Q,\varepsilon)\subset\Omega^C$. It is elementary to show that $\operatorname{cap}_2(\overline{B(Q,s)}\setminus\Omega)\geq C(K)s^{n-2}$ for all $0<s<\varepsilon$, and so (4.1) holds and Q is regular. Regularity of such points was known prior to Wiener (see [Poi90], [Zar09], and [Leb13]) and provided inspiration for the formulation of the Wiener test.

By [LSW63], if $L=-\operatorname{div} A\nabla$ is a second-order divergence-form operator, where the matrix $A(X)$ is bounded, measurable, real, symmetric and elliptic, then $Q\in\partial\Omega$ is regular for L if and only if Q and Ω satisfy (4.1). In other words, $Q\in\partial\Omega$ is regular for the Laplacian if and only if it is regular for all such operators. Similar results hold for some other classes of second-order equations; see, for example, [FJK82], [DMM86], or [EG82].

One would like to consider the Wiener criterion for higher-order elliptic equations, and that immediately gives rise to the question of natural generalization of the concept of a regular point. The Wiener criterion for the second order PDEs ensures, in particular, that weak $\mathring{W}_1^2$ solutions are *classical*. That is, the solution approaches its boundary values in the pointwise sense (continuously). From that point of view, one would extend the concept of regularity of a boundary point as continuity of derivatives of order $m-1$ of the solution to an equation of order $2m$ up to the boundary. On the other hand, as we discussed in the previous section, even the boundedness of solutions cannot be guaranteed in general, and thus, in lower dimensions the study of the continuity up to the boundary for solutions themselves is also very natural. We begin with the latter question, as it is better understood.

Let us first define a regular point for an arbitrary differential operator L of order $2m$ analogously to the case of the Laplacian, by requiring that every solution u to

$$Lu=h \text{ in } \Omega, \quad u\in\mathring{W}_m^2(\Omega) \tag{4.2}$$

for $h\in C_0^\infty(\Omega)$ satisfy $\lim_{X\to Q}u(X)=0$. Note that by the Sobolev embedding theorem, if $\Omega\subset\mathbb{R}^n$ for $n\leq 2m-1$, then every $u\in\mathring{W}_m^2(\Omega)$ is Hölder continuous on $\overline{\Omega}$ and so satisfies $\lim_{X\to Q} u(X) = 0$ at every point $Q \in \partial\Omega$. Thus, we are only interested in continuity of the solutions at the boundary when $n \geq 2m$.

In this context, the appropriate concept of capacity is the potential-theoretic Riesz capacity of order $2m$, given by

$$\operatorname{cap}_{2m}(K) = \inf\Big\{ \sum_{0\leq|\alpha|\leq m} \|\partial^\alpha u\|^2_{L^2(\mathbb{R}^n)} : u \in C_0^\infty(\mathbb{R}^n),\ u \geq 1 \text{ on } K \Big\}. \tag{4.3}$$

The following is known. If $m \geq 3$, and if $\Omega \subset \mathbb{R}^n$ for $n = 2m$, $2m+1$ or $2m+2$, or if $m = 2$ and $n = 4$, 5, 6 or 7, then $Q \in \partial\Omega$ is regular for Δ^m if and only if

$$\int_0^1 \operatorname{cap}_{2m}(\overline{B(Q,s)} \setminus \Omega) s^{2m-n-1}\, ds = \infty. \tag{4.4}$$

The biharmonic case was treated in [Maz77] and [Maz79], and the polyharmonic case for $m \geq 3$ in [MD83] and [Maz99a].

Let us briefly discuss the method of the proof in order to explain the restrictions on the dimension. Let L be an arbitrary elliptic operator, and let F be the fundamental solution for L in $\mathbb{R}^n$ with pole at Q. We say that L is positive with weight F if, for all $u \in C_0^\infty(\mathbb{R}^n \setminus \{Q\})$, we have that

$$\int_{\mathbb{R}^n} Lu(X)\cdot u(X)\, F(X)\, dX \geq c \sum_{k=1}^m \int_{\mathbb{R}^n} \big|\nabla^k u(X)\big|^2 |X|^{2k-n}\, dX. \tag{4.5}$$

The biharmonic operator is positive with weight F in dimension n if $4 \leq n \leq 7$, and the polyharmonic operator Δ^m, $m \geq 3$, is positive with weight F in dimension $2m \leq n \leq 2m+2$. (The Laplacian Δ is positive with weight F in any dimension.) The biharmonic operator Δ^2 is not positive with weight F in dimensions $n \geq 8$, and Δ^m is not positive with weight F in dimension $n \geq 2m+3$. See [Maz99a, Propositions 1 and 2].

The proof of the Wiener criterion for the polyharmonic operator required positivity with weight F. In fact, it turns out that positivity with weight F suffices to provide a Wiener criterion for an *arbitrary* scalar elliptic operator with constant coefficients.

Theorem 4.6 ([Maz02, Theorems 1 and 2]). *Suppose $\Omega \subset \mathbb{R}^n$ and that L is a scalar elliptic operator of order $2m$ with constant real coefficients, as defined by Definition 2.1.*

If $n = 2m$, then $Q \in \partial\Omega$ is regular for L if and only if (4.4) *holds.*

If $n \geq 2m+1$, and if the condition (4.5) *holds, then again $Q \in \partial\Omega$ is regular for L if and only if* (4.4) *holds.*

This theorem is also valid for certain variable-coefficient operators in divergence form; see the remark at the end of [Maz99a, Section 5].

Similar results have been proven for some second-order elliptic *systems*. In particular, for the Lamé system $Lu = \Delta u + \alpha \operatorname{grad}\operatorname{div} u$, $\alpha > -1$, positivity with weight F and Wiener criterion have been established for a range of α close to zero, that is, when the underlying operator is close to the Laplacian ([LM10]). It was also shown that positivity with weight F may in general fail for the Lamé system. Since the present review is restricted to the higher order operators, we shall not elaborate on this point and instead refer the reader to [LM10] for more detailed discussion.

In the absense of the positivity condition (4.5), the situation is much more involved. Let us point out first that the condition (4.5) is *not* necessary for regularity of a boundary point, that is, the continuity of the solutions. There exist fourth-order elliptic operators that are not positive with weight F whose solutions exhibit nice behavior near the boundary; there exist other such operators whose solutions exhibit very bad behavior near the boundary.

Specifically, recall that (4.5) fails for $L = \Delta^2$ in dimension $n \geq 8$. Nonetheless, solutions to $\Delta^2 u = h$ are often well-behaved near the boundary. By [MP81], the vertex of a cone is regular for the bilaplacian in any dimension. Furthermore, if the capacity condition (4.4) holds with $m = 2$, then by [Maz02, Section 10], any solution u to

$$\Delta^2 u = h \text{ in } \Omega, \quad u \in \mathring{W}_2^2(\Omega)$$

for $h \in C_0^\infty(\Omega)$ satisfies $\lim_{X \to Q} u(X) = 0$ provided the limit is taken along a *nontangential* direction.

Conversely, if $n \geq 8$ and $L = \Delta^2 + a\partial_n^4$, then by [MN86], there exists a cone K and a function $h \in C_0^\infty(\overline{K} \setminus \{0\})$ such that the solution u to (3.10) is not only discontinuous but *unbounded* near the vertex of the cone. We remark that a careful examination of the proof in [MN86] implies that solutions to (3.10) are unbounded even along some nontangential directions.

Thus, conical points in dimension eight are regular for the bilaplacian and irregular for the operator $\Delta^2 + a\partial_n^4$. Hence, a relevant Wiener condition *must* use different capacities for these two operators. This is a striking contrast with the second-order case, where the same capacity condition implies regularity for all divergence-form operators, even with variable coefficients.

This concludes the discussion of regularity in terms of continuity of the solution. We now turn to regularity in terms of continuity of the $(m-1)$-st derivatives. Unfortunately, much less is known in this case. The first such result has recently appeared in [MM09a]. It pertains to the biharmonic equation in dimension three.

We say that $Q \in \partial\Omega$ is 1-regular for the operator Δ^2 if every solution u to

$$\Delta^2 u = h \text{ in } \Omega, \quad u \in \mathring{W}_2^2(\Omega) \tag{4.7}$$

for $h \in C_0^\infty(\Omega)$ satisfies $\lim_{X \to Q} \nabla u(X) = 0$. In [MM09a] the second author of this paper and Maz'ya proved that in a three-dimensional domain the following holds. If

$$\int_0^c \inf_{P \in \Pi_1} \operatorname{cap}_P(\overline{B(0, as) \setminus B(0, s)} \setminus \Omega)\, ds = \infty \tag{4.8}$$

for some $a \geq 4$ and some $c > 0$, then 0 is 1-regular. Conversely, if $0 \in \partial\Omega$ is 1-regular for Δ^2 then for every $c > 0$ and every $a \geq 8$,

$$\inf_{P \in \Pi_1} \int_0^c \operatorname{cap}_P(\overline{B(0, as) \setminus B(0, s)} \setminus \Omega)\, ds = \infty. \tag{4.9}$$

Here

$$\mathrm{cap}_P(K) = \inf\{\|\Delta u\|^2_{L^2(\mathbb{R}^3)} : u \in \mathring{W}^2_2(\mathbb{R}^3 \setminus \{0\}),\ u = P \text{ in a neighborhood of } K\}$$

and Π_1 is the space of functions $P(X)$ of the form $P(X) = b_0 + b_1 X_1 + b_2 X_2 + b_3 X_3$ with coefficients $b_k \in \mathbb{R}$ that satisfy $\sqrt{b_0^2 + b_1^2 + b_2^2 + b_3^2} = 1$.

Note that the notion of capacity cap_P is quite different from the classical analogues, and even from the Riesz capacity used in the context of the higher order elliptic operators before (cf. (4.3)). Its properties, as well as properties of 1-regular and 1-irregular points, can be different from classical analogous as well. For instance, for some domains 1-irregularity turns out to be unstable under affine transformations of coordinates.

The slight discrepancy between the sufficient condition (4.8) and the necessary condition (4.9) is needed, in the sense that (4.8) is not always necessary for 1-regularity. However, in an important particular case, there exists a single simpler condition for 1-regularity. To be precise, let $\Omega \subset \mathbb{R}^3$ be a domain whose boundary is the graph of a function φ, and let ω be its modulus of continuity. If

$$\int_0^1 \frac{t\,dt}{\omega^2(t)} = \infty, \tag{4.10}$$

then every solution to the biharmonic equation (4.7) satisfies $\nabla u \in C(\overline{\Omega})$. Conversely, for every ω such that the integral in (4.10) is convergent, there exists a $C^{0,\omega}$ domain and a solution u of the biharmonic equation such that $\nabla u \notin C(\overline{\Omega})$. In particular, as expected, the gradient of a solution to the biharmonic equation is always bounded in Lipschitz domains and is not necessarily bounded in a Hölder domain. Moreover, one can deduce from (4.10) that the gradient of a solution is always bounded, e.g., in a domain with $\omega(t) \approx t \log^{1/2} t$, which is not Lipschitz, and might fail to be bounded in a domain with $\omega(t) \approx t \log t$. More properties of the new capacity and examples can be found in [MM09a].

