NONTANGENTIAL ESTIMATES ON LAYER POTENTIALS AND THE NEUMANN PROBLEM FOR HIGHER ORDER ELLIPTIC EQUATIONS

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Abstract. We solve the Neumann problem, with nontangential estimates, for higher order divergence form elliptic operators with variable t-independent coefficients. Our results are accompanied by nontangential estimates on higher order layer potentials.

1. Introduction

Consider the higher order elliptic differential operator L given by

$$Lu = (-1)^m \sum_{|\alpha|=|\beta|=m} \partial^\alpha (A_{\alpha\beta} \partial^\beta u) \tag{1}$$

for $m \geq 1$ a positive integer. The study of such operators, for $2m \geq 4$, is still fairly new. However, some results are known in the case of constant coefficients; see, for example, [26, 54, 51, 43] for some results related to those of the present paper.

In this paper we will consider coefficients $\boldsymbol{A}$ that are variable, but are bounded, elliptic, and t-independent in the sense that

$$\boldsymbol{A}(x,t) = \boldsymbol{A}(x,s) = \boldsymbol{A}(x) \quad \text{for all } x \in \mathbb{R}^n \text{ and all } s,\, t \in \mathbb{R}. \tag{2}$$

Such t-independent coefficients have been studied extensively in the second order case. In particular, layer potentials have been used extensively in this case; see [32, 33, 34] for some recent examples. In [18, 17] we generalized layer potentials to the higher order case; we will continue to use them in the present paper.

The main result of the present paper (see Theorem 1.1 below) is existence of solutions to the Neumann problem

$$\begin{cases} Lw = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \dot{\mathbf{M}}^+_{\boldsymbol{A}}\, w \ni \dot{\boldsymbol{g}}, \\ \|\widetilde{N}_+(\nabla^m w)\|_{L^2(\mathbb{R}^n)} \leq C \|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)} \end{cases} \tag{3}$$

and the rough Neumann problem (or subregularity problem)

$$\begin{cases} Lv = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \dot{\mathbf{M}}^+_{\boldsymbol{A}}\, v \ni \dot{\boldsymbol{h}}, \\ \|\widetilde{N}_+(\nabla^{m-1} v)\|_{L^2(\mathbb{R}^n)} \leq C \|\dot{\boldsymbol{h}}\|_{\dot{W}^2_{-1}(\mathbb{R}^n)} \end{cases} \tag{4}$$

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where $\dot{\mathbf{M}}^+_{\boldsymbol{A}}$ denotes the Neumann boundary value operator (given in the second order case by the boundary trace of $-\vec{e}_{n+1}\cdot\boldsymbol{A}\nabla$, and by formula (14) below or by [21, formula (2.10)] in the general case), and where $\widetilde{N}_+$ denotes the modified nontangential maximal operator; this is the natural sharp estimate on solutions to boundary value problems. This work builds on our earlier results [18, 20, 21, 19], in which we established well posedness in terms of the Lusin area integral. We will solve the problems (3) and (4) by establishing nontangential bounds on the double layer potential; we will in the process establish nontangential bounds on the single layer potential.

1.1. **Known results for the second order Neumann problem.** We begin by reviewing the history of the Neumann problem, with L^2 or $\dot{W}^2_{-1}$ boundary data, for second order operators (that is, for operators with $2m=2$).

In the case of harmonic functions, solutions to the problem

$$\Delta u = 0 \text{ in } \Omega, \qquad \nu\cdot\nabla u = g \text{ on } \partial\Omega, \qquad \|N_\Omega(\nabla u)\|_{L^2(\partial\Omega)} \leq C\|g\|_{L^2(\partial\Omega)} \tag{5}$$

for an arbitrary bounded C^1 domain Ω were constructed using the method of layer potentials in [29]. Here ν denotes the unit outward normal vector to $\partial\Omega$, and N_Ω denotes the standard nontangential maximal function $N_\Omega F(X) = \sup\{|F(Y)| : Y\in\Omega,\ \operatorname{dist}(Y,\partial\Omega) < 2|X-Y|\}$.

By the divergence theorem, if ∇u is continuous up to the boundary, $\nu\cdot\nabla u = g$ on $\partial\Omega$, and $\Delta u = \nabla\cdot\nabla u = 0$ in $\Omega\subset\mathbb{R}^{n+1}$, then

$$\int_\Omega \nabla\varphi\cdot\nabla u = \int_{\partial\Omega}\varphi\, g\, d\sigma \qquad \text{for all } \varphi\in C_0^\infty(\mathbb{R}^{n+1}). \tag{6}$$

The left hand side converges provided only that ∇u is locally integrable up to the boundary; thus, if $\Delta u = 0$ in Ω then we say that $\nu\cdot\nabla u = g$ on $\partial\Omega$ in the weak sense if the above equation is satisfied for all nice test functions φ.

In bounded Lipschitz domains, the L^2 Neumann problem for harmonic functions was shown to be well posed in [36], and in [52] it was shown that the solution to the Neumann problem may be written as a single layer potential.

We now turn to operators of the form (1) of second order with real symmetric t-independent coefficients. In the case of second order equations (but not higher order equations), a simple change of variables allows one to pass from the the half space $\mathbb{R}^{n+1}_+$ to the domain above a Lipschitz graph. This change of variables changes the coefficients in a way that preserves symmetry (or self-adjointness) and t-independence. Thus, much recent work in the second order case has considered the Neumann problem in the half space

$$\nabla\cdot\boldsymbol{A}\nabla u = 0 \text{ in } \mathbb{R}^{n+1}_+, \qquad \mathrm{M}^+_{\boldsymbol{A}}\, u = g, \qquad \|\widetilde{N}_+(\nabla u)\|_{L^2(\mathbb{R}^n)} \leq C\|g\|_{L^2(\mathbb{R}^n)} \tag{7}$$

where the Neumann boundary values $\mathrm{M}^+_{\boldsymbol{A}}\, u$ of a solution u to $\operatorname{div}\boldsymbol{A}\nabla u = 0$ are given by

$$\int_{\mathbb{R}^{n+1}_+}\nabla\varphi\cdot\boldsymbol{A}\nabla u = \int_{\mathbb{R}^n}\varphi(x,0)\,\mathrm{M}^+_{\boldsymbol{A}}\, u(x)\,dx \qquad \text{for all } \varphi\in C_0^\infty(\mathbb{R}^{n+1}). \tag{8}$$

If u and $\boldsymbol{A}$ are sufficiently smooth, then $\mathrm{M}^+_{\boldsymbol{A}}\, u = -\vec{e}_{n+1}\cdot\boldsymbol{A}\nabla u = \nu\cdot\boldsymbol{A}\nabla u$.

If the coefficients $\boldsymbol{A}$ are real, symmetric, and t-independent, then well posedness of the problem (7) was essentially established in [38]. This problem is also known to be well posed for certain classes of complex t-independent coefficients. If $\boldsymbol{A}$ is complex and constant, then the problem may be solved using the Fourier transform.

Well posedness of the Neumann problem in $\mathbb{R}^{n+1}_+$ in the case where $\boldsymbol{A}$ is a t-independent matrix in "block" form follows from the resolution of the Kato square root problem in $\mathbb{R}^n$ established in [9]. See [37, Remark 2.5.6] and [3, 7] for a discussion of block matrices. Well posedness was extended from the case of block matrices to that of block lower triangular matrices in [11].

Furthermore, well posedness of the L^2 Neumann problem in the half-space was shown in [3, 8] to be stable under t-independent L^∞ perturbations; in particular, if $\boldsymbol{A}$ is t-independent and close in L^∞ to a constant matrix, a variable self-adjoint matrix, or a variable block or block lower triangular matrix, then the Neumann problem is well posed.

The Neumann problem with L^2 boundary data is known to be ill posed for some real nonsymmetric coefficients; see the appendix to [39]. (One of the two main results of [39] is that the Neumann problem with L^p boundary data, for $p > 1$ sufficiently small, is well posed in $\mathbb{R}^2_+$.)

We now turn to the second order Neumann problem with boundary data in a negative Sobolev space $\dot{W}^2_{-1}$, that is, the dual space to the space $\dot{W}^2_1$ of functions whose gradient is square integrable.

In [54, Proposition 4.2], Verchota established well posedness of the Neumann subregularity problem for harmonic functions in Lipschitz domains

$$\Delta u = 0 \text{ in } \Omega, \qquad \mathrm{M}^\Omega_I\, u = h, \qquad \|N_\Omega u\|_{L^2(\partial\Omega)} \leq C\|h\|_{\dot{W}^2_{-1}(\partial\Omega)}. \tag{9}$$

For subregular solutions, (that is, for u with $N_\Omega u \in L^2(\partial\Omega)$ rather than $N_\Omega(\nabla u) \in L^2(\partial\Omega)$,) the definition of Neumann boundary values $\mathrm{M}^\Omega_I\, u$ must be modified, as the integral on the right hand side of formula (6) need not converge for all test functions $\varphi \in C^\infty_0(\mathbb{R}^{n+1})$; we refer the reader to [54, Definition 4.1] for a precise definition.

In [12, Section 11], it was shown that if $\boldsymbol{A}$ is real or if the ambient dimension $n+1 = 3$, (or, more generally, if all solutions u to either $\operatorname{div} \boldsymbol{A}\nabla u = 0$ or $\operatorname{div} \boldsymbol{A}^* u = 0$ satisfy the De Giorgi-Nash-Moser condition of local Hölder continuity,) then solvability of the L^2 Neumann problem (7) implies solvability of the $\dot{W}^2_{-1}$-Neumann problem

$$\nabla\cdot\boldsymbol{A}\nabla u = 0 \text{ in } \mathbb{R}^{n+1}_+, \qquad \mathrm{M}^+_{\boldsymbol{A}}\, u = h, \qquad \|\mathcal{A}^+_2(t\nabla u)\|_{L^2(\mathbb{R}^n)} \leq C\|h\|_{\dot{W}^2_{-1}(\mathbb{R}^n)}. \tag{10}$$

In this case, the estimates on solutions are given not in terms of nontangential maximal functions, but in terms of area integral estimates. (See formula (47) below for a definition of $\mathcal{A}^+_2$.) In a few cases, it has proven more convenient to solve boundary value problems posed with area integral estimates; in addition to the Neumann problem (10), see the Dirichlet problems of [3, problem (D2)], [11, Theorem 6.6], and [12, Section 11]. However, it is much more common to phrase well posedness in terms of nontangential estimates.

1.2. **Formulation of Neumann boundary values and the biharmonic Neumann problem.** We now turn to the higher order L^2 Neumann problem. Higher order Neumann boundary values are defined based on a generalization of formulas (6) and (8). It is natural to replace the left hand side with the quantity

$$\sum_{|\alpha|=|\beta|=m} \int_\Omega \partial^\alpha\varphi\, A_{\alpha\beta}\, \partial^\beta u. \tag{11}$$

However, the right hand sides may be treated in several different ways. Specifically, observe that the right hand sides of formulas (6) and (8) both involve $\varphi\big|_{\partial\Omega}$. The Dirichlet boundary values of a solution u to a second order equation in Ω are also $u\big|_{\partial\Omega}$. The Dirichlet boundary values of a solution to a higher order equation necessarily involve more than one function; a higher order analogue to formulas (6) and (8) must use the full array of Dirichlet boundary values of φ.

In [23, 27, 50], and [44, Theorem 8.8], the Dirichlet boundary values of a solution u to the biharmonic equation $\Delta^2 u = 0$ in Ω were taken to be the two functions $(u\big|_{\partial\Omega}, \partial_\nu u)$, while in [1, 41, 42] and [44, Section 9] the boundary values were taken to be $(u\big|_{\partial\Omega}, \nabla u\big|_{\partial\Omega})$. Here $\partial_\nu u = \nu \cdot \nabla u$ is the directional derivative of u in the direction of ν.

Turning to operators of order $2m$, the Dirichlet boundary values of a solution u were taken in [48, 49] and [45, Theorem 1] to be $(\{\partial^\zeta u\big|_{\partial\Omega}\}_{|\zeta|\leq m-2}, \partial_\nu^{m-1} u)$, and in [43] and [45, Theorem 2] to be $\{\partial^\xi u\big|_{\partial\Omega}\}_{|\xi|\leq m-1}$. (In smooth domains, the Dirichlet boundary values may be taken to be $(u\big|_{\partial\Omega}, \partial_\nu u, \dots, \partial_\nu^{m-1} u)$, where $\partial_\nu^j u$ is the jth directional derivative of u in the direction of ν. This formulation is somewhat more intuitive; however, in the generality of Lipschitz domains and of boundary data in Lebesgue and Sobolev spaces, complications arise if $2m > 4$. Indeed the papers [23, 53, 40], which state their main results in terms of the Dirichlet boundary values $(u\big|_{\partial\Omega}, \partial_\nu u, \dots, \partial_\nu^{m-1} u)$, stated their Dirichlet boundary condition with the cumbersome formulation $\partial_\nu^j u = \sum_{|\xi|=j} \frac{j!}{\xi!} \nu^\xi f_\xi$ for a Whitney array of functions $\dot{\boldsymbol{f}}$; this condition implies that $\partial^\xi u\big|_{\partial\Omega} = f_\xi$, and so is implicitly taking $\{\partial^\xi u\big|_{\partial\Omega}\}_{|\xi|\leq m-1}$ to be the Dirichlet boundary values.)

The authors of the present paper hope one day to solve the Dirichlet problem for higher order operators with self-adjoint t-independent coefficients in the half space; we believe that in that situation, the most convenient formulation of Dirichlet boundary values will be $\{\partial^\gamma u\big|_{\partial\Omega}\}_{|\gamma|=m-1}$.

We remark that all of the above formulations of Dirichlet boundary values are equivalent; different formulations may be chosen for reasons of convenience or clarity.

Thus, the Neumann boundary values of a solution u to $Lu = 0$ in Ω, for L an operator of order $2m$ associated to coefficients $\boldsymbol{A}$, and for solutions u with $\nabla^m u$ locally integrable up to the boundary, may be taken to be $\vec{\mathrm{M}}^\Omega_{\boldsymbol{A}} u$, $\ddot{\mathbf{M}}^\Omega_{\boldsymbol{A}}\, u$, or $\dot{\mathbf{M}}^\Omega_{\boldsymbol{A}}\, u$, where

$$\sum_{|\alpha|=|\beta|=m} \int_\Omega \partial^\alpha \varphi\, A_{\alpha\beta}\, \partial^\beta u = \sum_{j=0}^{m-1} \int_{\partial\Omega} \partial_\nu^j \varphi\, (\vec{\mathrm{M}}^\Omega_{\boldsymbol{A}} u)_j \, d\sigma, \tag{12}$$

$$\ddot{\boldsymbol{M}} \in \ddot{\mathbf{M}}^\Omega_{\boldsymbol{A}}\, u \quad \text{if} \quad \sum_{|\alpha|=|\beta|=m} \int_\Omega \partial^\alpha \varphi\, A_{\alpha\beta}\, \partial^\beta u = \sum_{|\xi|\leq m-1} \int_{\partial\Omega} \partial^\xi \varphi\, M_\xi \, d\sigma, \tag{13}$$

$$\dot{\boldsymbol{M}} \in \dot{\mathbf{M}}^\Omega_{\boldsymbol{A}}\, u \quad \text{if} \quad \sum_{|\alpha|=|\beta|=m} \int_\Omega \partial^\alpha \varphi\, A_{\alpha\beta}\, \partial^\beta u = \sum_{|\gamma|=m-1} \int_{\partial\Omega} \partial^\gamma \varphi\, M_\gamma \, d\sigma, \tag{14}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$.

As observed above, in the generality of Lipschitz domains, the formulation (12) is impractical unless $2m = 2$ or $2m = 4$. However, one significant benefit is that $\vec{\mathrm{M}}^\Omega_{\boldsymbol{A}} u$ as given by formula (12) is a single vector of distributions. It was shown in [55, formulas (4) and (5)] that if $m = 2$ and $\boldsymbol{A}$ is constant, and if $Lu = 0$ in $\Omega \subset \mathbb{R}^{n+1}$,

then

$$(\vec{\mathrm{M}}^{\Omega}_{\boldsymbol{A}} u)_0 = -\sum_{|\alpha|=|\beta|=2} A_{\alpha\beta}\Big(\nu_{\alpha'}\partial^{\alpha''}\partial^{\beta}u + \sum_{k=1}^{n+1}\frac{\partial}{\partial\tau_{k\alpha'}}(\nu_k\nu_{\alpha''}\partial^{\beta}u)\Big), \tag{15}$$
$$(\vec{\mathrm{M}}^{\Omega}_{\boldsymbol{A}} u)_1 = \sum_{|\alpha|=|\beta|=2} A_{\alpha\beta}\nu^{\alpha}\partial^{\beta}u$$

where $\alpha = \alpha' + \alpha''$ for some unit coordinate vectors α' and α'', and where $\frac{\partial}{\partial\tau_{k\ell}}$ satisfies $\frac{\partial}{\partial\tau_{k\ell}}\varphi = \nu_k\partial_\ell\varphi - \nu_\ell\partial_k\varphi$ for all functions φ smooth in a neighborhood of $\partial\Omega$ and extends in a natural way to Sobolev functions on the boundary. It was observed in [55] that $(\vec{\mathrm{M}}^{\Omega}_{\boldsymbol{A}} u)_0$ is independent of the particular choice of α', α''.

The Neumann boundary values of [55] generalize the biharmonic Neumann boundary values of [54]. Specifically, the boundary values of [54] are given by $\vec{\mathrm{M}}^{\Omega}_{\boldsymbol{A}_\rho} u$, where $\boldsymbol{A}_\rho$ is the constant coefficient matrix that satisfies

$$\sum_{|\alpha|=|\beta|=2} \partial^{\alpha}\varphi\,(A_\rho)_{\alpha\beta}\,\partial^{\beta}u = \rho\,\Delta\varphi\,\Delta u + (1-\rho)\sum_{j=1}^{n+1}\sum_{k=1}^{n+1}\partial^2_{x_jx_k}\varphi\,\partial^2_{x_jx_k}u \tag{16}$$

for all twice differentiable functions φ and u. We observe that if $\rho \in \mathbb{R}$ is any constant, then the operator L given by formula (1) with $\boldsymbol{A} = \boldsymbol{A}_\rho$ is the biharmonic operator $L = \Delta^2$.

The biharmonic L^2-Neumann problem

$$\begin{cases} \Delta^2 u = 0 \text{ in } \Omega, \qquad \vec{\mathrm{M}}^{\Omega}_{\boldsymbol{A}_\rho} u = (\Lambda, f) \text{ on } \partial\Omega, \\ \|N_\Omega(\nabla^2 u)\|_{L^2(\partial\Omega)} \le C\|f\|_{L^2(\partial\Omega)} + C\|\Lambda\|_{\dot{W}^2_{-1}(\partial\Omega)}, \end{cases} \tag{17}$$

was shown to be well posed in [54] in Lipschitz domains $\Omega \subset \mathbb{R}^{n+1}$ of arbitrary dimension for $-1/n \le \rho < 1$. The case of planar C^1 domains, with $\rho = -1$ and with Neumann boundary values as in formula (13), was studied earlier in [24]. The biharmonic subregularity problem

$$\begin{cases} \Delta^2 u = 0 \text{ in } \Omega, \qquad \vec{\mathrm{M}}^{\Omega}_{\boldsymbol{A}_\rho} u = (\Lambda, f) \text{ on } \partial\Omega, \\ \|N(\nabla u)\|_{L^2(\partial\Omega)} \le C\|\vec{f}\|_{\dot{W}^2_{-1}(\partial\Omega)} + C\|\Lambda\|_{\dot{W}^2_{-2}(\partial\Omega)} \end{cases} \tag{18}$$

was solved in [54, Section 22] in Lipschitz domains; as in the problem (9), the definition of $\vec{\mathrm{M}}^{\Omega}_{\boldsymbol{A}_\rho}$ was modified to allow for u with $\nabla^2 u$ not locally integrable up to the boundary.

We now consider the formulations of Neumann boundary values (13) and (14). These formulations may be used for operators of arbitrary order. However, if $2m \ge 4$ then $\ddot{\mathbf{M}}^{\Omega}_{\boldsymbol{A}} u$ and $\dot{\mathbf{M}}^{\Omega}_{\boldsymbol{A}} u$ as given by (13) and (14) are both sets of many arrays of distributions.

In particular, consider the upper half space. (This is the setting of [18, 20, 21, 19], and of the present paper.) Let $\dot{\mathbf{M}}^{+}_{\boldsymbol{A}} = \dot{\mathbf{M}}^{\Omega}_{\boldsymbol{A}}$ and $\ddot{\mathbf{M}}^{+}_{\boldsymbol{A}} = \ddot{\mathbf{M}}^{\Omega}_{\boldsymbol{A}}$ for $\Omega = \mathbb{R}^{n+1}_{+}$. Then

$\dot{\boldsymbol{M}} \in \dot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$ and $\ddot{\boldsymbol{M}} \in \ddot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$ if and only if

$$(19)\quad -\sum_{\substack{\zeta\in(\mathbb{N}_0)^n\\ |\zeta|=m-1-\ell}} \partial_\parallel^\zeta(\dot{\boldsymbol{M}})_{(\zeta,\ell)} = \sum_{\substack{\zeta\in(\mathbb{N}_0)^n\\ |\zeta|\le m-1-\ell}} (-1)^{m-\ell-|\zeta|}\partial_\parallel^\zeta(\ddot{\boldsymbol{M}})_{(\zeta,\ell)}$$
$$= \lim_{t\to 0^+}\sum_{\substack{\xi\in(\mathbb{N}_0)^n\\ |\xi|\le m-1-\ell}} \partial_\parallel^\xi\big(\boldsymbol{A}\nabla^m\partial_t^{m-1-\ell-|\xi|}u(\,\cdot\,,t)\big)_{(\xi,m-|\xi|)}$$

in the weak distributional sense on $\mathbb{R}^n$, for all integers ℓ with $0\le\ell\le m-1$. See Appendix A. For example, if

$$(20)\quad (\ddot{\boldsymbol{N}})_{(\zeta,\ell)} = (-1)^{m-\ell-|\zeta|}\lim_{t\rightharpoonup 0^+}\big(\boldsymbol{A}\nabla^m\partial_t^{m-1-\ell-|\zeta|}u(\,\cdot\,,t)\big)_{(\zeta,m-|\zeta|)},$$

$$(21)\quad (\ddot{\boldsymbol{P}})_{(\vec{0},\ell)} = (-1)^{m-\ell}\lim_{t\rightharpoonup 0^+}\sum_{\substack{\xi\in(\mathbb{N}_0)^n\\ |\xi|\le m-1-\ell}} \partial_\parallel^\xi\Big(\big(\boldsymbol{A}\nabla^m\partial_t^{m-1-\ell-|\xi|}u(\,\cdot\,,t)\big)_{(\xi,m-|\xi|)}\Big),$$

$$(\ddot{\boldsymbol{P}})_{(\zeta,\ell)} = 0 \text{ if } |\zeta|\ge 1,$$

then both $\ddot{\boldsymbol{N}}$ and $\ddot{\boldsymbol{P}}$ lie in $\ddot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$. Many other representatives may be easily constructed.