5. Boundary value problems in Lipschitz domains for elliptic operators with constant coefficients

The maximum principle (3.2) provides estimates on solutions whose boundary data lies in L^∞. Recall that for second-order partial differential equations with real coefficients, the maximum principle is valid in arbitrary bounded domains. The corresponding sharp estimates for boundary data in L^p, $1 < p < \infty$, are much more delicate. They are *not* valid in arbitrary domains, even for harmonic functions, and they depend in a delicate way on the geometry of the boundary. At present, boundary-value problems for the Laplacian and for general real symmetric elliptic operators of the second order are fairly well understood on Lipschitz domains. See, in particular, [Ken94].

We consider biharmonic functions and more general higher-order elliptic equations. The question of estimates on biharmonic functions with data in L^p

was raised by Rivière in the 1970s ([CFS79]), and later Kenig redirected it towards Lipschitz domains in [Ken90, Ken94]. The sharp range of well-posedness in L^p, even for biharmonic functions, remains an open problem (see [Ken94, Problem 3.2.30]). In this section we shall review the current state of the art in the subject, the main techniques that have been successfully implemented, and their limitations in the higher-order case.

Most of the results we will discuss are valid in Lipschitz domains, defined as follows.

Definition 5.1. A domain $\Omega \subset \mathbb{R}^n$ is called a *Lipschitz domain* if, for every $Q \in \partial\Omega$, there is a number $r > 0$, a Lipschitz function $\varphi : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ with $\|\nabla\varphi\|_{L^\infty} \leq M$, and a rectangular coordinate system for $\mathbb{R}^n$ such that

$$B(Q,r) \cap \Omega = \{(x,s) : x \in \mathbb{R}^{n-1},\ s \in \mathbb{R},\ |(x,s) - Q)| < r,\ \text{and}\ s > \varphi(x)\}.$$

If we may take the functions φ to be C^k (that is, to possess k continuous derivatives), we say that Ω is a C^k *domain.*

The outward normal vector to Ω will be denoted ν. The surface measure will be denoted σ, and the tangential derivative along $\partial\Omega$ will be denoted ∇_τ.

In this paper, we will assume that all domains under consideration have connected boundary. Furthermore, if $\partial\Omega$ is unbounded, we assume that there is a single Lipschitz function φ and coordinate system that satisfies the conditions given above; that is, we assume that Ω is the domain above (in some coordinate system) the graph of a Lipschitz function.

In order to properly state boundary-value problems on Lipschitz domains, we will need the notions of non-tangential convergence and non-tangential maximal function.

In this and subsequent sections we say that $u\big|_{\partial\Omega} = f$ if f is the *nontangential limit* of u, that is, if

$$\lim_{X \to Q,\ X \in \Gamma(Q)} u(X) = f(Q)$$

for almost every $(d\sigma)$ $Q \in \partial\Omega$, where $\Gamma(Q)$ is the *nontangential cone*

$$\Gamma(Q) = \{Y \in \Omega : \operatorname{dist}(Y, \partial\Omega) < (1+a)|X - Y|\}. \tag{5.2}$$

Here $a > 0$ is a positive parameter; the exact value of a is usually irrelevant to applications. The *nontangential maximal function* is given by

$$NF(Q) = \sup\{|F(X)| : X \in \Gamma(Q)\}. \tag{5.3}$$

The normal derivative of u of order m is defined as

$$\partial_\nu^m u(Q) = \sum_{|\alpha|=m} \nu(Q)^\alpha \frac{m!}{\alpha!} \partial^\alpha u(Q),$$

where $\partial^\alpha u(Q)$ is taken in the sense of nontangential limits as usual.

5.1. The Dirichlet problem: definitions, layer potentials, and some well-posedness results

We say that the L^p-Dirichlet problem for the biharmonic operator Δ^2 in a domain Ω is well-posed if there exists a constant $C > 0$ such that, for every $f \in W_1^p(\partial\Omega)$ and every $g \in L^p(\partial\Omega)$, there exists a unique function u that satisfies

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \\ \partial_\nu u = g & \text{on } \partial\Omega, \\ \|N(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)} + C\|\nabla_\tau f\|_{L^p(\partial\Omega)}. \end{cases} \tag{5.4}$$

The L^p-Dirichlet problem for the polyharmonic operator Δ^m is somewhat more involved, because the notion of boundary data is necessarily more subtle. We say that the L^p-Dirichlet problem for Δ^m in a domain Ω is well-posed if there exists a constant $C > 0$ such that, for every $g \in L^p(\partial\Omega)$ and every $\dot{f}$ in the Whitney-Sobolev space $WA^p_{m-1}(\partial\Omega)$, there exists a unique function u that satisfies

$$\begin{cases} \Delta^m u = 0 & \text{in } \Omega, \\ \partial^\alpha u\big|_{\partial\Omega} = f_\alpha & \text{for all } 0 \leq |\alpha| \leq m-2, \\ \partial_\nu^{m-1} u = g & \text{on } \partial\Omega, \\ \|N(\nabla^{m-1} u)\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega} + C \displaystyle\sum_{|\alpha|=m-2} \|\nabla_\tau f_\alpha\|_{L^p(\partial\Omega)}. \end{cases} \tag{5.5}$$

The space $WA^p_m(\partial\Omega)$ is defined as follows.

Definition 5.6. Suppose that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, and consider arrays of functions $\dot{f} = \{f_\alpha : |\alpha| \leq m-1\}$ indexed by multiindices α of length n, where $f_\alpha : \partial\Omega \mapsto \mathbb{C}$. We let $WA^p_m(\partial\Omega)$ be the completion of the set of arrays $\dot{\psi} = \{\partial^\alpha \psi : |\alpha| \leq m-1\}$, for $\psi \in C_0^\infty(\mathbb{R}^n)$, under the norm

$$\sum_{|\alpha|\leq m-1} \|\partial^\alpha \psi\|_{L^p(\partial\Omega)} + \sum_{|\alpha|=m-1} \|\nabla_\tau \partial^\alpha \psi\|_{L^p(\partial\Omega)}. \tag{5.7}$$

If we prescribe $\partial^\alpha u = f_\alpha$ on $\partial\Omega$ for some $f \in WA^p_m(\partial\Omega)$, then we are prescribing the values of $u, \nabla u, \ldots, \nabla^{m-1} u$ on $\partial\Omega$, and requiring that (the prescribed part of) $\nabla^m u\big|_{\partial\Omega}$ lie in $L^p(\partial\Omega)$.

The study of these problems began with biharmonic functions in C^1 domains. In [SS81], Selvaggi and Sisto proved that, if Ω is the domain above the graph of a compactly supported C^1 function φ, with $\|\nabla\varphi\|_{L^\infty}$ small enough, then solutions to the Dirichlet problem exist provided $1 < p < \infty$. Their method used certain biharmonic layer potentials composed with the Riesz transforms.

In [CG83], Cohen and Gosselin proved that, if Ω is a bounded, simply connected C^1 domain contained in the plane $\mathbb{R}^2$, then the L^p-Dirichlet problem is well-posed in Ω for any $1 < p < \infty$. In [CG85], they extended this result to the complements of such domains. Their proof used multiple layer potentials introduced by Agmon in [Agm57] in order to solve the Dirichlet problem with continuous boundary data. The general outline of their proof parallelled that of the proof of the corresponding result [FJR78] for Laplace's equation.

As in the case of Laplace's equation, a result in Lipschitz domains soon followed. In [DKV86], Dahlberg, Kenig and Verchota showed that the L^p-Dirichlet problem for the biharmonic equation is well-posed in any bounded simply connected Lipschitz domain $\Omega \subset \mathbb{R}^n$, provided $2-\varepsilon < p < 2+\varepsilon$ for some $\varepsilon > 0$ depending on the domain Ω.

In [Ver87], Verchota used the construction of [DKV86] to extend Cohen and Gosselin's results from planar C^1 domains to C^1 domains of arbitrary dimension. Thus, the L^p-Dirichlet problem for the bilaplacian is well-posed for $1 < p < \infty$ in C^1 domains.

In [Ver90], Verchota showed that the L^p-Dirichlet problem for the polyharmonic operator Δ^m could be solved for $2-\varepsilon < p < 2+\varepsilon$ in starlike Lipschitz domains by induction on the exponent m. He simultaneously proved results for the L^p-regularity problem in the same range; we will thus delay discussion of his methods to Section 5.3.

All three of the papers [SS81], [CG83] and [DKV86] constructed biharmonic functions as potentials. However, the potentials used differ. [SS81] constructed their solutions as

$$u(X) = \int_{\partial\Omega} \partial_n^2 F(X-Y)\, f(Y)\, d\sigma(Y) + \sum_{i=1}^{n-1} \int_{\partial\Omega} \partial_i \partial_n F(X-Y) R_i g(Y)\, d\sigma(Y)$$

where R_i are the Riesz transforms. Here $F(X)$ is the fundamental solution to the biharmonic equation; thus, u is biharmonic in $\mathbb{R}^n \setminus \partial\Omega$. As in the case of Laplace's equation, well-posedness of the Dirichlet problem follows from the boundedness relation $\|N(\nabla u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega)} + C\|g\|_{L^p(\partial\Omega)}$ and from invertibility of the mapping $(f,g) \mapsto (u\big|_{\partial\Omega}, \partial_\nu u)$ on $L^p(\partial\Omega) \times L^p(\partial\Omega) \mapsto W_1^p(\partial\Omega) \times L^p(\partial\Omega)$.

The multiple layer potential of [CG83] is an operator of the form

$$\mathcal{L}\dot{f}(P) = \text{p.v.} \int_{\partial\Omega} \mathcal{L}(P,Q) \dot{f}(Q)\, d\sigma(Q) \tag{5.8}$$

where $\mathcal{L}(P,Q)$ is a 3×3 matrix of kernels, also composed of derivatives of the biharmonic equation, and $\dot{f} = (f, f_x, f_y)$ is a "compatible triple" of boundary data, that is, an element of $W^{1,p}(\partial\Omega) \times L^p(\partial\Omega) \times L^p(\partial\Omega)$ that satisfies $\partial_\tau f = f_x \tau_x + f_y \tau_y$. Thus, the input is essentially a function and its gradient, rather than two functions, and the Riesz transforms are not involved.