The paper [43] considered the Neumann problem, with the formulation (13), in Lipschitz domains. They also produced representatives of Neumann boundary data. By [43, formula (4.54)], if $\boldsymbol{A}$ is constant, and if $Lu=0$ in a bounded Lipschitz domain Ω, then

$$\sum_{|\alpha|=|\beta|=m}\int_\Omega \partial^\alpha\varphi\, A_{\alpha\beta}\,\partial^\beta u$$
$$= \sum_{|\alpha|=|\beta|=m}\sum_{k=1}^m (-1)^{k+1}\int_{\partial\Omega}\nu_{j_k^\alpha}\partial^{\sum_{r=k+1}^m \vec{e}_{j_r^\alpha}}\varphi\,\partial^{\sum_{r=1}^{k-1}\vec{e}_{j_r^\alpha}}A_{\alpha\beta}\,\partial^\beta u\,d\sigma$$

where $\alpha=\sum_{r=1}^m \vec{e}_{j_r^\alpha}$ and each $\vec{e}_{j_r^\alpha}$ is a unit coordinate vector. Letting $\theta_{m-k}(\alpha)=\sum_{r=k+1}^m \vec{e}_{j_r^\alpha}$ and $\ell_r^\alpha = j_{m-r}^\alpha$, this implies that if

$$(22)\quad (M_\theta)_\gamma = \sum_{\alpha\in(\theta_{|\gamma|})^{-1}(\gamma)}\ \sum_{|\beta|=m}(-1)^{m-|\gamma|+1}\nu_{\ell^\alpha_{|\gamma|}}\partial^\gamma\varphi\,\partial^{\alpha-\gamma-\vec{e}_{\ell^\alpha_{|\gamma|}}}A_{\alpha\beta}\,\partial^\beta u$$

then $\ddot{\boldsymbol{M}}_\theta \in \ddot{\mathbf{M}}^\Omega_{\boldsymbol{A}}\, u$. We remark that different choices of the functions θ_ℓ yield different representatives $\ddot{\boldsymbol{M}}_\theta$ of $\ddot{\mathbf{M}}^\Omega_{\boldsymbol{A}}\, u$. Indeed arbitrary affine combinations of the arrays $\ddot{\boldsymbol{M}}_\theta$ are also representatives of $\ddot{\mathbf{M}}^\Omega_{\boldsymbol{A}}\, u$. (The representative used in [43, Proposition 4.3] and the definition of the double layer potential in [43, formula (4.57)] may be obtained by averaging over all possible choices of θ.)

We would like to emphasize that neither $\ddot{\boldsymbol{N}}$ nor $\ddot{\boldsymbol{P}}$, nor any specific $\ddot{\boldsymbol{M}}_\theta$, may be taken as the Neumann boundary values of u. By [43, Theorem 6.36], if $\boldsymbol{A}$ is constant and elliptic and Ω is a bounded, simply connected Lipschitz domain, then for every $\ddot{\boldsymbol{\Lambda}}$ in an appropriate Besov space, there is exactly one solution u with $Lu=0$ in Ω and $\ddot{\boldsymbol{\Lambda}}\in\ddot{\mathbf{M}}^\Omega_{\boldsymbol{A}}\, u$. In particular, if $\ddot{\mathbf{M}}^\Omega_{\boldsymbol{A}}\, u$ has multiple representatives then $\ddot{\boldsymbol{\Lambda}}$ cannot be presumed equal to any specific representative $\ddot{\boldsymbol{M}}_\theta$ of $\ddot{\mathbf{M}}^\Omega_{\boldsymbol{A}}\, u$. The

same result may be easily proven in $\mathbb{R}^{n+1}_+$ for variable coefficients using the Lax-Milgram lemma. A similar issue appears in the statements of [24, Theorems (3.3.4) and (3.4.2)].

In [19] and the present paper, we used the formulation (14) of Neumann boundary values. In using $\dot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$ rather than $\ddot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$, we lose natural explicit formulas for representatives such as (20), (21) or (22). However, we do gain some homogeneity.

The different components of $\vec{\mathrm{M}}^{\Omega}_{\boldsymbol{A}} u$ and $\ddot{\boldsymbol{N}}$, $\ddot{\boldsymbol{P}}$ and $\ddot{\boldsymbol{M}}_\theta$ in formulas (15), (20), (21), and (22) involve different numbers of derivatives applied to the solution u, and thus must in principle lie in different smoothness spaces. (Observe the presence of the Lebesgue and Sobolev spaces $L^2(\partial\Omega)$ and $\dot{W}^2_{-1}(\partial\Omega)$ in the problem (17), and of the two different Sobolev spaces $\dot{W}^2_{-1}(\partial\Omega)$ and $\dot{W}^2_{-2}(\partial\Omega)$ in the problem (18).) Furthermore, many components involve more than m derivatives applied to u, and so negative smoothness spaces must be invoked even for the standard Neumann problem, and not only for the subregular Neumann problem.

By using $\dot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$, we may expect to find representatives whose components all lie in the same smoothness space. In particular, it was shown in [21] that if $Lu = 0$ in $\mathbb{R}^{n+1}_+$, where L is given by formula (1) with t-independent coefficients $\boldsymbol{A}$, and if $\mathcal{A}^+_2(t\nabla^m\partial_t u) + \widetilde{N}_+(\nabla^m u) \in L^2(\mathbb{R}^n)$, then

$$\Big|\sum_{|\alpha|=|\beta|=m}\int_{\mathbb{R}^{n+1}_+}\partial^\alpha\varphi\, A_{\alpha\beta}\,\partial^\beta u\Big| \le C\|\nabla^{m-1}\varphi(\,\cdot\,,0)\|_{L^2(\mathbb{R}^n)}\|\mathcal{A}^+_2(t\nabla^m\partial_t u) + \widetilde{N}_+(\nabla^m u)\|_{L^2(\mathbb{R}^n)} \tag{23}$$

and so $\dot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$ has a representative in $L^2(\mathbb{R}^n)$.

1.3. **The Neumann problem for t-independent operators in the half space.** In [19], we established well posedness of the L^2 and $\dot{W}^2_{-1}$ Neumann problems. Specifically, suppose that L is an operator of the form (1) associated to coefficients $\boldsymbol{A}$ that are bounded, t-independent in the sense of formula (2), self-adjoint in the sense that $A_{\alpha\beta}(x) = \overline{A_{\beta\alpha}(x)}$ for all $|\alpha| = |\beta| = m$, and satisfy the ellipticity condition

$$\mathrm{Re}\sum_{|\alpha|=|\beta|=m}\int_{\mathbb{R}^n}\overline{\partial^\alpha\varphi(x,t)}\,A_{\alpha\beta}(x)\,\partial^\beta\varphi(x,t)\,dx \ge \lambda\|\nabla^m\varphi(\,\cdot\,,t)\|^2_{L^2(\mathbb{R}^n)} \tag{24}$$

for all φ smooth and compactly supported in $\mathbb{R}^{n+1}$ and all $t \in \mathbb{R}$. Then by [19, Theorems 1.1 and 1.2], the Neumann problem

$$\begin{cases} Lw = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \dot{\mathbf{M}}^+_{\boldsymbol{A}}\, w \ni \dot{\boldsymbol{g}}, \\ \|\mathcal{A}^+_2(t\nabla^m\partial_t w)\|_{L^2(\mathbb{R}^n)} + \sup\limits_{t>0}\|\nabla^m w(,\cdot\,,t)\|_{L^2(\mathbb{R}^n)} \le C\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)} \end{cases} \tag{25}$$

and the rough Neumann problem

$$\begin{cases} Lv = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ \dot{\mathbf{M}}^+_{\boldsymbol{A}}\, v \ni \dot{\boldsymbol{h}}, \\ \|\mathcal{A}^+_2(t\nabla^m v)\|_{L^2(\mathbb{R}^n)} \le C\|\dot{\boldsymbol{h}}\|_{\dot{W}^2_{-1}(\mathbb{R}^n)}. \end{cases} \tag{26}$$

are well posed. That is, if $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n)$ or $\dot{\boldsymbol{h}} \in \dot{W}^2_{-1}(\mathbb{R}^n)$, then there is a solution w or v to the problem (25) or (26), and this solution is unique up to adding polynomials of degree $m-1$.

Observe that the gradient $\nabla^m w$ of the solution w to the problem (25) is locally integrable up to $\partial\mathbb{R}^{n+1}_+$, and so $\dot{\mathbf{M}}^+_{\boldsymbol{A}}\, w$ is given by formula (14) with $\Omega = \mathbb{R}^{n+1}_+$. The gradient $\nabla^m v$ of the solution v to the problem (26) is not assumed to be locally integrable up to the boundary; it is only assumed to satisfy

$$\int_{\mathbb{R}^n} \mathcal{A}^+_2(t\nabla^m v)^2 = c_n \int_{\mathbb{R}^n}\int_0^\infty |\nabla^m v(x,t)|^2\, t\, dt\, dx < \infty.$$

In this case, like the harmonic and biharmonic subregularity problems (9) and (18), the notion of Neumann boundary value must be modified somewhat; we refer the reader to [21, Section 2.3.2] or [19, formula (2.7)] for the necessary generalization.

We remind the reader that it is somewhat unusual to formulate boundary value problems in terms of area integrals. Of the results mentioned above, [29, 36, 52, 27, 53, 44, 45, 38, 39, 54, 48, 49, 50, 51, 32, 33, 34] stated their main theorems in terms of nontangential maximal estimates, while [8] established both square function and area integral estimates, and [3, 11, 12] formulated solutions for some problems in terms of nontangential estimates and others in terms of square function estimates.

Thus, one of the two main results of this paper is the addition of nontangential estimates to the higher order Neumann problem.

Theorem 1.1. *Suppose that L is an elliptic operator of the form* (1) *of order $2m$ associated with coefficients $\boldsymbol{A}$ that satisfy $\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^n)} = \Lambda < \infty$ and the ellipticity condition* (24)*, are t-independent in the sense of formula* (2)*, and are self-adjoint, that is, satisfy $A_{\alpha\beta}(x) = \overline{A_{\beta\alpha}(x)}$.*

Let $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n)$ and $\dot{\boldsymbol{h}} \in \dot{W}^2_{-1}(\mathbb{R}^n)$, and let w and v be the solutions to the problems (25) *and* (26)*, respectively.*

There is a constant C, depending only on Λ, the ambient dimension $n+1$, order $2m$ of the operator L, and the ellipticity constant λ in the bound (24)*, such that*

$$\|\widetilde{N}_+(\nabla^m w)\|_{L^2(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)}.$$

Recall that v is unique up to adding polynomials of degree $m-1$. There is some such additive normalization of v such that the bound

$$\|\widetilde{N}_+(\nabla^{m-1} v)\|_{L^2(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{h}}\|_{\dot{W}^2_{-1}(\mathbb{R}^n)}$$

is valid.

1.4. **Layer potentials.** The proof of Theorem 1.1 is as follows. An examination of the proofs of [19, Theorems 1.1 and 1.2] in [19, Section 7] reveals that

$$w = \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}} \quad \text{and} \quad v = \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}},$$

where $\mathcal{D}^{\boldsymbol{A}}$ is the higher order double layer potential introduced in [18, 17] (and defined in formula (52) below), and where $\dot{\boldsymbol{\varphi}} = (\dot{\mathbf{M}}^+_{\boldsymbol{A}}\,\mathcal{D}^{\boldsymbol{A}})^{-1}\dot{\boldsymbol{g}}$ and $\dot{\boldsymbol{f}} = (\dot{\mathbf{M}}^+_{\boldsymbol{A}}\,\mathcal{D}^{\boldsymbol{A}})^{-1}\dot{\boldsymbol{h}}$ lie in the Whitney spaces $\dot{W}A^2_{m-1,1}(\mathbb{R}^n)$ and $\dot{W}A^2_{m-1,0}(\mathbb{R}^n)$, respectively, used in [19] (see Definition 50 below). We remark that this is a departure from tradition, as the Neumann problem has often been solved using the single layer potential.

Theorem 1.1 then follows from the bounds

$$\|\widetilde{N}_+(\nabla^m \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}})\|_{L^2(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^2_{m-1,1}(\mathbb{R}^n)},$$
$$\|\widetilde{N}_+(\nabla^{m-1} \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}})\|_{L^2(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{f}}\|_{\dot{W}A^2_{m-1,0}(\mathbb{R}^n)}.$$

Thus, the double layer potential is of great interest in the theory of the higher order Neumann problem.

The related single layer potential $\mathcal{S}^L$ is also of interest. It is often possible to use bounds on the single layer potential $\mathcal{S}^L$ to establish bounds on the double layer potential $\mathcal{D}^{\boldsymbol{A}}$; see, for example, Section 5.1 below. Bounds on the single layer potential were used to establish the bound (23), and in fact all of the Fatou type results of [21]. In the second order case, the single layer potential has long been used to establish well posedness of boundary value problems. In particular, invertibility of the operator $\nu \cdot \nabla \mathcal{S}^\Delta$ was used to solve the harmonic Neumann problem (5) in [29]. Invertibility of the operators $\mathrm{M}^+_{\boldsymbol{A}}\mathcal{S}^L$ and $\mathcal{S}^L\big|_{\partial\mathbb{R}^{n+1}_+}$ have been used in [52, 3, 15, 34, 33] to solve the second order Neumann problem (7) and the Dirichlet regularity problem

$$\nabla \cdot \boldsymbol{A}\nabla u = 0 \text{ in } \mathbb{R}^{n+1}_+, \quad u = f \text{ on } \partial\mathbb{R}^{n+1}_+, \quad \|\widetilde{N}_+(\nabla u)\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{\dot{W}^2_1(\mathbb{R}^n)}$$

for certain coefficients $\boldsymbol{A}$. In many of the above papers, the double layer potential was used to solve the Dirichlet problem with boundary data in L^2 and with the estimate $\|\widetilde{N}_+ u\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}$; however, in recent years it has been recognized that it is somewhat more natural to use the double layer potential to solve the Neumann problem and the single layer potential to solve the Dirichlet problem, regardless of the smoothness of the boundary data. See [32, Theorem 1.1], [17, Section 6], [4, Section 7.4], and [13, Theorem 1.11]. We hope that in future work, we may use the single layer potential to solve the higher order Dirichlet problem.

Thus, nontangential bounds on layer potentials are of independent interest. The following theorem is the second main result of this paper; note that Theorem 1.1 follows from Theorem 1.2, and in particular from the bounds (29) and (30).

Theorem 1.2. *Suppose that L is an elliptic operator of the form* (1) *of order $2m$ associated with coefficients $\boldsymbol{A}$ that satisfy the ellipticity conditions* (43) *and* (44) *and are t-independent in the sense of formula* (2).

Then there is an $\varepsilon > 0$, depending only on the dimension $n+1$, the order $2m$ of the operator L, and the constants λ and Λ in the bounds (43) *and* (44)*, with the following significance.*

If $2-\varepsilon < p < 2+\varepsilon$, then there is a constant C_p such that the bounds

$$\|\widetilde{N}_+(\nabla^m \mathcal{S}^L \dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}, \qquad 2-\varepsilon < p < 2+\varepsilon, \tag{27}$$

$$\|\widetilde{N}_+(\nabla^{m-1} \mathcal{S}^L_\nabla \dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}, \qquad 2-\varepsilon < p < 2+\varepsilon, \tag{28}$$

$$\|\widetilde{N}_+(\nabla^m \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)}, \qquad 2-\varepsilon < p < 2+\varepsilon, \tag{29}$$

$$\|\widetilde{N}_+(\nabla^{m-1} \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{f}}\|_{\dot{W}A^p_{m-1,0}(\mathbb{R}^n)}, \qquad 2-\varepsilon < p < 2+\varepsilon \tag{30}$$

are valid for all $\dot{\boldsymbol{g}} \in L^p(\mathbb{R}^n)$, $\dot{\boldsymbol{h}} \in L^p(\mathbb{R}^n)$, $\dot{\boldsymbol{f}} \in \dot{W}A^p_{m-1,0}(\mathbb{R}^n)$ and $\dot{\boldsymbol{\varphi}} \in \dot{W}A^p_{m-1,1}(\mathbb{R}^n)$.

Here $\dot{W}A^p_{m-1,1}(\mathbb{R}^n)$ and $\dot{W}A^p_{m-1,0}(\mathbb{R}^n)$ are closed proper subsets of $\dot{W}^p_1(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$, respectively; these subsets are the natural domain of $\mathcal{D}^{\boldsymbol{A}}$ in those spaces.

See Definition 50. The modified single layer potential $\mathcal{S}^L_\nabla$ is the higher order analogue of the operator $\mathcal{S}^L\nabla$ used in [3, 34, 33]. We will define $\mathcal{S}^L$ and $\mathcal{S}^L_\nabla$ in Section 2.5. We remark (see formula (74) below) that if $\dot{\boldsymbol{g}} \in \dot{W}^2_{-1}(\mathbb{R}^n)$, then there is an $\dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n)$ with $\|\dot{\boldsymbol{g}}\|_{\dot{W}^2_{-1}(\mathbb{R}^n)} \approx \|\dot{\boldsymbol{h}}\|_{L^2(\mathbb{R}^n)}$ and with $\nabla^m \mathcal{S}^L \dot{\boldsymbol{g}} = -\nabla^m \mathcal{S}^L_\nabla \dot{\boldsymbol{h}}$. Thus, formula (28) gives a bound on the standard single layer potential with inputs in a negative smoothness space.

We now summarize the known bounds on higher order layer potentials. We will use these bounds to establish the nontangential estimates of Theorem 1.2. By definition (see formulas (52) and (54) below), we have the bounds

$$\|\nabla^m \mathcal{S}^L \dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^{n+1})} \leq C\|\dot{\boldsymbol{g}}\|_{\dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)}, \qquad \|\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}}\|_{L^2(\mathbb{R}^{n+1}_\pm)} \leq C\|\dot{\boldsymbol{f}}\|_{\dot{W}A^2_{m-1,1/2}(\mathbb{R}^n)}$$

for all $\dot{\boldsymbol{g}} \in \dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$ and all $\dot{\boldsymbol{f}} \in \dot{W}A^2_{m-1,1/2}(\mathbb{R}^n)$. The main result of [18] is that the double and single layer potentials extend by density to operators that satisfy the bounds

$$\|\mathcal{A}_2^+(t\nabla^m \partial_t \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}})\|_{L^2(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^2_{m-1,1}(\mathbb{R}^n)}, \tag{31}$$

$$\|\mathcal{A}_2^+(t\nabla^m \partial_t \mathcal{S}^L \dot{\boldsymbol{g}})\|_{L^2(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)} \tag{32}$$

for all $\dot{\boldsymbol{\varphi}} \in \dot{W}A^2_{m-1,1}(\mathbb{R}^n)$ and all $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n)$.

In [20, Theorem 1.6], it was shown that if $\dot{\boldsymbol{f}} \in \dot{W}A^2_{m-1,0}(\mathbb{R}^n)$, then

$$\|\mathcal{A}_2^+(t\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}})\|_{L^2(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{f}}\|_{\dot{W}A^2_{m-1,0}(\mathbb{R}^n)}. \tag{33}$$

Finally, in [20, Theorem 1.13], the bound (32) was extended to $\dot{\boldsymbol{g}} \in L^p$ for some $p < 2$, and a bound on $\mathcal{S}^L_\nabla$ was established. Specifically, it was shown that there was some $\varepsilon > 0$ such that, if $2 - \varepsilon < p \leq 2$, then there is a C_p such that for all $\dot{\boldsymbol{g}} \in L^p(\mathbb{R}^n)$ and all $\dot{\boldsymbol{h}} \in L^p(\mathbb{R}^n)$,

$$\|\mathcal{A}_2^+(t\nabla^m \partial_t \mathcal{S}^L \dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}, \qquad 2 - \varepsilon < p \leq 2, \tag{34}$$

$$\|\mathcal{A}_2^+(t\nabla^m \mathcal{S}^L_\nabla \dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}, \qquad 2 - \varepsilon < p \leq 2. \tag{35}$$

Observe that these known bounds all involve inputs in L^p for $p = 2$ or $p < 2$. In the course of proving Theorem 1.2, we will also establish the following area integral estimates for inputs in L^p with $p > 2$.

Theorem 1.3. *Let L and $\boldsymbol{A}$ be as in Theorem 1.2. Then there is an $\varepsilon > 0$ such that, if $2 < p < 2 + \varepsilon$, then there is a constant C_p such that*

$$\|\mathcal{A}_2^+(t\nabla^m \partial_t \mathcal{S}^L \dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}, \qquad 2 < p < 2 + \varepsilon, \tag{36}$$

$$\|\mathcal{A}_2^+(t\nabla^m \mathcal{S}^L_\nabla \dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}, \qquad 2 < p < 2 + \varepsilon, \tag{37}$$

$$\|\mathcal{A}_2^+(t\nabla^m \partial_t \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)}, \qquad 2 < p < 2 + \varepsilon, \tag{38}$$

$$\|\mathcal{A}_2^+(t\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{f}}\|_{\dot{W}A^p_{m-1,0}(\mathbb{R}^n)}, \qquad 2 < p < 2 + \varepsilon \tag{39}$$

whenever $\dot{\boldsymbol{g}} \in L^p(\mathbb{R}^n)$, $\dot{\boldsymbol{h}} \in L^p(\mathbb{R}^n)$, $\dot{\boldsymbol{f}} \in \dot{W}A^p_{m-1,0}(\mathbb{R}^n)$, and $\dot{\boldsymbol{\varphi}} \in \dot{W}A^p_{m-1,1}(\mathbb{R}^n)$.

In a forthcoming paper, we will show that the bounds (38) and (39) extend to the case $2 - \varepsilon < p < 2$ (that is, we will establish the analogues to the bounds (34) and (35) for the double layer potential).

The theory of layer potentials for higher order operators is still relatively new, and thus to our knowledge the above represents a nearly comprehensive survey

of bounds on layer potentials for operators of order $2m \geq 4$ with t-independent coefficients in the half-space. (Some additional bounds on $\partial_t^k \mathcal{S}^L$ and $\partial_t^k \mathcal{S}^L_\nabla$, for k large enough, were established in [20] and used in [21].)

However, the theory of variable coefficient higher order operators builds on the extensive and well developed theory of second order operators (that is, the case $2m = 2$) and the reasonably well developed theory of constant coefficient higher order operators.

In the special case of constant coefficient operators (in particular, in the theory of harmonic functions) in Lipschitz domains, boundedness of layer potentials follows from boundedness of the Cauchy integral on Lipschitz curves; the Cauchy integral was famously bounded by Coifmann, McIntosh and Meyer in [25]. Layer potentials for the Laplacian $-\Delta$ were used in [29, 52, 28, 44, 30, 56], for the biharmonic operator Δ^2 in [23, 24, 54, 42], and for general higher order constant coefficient equations in [2, 43].

In the case of second order operators with variable t-independent coefficients, bounds on layer potentials were established in [39, 47, 15] in two dimensions for real (or almost real) coefficients.

Turning to higher dimensions, in [3] the $p = 2$ cases of the bounds (27) and (36) were established for operators of order $2m = 2$ with real symmetric t-independent coefficients, and a stability result under L^∞ perturbation was established. (The authors also established numerous more specialized bounds on layer potentials.) In [46], Rosén showed that layer potentials coincide with certain operators appearing in the theory of semigroups investigated in [7, 8, 6]. In particular, numerous bounds in the $p = 2$ case follow.

The theory of boundary value problems and layer potentials for second order operators was subsequently investigated extensively in the case where $L = -\operatorname{div} \boldsymbol{A}\nabla$ and $L^* = -\operatorname{div} \boldsymbol{A}^*\nabla$ satisfy the De Giorgi-Nash-Moser condition; this condition is always satisfied if the ambient dimension $n + 1$ satisfies $n + 1 = 2$, if $n + 1 = 3$ and $\boldsymbol{A}$ is t-independent, or if $2m = 2$ and $\boldsymbol{A}$ is real valued. In these cases, it is often possible to establish at least some bounds on layer potentials using the theory of Calderón-Zygmund operators with kernels that satisfy Littlewood-Paley estimates. See, for example, [3, Section 8] or [39, Section 4].

In particular, the $p = 2 = 2m$ case of the bounds (36) and (27) on the single layer potential $\mathcal{S}^L$ were established in [31] directly using Tb theorems, without recourse to the theory of semigroups used in [46]. Building on this bound, the $2m = 2$ case of all eight of the bounds (27)–(30) and (36)–(39) may be found in [12, 34, 33, 32] for a fairly broad range of p.

Finally, returning to the theory of semigroups, if $2m = 2$ then these eight bounds were established in [14, Theorem 12.7] without assuming the De Giorgi-Nash-Moser condition, that is, using only boundedness, ellipticity and t-independence of the coefficients.

1.5. **Outline.** The organization of this paper is as follows.

In Section 2 we will define our terminology. In Section 3 we will recall some known estimates on solutions that we will use extensively throughout the paper, and will prove a few lemmas involving the nontangential and area integral estimates of a general solution u to $Lu = 0$. In particular, given the known area integral estimates (31)–(35) and the nontangential estimates of Theorem 1.2, most of the

work involved in establishing the area integral estimates of Theorem 1.3 is contained in Lemma 3.7.

Section 4 will be devoted to the nontangential bounds (27) and (28) on the single layer potential (and modified single layer potential). Section 5 will mainly be concerned with establishing the nontangential estimate (29) on the double layer potential; the bound (30) (and the area integral bounds (38) and (39)) follow fairly quickly once this bound is established. We remark that we will establish area integral bounds (36) and (37) on the single layer potential in Section 4 using the nontangential bounds (27) and (28), and will use these area integral bounds in order to establish preliminary estimates on the double layer potential.

2. Definitions

In this section, we will provide precise definitions of the notation and concepts used throughout this paper.

We will work with elliptic operators L of order $2m$ in the divergence form (1) acting on functions defined on $\mathbb{R}^{n+1}$.

As usual, we let $B(X,r)$ denote the ball in $\mathbb{R}^n$ of radius r and center X. We let $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^{n+1}_-$ denote the upper and lower half-spaces $\mathbb{R}^n \times (0,\infty)$ and $\mathbb{R}^n \times (-\infty,0)$; we will identify $\mathbb{R}^n$ with $\partial\mathbb{R}^{n+1}_\pm$. If $Q \subset \mathbb{R}^n$ or $Q \subset \mathbb{R}^{n+1}$ is a cube, we will let $\ell(Q)$ be its side length, and we let cQ be the concentric cube of side length $c\ell(Q)$. If E is a set of finite measure, we let

$$\fint_E f(x)\,dx = \frac{1}{|E|}\int_E f(x)\,dx.$$

If $f \in L^1_{loc}(\mathbb{R}^n)$, then the Hardy-Littlewood maximal function $\mathcal{M}f$ is given by

$$\mathcal{M}f(x) = \sup_{Q\ni x} \fint_Q |f| \tag{40}$$

where the supremum is over all cubes $Q \subset \mathbb{R}^n$ with $x \in Q$.

If E is a measurable set, we will let $\mathbf{1}_E$ denote the characteristic function of E; that is, $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ if $x \notin E$. We will use $\mathbf{1}_\pm$ as a shorthand for $\mathbf{1}_{\mathbb{R}^{n+1}_\pm}$.

2.1. Multiindices and arrays of functions.

We will routinely work with multiindices in $(\mathbb{N}_0)^{n+1}$. (We will occasionally work with multiindices in $(\mathbb{N}_0)^n$.) Here $\mathbb{N}_0$ denotes the nonnegative integers. If $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_{n+1})$ is a multiindex, then we define $|\zeta|$ and ∂^ζ in the usual ways, as $|\zeta| = \zeta_1 + \zeta_2 + \cdots + \zeta_{n+1}$ and $\partial^\zeta = \partial_{x_1}^{\zeta_1}\partial_{x_2}^{\zeta_2}\cdots\partial_{x_{n+1}}^{\zeta_{n+1}}$.

We will routinely deal with arrays $\dot{\boldsymbol{F}} = \{F_\zeta\}_{|\zeta|=k}$ of numbers or functions indexed by multiindices ζ with $|\zeta| = k$ for some $k \geq 0$. In particular, if φ is a function with weak derivatives of order up to k, then we view $\nabla^k\varphi$ as such an array.

If $\dot{\boldsymbol{F}}$ and $\dot{\boldsymbol{G}}$ are two such arrays of numbers, then their inner product is given by

$$\langle \dot{\boldsymbol{F}}, \dot{\boldsymbol{G}} \rangle = \sum_{|\zeta|=k} \overline{F_\zeta}\, G_\zeta.$$

If $\dot{\boldsymbol{F}}$ and $\dot{\boldsymbol{G}}$ are two arrays of functions defined in a set Ω in Euclidean space, then their inner product is given by

$$\big\langle \dot{\boldsymbol{F}}, \dot{\boldsymbol{G}} \big\rangle_\Omega = \int_\Omega \langle \dot{\boldsymbol{F}}(X), \dot{\boldsymbol{G}}(X) \rangle \, dX = \sum_{|\zeta|=k} \int_\Omega \overline{F_\zeta(X)} \, G_\zeta(X) \, dX.$$

We let $\vec{e}_j$ be the unit vector in $\mathbb{R}^{n+1}$ in the jth direction; notice that $\vec{e}_j$ is a multiindex with $|\vec{e}_j| = 1$. We let $\dot{\boldsymbol{e}}_\zeta$ be the unit array corresponding to the multiindex ζ, so that

$$\langle \dot{\boldsymbol{e}}_\zeta, \dot{\boldsymbol{F}} \rangle = F_\zeta \quad \text{for all arrays } \dot{\boldsymbol{F}} \text{ indexed by multiindices of length } |\zeta|. \tag{41}$$

We will let $\nabla_\parallel$ denote either the gradient in $\mathbb{R}^n$, or the n horizontal components of the full gradient ∇ in $\mathbb{R}^{n+1}$. (Because we identify $\mathbb{R}^n$ with $\partial\mathbb{R}^{n+1}_\pm \subset \mathbb{R}^{n+1}$, the two uses are equivalent.) If ζ is a multiindex in $(\mathbb{N}_0)^n$, or a multiindex in $(\mathbb{N}_0)^{n+1}$ with $\zeta_{n+1} = 0$, we will occasionally use the terminology $\partial_\parallel^\zeta$ to emphasize that the derivatives are taken purely in the horizontal directions.

2.2. **Elliptic differential operators.** Let $\boldsymbol{A} = \big(A_{\alpha\beta}\big)$ be a matrix of measurable coefficients defined on $\mathbb{R}^{n+1}$, indexed by multtiindices α, β with $|\alpha| = |\beta| = m$. If $\dot{\boldsymbol{F}}$ is an array indexed by multiindices of length m, then $\boldsymbol{A}\dot{\boldsymbol{F}}$ is the array given by

$$(\boldsymbol{A}\dot{\boldsymbol{F}})_\alpha = \sum_{|\beta|=m} A_{\alpha\beta} F_\beta.$$

We let L be the $2m$th-order divergence form operator associated with $\boldsymbol{A}$. That is, we say that

$$Lu = 0 \text{ in } \Omega \text{ in the weak sense if } \langle \nabla^m \varphi, \boldsymbol{A}\nabla^m u \rangle_\Omega = 0 \text{ for all } \varphi \in C_0^\infty(\Omega). \tag{42}$$

Throughout we consider coefficients that satisfy the bound

$$\|\boldsymbol{A}\|_{L^\infty(\mathbb{R}^{n+1})} \leq \Lambda \tag{43}$$

and the Gårding inequality

$$\operatorname{Re} \big\langle \nabla^m \varphi, \boldsymbol{A}\nabla^m \varphi \big\rangle_{\mathbb{R}^{n+1}} \geq \lambda \|\nabla^m \varphi\|^2_{L^2(\mathbb{R}^{n+1})} \quad \text{for all } \varphi \in \dot{W}^2_m(\mathbb{R}^{n+1}) \tag{44}$$

for some $\Lambda > \lambda > 0$. (The stronger Gårding inequality (24) will not be used in the proof of Theorem 1.2 or 1.3; it was used only in the statement and proof of Theorem 1.1.)

The numbers C and ε denote constants whose value may change from line to line, but which are always positive and depend only on the dimension $n+1$, the order $2m$ of any relevant operators, and the numbers λ and Λ in the bounds (44) (or (24)) and (43).

2.3. **Function spaces and boundary data.** Let $\Omega \subseteq \mathbb{R}^n$ or $\Omega \subseteq \mathbb{R}^{n+1}$ be a measurable set in Euclidean space. We let $C_0^\infty(\Omega)$ be the space of all smooth functions that are compactly supported in Ω. We let $L^p(\Omega)$ denote the usual Lebesgue space with respect to Lebesgue measure with norm given by

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}.$$

If Ω is a connected open set, then we let the homogeneous Sobolev space $\dot{W}^p_k(\Omega)$ be the space of equivalence classes of functions u that are locally integrable in Ω and have weak derivatives in Ω of order up to k in the distributional sense, and whose

kth gradient $\nabla^k u$ lies in $L^p(\Omega)$. Two functions are equivalent if their difference is a polynomial of order $k-1$. We impose the norm

$$\|u\|_{\dot{W}^p_k(\Omega)} = \|\nabla^k u\|_{L^p(\Omega)}.$$

Then u is equal to a polynomial of order $k-1$ (and thus equivalent to zero) if and only if its $\dot{W}^p_k(\Omega)$-norm is zero. We say that $u \in L^p_{loc}(\Omega)$ or $u \in \dot{W}^p_{k,loc}(\Omega)$ if $u \in L^p(U)$ or $u \in \dot{W}^p_k(U)$ for any bounded open set U with $\overline{U} \subset \Omega$. In particular, if Ω is open, then functions in $L^1_{loc}(\overline{\Omega})$ are necessarily locally integrable up to the boundary $\partial\Omega$, but functions in $L^1_{loc}(\Omega)$ need not be.

We will need a number of more specialized norms on functions. The modified nontangential operator $\widetilde{N}_+$ was introduced in [38] and (in the half space) is given by

$$\widetilde{N}_+ H(x) = \sup\left\{ \left(\fint_{B((y,s),s/2)} |H|^2 \right)^{1/2} : s>0,\ y \in \mathbb{R}^n,\ |x-y|<s \right\}. \tag{45}$$

We will also use a two-sided nontangential maximal function, which we define as

$$\widetilde{N}_* H(x) = \sup\left\{ \left(\fint_{B((y,s),|s|/2)} |H|^2 \right)^{1/2} : s \in \mathbb{R},\ y \in \mathbb{R}^n,\ |x-y|<|s| \right\}. \tag{46}$$

Finally, we will use the Lusin area integral operator $\mathcal{A}_2^+$ given by

$$\mathcal{A}_2^+ H(x) = \left(\int_0^\infty \int_{|x-y|<t} |H(y,t)|^2 \frac{dy\,dt}{t^{n+1}} \right)^{1/2}. \tag{47}$$

2.3.1. *Boundary values and boundary function spaces.* Following [21], we define the boundary values $\operatorname{Tr}^\pm u$ of a function u defined in $\mathbb{R}^{n+1}_\pm$ by

$$\operatorname{Tr}^\pm u = f \quad \text{if} \quad \lim_{t\to 0^\pm} \|u(\,\cdot\,,t) - f\|_{L^1(K)} = 0 \tag{48}$$

for all compact sets $K \subset \mathbb{R}^n$. If $\operatorname{Tr}^+ u = \operatorname{Tr}^- u$, then we let $\operatorname{Tr} u = \operatorname{Tr}^\pm u$.We define

$$\dot{\mathbf{Tr}}^\pm_j u = \operatorname{Tr}^\pm \nabla^j u, \quad \dot{\mathbf{Tr}}_j u = \operatorname{Tr} \nabla^j u. \tag{49}$$

We remark that if ∇u is locally integrable up to the boundary, then $\operatorname{Tr}^\pm u$ exists, and furthermore $\operatorname{Tr}^\pm u$ coincides with the traditional trace in the sense of Sobolev spaces.

We are interested in functions with boundary data in Lebesgue or Sobolev spaces. However, observe that if $j \geq 1$, then the components of $\dot{\mathbf{Tr}}^\pm_j u$ are derivatives of a common function and so must satisfy certain compatibility conditions. We thus define the following Whitney-Lebesgue, Whitney-Sobolev and Whitney-Besov spaces of arrays that satisfy these conditions.

Definition 50. Let

$$\mathfrak{D} = \{\dot{\mathbf{Tr}}_{m-1}\,\varphi : \varphi \text{ smooth and compactly supported in } \mathbb{R}^{n+1}\}.$$

If $1 \leq p < \infty$, then we let $\dot{W}A^p_{m-1,0}(\mathbb{R}^n)$ be the completion of the set $\mathfrak{D}$ under the L^p norm. We let $\dot{W}A^p_{m-1,1}(\mathbb{R}^n)$ be the completion of $\mathfrak{D}$ under the $\dot{W}^p_1(\mathbb{R}^n)$ norm, that is, under the norm $\|\dot{\boldsymbol{f}}\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)} = \|\nabla_\parallel \dot{\boldsymbol{f}}\|_{L^p(\mathbb{R}^n)}$. Finally,

we let $\dot{W}A^2_{m-1,1/2}(\mathbb{R}^n)$ be the completion of $\mathfrak{D}$ under the norm in the Besov space $\dot{B}^{2,2}_{1/2}(\mathbb{R}^n)$; this norm may be written as

$$\|\dot{\boldsymbol{f}}\|_{\dot{B}^{2,2}_{1/2}(\mathbb{R}^n)} = \|\dot{\boldsymbol{f}}\|_{\dot{W}A^2_{m-1,1/2}(\mathbb{R}^n)} = \Bigg(\sum_{|\gamma|=m-1}\int_{\mathbb{R}^n}|\widehat{f_\gamma}(\xi)|^2|\xi|\,d\xi\Bigg)^{1/2}.$$

It is widely known that $\dot{\boldsymbol{f}} \in \dot{W}A^2_{m-1,1/2}(\mathbb{R}^n)$ if and only if $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}^+_{m-1}F$ for some F with $\nabla^m F \in L^2(\mathbb{R}^{n+1}_+)$.

Recall that Theorem 1.1 is concerned with Neumann boundary values $\dot{\mathbf{M}}^+_{\boldsymbol{A}}u$ of solutions u to $Lu = 0$. However, as discussed at the beginning of Section 1.4, Theorem 1.1 follows from Theorem 1.2 and the proof of [19, Theorems 1.1 and 1.2], and thus we will not use any particular properties of $\dot{\mathbf{M}}^+_{\boldsymbol{A}}u$ in the proof. We refer the reader to [21, Section 2.3.2] or [19, formula (2.7)] for a definition of $\dot{\mathbf{M}}^+_{\boldsymbol{A}}$.

In the proof of Lemma 5.1 below we will use some properties of $\dot{\mathbf{M}}^-_{\boldsymbol{A}}$ from [17] and [21]. In these cases we refer the reader to [21] for a definition of $\dot{\mathbf{M}}^\pm_{\boldsymbol{A}}u$; we remark only that if $u \in \dot{W}^2_m(\mathbb{R}^{n+1}_\pm)$ and $Lu = 0$ in $\mathbb{R}^{n+1}_\pm$, then the definitions of $\dot{\mathbf{M}}^\pm_{\boldsymbol{A}}u$ used in [17] and in [21] coincide.

2.4. **The double layer potential and the Newton potential.** In this section we define the double layer potential mentioned in Theorem 1.2.

We begin by defining the related Newton potential. For any $\dot{\boldsymbol{H}} \in L^2(\mathbb{R}^{n+1})$, by the Lax-Milgram lemma there is a unique function $\Pi^L\dot{\boldsymbol{H}}$ in $\dot{W}^2_m(\mathbb{R}^{n+1})$ that satisfies

$$\langle\nabla^m\varphi, \boldsymbol{A}\nabla^m\Pi^L\dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}} = \langle\nabla^m\varphi, \dot{\boldsymbol{H}}\rangle_{\mathbb{R}^{n+1}} \tag{51}$$

for all $\varphi \in \dot{W}^2_m(\mathbb{R}^{n+1})$. We will refer to the operator Π^L as the Newton potential. (This represents a break from tradition; in the classical case, the Newton potential $\mathcal{N}^L$ is generally taken to satisfy $\langle\nabla^m\varphi, \boldsymbol{A}\nabla^m\mathcal{N}^L H\rangle_{\mathbb{R}^{n+1}} = \langle\varphi, H\rangle_{\mathbb{R}^{n+1}}$.)

Now, suppose that $\dot{\boldsymbol{f}} \in \dot{W}A^2_{m-1,1/2}(\mathbb{R}^n)$. Recall that $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}^+_{m-1}F$ for some $F \in \dot{W}^2_m(\mathbb{R}^{n+1}_+)$. We define

$$\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}} = -\mathbf{1}_+F + \Pi^L(\mathbf{1}_+\boldsymbol{A}\nabla^m F). \tag{52}$$

This operator is well-defined, that is, does not depend on the choice of F. See [17, Lemma 4.2] or [18, section 2.4]. Furthermore, it is antisymmetric about exchange of $\mathbb{R}^{n+1}_+$ and $\mathbb{R}^{n+1}_-$; that is, if $\dot{\mathbf{Tr}}^-_{m-1}F = \dot{\boldsymbol{f}}$ and $F \in \dot{W}^2_m(\mathbb{R}^{n+1}_-)$, then

$$\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}} = -\Pi^L(\mathbf{1}_-\boldsymbol{A}\nabla^m F) + \mathbf{1}_-F. \tag{53}$$

See [18, formula (2.27)] or [17, formula (4.8)].

2.5. **The single layer potential.** Let $\dot{\boldsymbol{g}}$ be a bounded linear operator on the space $\dot{W}A^2_{m-1,1/2}(\mathbb{R}^n)$. Observe that $\dot{\boldsymbol{g}}$ extends to an operator on $\dot{B}^{2,2}_{1/2}(\mathbb{R}^n)$, and so $\dot{\boldsymbol{g}} \in (\dot{W}A^2_{m-1,1/2}(\mathbb{R}^n))^*$ if and only if there is a representative of $\dot{\boldsymbol{g}}$ that lies in $\dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$, that is, that satisfies

$$\Bigg(\sum_{|\gamma|=m-1}\int_{\mathbb{R}^n}\frac{1}{|\xi|}|\widehat{g_\gamma}(\xi)|^2\,d\xi\Bigg)^{1/2} = \|\dot{\boldsymbol{g}}\|_{\dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)} < \infty.$$

The operator $F \mapsto \langle \dot{\mathbf{T}}\mathbf{r}_{m-1} F, \dot{\boldsymbol{g}}\rangle_{\mathbb{R}^n}$ is a bounded linear operator on $\dot{W}^2_m(\mathbb{R}^{n+1})$, and so by the Lax-Milgram lemma there is a unique function $\mathcal{S}^L\dot{\boldsymbol{g}} \in \dot{W}^2_m(\mathbb{R}^{n+1})$ that satisfies

$$\langle \nabla^m \varphi, \boldsymbol{A}\nabla^m \mathcal{S}^L\dot{\boldsymbol{g}}\rangle_{\mathbb{R}^{n+1}} = \langle \dot{\mathbf{T}}\mathbf{r}_{m-1}\,\varphi, \dot{\boldsymbol{g}}\rangle_{\mathbb{R}^n} \qquad \text{for all } \varphi \in \dot{W}^2_m(\mathbb{R}^{n+1}). \tag{54}$$

See [17]. We remark that this definition coincides with the definition of $\mathcal{S}^L\dot{\boldsymbol{g}}$ given in [18, 20]. This defines $\mathcal{S}^L$ as a bounded operator $\dot{B}^{2,2}_{-1/2}(\mathbb{R}^n) \mapsto \dot{W}^2_m(\mathbb{R}^{n+1})$; by [19, formula (4.3)] (see Section 4.1 below) we have that $\mathcal{S}^L$ extends by density to an operator that is bounded $L^2(\mathbb{R}^n) \mapsto \dot{W}^2_m(\mathbb{R}^n \times (a,b))$ for any numbers $-\infty < a < b < \infty$.

As observed in [20, formula (2.21)], if $\dot{\boldsymbol{g}} \in \dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$ and if $|\alpha| = m$, then for almost every $(x,t) \in \mathbb{R}^{n+1}$ we have that

$$\partial^\alpha \mathcal{S}^L\dot{\boldsymbol{g}}(x,t) = \sum_{|\zeta|=m-1} \int_{\mathbb{R}^n} \partial^\alpha_{x,t}\partial^\zeta_{y,s} E^L(x,t,y,0)\, g_\zeta(y)\, dy \tag{55}$$

where E^L is the fundamental solution for the operator L constructed in [16]. By the bound [16, formula (63)] (reproduced as formula (59) below), for almost every $(x,t) \in \mathbb{R}^{n+1}_\pm$ we have that $\partial^\alpha_{x,t} E^L(x,t,\,\cdot\,,\,\cdot\,) \in \dot{W}^2_m(\mathbb{R}^{n+1}_\mp)$ and so $v(y) = \partial^\alpha_{x,t}\partial^\zeta_{y,s} E^L(x,t,y,0)$ lies in $\dot{B}^{2,2}_{1/2}(\mathbb{R}^n)$. Thus, the right hand side converges provided $g_\zeta \in \dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$.