The method of [DKV86] is to compose two potentials. First, the function $f \in L^2(\partial\Omega)$ is mapped to its Poisson extension v. Next, u is taken to be the solution of the inhomogeneous equation $\Delta u(Y) = (n + 2Y\cdot\nabla) v(Y)$ with $u = 0$ on $\partial\Omega$. If $G(X,Y)$ is the Green's function for Δ in Ω and k^Y is the harmonic measure density at Y, we may write the map $f \mapsto u$ as

$$u(X) = \int_\Omega G(X,Y)(n+2Y\cdot\nabla)\int_{\partial\Omega} k^Y(Q)\, f(Q)\, d\sigma(Q)\, dY. \tag{5.9}$$

Since $(n+2Y\cdot\nabla)v(Y)$ is harmonic, u is biharmonic, and so u solves the Dirichlet problem.

5.2. The L^p-Dirichlet problem: the summary of known results on well-posedness and ill-posedness

Recall that by [Ver90], the L^p-Dirichlet problem is well-posed in Lipschitz domains provided $2-\varepsilon < p < 2+\varepsilon$. As in the case of Laplace's equation (see [FJL77]), the range $p > 2-\varepsilon$ is sharp. That is, for any $p < 2$ and any integers $m \geq 2$, $n \geq 2$, there exists a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ such that the L^p-Dirichlet problem for Δ^m is ill-posed in Ω. See [DKV86, Section 5] for the case of the biharmonic operator Δ^2, and the proof of Theorem 2.1 in [PV95b] for the polyharmonic operator Δ^m.

The range $p < 2+\varepsilon$ is not sharp and has been studied extensively. Proving or disproving well-posedness of the L^p-Dirichlet problem for $p > 2$ in general Lipschitz domains has been an open question since [DKV86], and was formally stated as such in [Ken94, Problem 3.2.30]. (Earlier in [CFS79, Question 7], the authors had posed the more general question of what classes of boundary data give existence and uniqueness of solutions.)

In [PV92, Theorem 10.7], Pipher and Verchota constructed Lipschitz domains Ω such that the L^p-Dirichlet problem for Δ^2 was ill-posed in Ω, for any given $p > 6$ (in four dimensions) or any given $p > 4$ (in five or more dimensions). Their counterexamples built on the study of solutions near a singular point, in particular upon [MNP83] and [MP81]. In [PV95b], they provided other counterexamples to show that the L^p-Dirichlet problem for Δ^m is ill-posed, provided $p > 2(n-1)/(n-3)$ and $4 \leq n < 2m+1$. They remarked that if $n \geq 2m+1$, then ill-posedness follows from the results of [MNP83] provided $p > 2m/(m-1)$.

The endpoint result at $p = \infty$ is the Agmon-Miranda maximum principle (3.2) discussed above. We remark that if $2 < p_0 \leq \infty$, and the L^{p_0}-Dirichlet problem is well-posed (or (3.2) holds) then by interpolation, the L^p-Dirichlet problem is well-posed for any $2 < p < p_0$.

We shall adopt the following definition (justified by the discussion above).

Definition 5.10. Suppose that $m \geq 2$ and $n \geq 4$. Then $p_{m,n}$ is defined to be the extended real number that satisfies the following properties. If $2 \leq p \leq p_{m,n}$, then the L^p-Dirichlet problem for Δ^m is well-posed in any bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. Conversely, if $p > p_{m,n}$, then there exists a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ such that the L^p-Dirichlet problem for Δ^m is ill-posed in Ω. Here, well-posedness for $1 < p < \infty$ is meant in the sense of (5.5), and well-posedness for $p = \infty$ is meant in the sense of the maximum principle (see (5.24) below).

As in [DKV86], we expect the range of solvability for any *particular* Lipschitz domain Ω to be $2-\varepsilon < p < p_{m,n}+\varepsilon$ for some ε depending on the Lipschitz character of Ω.

Let us summarize here the results currently known for $p_{m,n}$. More details will follow in Section 5.3.

For any $m \geq 2$, we have that

- If $n = 2$ or $n = 3$, then the L^p-Dirichlet problem for Δ^m is well-posed in any Lipschitz domain Ω for any $2 \leq p < \infty$. ([PV92, PV95b])
- If $4 \leq n \leq 2m + 1$, then $p_{m,n} = 2(n - 1)/(n - 3)$. ([She06a, PV95b].)
- If $n = 2m+2$, then $p_{m,n} = 2m/(m-1) = 2(n-2)/(n-4)$. ([She06b, MNP83].)
- If $n \geq 2m+3$, then $2(n-1)/(n-3) \leq p_{m,n} \leq 2m/(m-1)$. ([She06a, MNP83].)

The value of $p_{m,n}$, for $n \geq 2m + 3$, is open.

In the special case of biharmonic functions ($m = 2$), more is known.

- $p_{2,4} = 6$, $p_{2,5} = 4$, $p_{2,6} = 4$, and $p_{2,7} = 4$. ([She06a] and [She06b])
- If $n \geq 8$, then
$$2 + \frac{4}{n - \lambda_n} < p_{2,n} \leq 4$$
where
$$\lambda_n = \frac{n + 10 + 2\sqrt{2(n^2 - n + 2)}}{7}.$$
([She06c])
- If Ω is a C^1 or convex domain of arbitrary dimension, then the L^p-Dirichlet problem for Δ^2 is well-posed in Ω for any $1 < p < \infty$. ([Ver90, She06c, KS11a].)

We comment on the nature of ill-posedness. The counterexamples of [DKV86] and [PV95b] for $p < 2$ are failures of uniqueness. That is, those counterexamples are nonzero functions u, satisfying $\Delta^m u = 0$ in Ω, such that $\partial_\nu^k u = 0$ on $\partial\Omega$ for $0 \leq k \leq m - 1$, and such that $N(\nabla^{m-1} u) \in L^p(\partial\Omega)$.

Observe that if Ω is bounded and $p > 2$, then $L^p(\partial\Omega) \subset L^2(\partial\Omega)$. Because the L^2-Dirichlet problem is well-posed, the failure of well-posedness for $p > 2$ can only be a failure of the optimal estimate $N(\nabla^{m-1} u) \in L^p(\partial\Omega)$. That is, if the L^p-Dirichlet problem for Δ^m is ill-posed in Ω, then for some Whitney array $\dot f \in WA^p_{m-1}(\partial\Omega)$ and some $g \in L^p(\partial\Omega)$, the unique function u that satisfies $\Delta^m u = 0$ in Ω, $\partial^\alpha u = f_\alpha$, $\partial_\nu^{m-1} u = g$ and $N(\nabla^{m-1} u) \in L^2(\partial\Omega)$ does not satisfy $N(\nabla^{m-1} u) \in L^p(\partial\Omega)$.

5.3. The regularity problem and the L^p-Dirichlet problem

In this section we elaborate on some of the methods used to prove the Dirichlet well-posedness results listed above, as well as their historical context. This naturally brings up a consideration of a different boundary value problem, the L^q-regularity problem for higher order operators.

Recall that for second-order equations the regularity problem corresponds to finding a solution with prescribed tangential gradient along the boundary. In analogy, we say that the L^q-regularity problem for Δ^m is well-posed in Ω if there exists a constant $C > 0$ such that, whenever $\dot f \in WA^q_m(\partial\Omega)$, there exists a unique function u that satisfies

$$\begin{cases} \Delta^m u = 0 & \text{in } \Omega, \\ \partial^\alpha u\big|_{\partial\Omega} = f_\alpha & \text{for all } 0 \leq |\alpha| \leq m-1, \\ \|N(\nabla^m u)\|_{L^q(\partial\Omega)} \leq C \displaystyle\sum_{|\alpha|=m-1} \|\nabla_\tau f_\alpha\|_{L^q(\partial\Omega)}. \end{cases} \tag{5.11}$$

There is an important endpoint formulation at $q = 1$ for the regularity problem. We say that the H^1-regularity problem is well-posed if there exists a constant $C > 0$ such that, whenever $\dot f$ lies in the Whitney-Hardy space $H^1_m(\partial\Omega)$, there exists a unique function u that satisfies

$$\begin{cases} \Delta^m u = 0 & \text{in } \Omega, \\ \partial^\alpha u\big|_{\partial\Omega} = f_\alpha & \text{for all } 0 \leq |\alpha| \leq m-1, \\ \|N(\nabla^m u)\|_{L^1(\partial\Omega)} \leq C \displaystyle\sum_{|\alpha|=m-1} \|\nabla_\tau f_\alpha\|_{H^1(\partial\Omega)}. \end{cases}$$

The space $H^1_m(\partial\Omega)$ is defined as follows.

Definition 5.12. We say that $\dot a \in WA^q_m(\partial\Omega)$ is a $H^1_m(\partial\Omega)$-L^q atom if $\dot a$ is supported in a ball $B(Q,r) \cap \partial\Omega$ and if

$$\sum_{|\alpha|=m-1} \|\nabla_\tau a_\alpha\|_{L^q(\partial\Omega)} \leq \sigma(B(Q,r)\cap\partial\Omega)^{1/q-1}.$$

If $\dot f \in WA^1_m(\partial\Omega)$ and there are H^1_m-L^2 atoms $\dot a_k$ and constants $\lambda_k \in \mathbb{C}$ such that

$$\nabla_\tau f_\alpha = \sum_{k=1}^\infty \lambda_k \nabla_\tau (a_k)_\alpha \text{ for all } |\alpha| = m-1$$

and such that $\sum |\lambda_k| < \infty$, we say that $\dot f \in H^1_m(\partial\Omega)$, with $\|\dot f\|_{H^1_m(\partial\Omega)}$ being the smallest $\sum|\lambda_k|$ among all such representations.

In [Ver90], Verchota proved well-posedness of the L^2-Dirichlet problem and the L^2-regularity problem for the polyharmonic operator Δ^m in any bounded starlike Lipschitz domain by simultaneous induction.

The base case $m = 1$ is valid in all bounded Lipschitz domains by [Dah79] and [JK81b]. The inductive step is to show that well-posedness for the Dirichlet problem for Δ^{m+1} follows from well-posedness of the lower-order problems. In particular, solutions with $\partial^\alpha u = f_\alpha$ may be constructed using the regularity problem for Δ^m, and the boundary term $\partial^m_\nu u = g$, missing from the regularity data, may be attained using the *inhomogeneous* Dirichlet problem for Δ^m. On the other hand, it was shown that the well-posedness for the regularity problem for Δ^{m+1} follows from well-posedness of the lower-order problems and from the Dirichlet problem for Δ^{m+1}, in some sense, by realizing the solution to the regularity problem as an integral of the solution to the Dirichlet problem.