As in [20, formula (2.27)], if $|\gamma| = m-1$, then we define

$$\partial^\gamma \mathcal{S}^L_\nabla \dot{\boldsymbol{h}}(x,t) = \sum_{|\beta|=m} \int_{\mathbb{R}^n} \partial^\gamma_{x,t}\partial^\beta_{y,s} E^L(x,t,y,0)\, h_\beta(y)\, dy. \tag{56}$$

We will see (Lemma 4.1 below) that if $\dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n)$, then the integral converges absolutely for almost every $(x,t) \in \mathbb{R}^{n+1}$, and the functions $\partial^\gamma \mathcal{S}^L_\nabla \dot{\boldsymbol{h}}$ given by formula (56) are indeed derivatives of a common $\dot{W}^2_{m-1,loc}(\mathbb{R}^{n+1})$-function that we may call $\mathcal{S}^L_\nabla \dot{\boldsymbol{h}}$.

3. Preliminaries

In Section 4, we will establish the bounds (27), (28), (36), and (37) on the single layer potential. In Section 5, we will establish the bounds (29), (30), (38), and (39) on the double layer potential. In this section, we will collect some known results and establish some preliminary estimates that will be of use in both Section 4 and Section 5.

3.1. Regularity results.

We begin by recalling some known regularity results for solutions to elliptic differential equations.

The following lemma is the higher order analogue of the Caccioppoli inequality. It was proven in full generality in [16] and some important preliminary versions were established in [22, 5].

Lemma 3.1 (The Caccioppoli inequality)**.** *Let L be an operator of the form* (1) *of order $2m$ associated to coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44)*. Let $u \in \dot{W}^2_m(B(X_0,2r))$ with $Lu = 0$ in $B(X_0,2r)$.*

Then we have the bound

$$\fint_{B(X,r)} |\nabla^j u(x,s)|^2 \, dx \, ds \leq \frac{C}{r^2} \fint_{B(X,2r)} |\nabla^{j-1} u(x,s)|^2 \, dx \, ds$$

for any j with $1 \leq j \leq m$.

We next state the higher order generalization of Meyers's reverse Hölder inequality for gradients. The $k = 0$ case of the bound (57) was established in [22, 5]. The $k \geq 1$ case was established in [16] and is a relatively straightforward consequence of the $k = 0$ case and the Gagliardo-Nirenberg-Sobolev inequality.

Theorem 3.2. *Let L be an operator of the form* (1) *of order $2m$ associated to coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44)*. Let $X_0 \in \mathbb{R}^{n+1}$ and let $r > 0$. Suppose that $u \in \dot{W}^2_m(B(X_0, 2r))$ with $Lu = 0$ in $B(X_0, 2r)$.*

If k is an integer with $0 \leq k \leq m$ and $2k < n+1$, then there is a number $p^+_{k,L} > 2(n+1)/(n+1-2k)$, depending only on the standard constants, such that if $0 < p < q < p^+_{k,L}$, then

$$\left(\int_{B(X_0,r)} |\nabla^{m-k} u|^q \right)^{1/q} \leq \frac{C_{p,q}}{r^{(n+1)/p-(n+1)/q}} \left(\int_{B(X_0,2r)} |\nabla^{m-k} u|^p \right)^{1/p} \tag{57}$$

for some constant $C_{p,q}$ depending only on p, q and the standard constants.

If $0 \leq m-k \leq m-(n+1)/2$, then $\nabla^{m-k} u$ is Hölder continuous and satisfies the bound

$$\sup_{B(X_0,r)} |\nabla^{m-k} u| \leq \frac{C_p}{r^{(n+1)/p}} \left(\int_{B(X_0,2r)} |\nabla^{m-k} u|^p \right)^{1/p} \tag{58}$$

for all $0 < p \leq \infty$.

Finally, if $\boldsymbol{A}$ is t-independent then solutions to $Lu = 0$ have additional regularity. The following lemma was proven in the case $m = 1$ in [3, Proposition 2.1] and generalized to the case $m \geq 2$ in [20, Lemma 3.20].

Lemma 3.3. *Let L be an operator of the form* (1) *of order $2m$ associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44)*. Let $t \in \mathbb{R}$ be a constant, and let $Q \subset \mathbb{R}^n$ be a cube.*

If $Lu = 0$ in $2Q \times (t - \ell(Q), t + \ell(Q))$, then

$$\int_Q |\nabla^j \partial_t^k u(x,t)|^p \, dx \leq \frac{C_p}{\ell(Q)} \int_{2Q} \int_{t-\ell(Q)}^{t+\ell(Q)} |\nabla^j \partial_s^k u(x,s)|^p \, ds \, dx$$

for any $0 \leq j \leq m$, any $0 < p < p^+_{m-j,L}$, and any integer $k \geq 0$, where $p^+_{m-j,L}$ is as in Theorem 3.2.

3.2. **The fundamental solution.** Recall from formula (55) that the single layer potential, originally constructed via the Lax-Milgram lemma, has an explicit representation as an integral operator involving the fundamental solution. We will often make use of this representation; thus, we now state the following result of [16] concerning the fundamental solution for higher order operators.

Theorem 3.4 ([16, Theorem 62 and Lemma 69])**.** *Let L be an operator of the form* (1) *of order $2m$ associated to coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44)*. Then there exists a function $E^L(X,Y)$ with the following properties.*

Let $s=0$ or $s=1$ and let $q=0$ or $q=1$. There is some $\varepsilon>0$ such that if X_0, $Y_0\in\mathbb{R}^{n+1}$, if $0<r<R<|X_0-Y_0|/3$, and if either $q=0$ or $n+1\geq 3$, then

$$\int_{B(Y_0,r)}\int_{B(X_0,R)}|\nabla_X^{m-s}\nabla_Y^{m-q}E^L(X,Y)|^2\,dX\,dY\leq Cr^{2q}R^{2s}\left(\frac{r}{R}\right)^{\varepsilon}. \tag{59}$$

If $2q=2=n+1$ and $s=0$, then we instead have the bound

$$\int_{B(Y_0,r)}\int_{B(X_0,R)}|\nabla_X^{m}\nabla_Y^{m-1}E^L(X,Y)|^2\,dX\,dY\leq C_\delta\, r^2\left(\frac{R}{r}\right)^{\delta} \tag{60}$$

for all $\delta>0$ and some constant C_δ depending on δ.

We have the symmetry property

$$\partial_X^\zeta\partial_Y^\xi E^L(X,Y)=\overline{\partial_X^\zeta\partial_Y^\xi E^{L^*}(Y,X)} \tag{61}$$

as locally L^2 functions, for all multiindices ζ, ξ with $m-1\leq|\zeta|\leq m$, $m-1\leq|\xi|\leq m$.

Furthermore, if $|\alpha|=m$ then

$$\partial^\alpha\Pi^L\dot{\boldsymbol{H}}(X)=\sum_{|\beta|=m}\int_{\mathbb{R}^{n+1}}\partial_X^\alpha\partial_Y^\beta E^L(X,Y)\,H_\beta(Y)\,dY \tag{62}$$

for almost every $X\notin\operatorname{supp}\dot{\boldsymbol{H}}$, and for all $\dot{\boldsymbol{H}}\in L^2(\mathbb{R}^{n+1})$ whose support is not all of $\mathbb{R}^{n+1}$.

Finally, if $\widetilde{E}^L$ is any other function that satisfies the bounds (59), (60) *and formula* (62), *then*

$$\nabla_X^{m-q}\nabla_Y^{m-s}\widetilde{E}^L(X,Y)=\nabla_X^{m-q}\nabla_Y^{m-s}E^L(X,Y) \tag{63}$$

as locally L^2 functions provided $0\leq q\leq 1$, $0\leq s\leq 1$ and either $n+1\geq 3$ or $q+s\leq 1$.

Here Π^L is the Newton potential defined by formula (51).

We remark that in particular, if ξ is a multiindex with $m-1\leq|\xi|\leq m$, and if we let

$$u(z,r)=\partial_{y,s}^\xi E^L(z,r,y,s),\quad v(z,r)=\overline{\partial_{x,t}^\xi E^L(x,t,z,r)}$$

then $u\in\dot{W}^2_{m,loc}(\mathbb{R}^{n+1}\setminus\{(y,s)\})$ and $v\in\dot{W}^2_{m,loc}(\mathbb{R}^{n+1}\setminus\{(x,t)\})$ for almost every $(x,t)\in\mathbb{R}^{n+1}$ and $(y,s)\in\mathbb{R}^{n+1}$, and furthermore

$$Lu=0\text{ in }\mathbb{R}^{n+1}\setminus\{(y,s)\},\qquad L^*v=0\text{ in }\mathbb{R}^{n+1}\setminus\{(x,t)\}.$$

In particular, we may apply Lemma 3.3 to the fundamental solution in either the first or second variables.

By uniqueness of the fundamental solution, if $\boldsymbol{A}$ is t-independent, and if $m-1\leq|\zeta|\leq m$ and $m-1\leq|\xi|\leq m$, then

$$\partial_{x,t}^\xi\partial_{y,s}^\zeta E^L(x,t+r,y,s+r)=\partial_{x,t}^\xi\partial_{y,s}^\zeta E^L(x,t,y,s) \tag{64}$$

and so

$$\partial_t\partial_{x,t}^\xi\partial_{y,s}^\zeta E^L(x,t,y,s)=-\partial_s\partial_{x,t}^\xi\partial_{y,s}^\zeta E^L(x,t,y,s). \tag{65}$$

3.3. **The lower half space.** Recall that Theorems 1.2 and 1.3 involve bounds on the quantities $\widetilde{N}_+(\nabla^{m-k}u)$ and $\mathcal{A}_2^+(t\nabla^m\partial_t^k u)$, where $k=0$ or $k=1$ and where u denotes various potentials. It is notationally convenient to work only in the upper half space.

However, estimates in terms of the two-sided nontangential maximal operator $\widetilde{N}_*$ defined in formula (46) will also be of use. In particular, in Lemma 3.7 we will pass from bounds on $\widetilde{N}_*(\nabla^{m-1}u)$ to bounds on $\mathcal{A}_2^+(t\nabla^m u)$, and in Lemma 4.5 we will pass from bounds on $\widetilde{N}_*(\partial_{n+1}^m\mathcal{S}^L\dot{\boldsymbol{g}})$ to bounds on $\widetilde{N}_+(\nabla^m\mathcal{S}^L\dot{\boldsymbol{g}})$.

We may easily translate bounds valid in the upper half space to bounds valid in the lower half space, using the following argument.

Let $A^-_{\alpha\beta}=(-1)^{\alpha_{n+1}+\beta_{n+1}}A_{\alpha\beta}$. Observe that if $\boldsymbol{A}$ is bounded or t-independent then so is $\boldsymbol{A}^-$. Let φ and u be scalar valued functions defined on $\mathbb{R}^{n+1}$ and let $\varphi^-(x,t)=\varphi(x,-t)$, $u^-(x,t)=u(x,-t)$. A straightforward change of variables argument establishes that

$$\langle\nabla^m\varphi,\boldsymbol{A}\nabla^m u\rangle_{\mathbb{R}^{n+1}}=\langle\nabla^m\varphi^-,\boldsymbol{A}^-\nabla^m u^-\rangle_{\mathbb{R}^{n+1}}.$$

Choosing $u=\varphi$, we see that if $\boldsymbol{A}$ satisfies the ellipticity condition (44) then so does $\boldsymbol{A}^-$.

Let $\dot{\boldsymbol{H}}\in L^2(\mathbb{R}^{n+1})$ and let $u=\Pi^L\dot{\boldsymbol{H}}$. Because $\Pi^L\dot{\boldsymbol{H}}$ is the unique solution to the problem (51), we have that if $H^-_\alpha(x,t)=(-1)^{\alpha_{n+1}}H_\alpha(x,-t)$, then

$$\Pi^L\dot{\boldsymbol{H}}(x,-t)=\Pi^{L^-}\dot{\boldsymbol{H}}^-(x,t). \tag{66}$$

By the definition (52) of the double layer potential and formula (53),

$$\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}(x,-t)=-\mathcal{D}^{\boldsymbol{A}^-}\dot{\boldsymbol{f}}^-(x,t) \tag{67}$$

where if $\dot{\boldsymbol{f}}=\dot{\mathbf{Tr}}_{m-1}F$, then $\dot{\boldsymbol{f}}^-=\dot{\mathbf{Tr}}_{m-1}F^-$. Similarly, by formula (54), if $g^-_\gamma(x)=(-1)^{\gamma_{n+1}}g_\gamma(x)$, then

$$\mathcal{S}^L\dot{\boldsymbol{g}}(x,-t)=\mathcal{S}^{L^-}\dot{\boldsymbol{g}}^-(x,t). \tag{68}$$

We may establish the similar formula

$$\mathcal{S}^L_\nabla\dot{\boldsymbol{h}}(x,-t)=\mathcal{S}^{L^-}_\nabla\dot{\boldsymbol{h}}^-(x,t), \tag{69}$$

where $h^-_\beta=(-1)^{\beta_{n+1}}h_\beta$, using either uniqueness of the fundamental solution, or using formulas (73) and (74) below.

Thus, we may easily pass from bounds in the upper half space to bounds in the lower half space.

3.4. **Nontangential bounds.** In Sections 4 and 5 we will use the following two lemmas to establish nontangential bounds.

Lemma 3.5. *If $F\in L^2_{loc}(\mathbb{R}^{n+1}_+)$ and $x_0\in\mathbb{R}^n$, then*

$$\widetilde{N}_+F(x_0)\le C\sup_{t_0>0}\left(\fint_{Q(x_0,t_0)}\fint_{t_0/6}^{t_0/2}|F|^2\right)^{1/2}$$

where $Q(x_0,t_0)$ is the cube in $\mathbb{R}^n$ with midpoint x_0 and side length t_0.

Proof. Recall from the definition (45) that

$$\widetilde{N}_+F(x_0)=\sup\left\{\left(\fint_{B((y,t_0/3),t_0/6)}|F|^2\right)^{1/2}:|x_0-y|<t_0/3\right\}.$$

But $B((y,t_0/3),t_0/6)\subset Q(x_0,t_0)\times(t_0/6,t_0/2)$ whenever $|x_0-y|<t_0/3$, and so

$$\widetilde{N}_+F(x_0)\leq \sup_{t_0>0}\bigg(\frac{6^{n+1}}{\omega_{n+1}t_0^{n+1}}\int_{Q(x_0,t_0)}\int_{t_0/6}^{t_0/2}|F|^2\bigg)^{1/2}$$

where ω_{n+1} is the volume of the unit ball in $\mathbb{R}^{n+1}$, as desired. □

The following lemma is very useful for bounding solutions in cubes, and in particular in $Q(x_0,t_0)\times(t_0/6,7t_0/6)$ or in $Q(x_0,t_0)\times(-t_0/2,t_0/2)$.

Lemma 3.6. *Let L be an operator of the form* (1) *of order $2m$ associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44).

Let $Q\subset\mathbb{R}^n$ be a cube and let $\widetilde{Q}=Q\times(s_0-\ell(Q)/2,s_0+\ell(Q)/2)$ be a cube in $\mathbb{R}^{n+1}$. Suppose that $u\in\dot{W}^2_m(2\widetilde{Q})$ and that $Lu=0$ in $2\widetilde{Q}$. Then

$$\fint_{\widetilde{Q}}|\nabla^j u(x,t)|^2\,dt\,dx\leq C\ell(Q)^2\bigg(\fint_{2\widetilde{Q}}|\partial_t^{j+1}u(x,t)|\,dt\,dx\bigg)^2+C\bigg(\fint_{2Q}|\nabla^j u(x,\tau)|\,dx\bigg)^2$$

whenever $0\leq j\leq m$ and $s_0-\ell(Q)/2\leq\tau\leq s_0+\ell(Q)/2$.

Proof. Let $0\leq k\leq j$. Let $\varepsilon>0$ be a small positive number and let $\widetilde{Q}_k=(1+k\varepsilon)\widetilde{Q}$. By Theorem 3.2,

$$\fint_{\widetilde{Q}_k}|\nabla^{j-k}\partial_t^k u(x,t)|^2\,dt\,dx\leq C_\varepsilon\bigg(\fint_{\widetilde{Q}_{k+1/2}}|\nabla^{j-k}\partial_t^k u(x,t)|\,dt\,dx\bigg)^2.$$

If $(x,t)\in\widetilde{Q}_{k+1/2}$, then

$$\begin{aligned}|\nabla^{j-k}\partial_t^k u(x,t)|&\leq|\nabla^{j-k}\partial_t^k u(x,t)-\nabla^{j-k}\partial_\tau^k u(x,\tau)|+|\nabla^{j-k}\partial_\tau^k u(x,\tau)|\\&\leq\int_{s_0-\ell(Q_{k+1/2})/2}^{s_0+\ell(Q_{k+1/2})/2}|\nabla^{j-k}\partial_s^{k+1}u(x,s)|\,ds+|\nabla^{j-k}\partial_\tau^k u(x,\tau)|.\end{aligned}$$

Thus, by Hölder's inequality

$$\begin{aligned}\fint_{\widetilde{Q}_k}|\nabla^{j-k}\partial_t^k u(x,t)|^2\,dt\,dx&\leq C_\varepsilon\ell(Q_{k+1/2})^2\fint_{\widetilde{Q}_{k+1/2}}|\nabla^{j-k}\partial_t^{k+1}u(x,t)|^2\,dt\,dx\\&\quad+C_\varepsilon\bigg(\fint_{Q_{k+1/2}}|\nabla^{j-k}\partial_\tau^k u(x,\tau)|\,dx\bigg)^2.\end{aligned}$$

If $k\leq j-1$, then by the Caccioppoli inequality,

$$\begin{aligned}\fint_{\widetilde{Q}_k}|\nabla^{j-k}\partial_t^k u(x,t)|^2\,dt\,dx&\leq C_\varepsilon\fint_{\widetilde{Q}_{k+1}}|\nabla^{j-k-1}\partial_t^{k+1}u(x,t)|^2\,dt\,dx\\&\quad+C_\varepsilon\bigg(\fint_{Q_{k+1}}|\nabla^{j-k}\partial_\tau^k u(x,\tau)|\,dx\bigg)^2.\end{aligned}$$

Iterating, we see that

$$\begin{aligned}\fint_{\widetilde{Q}}|\nabla^j u(x,t)|^2\,dt\,dx&\leq C_\varepsilon\ell(Q_{j+1/2})^2\fint_{\widetilde{Q}_{j+1/2}}|\partial_t^{j+1}u(x,t)|^2\,dt\,dx\\&\quad+C_\varepsilon\bigg(\fint_{Q_{j+1/2}}|\nabla^j u(x,\tau)|\,dx\bigg)^2.\end{aligned}$$

A final application of Theorem 3.2 yields that

$$\fint_{\widetilde{Q}} |\nabla^j u(x,t)|^2 \, dt \, dx \leq C_\varepsilon \ell(Q_{j+1/2})^2 \Big(\fint_{\widetilde{Q}_{j+1}} |\partial_t^{j+1} u(x,t)| \, dt \, dx \Big)^2 + C_\varepsilon \Big(\fint_{Q_{j+1/2}} |\nabla^j u(x,\tau)| \, dx \Big)^2 .$$

Letting $\varepsilon = 1/(j+1)$ and so $Q_{j+1} = 2Q$ completes the proof. □

3.5. **Area integral bounds.** We will use the following lemma to establish the area integral bounds in Theorem 1.3.

Lemma 3.7. *Let $\dot{\boldsymbol{u}} \in L^2_{loc}(\mathbb{R}^{n+1}_+)$ satisfy $\mathcal{A}_2^+(t\,\dot{\boldsymbol{u}}) \in L^2(\mathbb{R}^n)$. Suppose that there is a nonnegative real-valued function φ defined on $\mathbb{R}^n$, and a family of functions $\dot{\boldsymbol{u}}_Q$ indexed by cubes $Q \subset \mathbb{R}^n$, such that if $Q \subset \mathbb{R}^n$ is a cube, then*

$$\|\mathcal{A}_2^+(t\,\dot{\boldsymbol{u}}_Q)\|^2_{L^2(\mathbb{R}^n)} + \int_{(3/2)Q} \int_0^{\ell(Q)/4} |\dot{\boldsymbol{u}}(x,t) - \dot{\boldsymbol{u}}_Q(x,t)|^2 \, t \, dt \, dx \leq \|\varphi\|^2_{L^2(4Q)}.$$

Then there is some $\varepsilon > 0$ such that

$$\|\mathcal{A}_2^+(t\,\dot{\boldsymbol{u}})\|_{L^p(\mathbb{R}^n)} \leq C_p \|\varphi\|_{L^p(\mathbb{R}^n)}$$

for any $2 \leq p < 2 + \varepsilon$.

In particular, let L be an operator of the form (1) of order $2m$ associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds (43) and (44), and let $u \in \dot{W}^2_{m,loc}(\mathbb{R}^{n+1}_+ \cup \mathbb{R}^{n+1}_-)$ with $\mathcal{A}_2^+(t\nabla^m u) \in L^2(\mathbb{R}^n)$. If there is a function ψ and a family of functions u_Q indexed by cubes $Q \subset \mathbb{R}^n$, such that if $Q \subset \mathbb{R}^n$ is a cube then $u - u_Q \in \dot{W}^2_m(3Q \times (-\ell(Q), \ell(Q)))$, $L(u - u_Q) = 0$ in $3Q \times (-\ell(Q), \ell(Q))$, and

$$\|\mathcal{A}_2^+(t\nabla^m u_Q)\|^2_{L^2(\mathbb{R}^n)} + \|\widetilde{N}_*(\nabla^{m-1} u_Q)\|^2_{L^2(\mathbb{R}^n)} \leq \|\psi\|^2_{L^2(4Q)},$$

then there is some $\varepsilon > 0$ such that

$$\|\mathcal{A}_2^+(t\,\nabla^m u)\|_{L^p(\mathbb{R}^n)} \leq C_p \|\psi\|_{L^p(\mathbb{R}^n)} + C_p \|\widetilde{N}_*(\nabla^{m-1} u)\|_{L^p(\mathbb{R}^n)}$$

for any $2 < p < 2 + \varepsilon$.

We will use the following lemma in the proof of Lemma 3.7.

Lemma 3.8 ([35, Lemma 3.2])**.** *Suppose that g, $h \in L^q(\mathbb{R}^n)$ are nonnegative real-valued functions, $1 < q < \infty$, and that for some $C_0 > 0$ and for all cubes $Q \subset \mathbb{R}^n$,*

$$\Big(\fint_Q g^q \Big)^{1/q} \leq C_0 \fint_{4Q} g + \Big(\fint_{4Q} h^q \Big)^{1/q}.$$

Then there exist numbers $s > q$ and $C > 0$, depending only on n, q and C_0, such that if $h \in L^s(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} g^s \leq C \int_{\mathbb{R}^n} h^s.$$

We remark that the assumption $h \in L^q(\mathbb{R}^n)$ is not necessary; it suffices to require $h \in L^q_{loc}(\mathbb{R}^n)$. To see this, we may, for example, use a local version of this lemma (e.g., [35, Proposition 6.1]) in larger and larger localized regions.