As regards a broader range of p and q, Pipher and Verchota showed in [PV92] that the L^p-Dirichlet and L^q-regularity problems for Δ^2 are well-posed in all bounded Lipschitz domains $\Omega \subset \mathbb{R}^3$, provided $2 \leq p < \infty$ and $1 < q \leq 2$. Their method relied on duality. Using potentials similar to those of [DKV86], they constructed solutions to the L^2-Dirichlet problem in domains above Lipschitz graphs. The core of their proof was the invertibility on $L^2(\partial\Omega)$ of a certain potential operator T. They were able to show that the invertibility of its adjoint T^* on $L^2(\partial\Omega)$ implies that the L^2-regularity problem for Δ^2 is well-posed. Then, using the atomic decomposition of Hardy spaces, they analyzed the H^1-regularity problem. Applying interpolation and duality for T^* once again, now in the reverse regularity-to-Dirichlet direction, the full range for both regularity and Dirichlet problems was recovered in domains above graphs. Localization arguments then completed the argument in bounded Lipschitz domains.

In four or more dimensions, further progress relied on the following theorem of Shen.

Theorem 5.13 ([She06b]). *Suppose that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain. The following conditions are equivalent.*

- *The L^p-Dirichlet problem for L is well-posed, where L is a symmetric elliptic system of order $2m$ with real constant coefficients.*
- *There exists some constant $C > 0$ and some $p > 2$ such that*

$$\left(\fint_{B(Q,r)\cap\partial\Omega} N(\nabla^{m-1}u)^p \, d\sigma\right)^{1/p} \leq C \left(\fint_{B(Q,2r)\cap\partial\Omega} N(\nabla^{m-1}u)^2 \, d\sigma\right)^{1/2} \tag{5.14}$$

holds whenever u is a solution to the L^2-Dirichlet problem for L in Ω, with $\nabla u \equiv 0$ on $B(Q,3r)\cap\partial\Omega$.

For the polyharmonic operator Δ^m, this theorem was essentially proven in [She06a]. Furthermore, the reverse Hölder estimate (5.14) with $p = 2(n-1)/(n-3)$ was shown to follow from well-posedness of the L^2-regularity problem. Thus the L^p-Dirichlet problem is well-posed in bounded Lipschitz domains in $\mathbb{R}^n$ for $p = 2(n-1)/(n-3)$. By interpolation, and because reverse Hölder estimates have self-improving properties, well-posedness in the range $2 \leq p \leq 2(n-1)/(n-3) + \varepsilon$ for any particular Lipschtiz domain follows automatically.

Using regularity estimates and square-function estimates, Shen was able to further improve this range of p. He showed that with $p = 2 + 4/(n-\lambda)$, $0 < \lambda < n$, the reverse Hölder estimate (5.14) is true, provided that

$$\int_{B(Q,r)\cap\Omega} |\nabla^{m-1}u|^2 \leq C \left(\frac{r}{R}\right)^{\lambda} \int_{B(Q,R)\cap\Omega} |\nabla^{m-1}u|^2 \tag{5.15}$$

holds whenever u is a solution to the L^2-Dirichlet problem in Ω with $N(\nabla^{m-1}u) \in L^2(\partial\Omega)$ and $\nabla^k u\big|_{B(Q,R)\cap\Omega} \equiv 0$ for all $0 \leq k \leq m-1$.

It is illuminating to observe that the estimates arising in connection with the pointwise bounds on the solutions in arbitrary domains (cf. Section 3) and the

Wiener test (cf. Section 4), take essentially the form (5.15). Thus, Theorem 5.13 and its relation to (5.15) provide a direct way to transform results regarding local boundary regularity of solutions, obtained via the methods underlined in Sections 3 and 4, into well-posedness of the L^p-Dirichlet problem.

In particular, consider [Maz02, Lemma 5]. If u is a solution to $\Delta^m u = 0$ in $B(Q,R)\cap\Omega$, where Ω is a Lipschitz domain, then by [Maz02, Lemma 5] there is some constant $\lambda_0 > 0$ such that

$$\sup_{B(Q,r)\cap\Omega} |u|^2 \leq \left(\frac{r}{R}\right)^{\lambda_0} \frac{C}{R^n} \int_{B(Q,R)\cap\Omega} |u(X)|^2\, dX \tag{5.16}$$

provided that r/R is small enough, that u has zero boundary data on $B(Q,R)\cap\partial\Omega$, and where $\Omega\subset\mathbb{R}^n$ has dimension $n=2m+1$ or $n=2m+2$, or where $m=2$ and $n=7=2m+3$. (The bound on dimension comes from the requirement that Δ^m be positive with weight F; see equation (4.5).)

It is not difficult to see (cf., e.g., [She06b, Theorem 2.6]), that (5.16) implies (5.15) for some $\lambda > n-2m+2$, and thus implies well-posedness of the L^p-Dirichlet problem for a certain range of p. This provides an improvement on the results of [She06a] in the case $m=2$ and $n=6$ or $n=7$, and in the case $m\geq 3$ and $n=2m+2$. Shen has stated this improvement in [She06b, Theorems 1.4 and 1.5]: the L^p-Dirichlet problem for Δ^2 is well-posed for $2\leq p<4+\varepsilon$ in dimensions $n=6$ or $n=7$, and the L^p-Dirichlet problem for Δ^m is well-posed if $2\leq p<2m/(m-1)+\varepsilon$ in dimension $n=2m+2$.

The method of weighted integral identities, related to positivity with weight F (cf. (4.5)), can be further finessed in a particular case of the biharmonic equation. [She06c] uses this method (extending the ideas from [Maz79]) to show that if $n\geq 8$, then (5.15) is valid for solutions to Δ^2 with $\lambda=\lambda_n$, where

$$\lambda_n = \frac{n+10+2\sqrt{2(n^2-n+2)}}{7}. \tag{5.17}$$

We now return to the L^q-regularity problem. Recall that in [PV92], Pipher and Verchota showed that if $2<p<\infty$ and $1/p+1/q<1$, then the L^p-Dirichlet problem and the L^q-regularity problem for Δ^2 are both well-posed in three-dimensional Lipschitz domains. They proved this by showing that, in the special case of a domain above a Lipschitz graph, there is duality between the L^p-Dirichlet and L^q-regularity problems. Such duality results are common. See [KP93], [She07b], and [KR09] for duality results in the second-order case; although even in that case, duality is not always guaranteed. (See [May10].) Many of the known results concerning the regularity problem for the polyharmonic operator Δ^m are results relating the L^p-Dirichlet problem to the L^q-regularity problem.

In [MM10], I. Mitrea and M. Mitrea showed that if $1<p<\infty$ and $1/p+1/q=1$, and if the L^q-regularity problem for Δ^2 and the L^p-regularity problem for Δ were both well-posed in a particular bounded Lipschitz domain Ω, then the L^p-Dirichlet problem for Δ^2 was also well-posed in Ω. They proved this result (in arbitrary dimensions) using layer potentials and a Green representation formula for biharmonic equations. Observe that the extra requirement of well-posedness for the Laplacian is extremely unfortunate, since in bad domains it essentially restricts consideration to $p < 2 + \varepsilon$ and thus does not shed new light on well-posedness in the general class of Lipschitz domains. As will be discussed below, later Kilty and Shen established an optimal duality result for biharmonic Dirichlet and regularity problems.

Recall that the formula (5.14) provides a necessary and sufficient condition for well-posedness of the L^p-Dirichlet problem. In [KS11b], Kilty and Shen provided a similar condition for the regularity problem. To be precise, they demonstrated that if $q > 2$ and L is a symmetric elliptic system of order $2m$ with real constant coefficients, then the L^q-regularity problem for L is well-posed if and only if the estimate

$$\left(\fint_{B(Q,r)\cap\Omega} N(\nabla^m u)^q \, d\sigma\right)^{1/q} \leq C \left(\fint_{B(Q,2r)\cap\Omega} N(\nabla^m u)^2 \, d\sigma\right)^{1/2} \tag{5.18}$$

holds for all points $Q \in \partial\Omega$, all $r > 0$ small enough, and all solutions u to the L^2-regularity problem with $\nabla^k u\big|_{B(Q,3r)\cap\partial\Omega} = 0$ for $0 \leq k \leq m-1$. Observe that (5.18) is identical to (5.14) with p replaced by q and $m-1$ replaced by m.

As a consequence, well-posedness of the L^q-regularity problem in Ω for certain values of q implies well-posedness of the L^p-Dirichlet problem for some values of p. Specifically, arguments using interior regularity and fractional integral estimates (given in [KS11b, Section 5]) show that (5.18) implies (5.14) with $1/p = 1/q - 1/(n-1)$. But recall from [She06b] that (5.14) holds if and only if the L^p-Dirichlet problem for L is well-posed in Ω. Thus, if $2 < q < n-1$, and if the L^q-regularity problem for a symmetric elliptic system is well-posed in a Lipschitz domain Ω, then the L^p-Dirichlet problem for the same system and domain is also well-posed, provided $2 < p < p_0 + \varepsilon$ where $1/p_0 = 1/q - 1/(n-1)$.

For the bilaplacian, a full duality result is known. In [KS11a], Kilty and Shen showed that, if $1 < p < \infty$ and $1/p + 1/q = 1$, then well-posedness of the L^p-Dirichlet problem for Δ^2 in a Lipschitz domain Ω, and well-posedness of the L^q-regularity problem for Δ^2 in Ω, were both equivalent to the bilinear estimate

$$\begin{aligned}\left|\int_\Omega \Delta u \, \Delta v\right| \leq C &\left(\|\nabla_\tau \nabla f\|_{L^p} + |\partial\Omega|^{-1/(n-1)}\|\nabla f\|_{L^p} + |\partial\Omega|^{-2/(n-1)}\|f\|_{L^p}\right) \\ &\times \left(\|\nabla g\|_{L^q} + |\partial\Omega|^{-1/(n-1)}\|g\|_{L^q}\right)\end{aligned} \tag{5.19}$$

for all f, $g \in C_0^\infty(\mathbb{R}^n)$, where u and v are solutions of the L^2-regularity problem with boundary data $\partial^\alpha u = \partial^\alpha f$ and $\partial^\alpha v = \partial^\alpha g$. Thus, if $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, and if $1/p + 1/q = 1$, then the L^p-Dirichlet problem is well-posed in Ω if and only if the L^q-regularity problem is well-posed in Ω.

All in all, we see that the L^p-regularity problem for Δ^2 is well-posed in $\Omega \subset \mathbb{R}^n$ if

- Ω is C^1 or convex, and $1 < p < \infty$.
- $n = 2$ or $n = 3$ and $1 < p < 2 + \varepsilon$.
- $n = 4$ and $6/5 - \varepsilon < p < 2 + \varepsilon$.
- $n = 5$, 6 or 7, and $4/3 - \varepsilon < p < 2 + \varepsilon$.
- $n \geq 8$, and $2 - \frac{4}{4+n-\lambda_n} < p < 2 + \varepsilon$, where λ_n is given by (5.17). The above ranges of p are sharp, but this range is still open.