Proof of Lemma 3.7. We begin with the special case where $Lu = 0$. By the Caccioppoli inequality,

$$\int_{(3/2)Q}\int_0^{\ell(Q)/4} |\nabla^m (u-u_Q)(x,t)|^2\,dt\,dx \leq \frac{C}{\ell(Q)^2}\int_{3Q}\int_{-\ell(Q)}^{\ell(Q)} |\nabla^{m-1}(u-u_Q)(y,t)|^2\,dt\,dy.$$

It is straightforward to bound the right hand side by $\widetilde{N}_*(\nabla^{m-1}(u-u_Q))$, and so

$$\frac{\ell(Q)}{4}\int_{(3/2)Q}\int_0^{\ell(Q)/4} |\nabla^m (u-u_Q)(x,t)|^2\,dt\,dx \leq C\int_{3Q}(\widetilde{N}_*(\nabla^{m-1}u)^2+\widetilde{N}_*(\nabla^{m-1}u_Q)^2).$$

By assumption, and because $0<t<\ell(Q)/4$ in the region of integration, we have that

$$\|\mathcal{A}_2^+(t\nabla^m u_Q)\|^2_{L^2(\mathbb{R}^n)} + \int_{(3/2)Q}\int_0^{\ell(Q)/4} |\nabla^m (u-u_Q)(x,t)|^2\,t\,dt\,dx \leq C\int_{3Q}(\widetilde{N}_*(\nabla^{m-1}u)^2) + C\int_{4Q}\psi^2.$$

Choosing $\varphi^2 = C\psi^2 + C\widetilde{N}_*(\nabla^{m-1}u)^2$, $\dot{\boldsymbol{u}} = \nabla^m u$ and $\dot{\boldsymbol{u}}_Q = \nabla^m u_Q$, we may reduce to the general case.

We now turn to the general case. Let $Q\subset\mathbb{R}^n$ be a cube. By definition of $\mathcal{A}_2^+$,

$$\int_Q \mathcal{A}_2^+(t\,\dot{\boldsymbol{u}})(x)^2\,dx = \int_Q\int_0^\infty\int_{|x-y|<t} |\dot{\boldsymbol{u}}(y,t)|^2\frac{1}{t^{n-1}}\,dy\,dt\,dx.$$

We consider the cases $t>\ell(Q)/4$ and $t\leq\ell(Q)/4$ separately, so

$$\int_Q \mathcal{A}_2^+(t\,\dot{\boldsymbol{u}})(x)^2\,dx \leq \int_Q\int_0^{\ell(Q)/4}\int_{|x-y|<t} |\dot{\boldsymbol{u}}(y,t)|^2\frac{1}{t^{n-1}}\,dy\,dt\,dx + \int_Q\int_{\ell(Q)/4}^\infty\int_{|x-y|<t} |\dot{\boldsymbol{u}}(y,t)|^2\frac{1}{t^{n-1}}\,dy\,dt\,dx.$$

The first term satisfies

$$\begin{aligned}\int_Q\int_0^{\ell(Q)/4}\int_{|x-y|<t} |\dot{\boldsymbol{u}}(y,t)|^2\frac{1}{t^{n-1}}\,dy\,dt\,dx &\leq 2\int_Q\int_0^{\ell(Q)/4}\int_{|x-y|<t} |\dot{\boldsymbol{u}}_Q(y,t)|^2\frac{1}{t^{n-1}}\,dy\,dt\,dx \\ &\quad + 2\int_Q\int_0^{\ell(Q)/4}\int_{|x-y|<t} |\dot{\boldsymbol{u}}(y,t)-\dot{\boldsymbol{u}}_Q(y,t)|^2\frac{1}{t^{n-1}}\,dy\,dt\,dx.\end{aligned}$$

But

$$\int_Q\int_0^{\ell(Q)/4}\int_{|x-y|<t} |\dot{\boldsymbol{u}}_Q(y,t)|^2\frac{1}{t^{n-1}}\,dy\,dt\,dx \leq \int_{\mathbb{R}^n}\mathcal{A}_2^+(t\,\dot{\boldsymbol{u}}_Q)^2 \leq \int_{4Q}\varphi^2.$$

We have that

$$\int_Q\int_0^{\ell(Q)/4}\int_{|x-y|<t} |\dot{\boldsymbol{u}}(y,t)-\dot{\boldsymbol{u}}_Q(y,t)|^2\frac{1}{t^{n-1}}\,dy\,dt\,dx \leq C_n\int_{(3/2)Q}\int_0^{\ell(Q)/4} |\dot{\boldsymbol{u}}(y,t)-\dot{\boldsymbol{u}}_Q(y,t)|^2\,t\,dy\,dt.$$

By assumption the right hand side is at most $C_n\|\varphi\|^2_{L^2(4Q)}$.Thus,

$$\int_Q \mathcal{A}_2^+(t\,\dot{\boldsymbol{u}})(x)^2\,dx \le C\int_{4Q}\varphi^2 + \int_Q\int_{\ell(Q)/4}^\infty\int_{|x-y|<t}|\dot{\boldsymbol{u}}(y,t)|^2\frac{1}{t^{n-1}}\,dy\,dt\,dx.$$

Suppose that $x\in Q$, that $t>0$, and that $|x-y|<t$. Then $\operatorname{dist}(y,(3/2)Q)\le \max(0,t-\ell(Q)/4)$, and so

$$\int_Q\int_{\ell(Q)/4}^\infty\int_{|x-y|<t}|\dot{\boldsymbol{u}}(y,t)|^2\,\frac{1}{t^{n-1}}\,dy\,dt\,dx \le |Q|\int_{\ell(Q)/4}^\infty\int_{\operatorname{dist}(y,(3/2)Q)<t-\ell(Q)/4}|\dot{\boldsymbol{u}}(y,t)|^2\,\frac{1}{t^{n-1}}\,dy\,dt.$$

Let $\mathcal{G}$ be a grid of $(3N/2)^n$ cubes contained in $(3/2)Q$ with side length $\ell(Q)/N$ and pairwise-disjoint interiors, for N a large even integer to be chosen momentarily. Then

$$|Q|\int_{\ell(Q)/4}^\infty\int_{\operatorname{dist}(y,(3/2)Q)<t-\ell(Q)/4}|\dot{\boldsymbol{u}}(y,t)|^2\,\frac{1}{t^{n-1}}\,dy\,dt \le |Q|\sum_{R\in\mathcal{G}}\int_{\ell(Q)/4}^\infty\int_{\operatorname{dist}(y,R)<t-\ell(Q)/4}|\dot{\boldsymbol{u}}(y,t)|^2\,\frac{1}{t^{n-1}}\,dy\,dt.$$

If $z\in R$ and $\operatorname{dist}(y,R)<t-\ell(Q)/4$, then $|z-y|<t-\ell(Q)/4+\ell(R)\sqrt{n} = t+\ell(Q)(\sqrt{n}/N-1/4)$. Choosing $N\ge 4\sqrt{n}$, we see that for any $z\in R$,

$$\int_{\ell(Q)/4}^\infty\int_{\operatorname{dist}(y,R)<t-\ell(Q)/4}|\dot{\boldsymbol{u}}(y,t)|^2\,\frac{1}{t^{n-1}}\,dy\,dt \le \int_{\ell(Q)/4}^\infty\int_{|z-y|<t}|\dot{\boldsymbol{u}}(y,t)|^2\,\frac{1}{t^{n-1}}\,dy\,dt \le \mathcal{A}_2^+(t\,\dot{\boldsymbol{u}})(z)^2.$$

Averaging over all $z\in R$, we see that

$$|Q|\sum_{R\in\mathcal{G}}\int_{\ell(Q)/4}^\infty\int_{\operatorname{dist}(y,R)<t-\ell(Q)/4}|\dot{\boldsymbol{u}}(y,t)|^2\,\frac{1}{t^{n-1}}\,dy\,dt \le |Q|\sum_{R\in\mathcal{G}}\Big(\fint_R\mathcal{A}_2^+(t\,\dot{\boldsymbol{u}})(z)\,dz\Big)^2 \le |Q|(3N/2)^n\Big(\frac{|2Q|}{|R|}\fint_{2Q}\mathcal{A}_2^+(t\,\dot{\boldsymbol{u}})(z)\,dz\Big)^2$$

and so

$$\int_Q\int_{\ell(Q)/4}^\infty\int_{|x-y|<t}|\dot{\boldsymbol{u}}(y,t)|^2\,\frac{1}{t^{n-1}}\,dy\,dt\,dx \le C|Q|\Big(\fint_{2Q}\mathcal{A}_2^+(t\,\dot{\boldsymbol{u}})(z)\,dz\Big)^2.$$

Thus,

$$\int_Q\mathcal{A}_2^+(t\,\dot{\boldsymbol{u}})(x)^2\,dx \le C\int_{4Q}\varphi^2 + C|Q|\Big(\fint_{2Q}\mathcal{A}_2^+(t\,\dot{\boldsymbol{u}})(z)\,dz\Big)^2.$$

By Lemma 3.8, there is some $p>2$ depending on n and C such that

$$\int_{\mathbb{R}^n}\mathcal{A}_2^+(t\,\dot{\boldsymbol{u}})(x)^p\,dx \le C\int_{\mathbb{R}^n}\varphi^p$$

as desired. □

4. The single layer potential

In this section we will establish the nontangential estimates (27–28) and area integral estimates (36–37) on the single layer potential (and modified single layer potential).

We will begin (Sections 4.1 and 4.2) by showing that $\mathcal{S}^L$ and $\mathcal{S}^L_\nabla$ are well defined operators from $L^2(\mathbb{R}^n)$ to $\dot{W}^2_{m,loc}(\mathbb{R}^{n+1}_\pm)$ and $\dot{W}^2_{m-1,loc}(\mathbb{R}^{n+1}_\pm)$, respectively, and recalling or establishing some bounds on $\mathcal{S}^L\dot{\boldsymbol{g}}$ and $\mathcal{S}^L_\nabla\dot{\boldsymbol{h}}$ in the cases $\dot{\boldsymbol{g}}, \dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n)$. In particular, we will show that the boundary operators $\dot{\mathbf{Tr}}^\pm_m\,\mathcal{S}^L$ and $\dot{\mathbf{Tr}}^\pm_{m-1}\,\mathcal{S}^L_\nabla$ are bounded from $L^2(\mathbb{R}^n)$ to itself.

In Section 4.3, we will show that if the order $2m$ of the operator L is high enough, then the boundary operators $\dot{\mathbf{Tr}}^\pm_m\,\mathcal{S}^L$ and $\dot{\mathbf{Tr}}^\pm_{m-1}\,\mathcal{S}^L_\nabla$ are also bounded from $L^p(\mathbb{R}^n)$ to itself, for p near but not necessarily equal to 2. In Section 4.4 we will pass to the case of operators L of lower order, and finally in Section 4.5 will pass from boundary estimates to nontangential (and area integral) estimates.

4.1. $\mathcal{S}^L$ as an operator on $L^2(\mathbb{R}^n)$.

Recall from the definition (54) that the single layer potential $\mathcal{S}^L$ was originally defined as an operator from $(\dot{W}A^2_{m-1,1/2}(\mathbb{R}^n))^*$ (or $\dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$) to $\dot{W}^2_m(\mathbb{R}^{n+1})$. Suppose that $\dot{\boldsymbol{g}} \in \dot{B}^{2,2}_{-1/2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then by [19, formula (4.3)], we have that

$$\sup_{t\neq 0}\|\nabla^m\mathcal{S}^L\dot{\boldsymbol{g}}(\,\cdot\,,t)\|_{L^2(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)}. \tag{70}$$

Because $\dot{B}^{2,2}_{-1/2}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we have that $\mathcal{S}^L\dot{\boldsymbol{g}}$ extends to an operator that is bounded from $L^2(\mathbb{R}^n)$ to $\dot{W}^2_m(\mathbb{R}^n\times(a,b))$ for any $-\infty < a < b < \infty$.

We have some further properties of $\mathcal{S}^L\dot{\boldsymbol{g}}$ for $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n)$.

Recall that formula (55) is valid for almost every $(x,t) \in \mathbb{R}^{n+1}$ and all $\dot{\boldsymbol{g}} \in \dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$. If $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n)$ is compactly supported and integrates to zero, then $\dot{\boldsymbol{g}} \in \dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$ and so formula (55) is valid. By Lemma 3.3 and the bound (59), if $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n)$ is compactly supported then the integral on the right hand side of formula (55) converges absolutely for almost every $(x,t) \in \mathbb{R}^{n+1}_\pm$, even if $\dot{\boldsymbol{g}}$ does not integrate to zero. By density, formula (55) is valid for all compactly supported $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n)$.

By density, we have that $L(\mathcal{S}^L\dot{\boldsymbol{g}}) = 0$ in the weak sense in $\mathbb{R}^{n+1}_\pm$ for any $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n)$. By [21, Theorem 5.2], and by the bounds (34) and (70),

$$\lim_{t\to\pm\infty}\|\nabla^m\mathcal{S}^L\dot{\boldsymbol{g}}(\,\cdot\,,t)\|_{L^2(\mathbb{R}^n)} = 0 \quad \text{for all } \dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n). \tag{71}$$

Furthermore, the operators $\dot{\mathbf{Tr}}^\pm_m\,\mathcal{S}^L$ are bounded from $L^2(\mathbb{R}^n)$ to itself and satisfy

$$\lim_{t\to 0^\pm}\|\nabla^m\mathcal{S}^L\dot{\boldsymbol{g}}(\,\cdot\,,t) - \dot{\mathbf{Tr}}^\pm_m\,\mathcal{S}^L\dot{\boldsymbol{g}}\|_{L^2(\mathbb{R}^n)} = 0 \quad \text{for all } \dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n). \tag{72}$$

4.2. The modified single layer potential $\mathcal{S}^L_\nabla$.

The gradient $\nabla^{m-1}\mathcal{S}^L_\nabla$ of the modified single layer potential was defined by formula (56) as an integral operator. We begin this section by showing that there is a function $\mathcal{S}^L_\nabla\dot{\boldsymbol{h}}$ in $\dot{W}^2_{m-1,loc}(\mathbb{R}^{n+1}_\pm)$ whose gradient $\nabla^{m-1}\mathcal{S}^L_\nabla\dot{\boldsymbol{h}}$ is given by formula (56), and that $L(\mathcal{S}^L_\nabla\dot{\boldsymbol{h}}) = 0$ in $\mathbb{R}^{n+1}_\pm$.

Lemma 4.1. *Let L be an operator of the form* (1) *of order $2m$ associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44).

If $|\gamma| = m-1$, and if $\dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n)$, then the integral in the definition (56) *of $\partial^\gamma \mathcal{S}^L_\nabla \dot{\boldsymbol{h}}(x,t)$ converges absolutely for almost every $(x,t) \in \mathbb{R}^{n+1}_\pm$. Furthermore, if $K \subset \mathbb{R}^{n+1}_\pm$ is compact, then $\partial^\gamma \mathcal{S}^L_\nabla$ is bounded $L^2(\mathbb{R}^n) \mapsto L^2(K)$.*

If $|\beta| = m$ and $\beta_{n+1} \geq 1$, and if $h \in \dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$ or $h \in L^2(\mathbb{R}^n)$, then the function $\nabla^{m-1}\mathcal{S}^L_\nabla(h\dot{\boldsymbol{e}}_\beta)$ given by formula (56) *satisfies*

$$\nabla^{m-1}\mathcal{S}^L_\nabla(h\dot{\boldsymbol{e}}_\beta)(x,t) = -\nabla^{m-1}\partial_t \mathcal{S}^L(h\dot{\boldsymbol{e}}_\zeta)(x,t) \quad \text{where } \beta = \zeta + \vec{e}_{n+1} \tag{73}$$

where $\dot{\boldsymbol{e}}_\beta$ and $\dot{\boldsymbol{e}}_\zeta$ are as given by formula (41).

If $h \in L^2(\mathbb{R}^n) \cap \dot{B}^{2,2}_{1/2}(\mathbb{R}^n)$, and if $\beta_{n+1} < |\beta| = m$, then the gradient $\nabla^m \mathcal{S}^L_\nabla(h\dot{\boldsymbol{e}}_\beta)$ of the function $\nabla^{m-1}\mathcal{S}^L_\nabla(h\dot{\boldsymbol{e}}_\beta)$ given by formula (56) *satisfies*

$$\nabla^m \mathcal{S}^L_\nabla(h\dot{\boldsymbol{e}}_\beta)(x,t) = -\nabla^m \mathcal{S}^L(\partial_{x_j} h\dot{\boldsymbol{e}}_\zeta)(x,t) \quad \text{where } \beta = \zeta + \vec{e}_j \tag{74}$$

where j is any number with $1 \leq j \leq n$ and with $\vec{e}_j \leq \beta$.

Thus by density, if $\dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n)$, then there is a function $\mathcal{S}^L_\nabla \dot{\boldsymbol{h}} \in \dot{W}^2_{m-1,loc}(\mathbb{R}^n)$ such that, if $|\gamma| = m-1$, then $\partial^\gamma \mathcal{S}^L_\nabla \dot{\boldsymbol{h}}$ is given by formula (56). *Furthermore,*

$$L(\mathcal{S}^L_\nabla \dot{\boldsymbol{h}}) = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_\pm.$$

Proof. By Lemma 3.3 and the bound (59) or (60), if $Q \subset \mathbb{R}^n$ is a cube of side length $\ell > 0$, then

$$\int_Q \int_\ell^{2\ell} \int_{\mathbb{R}^n} |\partial^\gamma_{x,t} \nabla^m_{y,s} E^L(x,t,y,0)|^2 \, dy \, dt \, dx + \int_Q \int_{-2\ell}^{-\ell} \int_{\mathbb{R}^n} |\partial^\gamma_{x,t} \nabla^m_{y,s} E^L(x,t,y,0)|^2 \, dy \, dt \, dx \leq C\ell. \tag{75}$$

In particular, $\partial^\gamma_{x,t} \nabla^m_{y,s} E^L(x,t,\cdot,0) \in L^2(\mathbb{R}^n)$ for almost every $(x,t) \in \mathbb{R}^{n+1}_\pm$. A straightforward covering argument establishes the local boundedness of $\partial^\gamma \mathcal{S}^L_\nabla$.

We now turn to formula (73). Choose some β with $|\beta| = m$ and $\beta_{n+1} \geq 1$. Let $\zeta + \vec{e}_{n+1} = \beta$. If $h \in \dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$, then by formula (55) the function $-\partial_t \mathcal{S}^L(h\dot{\boldsymbol{e}}_\zeta)$ satisfies

$$-\partial^\gamma \partial_t \mathcal{S}^L(h\dot{\boldsymbol{e}}_\zeta)(x,t) = -\int_{\mathbb{R}^n} \partial^\gamma_{x,t} \partial_t \partial^\zeta_{y,s} E^L(x,t,y,0) \, h(y) \, dy.$$

By formula (65),

$$-\partial^\gamma \partial_t \mathcal{S}^L(h\dot{\boldsymbol{e}}_\zeta)(x,t) = \int_{\mathbb{R}^n} \partial^\gamma_{x,t} \partial^\beta_{y,s} E^L(x,t,y,0) \, h(y) \, dy.$$

Thus, by formula (56), formula (73) is valid for all $h \in \dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$; because $\partial^\gamma \mathcal{S}^L_\nabla$ and $\nabla^m \mathcal{S}^L$ are both bounded $L^2(\mathbb{R}^n) \mapsto L^2_{loc}(\mathbb{R}^{n+1}_\pm)$, by density formula (73) is valid for all $h \in L^2(\mathbb{R}^n)$.

Finally, we turn to formula (74). If $\beta_{n+1} < |\beta| = m$, then there is some j with $1 \leq j \leq n$ such that $\vec{e}_j \leq \beta$. Let $\zeta + \vec{e}_j = \beta$. If $h \in \dot{B}^{2,2}_{1/2}(\mathbb{R}^n)$, then the (formal) derivative $\partial_j h$ lies in $\dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$, and

$$-\partial^\alpha \mathcal{S}^L(\partial_j h\dot{\boldsymbol{e}}_\zeta)(x,t) = -\int_{\mathbb{R}^n} \partial^\alpha_{x,t} \partial^\zeta_{y,s} E^L(x,t,y,0) \, \partial_{y_j} h(y) \, dy.$$

Recall from the remarks following formula (55) that for almost every $(x,t) \in \mathbb{R}^{n+1}_{\pm}$, the right hand side converges provided $h \in \dot{B}^{2,2}_{1/2}(\mathbb{R}^n)$. But if $h \in L^2(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} \partial^{\alpha}_{x,t}\partial^{\beta}_{y,s} E^L(x,t,y,0)\, h(y)\, dy$$

converges absolutely for almost every $(x,t) \in \mathbb{R}^{n+1}_{\pm}$. Thus, we may integrate by parts to see that

$$-\partial^{\alpha}\mathcal{S}^L(\partial_j h \dot{\boldsymbol{e}}_{\zeta})(x,t) = \int_{\mathbb{R}^n} \partial^{\alpha}_{x,t}\partial^{\beta}_{y,s} E^L(x,t,y,0)\, h(y)\, dy.$$

If $|\gamma| = m-1$ and $1 \le k \le n+1$, then

$$-\partial_k\partial^{\gamma}\mathcal{S}^L(\partial_j h \dot{\boldsymbol{e}}_{\zeta})(x,t) = \partial_k \int_{\mathbb{R}^n} \partial^{\gamma}_{x,t}\partial^{\beta}_{y,s} E^L(x,t,y,0)\, h(y)\, dy.$$

Thus, formula (74) is valid. This completes the proof. □

We now establish bounds similar to the bounds (70), (71), and (72).

Lemma 4.2. *Let L be an operator of the form* (1) *of order $2m$ associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44).

For all $\dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n)$, we have that

$$\sup_{t\neq 0} \|\nabla^{m-1}\mathcal{S}^L_{\nabla}\dot{\boldsymbol{h}}(\,\cdot\,,t)\|_{L^2(\mathbb{R}^n)} \le C\|\dot{\boldsymbol{h}}\|_{L^2(\mathbb{R}^n)}, \tag{76}$$

$$\lim_{t\to\pm\infty} \|\nabla^{m-1}\mathcal{S}^L_{\nabla}\dot{\boldsymbol{h}}(\,\cdot\,,t)\|_{L^2(\mathbb{R}^n)} = 0. \tag{77}$$

Furthermore, the boundary operator $\dot{\mathbf{Tr}}^{\pm}_{m-1}\,\mathcal{S}^L_{\nabla}$ is bounded from $L^2(\mathbb{R}^n)$ to itself and satisfies

$$\lim_{t\to 0^{\pm}} \|\nabla^{m-1}\mathcal{S}^L_{\nabla}\dot{\boldsymbol{h}}(\,\cdot\,,t) - \dot{\mathbf{Tr}}^{\pm}_{m-1}\,\mathcal{S}^L_{\nabla}\dot{\boldsymbol{h}}\|_{L^2(\mathbb{R}^n)} = 0 \tag{78}$$

for all $\dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n)$.

Proof. By formula (73), if $\dot{\boldsymbol{h}} = h\dot{\boldsymbol{e}}_{\beta}$ for some β with $\beta_{n+1} \ge 1$, then the theorem follows from the bounds (70–72) and Lemma 4.1.

More generally, by [21, Theorem 5.1] and the bound (35), if $\dot{\boldsymbol{h}} \in L^2(\mathbb{R}^n)$, then there are two polynomials $P_{\pm}$ of degree $m-1$ that satisfy

$$\sup_{\pm t>0} \|\nabla^{m-1}\mathcal{S}^L_{\nabla}\dot{\boldsymbol{h}}(\,\cdot\,,t) - \nabla^{m-1}P_{\pm}\|_{L^2(\mathbb{R}^n)} \le C\|\dot{\boldsymbol{h}}\|_{L^2(\mathbb{R}^n)},$$

$$\|\dot{\mathbf{Tr}}^{\pm}_{m-1}\,\mathcal{S}^L_{\nabla}\dot{\boldsymbol{h}} - \nabla^{m-1}P_{\pm}\|_{L^2(\mathbb{R}^n)} \le C\|\dot{\boldsymbol{h}}\|_{L^2(\mathbb{R}^n)},$$

$$\lim_{t\to\pm\infty} \|\nabla^{m-1}\mathcal{S}^L_{\nabla}\dot{\boldsymbol{h}}(\,\cdot\,,t) - \nabla^{m-1}P_{\pm}\|_{L^2(\mathbb{R}^n)} = 0,$$

$$\lim_{t\to 0^{\pm}} \|\nabla^{m-1}\mathcal{S}^L_{\nabla}\dot{\boldsymbol{h}}(\,\cdot\,,t) - \dot{\mathbf{Tr}}^{\pm}_{m-1}\,\mathcal{S}^L_{\nabla}\dot{\boldsymbol{h}}\|_{L^2(\mathbb{R}^n)} = 0.$$

We need only show that $\nabla^{m-1}P_{\pm} = 0$.

We will consider only P_+. Let Q be a cube in $\mathbb{R}^n$ of side length t. Then

$$\begin{aligned}|\nabla^{m-1}P_+|^2 &= \fint_Q |\nabla^{m-1}P_+|^2 \\ &\le 2\fint_Q |\nabla^{m-1}\mathcal{S}^L_{\nabla}\dot{\boldsymbol{h}}(x,t) - \nabla^{m-1}P_+|^2\, dx + 2\fint_Q |\nabla^{m-1}\mathcal{S}^L_{\nabla}\dot{\boldsymbol{h}}(x,t)|^2\, dx.\end{aligned}$$

By the given bound on $\nabla^{m-1}\mathcal{S}^L_\nabla \dot{\boldsymbol{h}}(x,t) - \nabla^{m-1}P_+$,

$$|\nabla^{m-1}P_+|^2 \leq \frac{C}{t^n}\|\dot{\boldsymbol{h}}\|^2_{L^2(\mathbb{R}^n)} + 2\fint_Q |\nabla^{m-1}\mathcal{S}^L_\nabla \dot{\boldsymbol{h}}(x,t)|^2\,dx.$$

By the bound (75) and Lemma 3.3,

$$|\nabla^{m-1}P_+|^2 \leq \frac{C}{t^n}\|\dot{\boldsymbol{h}}\|_{L^2(\mathbb{R}^n)}.$$

Letting $t \to \infty$, we see that $\nabla^{m-1}P_+ = 0$, as desired. □

4.3. **Boundary values and operators of high order.** In this section, we will show that if $2m \geq n+3$, then the boundary operators $\dot{\mathbf{Tr}}^\pm_m \mathcal{S}^L$ and $\dot{\mathbf{Tr}}^\pm_{m-1}\mathcal{S}^L_\nabla$ are bounded on $L^p(\mathbb{R}^n)$ for some values of $p \neq 2$. We will also establish some preliminary nontangential estimates. In Section 4.4 we will show how to generalize to operators of low order, and in Section 4.5 we will pass to nontangential and area integral estimates.

We begin with the purely vertical derivatives.

Lemma 4.3. *Let L be an operator of the form* (1) *of order $2m$, with $2m \geq n+3$, associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44). *Suppose that $2 \leq p < \infty$. Then*

$$\|\mathrm{Tr}^\pm\, \partial_t^{m-1}\mathcal{S}^L_\nabla \dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)},$$
$$\|\mathrm{Tr}^\pm\, \partial_t^{m}\mathcal{S}^L \dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}$$

provided $\dot{\boldsymbol{g}} \in L^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and $\dot{\boldsymbol{h}} \in L^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$.

Proof. By the bounds (70), (72), (76), and (78), we have that $\mathrm{Tr}^\pm\, \partial_t^{m-1}\mathcal{S}^L_\nabla \dot{\boldsymbol{h}}$ and $\mathrm{Tr}^\pm\, \partial_t^{m}\mathcal{S}^L \dot{\boldsymbol{g}}$ do exist as L^2 functions. The bound on $\mathcal{S}^L$ follows from the bound on $\mathcal{S}^L_\nabla$ by formula (73). By formula (55) for $\mathcal{S}^{L^*}$, the definition (56) of $\mathcal{S}^L_\nabla$, and the symmetry relations (61) and (64) for the fundamental solution, we have the duality relation

$$\langle \dot{\boldsymbol{g}}, \nabla^{m-1}\mathcal{S}^L_\nabla \dot{\boldsymbol{h}}(\,\cdot\,,t)\rangle_{\mathbb{R}^n} = \langle \nabla^m \mathcal{S}^{L^*}\dot{\boldsymbol{g}}(\,\cdot\,,-t), \dot{\boldsymbol{h}}\rangle_{\mathbb{R}^n} \tag{79}$$

for all $t \neq 0$. Taking limits, we see that it suffices to show that the bound

$$\|\dot{\mathbf{Tr}}^\mp_m\, \mathcal{S}^{L^*}(g\dot{\boldsymbol{e}}_\perp)(\,\cdot\,,t)\|_{L^q(\mathbb{R}^n)} \leq C\|g\|_{L^q(\mathbb{R}^n)}$$

is valid for all $1 < q \leq 2$. Here $\dot{\boldsymbol{e}}_\perp = \dot{\boldsymbol{e}}_{(m-1)\vec{e}_{n+1}}$, so $\langle \dot{\boldsymbol{e}}_\perp, \nabla^{m-1}\varphi\rangle = \partial^{m-1}_{n+1}\varphi$ for all functions φ with weak derivatives of order up to $m-1$.

Let $|\alpha| = m$, and let

$$T_\alpha g = \mathrm{Tr}^\mp\, \partial^\alpha \mathcal{S}^{L^*}(g\dot{\boldsymbol{e}}_\perp).$$

Again by formulas (70) and (72), T_α is a well-defined, bounded operator on $L^2(\mathbb{R}^n)$.

We now show that T_α satisfies a weak bound on $L^1(\mathbb{R}^n)$; by interpolation this operator satisfies a strong bound on $L^q(\mathbb{R}^n)$ for any q in the range $1 < q < 2$.

Let $g \in L^1(\mathbb{R}^n)$. Fix some number $\mu > 0$. We seek to show that

$$|\{x : |T_\alpha g(x)| > \mu\}| < \frac{C\|g\|_{L^1(\mathbb{R}^n)}}{\mu}.$$

We apply a standard Calderón-Zygmund decomposition to g. That is, there exists a collection $\{Q_i\}$ of closed cubes with pairwise-disjoint interiors, a bounded

function s, and unbounded functions u_i such that

$$g = s + \sum_i u_i,$$

such that each u_i is supported in Q_i, and such that the following bounds are valid:

$$\|s\|_{L^\infty(\mathbb{R}^n)} \leq \mu, \qquad \int_{Q_i} u_i = 0, \qquad \int_{Q_i} |u_i| \leq 2\mu|Q_i|, \qquad \sum_i |Q_i| \leq \frac{2^n}{\mu}\int_{\mathbb{R}^n}|g|.$$

As usual, if $|T_\alpha g(x)| > \mu$ then either $|T_\alpha s(x)| > \mu/2$ or $|T_\alpha u(x)| > \mu/2$, where $u = \sum_i u_i$. Notice that

$$\|s\|_{L^2(\mathbb{R}^n)} \leq \left(\int_{\mathbb{R}^n\setminus\cup_i Q_i} |s|^2 + \sum_i |Q_i|\mu^2\right)^{1/2}.$$

For almost every $x \notin \cup_i Q_i$, we have that $s(x) = g(x)$ and $|s(x)| \leq \mu$; thus

$$\|s\|_{L^2(\mathbb{R}^n)} \leq \left(\int_{\mathbb{R}^n\setminus\cup_i Q_i} |g|\mu + \sum_i |Q_i|\mu^2\right)^{1/2} \leq C\mu^{1/2}\|g\|^{1/2}_{L^1(\mathbb{R}^n)}.$$

Applying boundedness of T_α on $L^2(\mathbb{R}^n)$, we see that

$$|\{x \in \mathbb{R}^n : |T_\alpha s(x)| > \mu/2\}| \leq 4\frac{\|T_\alpha s\|^2_{L^2(\mathbb{R}^n)}}{\mu^2} \leq C\frac{\|s\|^2_{L^2(\mathbb{R}^n)}}{\mu^2} \leq C^2\frac{\|g\|_{L^1(\mathbb{R}^n)}}{\mu}$$

as desired.

We now turn to the set $|\{x \in \mathbb{R}^n : |T_\alpha u(x)| > \mu/2\}|$. We have that $\sum_i |8Q_i| \leq C\|g\|_{L^1(\mathbb{R}^n)}/\mu$, and so we will consider only the set

$$\{x \in \mathbb{R}^n : |T_\alpha u(x)| > \mu/2\} \setminus \bigcup_i 8Q_i.$$

If $x \notin Q_i$, then by formula (55),

$$T_\alpha u_i(x) = \int_{Q_i} (\partial^\alpha_{x,t}\partial^{m-1}_s E^{L^*}(x,t,y,s) - \partial^\alpha_{x,t}\partial^{m-1}_s E^{L^*}(x,t,y_0,s))\big|_{s=t=0}\, u_i(y)\, dy$$

for any y_0; in particular, we choose y_0 to be the midpoint of Q_i. Let $A_j = 2^{j+1}Q_i \setminus 2^jQ_i$. Suppose that $j \geq 3$. Then

$$\int_{A_j} |T_\alpha u_i(x)|\, dx \leq \int_{Q_i} |u_i(y)| \int_{A_j} |\partial^\alpha_{x,t}\partial^{m-1}_s(E^{L^*}(x,0,y,0) - E^{L^*}(x,0,y_0,0))|\, dx\, dy.$$

Let $w(y,s) = \partial^\alpha_{x,t}\partial^{m-1}_s E^{L^*}(x,t,y,s)$. We observe that $L^*w = 0$ away from the point (x,t). If $2m \geq n+3$, then by Theorem 3.2, ∇w is continuous and pointwise bounded away from (x,t), and so if $j \geq 3$ then

$$\int_{A_j} |\partial^\alpha_{x,t}\partial^{m-1}_s(E^{L^*}(x,0,y,0) - E^{L^*}(x,0,y_0,0))|\, dx$$
$$\leq C\ell(Q_i)\int_{A_j} \fint_{2^{j-2}\widetilde{Q}_i} |\partial^\alpha_{x,t}\nabla^m_{y,s}E^{L^*}(x,0,y,s)|\, dy\, ds\, dx$$

where $\widetilde{Q}_i = Q_i \times (-\ell(Q_i)/2, \ell(Q_i)/2)$ is a cube in $\mathbb{R}^{n+1}$. We change the order of integration and apply Lemma 3.3 to the function $v(x,t) = \nabla^m_{y,s}E^{L^*}(x,t,y,s)$ to see

that

$$\int_{A_j} |\partial_{x,t}^\alpha \partial_s^{m-1}(E^{L^*}(x,0,y,0) - E^{L^*}(x,0,y_0,0))|\,dx \\ \leq \frac{C}{2^j} \fint_{2^{j-2}\widetilde{Q}_i} \int_{\widetilde{A}_{j,1}} |\partial_{x,t}^\alpha \nabla_{y,s}^m E^{L^*}(x,t,y,s)|\,dx\,dt\,dy\,ds$$

where $\widetilde{A}_{j,1} = (A_j \cup A_{j-1} \cup A_{j+1}) \times (-2^j \ell(Q), 2^j \ell(Q))$.

By Hölder's inequality and the bound (59),

$$\int_{A_j} |\partial_{x,t}^\alpha \partial_s^{m-1}(E^{L^*}(x,0,y,0) - E^{L^*}(x,0,y_0,0))|\,dx \leq C2^{-j}$$

and so

$$\int_{\mathbb{R}^n \setminus 8Q_i} |T_\alpha u_i(x)|\,dx \leq C \int_{Q_i} |u_i(y)|\,dy.$$

Thus,

$$\int_{\mathbb{R}^n \setminus \cup_i 8Q_i} |T_\alpha u| \leq C \int_{\mathbb{R}^n} |g|$$

and so

$$|\{x : |T_\alpha u(x)| > \mu/2\}| \leq \frac{C}{\mu} \int_{\mathbb{R}^n} |g|$$

as desired. □

We will now establish nontangential estimates on the purely vertical derivatives of the single layer potential. We observe that the conditions of the following lemma are met when $2m \geq n+3$, $k = m$ and $2 \leq q < p < \infty$. We will later apply the lemma in the $k = 0$ and $q < 2$ cases.

Lemma 4.4. *Let L be an operator of the form* (1) *of order $2m$, with $2m \geq n+1$, associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44)*. Let p^+_{j,L^*} be as in Theorem 3.2 with L replaced by L^*, and let $1/p^+_{j,L^*} + 1/p^-_{j,L^*} = 1$.*

Let $0 \leq k \leq m$, let γ be a multiindex with $|\gamma| = m-1$, and let q and p satisfy $p^-_{j,L^*} < q < p \leq \infty$, where $j = \gamma_{n+1} + 1$. Suppose that the boundary value operator

$$g \mapsto \dot{\mathbf{Tr}}^+_{m-k}\, \partial_{n+1}^k \mathcal{S}^L(g\dot{\boldsymbol{e}}_\gamma),$$

(which is well defined for all $g \in L^2(\mathbb{R}^n)$) extends by density to an operator that is bounded $L^q(\mathbb{R}^n) \mapsto L^q(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n) \mapsto L^p(\mathbb{R}^n)$.

Then we have the bound

$$\|\widetilde{N}_+(\nabla^{m-k}\partial_{n+1}^k \mathcal{S}^L(g\dot{\boldsymbol{e}}_\gamma))\|_{L^p(\mathbb{R}^n)} \leq C_{p,q}\|g\|_{L^p(\mathbb{R}^n)}$$

for all $g \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, for some constant $C_{p,q}$ depending only on q, p and the standard parameters.

Similarly, if $1 \leq k \leq m$, if $|\alpha| = m$, and if

$$h \mapsto \dot{\mathbf{Tr}}^+_{m-k}\, \partial_{n+1}^{k-1} \mathcal{S}^L_\nabla(h\dot{\boldsymbol{e}}_\alpha)$$

is bounded $L^q(\mathbb{R}^n) \mapsto L^q(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n) \mapsto L^p(\mathbb{R}^n)$ for some $p^-_{j,L^*} < q < p \leq \infty$, where $j = \alpha_{n+1}$, then

$$\|\widetilde{N}_+(\nabla^{m-k}\partial_{n+1}^{k-1} \mathcal{S}^L_\nabla(h\dot{\boldsymbol{e}}_\alpha))\|_{L^p(\mathbb{R}^n)} \leq C_{p,q}\|h\|_{L^p(\mathbb{R}^n)}$$

for all $h \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Proof. Let $\zeta = \alpha - j\vec{e}_{n+1}$ or $\zeta = \gamma - (j-1)\vec{e}_{n+1}$, so $\alpha = (\zeta, j)$ or $\gamma = (\zeta, j-1)$. By formulas (55) and (56) for $\mathcal{S}^L$ and $\mathcal{S}^L_\nabla$, and by formula (65), we have that if $f \in L^2(\mathbb{R}^n)$ is compactly supported, then for almost every $(x,t) \in \mathbb{R}^{n+1}_\pm$,

$$-\nabla^{m-k}\partial_t^k \mathcal{S}^L(f\dot{\boldsymbol{e}}_\gamma)(x,t) = (-1)^j \int_{\mathbb{R}^n} \nabla^{m-k}_{x,t}\partial_t^{k+j-1}\partial_y^\zeta E^L(x,t,y,0)\, f(y)\, dy, \tag{80}$$

$$\nabla^{m-k}\partial_t^{k-1} \mathcal{S}^L_\nabla(f\dot{\boldsymbol{e}}_\alpha)(x,t) = (-1)^j \int_{\mathbb{R}^n} \nabla^{m-k}_{x,t}\partial_t^{k+j-1}\partial_y^\zeta E^L(x,t,y,0)\, f(y)\, dy. \tag{81}$$

Choose some $g \in L^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and let either $u = -\partial_{n+1}^k \mathcal{S}^L(g\dot{\boldsymbol{e}}_\gamma)$ or $u = \partial_{n+1}^{k-1}\mathcal{S}^L_\nabla(g\dot{\boldsymbol{e}}_\alpha)$.

In either case we wish to bound $\widetilde{N}_+(\nabla^{m-k}u)$. Let $x_0 \in \mathbb{R}^n$. By Lemma 3.5,

$$\widetilde{N}_+(\nabla^{m-k}u)(x_0) \leq C \sup_{t_0>0} \left(\fint_{Q(x_0,t_0)} \fint_{t_0/6}^{t_0/2} |\nabla^{m-k}u|^2 \right)^{1/2}.$$

Choose some $t_0 > 0$ and let $Q = Q(x_0,t_0)$. Let $\widetilde{Q} = Q \times (-\ell(Q)/2, \ell(Q)/2) = Q \times (-t_0/2, t_0/2)$. Let $u_Q = -\partial_{n+1}^k \mathcal{S}^L(\mathbf{1}_{4Q} g\dot{\boldsymbol{e}}_\gamma)$ or $u_Q = \partial_{n+1}^{k-1}\mathcal{S}^L_\nabla(\mathbf{1}_{4Q} g\dot{\boldsymbol{e}}_\alpha)$. Observe that

$$\left(\fint_Q \fint_{t_0/6}^{t_0/2} |\nabla^{m-k}u|^2 \right)^{1/2} \leq \left(\fint_Q \fint_{t_0/6}^{t_0/2} |\nabla^{m-k}u_Q|^2 \right)^{1/2} + \left(3 \fint_{\widetilde{Q}} |\nabla^{m-k}(u-u_Q)|^2 \right)^{1/2}.$$

By formulas (80) and (81) and by Hölder's inequality,

$$\begin{aligned} &\fint_Q \fint_{t_0/6}^{t_0/2} |\nabla^{m-k}u_Q|^2 \\ &\quad \leq \fint_Q \fint_{t_0/6}^{t_0/2} \left(\int_{4Q} |\nabla^{m-k}_{x,t}\partial_t^{k+j-1}\partial_y^\zeta E^L(x,t,y,0)|^{q'}\, dy \right)^{2/q'} dt\, dx \left(\int_{4Q} |g|^q \right)^{2/q}. \end{aligned}$$

If $q > p^-_{j,L^*}$, then $q' < p^+_{j,L^*}$ and so we may use Lemma 3.3, Theorem 3.2 and the bound (59) to bound the integral of E^L. Thus,

$$\left(\fint_Q \fint_{t_0/6}^{t_0/2} |\nabla^{m-k}u_Q|^2 \right)^{1/2} \leq C \left(\fint_{4Q} |g|^q \right)^{1/q} \leq C\mathcal{M}(|g|^q)(x_0)^{1/q}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function given by formula (40).

By Lemma 3.6,

$$\begin{aligned} \left(\fint_{\widetilde{Q}} |\nabla^{m-k}(u-u_Q)|^2 \right)^{1/2} &\leq C\ell(Q) \fint_{2\widetilde{Q}} |\partial_t^{m-k+1}(u-u_Q)(x,t)|\, dt\, dx \\ &\quad + C \fint_{2Q} |\dot{\mathbf{T}}\mathbf{r}^+_{m-k}\, u_Q| + C \fint_{2Q} |\dot{\mathbf{T}}\mathbf{r}^+_{m-k}\, u|. \end{aligned}$$

The last term is at most $C\mathcal{M}(\dot{\mathbf{T}}\mathbf{r}^+_{m-k}\, u)$. By Hölder's inequality,

$$\fint_{2Q} |\dot{\mathbf{T}}\mathbf{r}^+_{m-k}\, u_Q| \leq \left(\fint_{2Q} |\dot{\mathbf{T}}\mathbf{r}^+_{m-k}\, u_Q|^q \right)^{1/q}.$$

By assumption,

$$\left(\fint_{2Q} |\dot{\mathbf{T}}\mathbf{r}^+_{m-k}\, u_Q|^q \right)^{1/q} \leq C \left(\fint_{4Q} |g|^q \right)^{1/q} \leq C\mathcal{M}(|g|^q)(x_0)^{1/q}.$$

Finally, we consider the term involving $\partial_t^{m-k+1}(u-u_Q)$. By formula (80) or (81), for almost every $(x,t)\in\mathbb{R}^{n+1}_\pm$ we have that

$$\partial_t^{m-k+1}(u-u_Q)(x,t)=(-1)^j\int_{\mathbb{R}^n\setminus 4Q}\partial_t^{m+j}\partial_y^\zeta E^L(x,t,y,0)\,g(y)\,dy.$$

Let $A_\ell=2^{\ell+1}Q\setminus 2^\ell Q$. Then

$$\partial_t^{m-k+1}(u-u_Q)(x,t)=(-1)^j\sum_{\ell=2}^\infty\int_{A_\ell}\partial_t^{m+j}\partial_y^\zeta E^L(x,t,y,0)\,g(y)\,dy.$$

Let

$$u_\ell(x,t)=\int_{A_\ell}\partial_t^{m+j}\partial_y^\zeta E^L(x,t,y,0)\,g(y)\,dy.$$

Observe that $Lu_\ell=0$ away from $A_\ell\times\{0\}$. If $2m\geq n+1$ and $\ell\geq 2$, then by Theorem 3.2,

$$\sup_{(x,t)\in 2\widetilde{Q}}|u_\ell(x,t)|\leq C\left(\fint_{2^{\ell-1/2}\widetilde{Q}}|u_\ell|^2\right)^{1/2}.$$

As before, by Hölder's inequality, Lemma 3.3, Theorem 3.2 and the bound (59), if $q'<p^+_{j,L^*}$, then

$$\left(\fint_{2^{\ell-1/2}\widetilde{Q}}|u_\ell|^2\right)^{1/2}\leq\frac{C}{2^\ell t_0}\left(\fint_{A_\ell}|g|^q\right)^{1/q}\leq\frac{C}{2^\ell t_0}\mathcal{M}(|g|^q)(x_0)^{1/q}.$$

Thus,

$$\left(\fint_Q\fint_{t_0/6}^{t_0/2}|\nabla^{m-k}u|^2\right)^{1/2}\leq C\mathcal{M}(|g|^q)(x_0)^{1/q}+C\mathcal{M}(\dot{\mathbf{T}}\mathbf{r}^+_{m-k}\,u)(x_0)$$

and so by Lemma 3.5,

$$\widetilde{N}_+(\nabla^{m-k}u)(x_0)\leq C\mathcal{M}(|g|^q)(x_0)^{1/q}+C\mathcal{M}(\dot{\mathbf{T}}\mathbf{r}^+_{m-k}\,u)(x_0).$$

By assumption, if $g\in L^p(\mathbb{R}^n)$ then $\dot{\mathbf{T}}\mathbf{r}^+_{m-k}\,u\in L^p(\mathbb{R}^n)$. We have that $p>1$ and so $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$, and so $\mathcal{M}(\dot{\mathbf{T}}\mathbf{r}^+_{m-k}\,u)\in L^p(\mathbb{R}^n)$. Furthermore, if $g\in L^p(\mathbb{R}^n)$, then $|g|^q\in L^{p/q}(\mathbb{R}^n)$. If $p>q$, then $p/q>1$ and so $\mathcal{M}$ is bounded on $L^{p/q}(\mathbb{R}^n)$. Thus, the right-hand side is in $L^p(\mathbb{R}^n)$ and the proof is complete. □

We now extend from boundary values of the purely vertical derivatives to boundary values of the full gradient.