5.4. Higher-order elliptic systems

The polyharmonic operator Δ^m is part of a larger class of elliptic higher-order operators. Some study has been made of boundary-value problems for such operators and systems.

The L^p-Dirichlet problem for a strongly elliptic system L of order $2m$, as defined in Definition 2.1, is well-posed in Ω if there exists a constant C such that, for every $\dot{f} \in WA^p_{m-1}(\partial\Omega \mapsto \mathbb{C}^\ell)$ and every $\vec{g} \in L^p(\partial\Omega \mapsto \mathbb{C}^\ell)$, there exists a unique vector-valued function $\vec{u} : \Omega \mapsto \mathbb{C}^\ell$ such that

$$\begin{cases} (L\vec{u})_j = \displaystyle\sum_{k=1}^{\ell} \sum_{|\alpha|=|\beta|=m} \partial^\alpha a^{jk}_{\alpha\beta} \partial^\beta u_k = 0 & \text{in } \Omega \text{ for each } 1 \leq j \leq \ell, \\ \partial^\alpha \vec{u} = f_\alpha & \text{on } \partial\Omega \text{ for } |\alpha| \leq m-2, \\ \partial^{m-1}_\nu \vec{u} = \vec{g} & \text{on } \partial\Omega, \\ \|N(\nabla^{m-1} u)\|_{L^q(\partial\Omega)} \leq C \displaystyle\sum_{|\alpha|=m-2} \|\nabla_\tau f_\alpha\|_{L^q(\partial\Omega)} + C\|\vec{g}\|_{L^p(\partial\Omega)}. \end{cases} \tag{5.20}$$

The L^q-regularity problem is well-posed in Ω if there is some constant C such that, for every $\dot{f} \in WA^p_m(\partial\Omega \mapsto \mathbb{C}^\ell)$, there exists a unique $\vec{u}$ such that

$$\begin{cases} (L\vec{u})_j = \displaystyle\sum_{k=1}^{\ell} \sum_{|\alpha|=|\beta|=m} \partial^\alpha a^{jk}_{\alpha\beta} \partial^\beta u_k = 0 & \text{in } \Omega \text{ for each } 1 \leq j \leq \ell, \\ \partial^\alpha \vec{u} = f_\alpha & \text{on } \partial\Omega \text{ for } |\alpha| \leq m-1, \\ \|N(\nabla^m u)\|_{L^q(\partial\Omega)} \leq C \displaystyle\sum_{|\alpha|=m-1} \|\nabla_\tau f_\alpha\|_{L^q(\partial\Omega)}. \end{cases} \tag{5.21}$$

In [PV95a], Pipher and Verchota showed that the L^p-Dirichlet and L^p-regularity problems were well-posed for $2 - \varepsilon < p < 2 + \varepsilon$, for any higher-order elliptic partial differential equation with real constant coefficients, in Lipschitz domains of arbitrary dimension. This was extended to symmetric elliptic systems in [Ver96]. A key ingredient of the proof was the boundary Gårding inequality

$$\frac{\lambda}{4} \int_{\partial\Omega} |\nabla^m u|(-\nu_n)\, d\sigma \leq \sum_{j,k=1}^{\ell} \sum_{|\alpha|=|\beta|=m} \int_{\partial\Omega} \partial^\alpha a^{jk}_{\alpha\beta} \partial^\beta u_k (-\nu_n)\, d\sigma + C \int_{\partial\Omega} \left|\nabla^{m-1}\partial_n u\right|^2 d\sigma$$

valid if $u \in C_0^\infty(\mathbb{R}^n)^\ell$, if $L = \partial^\alpha a_{\alpha\beta}^{jk} \partial^\beta$ is a symmetric elliptic system with real constant coefficients, and if Ω is the domain above the graph of a Lipschitz function. We observe that in this case, $(-\nu_n)$ is a positive number bounded from below. Pipher and Verchota then used this Gårding inequality and a Green's formula to construct the nontangential maximal estimate. See [PV95b] and [Ver96, Sections 4 and 6].

As in the case of the polyharmonic operator Δ^m, this first result concerned the L^p-Dirichlet problem and L^q-regularity problem only for $2-\varepsilon < p < 2+\varepsilon$ and for $2-\varepsilon < q < 2+\varepsilon$. The polyharmonic operator Δ^m is an elliptic system, and so we cannot in general improve upon the requirement that $2-\varepsilon < p$ for well-posedness of the L^p-Dirichlet problem.

However, we can improve on the requirement $p < 2+\varepsilon$. Recall that Theorem 5.13 from [She06b], and its equivalence to (5.15), were proven in the general case of strongly elliptic systems with real symmetric constant coefficients. As in the case of the polyharmonic operator Δ^m, (5.14) follows from well-posedness of the L^2-regularity problem provided $p = 2(n-1)/(n-3)$, and so if L is such a system, the L^p-Dirichlet problem for L is well-posed in Ω provided $2-\varepsilon < p < 2(n-1)/(n-3)+\varepsilon$. This is [She06b, Corollary 1.3]. Again, by the counterexamples of [PV95b], this range cannot be improved if $m \geq 2$ and $4 \leq n \leq 2m+1$; the question of whether this range can be improved for general operators L if $n \geq 2m+2$ is still open.

Little is known concerning the regularity problem in a broader range of p. Recall that (5.18) from [KS11b] was proven in the general case of strongly elliptic systems with real symmetric constant coefficients. Thus, we known that for such systems, well-posedness of the L^q-regularity problem for $2 < q < n-1$ implies well-posedness of the L^p-Dirichlet problem for appropriate p. The question of whether the reverse implication holds, or whether this result can be extended to a broader range of q, is open.

5.5. The area integral

One of major tools in the theory of second-order elliptic differential equations is the Lusin area integral, defined as follows. If w lies in $W_{1,loc}^2(\Omega)$ for some domain $\Omega \subset \mathbb{R}^n$, then the area integral (or square function) of w is defined for $Q \in \partial\Omega$ as

$$Sw(Q) = \left(\int_{\Gamma(Q)} |\nabla w(X)|^2 \operatorname{dist}(X, \partial\Omega)^{2-n} dX\right)^{1/2}.$$

In [Dah80], Dahlberg showed that if u is harmonic in a bounded Lipschitz domain Ω, if $P_0 \in \Omega$ and $u(P_0) = 0$, then for any $0 < p < \infty$,

$$\frac{1}{C}\int_{\partial\Omega} Su^p \, d\sigma \leq \int_{\partial\Omega} (Nu)^p \, d\sigma \leq C \int_{\partial\Omega} (Su)^p \, d\sigma \tag{5.22}$$

for some constants C depending only on p, Ω and P_0. Thus, the Lusin area integral bears deep connections to the L^p-Dirichlet problem. In [DJK84], Dahlberg, Jerison and Kenig generalized this result to solutions to second-order divergence-form elliptic equations with real coefficients for which the L^r-Dirichlet problem is well-posed for at least one r.

If L is an operator of order $2m$, then the appropriate estimate is

$$\frac{1}{C}\int_{\partial\Omega} N(\nabla u^{m-1})^p\,d\sigma \leq \int_{\partial\Omega} S(\nabla u^{m-1})^p\,d\sigma \leq C\int_{\partial\Omega} N(\nabla u^{m-1})^p\,d\sigma. \qquad (5.23)$$

Before discussing their validity for particular operators, let us point out that such square-function estimates are very useful in the study of higher-order equations. In [She06b], Shen used (5.23) to prove the equivalence of (5.15) and (5.14), above. In [KS11a], Kilty and Shen used (5.23) to prove that well-posedness of the L^p-Dirichlet problem for Δ^2 implies the bilinear estimate (5.19). The proof of the maximum principle (3.2) in [Ver96, Section 8] (to be discussed in Section 5.6) also exploited (5.23). Estimates on square functions can be used to derive estimates on Besov space norms; see [AP98, Proposition S].

In [PV91], Pipher and Verchota proved that (5.23) (with $m = 2$) holds for solutions u to $\Delta^2 u = 0$, provided Ω is a bounded Lipschitz domain, $0 < p < \infty$, and $\nabla u(P_0) = 0$ for some fixed $P_0 \in \Omega$. Their proof was an adaptation of Dahlberg's proof [Dah80] of the corresponding result for harmonic functions. They used the L^2-theory for the biharmonic operator [DKV86], the representation formula (5.9), and the L^2-theory for harmonic functions to prove good-λ inequalities, which, in turn, imply L^p estimates for $0 < p < \infty$.

In [DKPV97], Dahlberg, Kenig, Pipher and Verchota proved that (5.23) held for solutions u to $Lu = 0$, for a symmetric elliptic system L of order $2m$ with real constant coefficients, provided as usual that Ω is a bounded Lipschitz domain, $0 < p < \infty$, and $\nabla^{m-1}u(P_0) = 0$ for some fixed $P_0 \in \Omega$. The argument is necessarily considerably more involved than the argument of [PV91] or [Dah80]. In particular, the bound $\|S(\nabla^{m-1}u)\|_{L^2(\partial\Omega)} \leq C\|N(\nabla^{m-1}u)\|_{L^2(\partial\Omega)}$ was proven in three steps.

The first step was to reduce from the elliptic system L of order $2m$ to the scalar elliptic operator $M = \det L$ of order $2\ell m$, where ℓ is as in formula (2.2). The second step was to reduce to elliptic equations of the form $\sum_{|\alpha|=m} a_\alpha \partial^{2\alpha} u = 0$, where $|a_\alpha| > 0$ for all $|\alpha| = m$. Finally, it was shown that for operators of this form

$$\sum_{|\alpha|=m} \int_\Omega a_\alpha\,\partial^\alpha u(X)^2\,\operatorname{dist}(X,\partial\Omega)\,dX \leq C\int_{\partial\Omega} N(\nabla^{m-1}u)^2\,d\sigma.$$

The passage to $0 < p < \infty$ in (5.23) was done, as usual, using good-λ inequalities. We remark that these arguments used the result of [PV95a] that the L^2-Dirichlet problem is well-posed for such operators L in Lipschitz domains.

It is quite interesting that for second-order elliptic systems, the only currently known approach to the square-function estimate (5.22) is this reduction to a higher-order operator.

5.6. The maximum principle in Lipschitz domains

We are now in a position to discuss the maximum principle (3.2) for higher-order equations in Lipschitz domains.