Lemma 4.5. *Let L be an operator of the form* (1) *of order $2m$, with $2m\geq n+3$, associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44).

Then there is some $p>2$ such that

$$\|\dot{\mathbf{T}}\mathbf{r}^+_m\,\mathcal{S}^L\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}\leq C\|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}$$

for all $\dot{\boldsymbol{g}}\in L^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$.

Similarly, there is some $p>2$ such that

$$\|\dot{\mathbf{T}}\mathbf{r}^+_{m-1}\,\mathcal{S}^L_\nabla\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}\leq C\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}$$

for all $\dot{\boldsymbol{h}}\in L^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$.

Proof. We follow the proof of a similar inequality in [34, pp. 697–699]. Choose some $\dot{\boldsymbol{g}} \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, and let either $u = \mathcal{S}^L \dot{\boldsymbol{g}}$ and $\ell = m$ or $u = \mathcal{S}^L_\nabla \dot{\boldsymbol{g}}$ and $\ell = m-1$. We wish to show that for some $p > 2$ we have that $\dot{\mathbf{T}}\mathbf{r}^+_\ell\, u \in L^p(\mathbb{R}^n)$.

As in the proof of Lemma 3.7, we will use Lemma 3.8. For each cube $Q \subset \mathbb{R}^n$, let $\dot{\boldsymbol{g}}_Q = \dot{\boldsymbol{g}} \mathbf{1}_{4Q}$ and $\dot{\boldsymbol{g}} = \dot{\boldsymbol{g}}_Q + \dot{\boldsymbol{g}}_{Q,f}$, and let $u_Q = \mathcal{S}^L \dot{\boldsymbol{g}}_Q$ or $\mathcal{S}^L_\nabla \dot{\boldsymbol{g}}_Q$. Then

$$\fint_Q |\dot{\mathbf{T}}\mathbf{r}^+_\ell\, u|^2 \leq 2 \fint_Q |\dot{\mathbf{T}}\mathbf{r}^+_\ell\, u_Q|^2 + 2 \fint_Q |\dot{\mathbf{T}}\mathbf{r}^+_\ell (u - u_Q)|^2.$$

By the bounds (70) and (72) or (76) and (78),

$$\fint_Q |\dot{\mathbf{T}}\mathbf{r}^+_\ell\, u|^2 \leq C \fint_{4Q} |\dot{\boldsymbol{g}}|^2 + 2 \fint_Q |\dot{\mathbf{T}}\mathbf{r}^+_\ell (u - u_Q)|^2.$$

Observe that $L(u - u_Q) = 0$ in a neighborhood of $Q \times \{0\}$, and so we may write

$$\fint_Q |\dot{\mathbf{T}}\mathbf{r}^+_\ell (u - u_Q)|^2 = \fint_Q |\nabla^\ell (u - u_Q)(x, 0)|^2 \, dx.$$

By Lemma 3.3,

$$\fint_Q |\dot{\mathbf{T}}\mathbf{r}^+_\ell (u - u_Q)|^2 \leq C \fint_{(3/2)Q} \fint_{-\ell(Q)/2}^{\ell(Q)/2} |\nabla^\ell (u - u_Q)(x,t)|^2 \, dt \, dx.$$

By Lemma 3.6 and Hölder's inequality,

$$\begin{aligned} \fint_Q |\dot{\mathbf{T}}\mathbf{r}^+_\ell (u - u_Q)|^2 \leq C \ell(Q)^2 \fint_{2Q} \fint_{-\ell(Q)}^{\ell(Q)} |\partial_t^{\ell+1} (u - u_Q)(x,t)|^2 \, dt \, dx \\ + C \bigg(\fint_{2Q} |\nabla^\ell (u - u_Q)(x,0)| \, dx \bigg)^2. \end{aligned}$$

By the Caccioppoli inequality,

$$\ell(Q)^2 \fint_{2Q} \fint_{-\ell(Q)}^{\ell(Q)} |\partial_t^{\ell+1} (u - u_Q)(x,t)|^2 \, dt \, dx \leq C \fint_{3Q} \fint_{-(3/2)\ell(Q)}^{(3/2)\ell(Q)} |\partial_t^{\ell} (u - u_Q)(x,t)|^2 \, dt \, dx$$

Now,

$$\fint_{3Q} \fint_{-(3/2)\ell(Q)}^{(3/2)\ell(Q)} |\partial_t^\ell u_Q(x,t)|^2 \, dt \, dx \leq \frac{C}{|Q|} \sup_{t \neq 0} \|\partial_t^\ell u_Q(\,\cdot\,, t)\|^2_{L^2(\mathbb{R}^n)}$$

which by the bound (70) or (76) is at most $C|Q|^{-1} \|\dot{\boldsymbol{g}}_Q\|^2_{L^2(\mathbb{R}^n)}$. Thus,

$$\begin{aligned} \fint_Q |\dot{\mathbf{T}}\mathbf{r}^+_\ell (u - u_Q)|^2 \leq C \fint_{3Q} \fint_{-(3/2)\ell(Q)}^{(3/2)\ell(Q)} |\partial_t^\ell u(x,t)|^2 \, dt \, dx + C \fint_{4Q} |\dot{\boldsymbol{g}}(x)|^2 \, dx \\ + C \bigg(\fint_{2Q} |\nabla^\ell (u - u_Q)(x,0)| \, dx \bigg)^2. \end{aligned}$$

An elementary argument shows that

$$\fint_{3Q} \fint_{-(3/2)\ell(Q)}^{(3/2)\ell(Q)} |\partial_t^\ell u(x,t)|^2 \, dt \, dx \leq C \fint_{3Q} \widetilde{N}_*(\partial_{n+1}^\ell u)(x)^2 \, dx.$$

By Hölder's inequality,

$$\fint_{2Q} |\nabla^\ell (u - u_Q)(x,0)| \, dx \leq \fint_{2Q} |\dot{\mathbf{T}}\mathbf{r}^+_\ell\, u(x)| \, dx + \bigg(\fint_{2Q} |\dot{\mathbf{T}}\mathbf{r}^+_\ell\, u_Q(x)|^2 \, dx \bigg)^{1/2}$$

which by the bound (72) or (78) is at most

$$\fint_{2Q} |\dot{\mathbf{Tr}}_\ell^+ u(x)|\,dx + C\Big(\fint_{4Q} |\dot{\boldsymbol{g}}(x)|^2\,dx\Big)^{1/2}.$$

Thus, we see that

$$\fint_Q |\dot{\mathbf{Tr}}_\ell^+ u|^2 \leq C\fint_{4Q} |\dot{\boldsymbol{g}}|^2 + C\fint_{3Q} \widetilde{N}_*(\partial_{n+1}^\ell u)(x)^2\,dx + C\Big(\fint_{2Q} |\dot{\mathbf{Tr}}_\ell^+ u(x)|\,dx\Big)^2.$$

We will use Lemma 3.8. Let $g = |\dot{\mathbf{Tr}}_\ell^+ u|$, let $h = |\dot{\boldsymbol{g}}| + \widetilde{N}_*(\partial_{n+1}^\ell u)$ and let $q = 2$. Then there is some $p > 2$ such that

$$\int_{\mathbb{R}^n} |\dot{\mathbf{Tr}}_\ell^+ u(x)|^p\,dx \leq C\int_{\mathbb{R}^n} |\dot{\boldsymbol{g}}(x)|^p + \widetilde{N}_*(\partial_{n+1}^\ell u)(x)^p\,dx. \tag{82}$$

Because $2m \geq n+3$, we may apply Lemmas 4.3 and 4.4 to bound $\|\widetilde{N}_*(\partial_{n+1}^\ell u)\|_{L^p(\mathbb{R}^n)}$ and complete the proof. □

4.4. **Reduction to operators of high order.** The following formulas were established in [16, 18, 20] and inspired by an argument in [10, Section 2.2]; we will use them to pass from the case $2m \geq n+3$ to the general case.

Choose some large number M. There are constants κ_ζ such that

$$\Delta^M = \sum_{|\zeta|=M} \kappa_\zeta\,\partial^{2\zeta}.$$

In fact, by the multinomial theorem $\kappa_\zeta = M!/\zeta! = M!/(\zeta_1!\zeta_2!\dots\zeta_{n+1}!)$, and so we have that $\kappa_\zeta \geq 1$ for all $|\zeta| = M$.

Define the differential operator $\widetilde{L} = \Delta^M L \Delta^M$; that is, $\widetilde{L}$ is the operator of order $4M + 2m$ associated to coefficients $\widetilde{\boldsymbol{A}}$ that satisfy

$$\langle \nabla^{m+2M}\varphi, \widetilde{\boldsymbol{A}}\nabla^{m+2M}\psi\rangle = \langle \nabla^m\Delta^M\varphi, \boldsymbol{A}\nabla^m\Delta^M\psi\rangle \tag{83}$$

for all nice test functions φ and ψ. Observe that $\widetilde{\boldsymbol{A}}$ is t-independent and satisfies the bounds (43) and (44). A precise formula for $\widetilde{A}$ may be found in [18, formula 11.1].

Let $\widetilde{g}_\varepsilon(x) = \sum_{\gamma+2\xi=\varepsilon} \kappa_\xi\, g_\gamma(x)$. By [18, formula (11.2)], if $|\alpha| = m$ then

$$\partial^\alpha \mathcal{S}^L \dot{\boldsymbol{g}}(x,t) = \sum_{|\zeta|=M} \kappa_\zeta \partial^{\alpha+2\zeta}\mathcal{S}^{\widetilde{L}}\dot{\widetilde{\boldsymbol{g}}}(x,t) = \partial^\alpha\Delta^M\mathcal{S}^{\widetilde{L}}\dot{\widetilde{\boldsymbol{g}}}(x,t). \tag{84}$$

Similarly, let $\widetilde{h}_\varepsilon = \sum_{\alpha+2\xi=\varepsilon} \kappa_\xi h_\alpha$. If $|\gamma| = m-1$, then by [20, formula (3.10)],

$$\partial^\gamma \mathcal{S}_\nabla^L \dot{\boldsymbol{h}}(x,t) = \sum_{|\zeta|=M} \kappa_\zeta \partial^{\gamma+2\zeta}\mathcal{S}_\nabla^{\widetilde{L}}\dot{\widetilde{\boldsymbol{h}}}(x,t) = \partial^\gamma\Delta^M\mathcal{S}_\nabla^{\widetilde{L}}\dot{\widetilde{\boldsymbol{h}}}(x,t). \tag{85}$$

4.5. **Nontangential and area integral estimates.** We now establish the nontangential bounds (27) and (28) on the single layer potential.

Lemma 4.6. *Let L be an operator of the form* (1) *of order $2m$ associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44).

Then the bounds (27) *and* (28) *are valid. That is, there is some number $\varepsilon > 0$ such that the bounds*

$$\|\widetilde{N}_*(\nabla^m\mathcal{S}^L\dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}$$

and

$$\|\widetilde{N}_*(\nabla^{m-1}\mathcal{S}^L_\nabla \dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)}$$

are valid whenever $2-\varepsilon < p < 2+\varepsilon$.

Proof. Let M be large enough that $2\widetilde{m} = 2m+4M \geq n+3$, and let $\widetilde{L}$ be the operator of order $2\widetilde{m}$ associated to the coefficients $\widetilde{\boldsymbol{A}}$ given by formula (83).

By Lemmas 4.3 and 4.4 (with $k=\widetilde{m}$ or $k=\widetilde{m}-1$), and by Section 3.3, we have that the bounds

$$\|\widetilde{N}_*(\partial^{\widetilde{m}}_{n+1}\mathcal{S}^{\widetilde{L}}\dot{\tilde{\boldsymbol{g}}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\tilde{\boldsymbol{g}}}\|_{L^p(\mathbb{R}^n)}, \qquad \|\widetilde{N}_*(\partial^{\widetilde{m}-1}_{n+1}\mathcal{S}^{\widetilde{L}}_\nabla\dot{\tilde{\boldsymbol{h}}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\tilde{\boldsymbol{h}}}\|_{L^p(\mathbb{R}^n)}$$

are valid for all $2<p<\infty$ and all $\dot{\tilde{\boldsymbol{g}}}$, $\dot{\tilde{\boldsymbol{h}}} \in L^p(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$.

Thus, by Lemma 4.5, there is some $\tilde{p}>2$ such that if $p=\tilde{p}$, then

$$\|\dot{\mathbf{Tr}}^{\pm}_{\widetilde{m}}\,\mathcal{S}^{\widetilde{L}}\dot{\tilde{\boldsymbol{g}}}\|_{L^p(\mathbb{R}^n)} \leq C\|\dot{\tilde{\boldsymbol{g}}}\|_{L^p(\mathbb{R}^n)}, \quad \|\dot{\mathbf{Tr}}^{\pm}_{\widetilde{m}-1}\,\mathcal{S}^{\widetilde{L}}_\nabla\dot{\tilde{\boldsymbol{h}}}\|_{L^p(\mathbb{R}^n)} \leq C\|\dot{\tilde{\boldsymbol{h}}}\|_{L^p(\mathbb{R}^n)}. \tag{86}$$

By interpolation, the inequalities (86) are valid for all p with $2\leq p\leq\tilde{p}$.

The adjoint operator $\widetilde{L}^*$ to $\widetilde{L}$ is also of the form (1), of order $2\widetilde{m}$, and associated to t-independent coefficients $\widetilde{\boldsymbol{A}}^*$ that satisfy the bounds (43) and (44). Thus, there is some $\tilde{p}_*>0$ such that the inequalities (86) are valid, with $\widetilde{L}$ replaced by $\widetilde{L}^*$, for all p with $2\leq p\leq\tilde{p}_*$.

By the duality relation (79), the inequalities (86) (with the original $\widetilde{L}$) are valid for all $\tilde{p}_*' \leq p \leq \tilde{p}$.

By Lemma 4.4 (with $k=0$), we have that if $\max(\tilde{p}_*', p^-_{0,L^*}) < p \leq \tilde{p}$, then

$$\|\widetilde{N}_*(\nabla^{\widetilde{m}}\mathcal{S}^{\widetilde{L}}\dot{\tilde{\boldsymbol{g}}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\tilde{\boldsymbol{g}}}\|_{L^p(\mathbb{R}^n)}, \qquad \|\widetilde{N}_*(\nabla^{\widetilde{m}-1}\mathcal{S}^{\widetilde{L}}_\nabla\dot{\tilde{\boldsymbol{h}}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\tilde{\boldsymbol{h}}}\|_{L^p(\mathbb{R}^n)}.$$

An application of formulas (84) and (85) completes the proof. □

As an immediate corollary we have area integral estimates.

Lemma 4.7. *Let L be an operator of the form* (1) *of order $2m$ associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44).

Then the bounds (36) *and* (37) *are valid. That is, there is some number $\varepsilon>0$ such that the bounds*

$$\|\mathcal{A}_2^+(t\,\nabla^m\partial_t\mathcal{S}^L\dot{\boldsymbol{g}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{g}}\|_{L^p(\mathbb{R}^n)}, \tag{87}$$

$$\|\mathcal{A}_2^+(t\,\nabla^m\mathcal{S}^L_\nabla\dot{\boldsymbol{h}})\|_{L^p(\mathbb{R}^n)} \leq C_p\|\dot{\boldsymbol{h}}\|_{L^p(\mathbb{R}^n)} \tag{88}$$

are valid whenever $2-\varepsilon<p<2+\varepsilon$.

Proof. The case $2-\varepsilon<p\leq 2$ is known (see formulas (34) and (35) above). The $p>2$ case follows from Lemma 3.7 with $u=\partial_t\mathcal{S}^L\dot{\boldsymbol{g}}$ or $u=\mathcal{S}^L_\nabla\dot{\boldsymbol{h}}$, $u_Q=\partial_t\mathcal{S}^L(\mathbf{1}_{4Q}\dot{\boldsymbol{g}})$ or $u_Q=\mathcal{S}^L_\nabla(\mathbf{1}_{4Q}\dot{\boldsymbol{h}})$; by the bounds (34) and (35) and Lemma 4.6, the conditions of the lemma are satisfied with $\psi=C|\dot{\boldsymbol{h}}|$ or $\psi=C|\dot{\boldsymbol{g}}|$. □

5. The double layer potential

In this section we will establish the nontangential estimates (29–30) and the area integral estimates (38–39) on the double layer potential.

We will begin (Section 5.1) by showing that the boundary values $\dot{\mathbf{Tr}}_m\,\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}$ and $\dot{\mathbf{Tr}}_{m-1}\,\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}$ lie in $L^p(\mathbb{R}^n)$, for p near 2 and for appropriate inputs $\dot{\boldsymbol{\varphi}}$ and $\dot{\boldsymbol{f}}$. We will then (Section 5.2) establish the nontangential estimate (29) on $\nabla^m\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}$ in the special case where $2m\geq n+1$. In Section 5.3 we will extend to the case

$2m < n+1$. Finally, in Section 5.4, we will complete the proof of Theorem 1.2 by establishing the bounds (30), (38), and (39).

5.1. **Boundary values of the double layer potential.** We begin by bounding the boundary values of the double layer potential.

Lemma 5.1. *Let L be an operator of the form* (1) *of order $2m$ associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44)*. Then there is an $\varepsilon > 0$ such that if $2-\varepsilon < p < 2+\varepsilon$, then*

$$\|\dot{\mathbf{Tr}}^+_{m-1}\,\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}\|_{L^p(\mathbb{R}^n)} \le C_p\|\dot{\boldsymbol{f}}\|_{\dot{W}A^p_{m-1,0}(\mathbb{R}^n)}, \tag{89}$$

$$\|\dot{\mathbf{Tr}}^+_{m}\,\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}\|_{L^p(\mathbb{R}^n)} \le C_p\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)} \tag{90}$$

whenever $\dot{\boldsymbol{f}} \in \dot{W}A^2_{m-1,1/2}(\mathbb{R}^n)\cap\dot{W}A^p_{m-1,0}(\mathbb{R}^n)$ and whenever $\dot{\boldsymbol{\varphi}} = \dot{\mathbf{Tr}}_{m-1}\,\Phi$ for some $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$.

Proof. By [17, formulas (5.4) and (5.6)], we have the duality relation

$$\langle \dot{\boldsymbol{g}}, \dot{\mathbf{Tr}}^+_{m-1}\,\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}\rangle_{\mathbb{R}^n} = -\langle \dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\,\mathcal{S}^{L^*}\dot{\boldsymbol{g}}, \dot{\boldsymbol{f}}\rangle_{\mathbb{R}^n} \tag{91}$$

for all $\dot{\boldsymbol{f}} \in \dot{W}A^2_{m-1,1/2}(\mathbb{R}^n)$ and all $\dot{\boldsymbol{g}} \in \dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$. Here $\dot{\mathbf{M}}^-_{\boldsymbol{A}^*}$ represents the Neumann boundary operator of [17]; by [21, Lemma 2.4], if $u \in \dot{W}^2_m(\mathbb{R}^{n+1}_-)$ then the definition of $\dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\,u$ in [17] coincides with that in [21].

Recall from the definition (54) of $\mathcal{S}^{L^*}$ that if $\dot{\boldsymbol{g}} \in \dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$ then $\mathcal{S}^{L^*}\dot{\boldsymbol{g}} \in \dot{W}^2_m(\mathbb{R}^{n+1})$. By [21, Theorem 6.2], if $\mathcal{S}^{L^*}\dot{\boldsymbol{g}} \in \dot{W}^2_m(\mathbb{R}^{n+1}_-)$ and $1<p<\infty$, then

$$\begin{aligned}|\langle \dot{\boldsymbol{g}}, \dot{\mathbf{Tr}}^+_{m-1}\,\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}\rangle_{\mathbb{R}^n}| &= |\langle \dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\,\mathcal{S}^{L^*}\dot{\boldsymbol{g}}, \dot{\boldsymbol{f}}\rangle_{\mathbb{R}^n}| \\ &\le C\|\dot{\boldsymbol{f}}\|_{\dot{W}A^p_{m-1,0}(\mathbb{R}^n)}\big(\|\mathcal{A}_2^-(t\nabla^m\partial_t\mathcal{S}^{L^*}\dot{\boldsymbol{g}})\|_{L^{p'}(\mathbb{R}^n)} + \|\widetilde{N}_-(\nabla^m\mathcal{S}^{L^*}\dot{\boldsymbol{g}})\|_{L^{p'}(\mathbb{R}^n)}\big).\end{aligned}$$

Here $\mathcal{A}_2^-$ and $\widetilde{N}_-$ are defined analogously to $\mathcal{A}_2^+$ and $\widetilde{N}_+$ in the lower half space.

By Lemmas 4.6 and 4.7, if $\dot{\boldsymbol{g}} \in L^{p'}(\mathbb{R}^n)\cap\dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$ for some $2-\varepsilon < p' < 2+\varepsilon$, then

$$|\langle \dot{\boldsymbol{g}}, \dot{\mathbf{Tr}}^+_{m-1}\,\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{f}}\rangle_{\mathbb{R}^n}| \le C\|\dot{\boldsymbol{f}}\|_{\dot{W}A^p_{m-1,0}(\mathbb{R}^n)}\|\dot{\boldsymbol{g}}\|_{L^{p'}(\mathbb{R}^n)}.$$

Because $L^{p'}(\mathbb{R}^n)\cap\dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$ is dense in $L^{p'}(\mathbb{R}^n)$, the bound (89) is valid.