We say that the maximum principle for an operator L of order $2m$ holds in the bounded Lipschitz domain Ω if there exists a constant $C > 0$ such that, whenever $f \in WA^\infty_{m-1}(\partial\Omega) \subset WA^2_{m-1}(\partial\Omega)$ and $g \in L^\infty(\partial\Omega) \subset L^2(\partial\Omega)$, the solution u to the Dirichlet problem (5.20) with boundary data f and g satisfies

$$\|\nabla^{m-1}u\|_{L^\infty} \leq C\|g\|_{L^\infty(\partial\Omega)} + C \sum_{|\alpha|=m-2} \|\nabla_\tau f_\alpha\|_{L^\infty(\partial\Omega)}. \tag{5.24}$$

The maximum principle (5.24) was proven to hold in three-dimensional Lipschitz domains by Pipher and Verchota in [PV93] (for biharmonic functions), in [PV95b] (for polyharmonic functions), and by Verchota in [Ver96, Section 8] (for solutions to symmetric systems with real constant coefficients). Pipher and Verchota also proved in [PV93] that the maximum principle was valid for biharmonic functions in C^1 domains of arbitrary dimension. In [KS11a, Theorem 1.5], Kilty and Shen observed that the same techinque gives validity of the maximum principle for biharmonic functions in convex domains of arbitrary dimension.

The proof of [PV93] uses the L^2-regularity problem in the domain Ω to construct the Green's function $G(X,Y)$ for Δ^2 in Ω. Then if u is biharmonic in Ω with $N(\nabla u) \in L^2(\partial\Omega)$, we have that

$$u(X) = \int_{\partial\Omega} u(Q)\, \partial_\nu \Delta G(X,Q)\, d\sigma(Q) + \int_{\partial\Omega} \partial_\nu u(Q)\, \Delta G(X,Q)\, d\sigma(Q)$$

where all derivatives of G are taken in the second variable Q. If the H^1-regularity problem is well-posed in appropriate subdomains of Ω, then $\nabla^2\nabla_X G(X,\cdot)$ is in $L^1(\partial\Omega)$ with L^1-norm independent of X, and so the second integral is at most $C\|\partial_\nu u\|_{L^\infty(\partial\Omega)}$. By taking Riesz transforms, the normal derivative $\partial_\nu \Delta G(X,Q)$ may be transformed to tangential derivatives $\nabla_\tau \Delta G(X,Q)$; integrating by parts transfers these derivatives to u. The square-function estimate (5.23) implies that the Riesz transforms of $\nabla_X \Delta_Q G(X,Q)$ are bounded on $L^1(\partial\Omega)$. This completes the proof of the maximum principle.

Similar arguments show that the maximum principle is valid for more general operators. See [PV95b] for the polyharmonic operator, or [Ver96, Section 8] for arbitrary symmmetric operators with real constant coefficients.

An important transitional step is the well-posedness of the H^1-regularity problem. It was established in three-dimensional (or C^1) domains in [PV93, Theorem 4.2] and [PV95b, Theorem 1.2] and discussed in [Ver96, Section 7]. In each case, well-posedness was proven by analyzing solutions with atomic data $\dot{f}$ using a technique from [DK90]. A crucial ingredient in this technique is the well-posedness of the L^p-Dirichlet problem for some $p < (n-1)/(n-2)$; the latter is valid if $n = 3$ by [DKV86], and (for Δ^2) in C^1 and convex domains by [Ver90] and [KS11a], but fails in general Lipschitz domains for $n \geq 4$.

5.7. Biharmonic functions in convex domains

We say that a domain Ω is *convex* if, whenever X, $Y \in \Omega$, the line segment connecting X and Y lies in Ω. Observe that all convex domains are necessarily

Lipschitz domains but the converse does not hold. Moreover, while convex domains are in general no smoother than Lipschitz domains, the extra geometrical structure often allows for considerably stronger results.

Recall that in [MM09a], the second author of this paper and Maz'ya showed that the gradient of a biharmonic function is bounded in a three-dimensional domain. This is a sharp property in dimension three, and in higher dimensional domains the solutions can be even less regular (cf. Section 3). However, using some intricate linear combination of weighted integrals, the same authors showed in [MM08] that *second* derivatives to biharmonic functions were locally bounded when the domain was convex. To be precise, they showed that if Ω is convex, and $u \in \mathring{W}_2^2(\Omega)$ is a solution to $\Delta^2 u = h$ for some $h \in C_0^\infty(\Omega \setminus B(Q, 10R))$, $R > 0$, $Q \in \partial\Omega$, then

$$\sup_{B(Q,R/5)\cap\Omega} |\nabla^2 u| \leq \frac{C}{R^2} \left(\fint_{\Omega\cap B(Q,5R)\setminus B(Q,R/2)} |u|^2 \right)^{1/2}. \tag{5.25}$$

In particular, not only are all boundary points of convex domains 1-regular, but the gradient ∇u is Lipschitz continuous near such points.

Kilty and Shen noted in [KS11a] that (5.25) implies that (5.18) holds in convex domains for any q; thus, the L^q-regularity problem for the bilaplacian is well-posed for any $2 < q < \infty$ in a convex domain. Well-posedness of the L^p-Dirichlet problem for $2 < p < \infty$ has been established by Shen in [She06c]. By the duality result (5.19), again from [KS11a], this implies that both the L^p-Dirichlet and L^q-regularity problems are well-posed, for any $1 < p < \infty$ and any $1 < q < \infty$, in a convex domain of arbitrary dimension. They also observed that, by the techniques of [PV93] (discussed in Section 5.6 above), the maximum principle (5.24) is valid in arbitrary convex domains.

It is interesting to note how, once again, the methods and results related to pointwise estimates, the Wiener criterion, and local regularity estimates near the boundary are intertwined with the well-posedness of boundary problems in L^p.

5.8. The Neumann problem for the biharmonic equation

So far we have only discussed the Dirichlet and regularity problems for higher order operators. Another common and important boundary-value problem that arises in applications is the Neumann problem. Indeed, the principal physical motivation for the inhomogeneous biharmonic equation $\Delta^2 u = h$ is that it describes the equilibrium position of a thin elastic plate subject to a vertical force h. The Dirichlet problem $u\big|_{\partial\Omega} = f$, $\nabla u\big|_{\partial\Omega} = g$ describes an elastic plate whose edges are clamped, that is, held at a fixed position in a fixed orientation. The Neumann problem, on the other hand, corresponds to the case of a free boundary. Guido Sweers has written an excellent short paper [Swe09] discussing the boundary conditions that correspond to these and other physical situations.

More precisely, if a thin two-dimensional plate is subject to a force h and the edges are free to move, then its displacement u satisfies the boundary value problem

$$\begin{cases} \Delta^2 u = h & \text{in } \Omega, \\ \rho\Delta u + (1-\rho)\partial_\nu^2 u = 0 & \text{on } \partial\Omega, \\ \partial_\nu \Delta u + (1-\rho)\partial_{\tau\tau\nu} u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here ρ is a physical constant, called the Poisson ratio. This formulation goes back to Kirchoff and is well known in the theory of elasticity; see, for example, Section 3.1 and Chapter 8 of the classic engineering text [Nad63]. We remark that by [Nad63, Formula (8-10)],

$$\partial_\nu \Delta u + (1-\rho)\partial_{\tau\tau\nu} u = \partial_\nu \Delta u + (1-\rho)\partial_\tau \left(\partial_{\nu\tau} u\right).$$

This suggests the following homogeneous boundary value problem in a Lipschitz domain Ω of arbitrary dimension. We say that the L^p-Neumann problem is well-posed if there exists a constant $C > 0$ such that, for every $f_0 \in L^p(\partial\Omega)$ and $\Lambda_0 \in W^p_{-1}(\partial\Omega)$, there exists a function u such that

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ M_\rho u := \rho\Delta u + (1-\rho)\partial_\nu^2 u = f_0 & \text{on } \partial\Omega, \\ K_\rho u := \partial_\nu \Delta u + (1-\rho)\dfrac{1}{2}\partial_{\tau_{ij}}\left(\partial_{\nu\tau_{ij}} u\right) = \Lambda_0 & \text{on } \partial\Omega, \\ \|N(\nabla^2 u)\|_{L^p(\partial\Omega)} \leq C\|f_0\|_{W^p_1(\partial\Omega)} + C\|\Lambda_0\|_{W^p_{-1}(\partial\Omega)}. \end{cases} \tag{5.26}$$

Here $\tau_{ij} = \nu_i \mathbf{e}_j - \nu_j \mathbf{e}_i$ is a vector orthogonal to the outward normal ν and lying in the $x_i x_j$-plane.

In addition to the connection to the theory of elasticity, this problem is of interest because it is in some sense adjoint to the Dirichlet problem (5.4). That is, if $\Delta^2 u = \Delta^2 w = 0$ in Ω, then $\int_{\partial\Omega} \partial_\nu w\, M_\rho u - w\, K_\rho u\, d\sigma = \int_{\partial\Omega} \partial_\nu u\, M_\rho w - u\, K_\rho w\, d\sigma$, where M_ρ and K_ρ are as in (5.26); this follows from the more general formula

$$\int_\Omega w\,\Delta^2 u = \int_\Omega \left(\rho\Delta u\,\Delta w + (1-\rho)\partial_{jk} u\,\partial_{jk} w\right) + \int_{\partial\Omega} w\, K_\rho u - \partial_\nu w\, M_\rho u\, d\sigma \tag{5.27}$$

valid for arbitrary smooth functions. This formula is analogous to the classical Green's identity for the Laplacian

$$\int_\Omega w\,\Delta u = -\int_\Omega \nabla u \cdot \nabla w + \int_{\partial\Omega} w\,\nu \cdot \nabla u\, d\sigma. \tag{5.28}$$

Observe that, contrary to the Laplacian or more general second order operators, there is a *family* of relevant Neumann data for the biharmonic equation. Moreover, different values (or, rather, ranges) of ρ correspond to different natural physical situations. We refer the reader to [Ver05] for a detailed discussion.

In [CG85], Cohen and Gosselin showed that the L^p-Neumann problem (5.26) was well-posed in C^1 domains contained in $\mathbb{R}^2$ for for $1 < p < \infty$, provided in addition that $\rho = -1$. The method of proof was as follows. Recall from (5.8) that Cohen and Gosselin showed that the L^p-Dirichlet problem was well-posed by constructing a multiple layer potential $\mathcal{L}\dot{f}$ with boundary values $(I+\mathcal{K})\dot{f}$, and showing that $I+\mathcal{K}$ is invertible. We remark that because Cohen and Gosselin preferred to work with Dirichlet boundary data of the form $(u, \partial_x u, \partial_y u)\big|_{\partial\Omega}$ rather than of the form $(u, \partial_\nu u)\big|_{\partial\Omega}$, the notation of [CG85] is somewhat different from that of the present paper. In the notation of the present paper, the method of proof of [CG85] was to observe that $(I+\mathcal{K})^*\dot\theta$ is equivalent to $(K_{-1}v\dot\theta, M_{-1}v\dot\theta)_{\partial\Omega^C}$, where v is another biharmonic layer potential and $(I+\mathcal{K})^*$ is the adjoint to $(I+\mathcal{K})$. Well-posedness of the Neumann problem then follows from invertibility of $I+\mathcal{K}$ on $\partial\Omega^C$.

In [Ver05], Verchota investigated the Neumann problem (5.26) in full generality. He considered Lipschitz domains with compact, connected boundary contained in $\mathbb{R}^n$, $n \geq 2$. He showed that if $-1/(n-1) \leq \rho < 1$, then the Neumann problem is well-posed provided $2-\varepsilon < p < 2+\varepsilon$. That is, the solutions exist, satisfy the desired estimates, and are unique either modulo functions of an appropriate class, or (in the case where Ω is unbounded) when subject to an appropriate growth condition. See [Ver05, Theorems 13.2 and 15.4]. Verchota's proof also used boundedness and invertibility of certain potentials on $L^p(\partial\Omega)$; a crucial step was a coercivity estimate $\|\nabla^2 u\|_{L^2(\partial\Omega)} \leq C\|K_\rho u\|_{W^2_{-1}(\partial\Omega)} + C\|M_\rho u\|_{L^2(\partial\Omega)}$. (This estimate is valid provided u is biharmonic and satisfies some mean-value hypotheses; see [Ver05, Theorem 7.6]).

In [She07a], Shen improved upon Verchota's results by extending the range on p (in bounded simply connected Lipschitz domains) to $2(n-1)/(n+1)-\varepsilon < p < 2+\varepsilon$ if $n \geq 4$, and $1 < p < 2+\varepsilon$ if $n=2$ or $n=3$. This result again was proven by inverting layer potentials. Observe that the L^p-regularity problem is also known to be well-posed for p in this range, and (if $n \geq 6$) in a broader range of p; see Section 5.3. The question of the sharp range of p for which the L^p-Neumann problem is well-posed is still open.

It turns out that extending the well-posedness results for the Neumann problem beyond the case of the bilaplacian is an excruciatingly difficult problem, even if one considers only fourth-order operators with constant coefficients.

The solutions to (5.26) in [CG85], [Ver05] and [She07a] were constructed using layer potentials. It is possible to construct layer potential operators, and to prove their boundedness, for a fairly general class of higher order operators. However, the problems arise at a much more fundamental level.

In analogy to (5.27) and (5.28), one can write

$$\int_\Omega w\,Lu = A[u,w] + \int_{\partial\Omega} w\,K_A u - \partial_\nu w\,M_A u\,d\sigma, \tag{5.29}$$

where $A[u,w] = \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta} \int_\Omega D^\beta u\,D^\alpha w$ is an energy form associated to the operator $L = \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta} D^\alpha D^\beta$. Note that in the context of fourth-order operators, the pair $(w, \partial_\nu w)$ constitutes the Dirichlet data for w on the boundary, and so one can say that the operators $K_A u$ and $M_A u$ define the Neumann data for u. One immediately faces the problem that the same higher-order operator L

can be rewritten in many different ways and gives rise to different energy forms. The corresponding Neumann data will be different. (This is the reason why there is a family of Neumann data for the biharmonic operator.)

Furthermore, whatever the choice of the form, in order to establish well-posedness of the Neumann problem, one needs to be able to estimate all second derivatives of a solution on the boundary in terms of the Neumann data. In the analogous second-order case, such an estimate is provided by the Rellich identity, which shows that the tangential derivatives are equivalent to the normal derivative in L^2 for solutions of elliptic PDEs. In the higher-order scenario, such a result calls for certain coercivity estimates which are still rather poorly understood. We refer the reader to [Ver10] for a detailed discussion of related results and problems.

5.9. Inhomogeneous problems for the biharmonic equation and other classes of boundary data

In [AP98], Adolfsson and Pipher investigated the inhomogeneous Dirichlet problem for the biharmonic equation with data in Besov and Sobolev spaces. While resting on the results for homogeneous boundary value problems discussed in Sections 5.1 and 5.3, such a framework presents a completely new setting, allowing for the inhomogeneous problem and for consideration of the classes of boundary data which are, in some sense, intermediate between the Dirichlet and the regularity problems.

They showed that if $f \in WA^p_{1+s}(\partial\Omega)$ and $h \in L^p_{s+1/p-3}(\Omega)$, then there exists a unique function u that satisfies

$$\begin{cases} \Delta^2 u = h & \text{in } \Omega, \\ \operatorname{Tr} \partial^\alpha u = f_\alpha, & \text{for } 0 \leq |\alpha| \leq 1 \end{cases} \tag{5.30}$$

subject to the estimate

$$\|u\|_{L^p_{s+1/p+1}(\Omega)} \leq C\|h\|_{L^p_{s+1/p-3}(\Omega)} + C\|\dot{f}\|_{WA^p_{1+s}(\partial\Omega)} \tag{5.31}$$

provided $2-\varepsilon < p < 2+\varepsilon$ and $0 < s < 1$. Here $\operatorname{Tr} w$ denotes the trace of w in the sense of Sobolev spaces; that these may be extended to functions $u \in L^p_{s+1+1/p}$, $s > 0$, was proven in [AP98, Theorem 1.12].

In Lipschitz domains contained in $\mathbb{R}^3$, they proved these results for a broader range of p and s, namely for $0 < s < 1$ and for

$$\max\left(1, \frac{2}{s+1+\varepsilon}\right) < p < \begin{cases} \infty, & s < \varepsilon, \\ \frac{2}{s-\varepsilon}, & \varepsilon \leq s < 1. \end{cases} \tag{5.32}$$

Finally, in C^1 domains, they proved these results for any p and s with $1 < p < \infty$ and $0 < s < 1$.

In [MMW11], I. Mitrea, M. Mitrea and Wright extended the three-dimensional results to $p = \infty$ (for $0 < s < \varepsilon$) or $2/(s+1+\varepsilon) < p \leq 1$ (for $1-\varepsilon < s < 1$). They also extended these results to data h and $\dot{f}$ in more general Besov or Triebel-Lizorkin spaces.

Let us define the function spaces appearing above. $L^p_\alpha(\mathbb{R}^n)$ is defined to be $\{g : (I-\Delta)^{\alpha/2} g \in L^p(\mathbb{R}^n)\}$; we say $g \in L^p_\alpha(\Omega)$ if $g = h\big|_\Omega$ for some $h \in L^p_\alpha(\mathbb{R}^n)$. If k is a nonnegative integer, then $L^p_k = W^p_k$. If m is an integer and $0 < s < 1$, then the Whitney-Besov space WA^p_{m-1+s} is defined analogously to WA^p_m (see Definition 5.6), except that we take the completion with respect to the Whitney-Besov norm

$$\sum_{|\alpha|\leq m-1} \|\partial^\alpha \psi\|_{L^p(\partial\Omega)} + \sum_{|\alpha|=m-1} \|\partial^\alpha \psi\|_{B^{p,p}_s(\partial\Omega)} \tag{5.33}$$

rather than the Whitney-Sobolev norm

$$\sum_{|\alpha|\leq m-1} \|\partial^\alpha \psi\|_{L^p(\partial\Omega)} + \sum_{|\alpha|=m-1} \|\nabla_\tau \partial^\alpha \psi\|_{L^p(\partial\Omega)}.$$

The general problem (5.30) was first reduced to the case $h = 0$ (that is, to a homogeneous problem) by means of trace/extension theorems, that is, subtracting $w(X) = \int_{\mathbb{R}^n} F(X,Y)\,\tilde{h}(Y)\,dY$, and showing that if $h \in L^p_{s+1/p-3}(\Omega)$ then $(\operatorname{Tr} w, \operatorname{Tr} \nabla w) \in WA^p_{1+s}(\partial\Omega)$. Next, the well-posedness of Dirichlet and regularity problems discussed in Sections 5.1 and 5.3 provide the endpoint cases $s = 0$ and $s = 1$, respectively. The core of the matter is to show that, if u is biharmonic, k is an integer and $0 \leq \alpha \leq 1$, then $u \in L^p_{k+\alpha}(\Omega)$ if and only if

$$\int_\Omega \left|\nabla^{k+1} u(X)\right|^p \operatorname{dist}(X,\partial\Omega)^{p-p\alpha} + \left|\nabla^k u(X)\right|^p + |u(X)|^p \, dX < \infty, \tag{5.34}$$

(cf. [AP98, Proposition S]). With this at hand, one can use square-function estimates to justify the aforementioned endpoint results. Indeed, observe that for $p = 2$ the first integral on the left-hand side of (5.34) is exactly the L^2 norm of $S(\nabla^k u)$. The latter, by [PV91] (discussed in Section 5.5), is equivalent to the L^2 norm of the corresponding non-tangential maximal function, connecting the estimate (5.31) to the nontangential estimates in the Dirichlet problem (5.4) and the regularity problem 5.11. Finally, one can build an interpolation-type scheme to pass to well-posedness in intermediate Besov and Sobolev spaces.

6. Boundary value problems with variable coefficients

In this section we discuss divergence-form operators with variable coefficients. At the moment, well-posedness results for such operators are restricted in two serious ways. First, the coefficients cannot oscillate too much. Secondly, the boundary problems treated fall *strictly* between the range of L^p-Dirichlet and L^p-regularity, in the sense of Section 5.9. That is, the L^p-Dirichlet, regularity, and Neumann problems on Lipschitz domains with the usual sharp estimates in terms of the non-tangential maximal function for these divergence-form operators seem to be completely open.

To be more precise, recall from the discussion in Section 5.9 that the classical Dirichlet and regularity problems, with boundary data in L^p, can be viewed as the

$s = 0$, 1 endpoints of the boundary problem studied in [AP98] and [MMW11]

$$\Delta^2 u = h \text{ in } \Omega, \quad \partial^\alpha u\big|_{\partial\Omega} = f_\alpha \text{ for all } |\alpha| \leq 1$$

with $\dot f$ lying in an *intermediate* smoothness space $WA^p_{1+s}(\partial\Omega)$, $0 \leq s \leq 1$. In the context of divergence-form higher-order operators with variable coefficients, essentially the known results pertain *only* to boundary data of intermediate smoothness.

We now establish some terminology. A divergence-form operator L, acting on $W^2_{m,loc}(\Omega \mapsto \mathbb{C}^\ell)$, may be defined weakly via (2.5); we say that $Lu = h$ if

$$\sum_{j=1}^{\ell} \int_\Omega \varphi_j\, h_j = (-1)^m \sum_{j,k=1}^{\ell} \sum_{|\alpha|=|\beta|=m} \int_\Omega \partial^\alpha \varphi_j(X)\, a^{jk}_{\alpha\beta}(X)\, \partial^\beta u_k(X)\, dX \tag{6.1}$$

for all φ smooth and compactly supported in Ω.

In [Agr07], Agranovich investigated the inhomogeneous Dirichlet problem, in Lipschitz domains, for such operators L that are elliptic (in the sense of (2.7)) and whose coefficients $a^{jk}_{\alpha\beta}$ are Lipschitz continuous in Ω.

He showed that if $h \in L^p_{-m-1+1/p+s}(\Omega)$ and $\dot f \in WA^p_{m-1+s}(\partial\Omega)$, for some $0 < s < 1$, and if $|p-2|$ is small enough, then the Dirichlet problem

$$\begin{cases} \quad Lu = h & \text{in } \Omega, \\ \operatorname{Tr} \partial^\alpha u = f_\alpha & \text{for all } 0 \leq |\alpha| \leq m-1 \end{cases} \tag{6.2}$$

has a unique solution u that satisfies the estimate

$$\|u\|_{L^p_{m-1+s+1/p}(\Omega)} \leq C\|h\|_{L^p_{-m-1+1/p+s}(\partial\Omega)} + C\|\dot f\|_{WA^p_{m-1+s}(\partial\Omega)}. \tag{6.3}$$

Agranovich also considered the Neumann problem for such operators. As we discussed in Section 5.8, defining Neumann problem is a delicate matter. In the context of zero boundary data, the situation is a little simpler as one can take a formal functional analytic point of view and avoid to some extent the discussion of estimates at the boundary. First, observe that if the test function φ does not have zero boundary data, then formula (6.1) becomes

$$\begin{aligned} \sum_{j=1}^{\ell} \int_\Omega (Lu)_j\, \varphi_j = (-1)^m \sum_{j,k=1}^{\ell} \sum_{|\alpha|=|\beta|=m} \int_\Omega \partial^\alpha \varphi_j(X)\, a^{jk}_{\alpha\beta}(X)\, \partial^\beta u_k(X)\, dX \\ + \sum_{i=0}^{m-1} \int_{\partial\Omega} B_{m-1-i} u\, \partial^i_\nu \varphi\, d\sigma \end{aligned}$$

where $B_i u$ is an appropriate linear combination of the functions $\partial^\alpha u$ where $|\alpha| = m + i$. The expressions $B_i u$ may then be regarded as the Neumann data for u. Notice that if L is a fourth-order constant-coefficient operator, then $B_0 = -M_A$ and $B_1 = K_A$, where K_A, M_A are given by (5.29).

We say that u solves the Neumann problem for L, with homogeneous boundary data, if (6.1) is valid for all test functions φ compactly supported in $\mathbb{R}^n$ (but not necessarily in Ω.) Agranovich showed that, if $h \in \mathring{L}^p_{-m-1+1/p+s}(\Omega)$, then there exists a unique function $u \in L^p_{m-1+1/p+s}(\Omega)$ that solves this Neumann problem with homogeneous boundary data, under the same conditions on p, s, L as for his results for the Dirichlet problem. He also provided some brief discussion (see [Agr07, Section 5.2]) of the conditions needed to resolve the Neumann problem with inhomogeneous boundary data. Here $h \in \mathring{L}^p_\alpha(\Omega)$ if $h = g\big|_\Omega$ for some $g \in L^p_\alpha(\mathbb{R}^n)$ that in addition is supported in $\bar{\Omega}$.

In [MMS10], Maz'ya, M. Mitrea and Shaposhnikova considered the Dirichlet problem, again with boundary data in intermediate Besov spaces, for much rougher coefficients. They showed that if $f \in WA^p_{m-1+s}$, for some $0 < s < 1$ and some $1 < p < \infty$, if h lies in an appropriate space, and if L is a divergence-form operator of order $2m$ (as defined by (2.5)), then under some conditions, there is a unique function u that satisfies (6.2) subject to the estimate

$$\sum_{|\alpha|\leq m} \int_\Omega |\partial^\alpha u(X)|^p \operatorname{dist}(X, \partial\Omega)^{p-ps-1}\, dX < \infty. \tag{6.4}$$

See [MMS10, Theorem 8.1]. The inhomogeneous data h is required to lie in the space $V^p_{-m,1-s-1/p}(\Omega)$, the dual space to $V^q_{m,s+1/p-1}(\Omega)$, where

$$\|w\|_{V^p_{m,a}} = \biggl(\sum_{|\alpha|\leq m} \int_\Omega |\partial^\alpha u(X)|^p \operatorname{dist}(X, \partial\Omega)^{pa+p|\alpha|-pm}\, dX\biggr)^{1/p}.$$

The conditions are that Ω be a Lipschitz domain whose normal vector ν lies in $VMO(\partial\Omega)$, and that the coefficients $a^{ij}_{\alpha\beta}$ lie in $VMO(\mathbb{R}^n)$. Recall that this condition on Ω has also arisen in [MM11] (it ensures the validity of formula (3.11)). The ellipticity condition they required was that the coefficients be bounded and that $\langle \varphi, L\varphi\rangle \geq \lambda \|\nabla^m \varphi\|^2_{L^2}$ for all smooth compactly supported functions φ, that is, that

$$\operatorname{Re} \sum_{|\alpha|=|\beta|=m} \sum_{j,k=1}^{\ell} \int_\Omega a^{jk}_{\alpha\beta}(X)\partial^\beta \varphi_k(X) \partial^\alpha \overline{\varphi_j}(X)\, dX \geq \lambda \sum_{|\alpha|=m} \sum_{k=1}^{\ell} \int_\Omega |\partial^\alpha \varphi_k|^2 \tag{6.5}$$

for all functions $\varphi \in C^\infty_0(\Omega \mapsto \mathbb{C}^\ell)$. This is a weaker requirement than condition (2.7).

In fact, [MMS10] provides a more intricate result, allowing one to deduce a well-posedness range of s and p, given information about the oscillation of the coefficients $a^{jk}_{\alpha\beta}$ and the normal to the domain ν. In the extreme case, when the oscillations for both are vanishing, the allowable range expands to $0 < s < 1$, $1 < p < \infty$, as stated above.

We comment on the estimate (6.4). First, by [AP98, Propositon S] (listed above as formula (5.34)), if u is biharmonic then the estimate (6.4) is equivalent to the estimate (6.3) of [Agr07]. Second, by (5.23), if the coefficients $a^{jk}_{\alpha\beta}$ are constant, one can draw connections between (6.4) for $s = 0, 1$ and the nontangential maximal estimates of the Dirichlet or regularity problems (5.20) or (5.21). However, as we pointed out earlier, this endpoint case, corresponding to the true L^p-Dirichlet and regularity problems, has not been achieved.

6.1. The Kato problem and the Riesz transforms

An important topic in elliptic theory, which formally stands somewhat apart from the well-posedness issues, is the Kato problem and the properties of the Riesz transform. In the framework of elliptic boundary problems, the related results can be viewed as the estimates for the solutions with data in L^p for certain operators in block form.

Suppose that L is a variable-coefficient operator in divergence form, that is, an operator defined by (2.5). Suppose that L satisfies the ellipticity estimate (6.5), and the bounds

$$\Big|\sum_{|\alpha|=|\beta|=m}\sum_{j,k=1}^{\ell}\int_{\mathbb{R}^n} a^{jk}_{\alpha\beta}\partial^\beta f_k\,\partial^\alpha g_j\Big| \leq C\|\nabla^m f\|_{L^2(\mathbb{R}^n)}\|\nabla^m g\|_{L^2(\mathbb{R}^n)}. \tag{6.6}$$

(This is a weaker condition than the assumption of [MMS10] that $a^{ij}_{\alpha\beta}$ be bounded pointwise.) Auscher, Hofmann, McIntosh and Tchamitchian [AHMT01] proved that under these conditions, the Kato estimate

$$\frac{1}{C}\|\nabla^m f\|_{L^2(\mathbb{R}^n)} \leq \|\sqrt{L} f\|_{L^2(\mathbb{R}^n)} \leq C\|\nabla^m f\|_{L^2(\mathbb{R}^n)} \tag{6.7}$$

is valid for some constant C. They also proved similar results for operators with lower-order terms.

It was later observed in [Aus04] that by the methods of [AT98], if $1 \leq n \leq 2m$, then the bound on the Riesz transform $\nabla^m L^{-1/2}$ in L^p (that is, the first inequality in (6.7)) extends to the range $1 < p < 2+\varepsilon$, and the reverse Riesz transform bound (that is, the second inequality in (6.7)) extends to the range $1 < p < \infty$. This also holds if the Schwartz kernel $W_t(X,Y)$ of the operator e^{-tL} satisfies certain pointwise bounds (e.g., if the coefficients of A are real).

In the case where $n > 2m$, the inequality $\|\nabla^m L^{-1/2} f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$ holds for $2n/(n+2m) - \varepsilon < p \leq 2$; see [BK04, Aus04]. The reverse inequality holds for $\max(2n/(n+4m)-\varepsilon, 1) < p < 2$ by [Aus04, Theorem 18], and for $2 < p < 2n/(n-2m)+\varepsilon$ by duality (see [Aus07, Section 7.2]).

Going further, let us consider the second-order divergence-form operator $\mathbb{L} = -\operatorname{div}\mathbb{A}\nabla$ in $\mathbb{R}^{n+1}$, where $\mathbb{A}$ is an $(n+1)\times(n+1)$ matrix in block form; that is, $\mathbb{A}_{j,n+1} = \mathbb{A}_{n+1,j} = 0$ for $1 \leq j \leq n$, and $\mathbb{A}_{n+1,n+1} = 1$. It is fairly easy to see that one can formally realize the solution to $\mathbb{L}u = 0$ in $\mathbb{R}^{n+1}_+$, $u\big|_{\mathbb{R}^n} = f$, as the Poisson semigroup $u(x,t) = e^{-t\sqrt{L}} f(x)$, $(x,t) \in \mathbb{R}^{n+1}_+$. Then (6.7) essentially provides an analogue of the Rellich identity-type estimate for the block operator $\mathbb{L}$, that is, the L^2-equivalence between normal and tangential derivatives of the solution on the boundary

$$\|\partial_t u(\,\cdot\,,0)\|_{L^2(\mathbb{R}^n)} \approx \|\nabla_x u(\,\cdot\,,0)\|_{L^2(\mathbb{R}^n)}.$$

As we discussed in Section 5, such a Rellich identity-type estimate is, in a sense, a core result needed to approach Neumann and regularity problems, and for second-order equations it was formally shown that it translates into familiar well-posedness results with the sharp non-tangential maximal function bounds. ([May10]; see also [AAA+11].)

Following the same line of reasoning, one can build a higher order "block-type" operator $\mathbb{L}$, for which the Kato estimate (6.7) would imply a certain comparison between normal and tangential derivatives on the boundary

$$\|\partial_t^m u(\,\cdot\,,0)\|_{L^2(\mathbb{R}^n)} \approx \|\nabla_x^m u(\,\cdot\,,0)\|_{L^2(\mathbb{R}^n)}.$$

It remains to be seen whether these bounds lead to standard well-posedness results. However, we would like to emphasize that such a result would be restricted to very special, block-type, operators.

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