We now turn to the bound (90). We wish to bound $\mathrm{Tr}^+\,\partial^\alpha\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}$ for all $|\alpha| = m$. We will need separate arguments for the cases $\alpha_{n+1} < m$ and $\alpha_{n+1} > 0$.

We begin with the case $\alpha_{n+1} < |\alpha| = m$; then there is some j with $1 \le j \le n$ such that $\alpha_j \ge 1$, and so $\alpha = \gamma + \vec{e}_j$ for some multiindex γ with nonnegative entries. Integrating by parts, we have that if $h \in C_0^\infty(\mathbb{R}^n)$ then

$$\langle h, \mathrm{Tr}^+\,\partial^\alpha\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}\rangle_{\mathbb{R}^n} = -\langle \partial_{x_j}h, \mathrm{Tr}^+\,\partial^\gamma\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}\rangle_{\mathbb{R}^n}.$$

By formula (91), if $\dot{\boldsymbol{\varphi}} \in \dot{W}A^2_{m-1,1/2}(\mathbb{R}^n)$ then

$$\langle h, \mathrm{Tr}^+\,\partial^\alpha\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}\rangle_{\mathbb{R}^n} = \langle \dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\,\mathcal{S}^{L^*}(\partial_{x_j}h\dot{\boldsymbol{e}}_\gamma), \dot{\boldsymbol{\varphi}}\rangle_{\mathbb{R}^n}.$$

The function $\partial_{x_j}h$ is in $\dot{B}^{2,2}_{-1/2}(\mathbb{R}^n)$, and so $\mathcal{S}^{L^*}(\partial_j h\dot{\boldsymbol{e}}_\gamma) \in \dot{W}^2_m(\mathbb{R}^{n+1})$. By [21, Theorem 6.1], if $1<p'<\infty$ then

$$|\langle \dot{\mathbf{M}}^-_{\boldsymbol{A}^*}\,\mathcal{S}^{L^*}(\partial_j h\dot{\boldsymbol{e}}_\gamma), \dot{\boldsymbol{\varphi}}\rangle_{\mathbb{R}^n}| \le C\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)}\|\mathcal{A}_2^-(t\nabla^m\mathcal{S}^{L^*}(\partial_j h\dot{\boldsymbol{e}}_\gamma))\|_{L^{p'}(\mathbb{R}^n)}.$$

By formula (74), $\nabla^m \mathcal{S}^{L^*}(\partial_j h \dot{\boldsymbol{e}}_\gamma) = -\nabla^m \mathcal{S}_\nabla^{L^*}(h\dot{\boldsymbol{e}}_\alpha)$. Thus,

$$|\langle h, \mathrm{Tr}^+ \partial^\alpha \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}}\rangle_{\mathbb{R}^n}| \leq C\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)} \|\mathcal{A}_2^-(t\nabla^m \mathcal{S}_\nabla^{L^*}(h\dot{\boldsymbol{e}}_\alpha))\|_{L^{p'}(\mathbb{R}^n)}.$$

By Lemma 4.7, if $h \in L^{p'}(\mathbb{R}^n)$ for some $2-\varepsilon < p' < 2+\varepsilon$, then

$$|\langle h, \mathrm{Tr}^+ \partial^\alpha \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}}\rangle_{\mathbb{R}^n}| \leq C\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)} \|h\|_{L^{p'}(\mathbb{R}^n)}.$$

By duality and by density, we have that

$$\|\mathrm{Tr}^+ \partial^\alpha \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}}\|_{L^p(\mathbb{R}^n)} \leq C_p \|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)}$$

whenever $\alpha_{n+1} < m$.

Finally, we turn to $\partial^m_{n+1}\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}$. In fact, we will bound $\partial^\alpha \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}$ for any α with $|\alpha| = m$ and $\alpha_{n+1} > 0$. Recall that $\dot{\boldsymbol{\varphi}} = \dot{\mathbf{Tr}}_{m-1}\,\Phi$ for some $\Phi \in \dot{W}^2_m(\mathbb{R}^{n+1})$. As in the proof of [20, formula (6.3)], by formulas (53) and (62) we have that

$$\partial^\alpha \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}(x,t) = -\sum_{|\xi|=|\beta|=m} \int_{\mathbb{R}^{n+1}_-} \partial^\alpha_{x,t}\partial^\xi_{y,s} E^L(x,t,y,s)\, A_{\xi\beta}(y)\, \partial^\beta \Phi(y,s)\, ds\, dy \tag{92}$$

for a.e. $(x,t) \in \mathbb{R}^{n+1}_+$. Let $\gamma = \alpha - \vec{e}_{n+1}$. By assumption, $\gamma \in (\mathbb{N}_0)^{n+1}$ is a multiindex with nonnegative entries. By formula (65),

$$\partial^\gamma \partial_t \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}(x,t) = \sum_{|\xi|=|\beta|=m} \int_{\mathbb{R}^{n+1}_-} \partial^\gamma_{x,t}\partial_s\partial^\xi_{y,s} E^L(x,t,y,s)\, A_{\xi\beta}(y)\, \partial^\beta \Phi(y,s)\, ds\, dy.$$

If $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$, then we may integrate by parts in s to see that

$$\begin{aligned}\partial^\gamma \partial_t \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}(x,t) = &-\sum_{|\xi|=|\beta|=m} \int_{\mathbb{R}^{n+1}_-} \partial^\gamma_{x,t}\partial^\xi_{y,s} E^L(x,t,y,s)\, A_{\xi\beta}(y)\, \partial^\beta \partial_s \Phi(y,s)\, ds\, dy \\ &+ \sum_{|\xi|=|\beta|=m} \int_{\mathbb{R}^n} \partial^\gamma_{x,t}\partial^\xi_{y,s} E^L(x,t,y,0)\, A_{\xi\beta}(y)\, \partial^\beta \Phi(y,0)\, dy.\end{aligned}$$

By formulas (53) and (56), we have that if $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$ and $|\gamma| = m-1$, then

$$\partial^\gamma \partial_t \mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}_{m-1}\,\Phi)(x,t) = \partial^\gamma \mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}_{m-1}\,\partial_{n+1}\Phi)(x,t) + \partial^\gamma \mathcal{S}^L_\nabla(\boldsymbol{A}\,\dot{\mathbf{Tr}}_m\,\Phi)(x,t). \tag{93}$$

Let $\Psi(x,t) = \Phi(x,t) - \eta(t)\, t^m\, \partial^m_{n+1}\Phi(x,0)/m!$ for a smooth cutoff function η equal to 1 near $t=0$. Then $\dot{\mathbf{Tr}}_{m-1}\,\Psi = \dot{\mathbf{Tr}}_{m-1}\,\Phi$ and $\partial^m_{n+1}\Psi(x,0) = 0$, and so

$$\|\dot{\mathbf{Tr}}^+_{m-1}\,\partial_{n+1}\Psi\|_{\dot{W}A^p_{m-1,0}(\mathbb{R}^n)} + \|\boldsymbol{A}\,\dot{\mathbf{Tr}}^+_m\,\Psi\|_{L^p(\mathbb{R}^n)} \leq C\|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)}.$$

Thus, by the bounds (89) and (86), we have that

$$\|\mathrm{Tr}^+ \partial^\alpha \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}}\|_{L^p(\mathbb{R}^n)} \leq C_p \|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)}$$

whenever $\alpha_{n+1} > 0$ and p is sufficiently close to 2. This completes the proof. □

5.2. **Nontangential estimates for operators of high order.** In this section we will establish the bound (29) in the special case $2m \geq n+1$. In Section 5.3 we will pass to the case of lower order operators, and in Section 5.4 we will establish the bounds (30), (38) and (39).

Lemma 5.2. *Let L be an operator of the form* (1) *of order $2m \geq n+1$ associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44)*. Then the bound* (29) *is valid; that is, there is some $\varepsilon > 0$ such that if $2-\varepsilon < p < 2+\varepsilon$, then*

$$\|\widetilde{N}_+(\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}})\|_{L^p(\mathbb{R}^n)} \leq C_p \|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)}$$

for any $\dot{\boldsymbol{\varphi}} = \dot{\mathbf{T}}\mathbf{r}_{m-1}\,\Phi$ for some Φ smooth and compactly supported.

The remainder of this section will be devoted to a proof of this lemma.

We will apply Lemma 3.5 to $\nabla^m u$, where $u = \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}$. Let $Q = Q(x_0, t_0) \subset \mathbb{R}^n$ be a cube of side length t_0 and with midpoint x_0. As in the proof of Lemma 4.4, by Lemma 3.6, if u_Q satisfies $L(u - u_Q) = 0$ in $2Q \times (-\ell(Q), \ell(Q))$, then

$$\begin{aligned}\Big(\fint_Q \fint_{t_0/6}^{t_0/2} |\nabla^m u|^2\Big)^{1/2} \leq{}& \Big(\fint_Q \fint_{t_0/6}^{t_0/2} |\nabla^m u_Q|^2\Big)^{1/2} \\ &+ C\ell(Q) \fint_{2Q} \fint_{-\ell(Q)}^{\ell(Q)} |\partial_t^{m+1}(u - u_Q)| \\ &+ C \fint_{2Q} |\dot{\mathbf{T}}\mathbf{r}_m^+ u_Q| + C \fint_{2Q} |\dot{\mathbf{T}}\mathbf{r}_m^+ u|.\end{aligned} \tag{94}$$

The final term is at most $C\mathcal{M}(\dot{\mathbf{T}}\mathbf{r}_m^+ u)(x_0) = C\mathcal{M}(\dot{\mathbf{T}}\mathbf{r}_m^+ \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}})(x_0)$, which we may control using Lemma 5.1 and boundedness of the Hardy-Littlewood maximal operator. We will bound the remaining terms much as in the proof of Lemma 4.4. Our first step is to construct an appropriate u_Q.

Definition 95. Suppose that $\dot{\boldsymbol{\varphi}} = \dot{\mathbf{T}}\mathbf{r}_{m-1}\,\Phi$ for some $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$, and let $R \subset \mathbb{R}^n$ be a cube. We define $\dot{\boldsymbol{\varphi}}_R$ as follows.

Let $\rho_R : \mathbb{R}^n \mapsto [0, \infty)$ be smooth, supported in $(4/3)R$ and identically equal to 1 in R, and let $\eta : \mathbb{R} \mapsto [0, \infty)$ be smooth, supported in $(-2, 2)$ and equal to 1 in $(-1, 1)$.

Let $\Phi_R(x,t) = \rho_R(x)\eta(2t/\ell(R))(\Phi(x,t) - P_R(x,t)) + P_R(x,t)$, where P_R is the polynomial of degree $m-1$ that satisfies $\int_{(4/3)R} \nabla^k \Phi(x,0) - \nabla^k P_R(x,0)\,dx = 0$ for any $0 \leq k \leq m-1$. Observe that $\nabla^m \Phi_R = 0$ outside of $(4/3)R \times (-\ell(R), \ell(R))$ and that $\Phi_R = \Phi$ in $R \times (-\ell(R)/2, \ell(R)/2)$.

Let $\dot{\boldsymbol{\varphi}}_R = \dot{\mathbf{T}}\mathbf{r}_{m-1}\,\Phi_R$. Observe that $\dot{\boldsymbol{\varphi}}_R = \dot{\boldsymbol{\varphi}}$ in R and $\dot{\boldsymbol{\varphi}}_R$ is constant outside $(4/3)R$.

By the Poincaré inequality, $\dot{\boldsymbol{\varphi}}_R \in \dot{W}A^p_{m-1,1}(\mathbb{R}^n)$ for any $1 \leq p < \infty$, with $\|\dot{\boldsymbol{\varphi}}_R\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)} \leq C\|\nabla_\parallel \dot{\boldsymbol{\varphi}}\|_{L^p((4/3)R)}$. Furthermore, by formula (53) for the double layer potential,

$$\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}} - \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}_R = \mathbf{1}_-\Phi - \mathbf{1}_-\Phi_R - \Pi^L(\mathbf{1}_-\boldsymbol{A}\nabla^m\Phi) + \Pi^L(\mathbf{1}_-\boldsymbol{A}\nabla^m\Phi_R)$$

and so $\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}} - \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}_R \in \dot{W}^2_m(R \times (-\ell(R)/2, \ell(R)/2))$ with $L(\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}} - \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}_R) = 0$ in $R \times (-\ell(R)/2, \ell(R)/2)$.

We will use this definition again in the proof of Corollary 98.

Let $u_Q = \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}_{8Q}$, so $\dot{\boldsymbol{\varphi}} = \dot{\boldsymbol{\varphi}}_{8Q}$ in $8Q$ and $L(u - u_Q) = 0$ in $8Q \times (-4\ell(Q), 4\ell(Q))$.

Let $q \geq 1$. We will impose further conditions on q throughout the proof. By Hölder's inequality,

$$\fint_{2Q} |\dot{\mathbf{T}}\mathbf{r}_m^+ u_Q(x)|\,dx \leq \Big(\fint_{2Q} |\dot{\mathbf{T}}\mathbf{r}_m^+ u_Q(x)|^q\,dx\Big)^{1/q}$$

and by Lemma 5.1 and the definition 95 of $\dot{\boldsymbol{\varphi}}_{8Q}$, if $|q-2|$ is small enough then

$$\fint_{2Q} |\dot{\mathbf{Tr}}_m^+ \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}}_{8Q}| \leq C \left(\frac{1}{|2Q|} \int_{\mathbb{R}^n} |\nabla_\parallel \dot{\boldsymbol{\varphi}}_{8Q}|^q \right)^{1/q} \leq C \mathcal{M}(|\nabla_\parallel \dot{\boldsymbol{\varphi}}|^q)(x_0)^{1/q}.$$

To contend with the remaining terms in the bound (94), we will need a decomposition $\dot{\boldsymbol{\varphi}} = \sum_{j=0}^\infty \dot{\boldsymbol{\varphi}}_j$ and functions Ψ_j such that $\dot{\mathbf{Tr}}_{m-1} \Psi_j = \dot{\boldsymbol{\varphi}}_j$.

Let $\Phi_0 = \Phi_{8Q}$ and $\dot{\boldsymbol{\varphi}}_0 = \dot{\boldsymbol{\varphi}}_{8Q} = \dot{\mathbf{Tr}}_{m-1} \Phi_0$, and for each $j \geq 1$, let $\Phi_j = \Phi_{2^{j+3}Q} - \Phi_{2^{j+2}Q}$ and $\dot{\boldsymbol{\varphi}}_j = \dot{\mathbf{Tr}}_{m-1} \Phi_j = \dot{\boldsymbol{\varphi}}_{2^{j+3}Q} - \dot{\boldsymbol{\varphi}}_{2^{j+2}Q}$, where Φ_R and $\dot{\boldsymbol{\varphi}}_R$ are as in Definition 95.

Then $\dot{\boldsymbol{\varphi}} = \sum_{j=0}^\infty \dot{\boldsymbol{\varphi}}_j$, $\nabla_\parallel \dot{\boldsymbol{\varphi}}_j = 0$ outside of $(4/3)2^{j+3}Q \subset 2^{j+4}Q$, and

$$\|\nabla_\parallel \dot{\boldsymbol{\varphi}}_j\|_{L^p(\mathbb{R}^n)} \leq C_p \|\nabla_\parallel \dot{\boldsymbol{\varphi}}\|_{L^p(2^{j+4}Q)}$$

for any $1 \leq p < \infty$. Furthermore, if $j \geq 1$ then $\dot{\boldsymbol{\varphi}}_j = 0$ in $2^{j+2}Q$.

We will need extensions Ψ_j with $\dot{\boldsymbol{\varphi}}_j = \dot{\mathbf{Tr}}_{m-1} \Psi_j$. We have that $\dot{\boldsymbol{\varphi}}_j = \dot{\mathbf{Tr}}_{m-1} \Phi_j$; however, we will need Ψ_j to satisfy some bounds in terms of the norms of the boundary values $\dot{\boldsymbol{\varphi}}_j$, and so we cannot use the obvious extensions $\Psi_j = \Phi_j$.

We define extensions Ψ_j as follows. Let $\varphi_{j,k}(x) = \partial_{n+1}^k \Phi_j(x,0)$; we have that $|\nabla_\parallel^{m-k} \varphi_{j,k}(x)| \leq |\dot{\boldsymbol{\varphi}}_k(x)|$. Let $\theta : \mathbb{R}^n \mapsto \mathbb{R}$ be smooth, nonnegative, and satisfy the conditions $\int_{\mathbb{R}^n} \theta = 1$, $\theta(x) = 0$ whenever $|x| > 1$, and $\int_{\mathbb{R}^n} x^\zeta \theta(x)\, dx = 0$ for all multiindices $\zeta \in (\mathbb{N}_0)^n$ with $1 \leq |\zeta| \leq m-1$. Let $\theta_t(x) = t^{-n}\theta(x/t)$. Define

$$H_j(x,t) = \sum_{k=0}^{m-1} \frac{1}{k!} t^k \varphi_{j,k} * \theta_t(x) = \sum_{k=0}^{m-1} \frac{1}{k!} t^k \int_{\mathbb{R}^n} \varphi_{j,k}(x-ty)\, \theta(y)\, dy.$$

By the proof of [18, Lemma 3.3], we have that $\dot{\mathbf{Tr}}_{m-1} H_j = \dot{\boldsymbol{\varphi}}_j$. Furthermore, if $x \in \mathbb{R}^n \setminus 2^{j+4}Q$ and $|t| < \operatorname{dist}(x, 2^{j+4}Q)$, then $\nabla^m H_j(x,t) = 0$. Finally, if $j \geq 1$, if $x \in 2^{j+2}Q$, and if $|t| < \operatorname{dist}(x, \mathbb{R}^n \setminus 2^{j+2}Q)$, then $\nabla^{m-1} H_j(x,t) = 0$.

Observe that if $\zeta \in (\mathbb{N}_0)^n$ is a multiindex, then $\|(\partial_\parallel^\zeta \theta)_t\|_{L^1(\mathbb{R}^n)}$ is bounded, uniformly in t, and so convolution with $(\partial_\parallel^\zeta \theta)_t$ represents a bounded operator on $L^q(\mathbb{R}^n)$ for any $1 \leq q \leq \infty$. Using this fact, it is elementary to show that

$$\sup_{t \in \mathbb{R}} \|\nabla^m H_j(\,\cdot\,,t)\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla_\parallel \dot{\boldsymbol{\varphi}}_j\|_{L^q(\mathbb{R}^n)} \leq C \|\nabla_\parallel \dot{\boldsymbol{\varphi}}\|_{L^q(2^{j+4}Q)}.$$

This is not true of the function Φ_j.

However, observe that $\nabla^m H_j$ is not compactly supported. Let

$$\Psi_j(x,t) = (H_j(x,t) - P_j(x,t))\, \eta\left(\frac{t}{2^{j+2}\ell(Q)} \right) + P_j(x,t),$$

where $\eta(t) = 1$ if $|t| \leq 1$ and $\eta(t) = 0$ if $|t| \geq 2$, and where $P_j(x,t)$ is the polynomial of degree $m-1$ with

$$\int_{2^{j+5}Q} \int_{-2^{j+3}\ell(Q)}^{-2^{j+2}\ell(Q)} \nabla^k (H_j(x,t) - P_j(x,t))\, dt\, dx = 0$$

for all $0 \leq k \leq m-1$. Observe that $H_j(x,t) = \Psi_j(x,t)$ whenever $|t| < 2^{j+2}\ell(Q)$. By the Poincaré inequality, $\int_{\mathbb{R}^{n+1}_-} |\nabla^m \Psi_j|^q \leq C 2^j \ell(Q) \|\nabla_\parallel \dot{\boldsymbol{\varphi}}\|^q_{L^q(2^{j+4}Q)}$ for any $1 \leq q < \infty$.

We now return to the terms in the bound (94). By Theorem 3.2 (if $q<2$) or by Hölder's inequality (if $q>2$),

$$\Big(\fint_Q \fint_{t_0/6}^{t_0/2} |\nabla^m u_Q|^2\Big)^{1/2} \leq \frac{C}{\ell(Q)^{(n+1)/q}} \|\nabla^m u_Q\|_{L^q(\mathbb{R}^{n+1})}.$$

By formula (53), if $|\alpha|=m$, $x\in\mathbb{R}^n$ and $t>0$, then

$$\partial^\alpha u_Q(x,t) = \partial^\alpha \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}}_0(x,t) = -\partial^\alpha \Pi^L(\mathbf{1}_- \boldsymbol{A}\nabla^m \Psi_0)(x,t).$$

By [16, Lemma 43], if $p^-_{0,L^*} < q < p^+_{0,L}$, where $p^+_{0,L}$ is as in Theorem 3.2 and where $1/p^+_{0,L^*} + 1/p^-_{0,L^*} = 1$, then $\nabla^m \Pi^L$ is bounded $L^q(\mathbb{R}^{n+1}) \mapsto L^q(\mathbb{R}^{n+1})$, and so

$$\Big(\fint_Q \fint_{t_0/6}^{t_0/2} |\nabla^m u_Q|^2\Big)^{1/2} \leq \frac{C}{\ell(Q)^{(n+1)/q}} \|\nabla^m \Psi_0\|_{L^q(\mathbb{R}^{n+1})}.$$

By our bounds on Ψ_0,

$$\begin{aligned}\Big(\fint_Q \fint_{t_0/6}^{t_0/2} |\nabla^m u_Q|^2\Big)^{1/2} &\leq \frac{C}{\ell(Q)^{n/q}} \|\nabla_\parallel \dot{\boldsymbol{\varphi}}_0\|_{L^q(\mathbb{R}^n)} \leq \frac{C}{\ell(Q)^{n/q}} \|\nabla_\parallel \dot{\boldsymbol{\varphi}}\|_{L^q(8Q)} \\ &\leq C\mathcal{M}(|\nabla_\parallel \dot{\boldsymbol{\varphi}}|^q)(x_0)^{1/q}.\end{aligned}$$

Finally, let $u_f = u - u_Q = \mathcal{D}^{\boldsymbol{A}}(\dot{\boldsymbol{\varphi}} - \dot{\boldsymbol{\varphi}}_0)$. By formula (53) for the double layer potential, and because $\dot{\boldsymbol{\varphi}} = \sum_{j=0}^\infty \dot{\boldsymbol{\varphi}}_j$, we have that

$$|\partial_t^{m+1} u_f(x,t)| = \Big|\partial_t^{m+1}\Pi^L\Big(\sum_{j=1}^\infty \mathbf{1}_- \boldsymbol{A}\nabla^m \Psi_j\Big)(x,t)\Big| \leq \sum_{j=1}^\infty |\partial_t^{m+1}\Pi^L(\mathbf{1}_- \boldsymbol{A}\nabla^m\Psi_j)(x,t)|.$$

Let $x\in 2Q$ and let $-\ell(Q)<t<\ell(Q)$. Recall that if $j\geq 1$, then $\nabla^m\Psi_j = 0$ in $\{(y,s) : |s| < \operatorname{dist}(y, \mathbb{R}^n\setminus 2^{j+2}Q)\}$, and so $\mathbf{1}_- \boldsymbol{A}\nabla^m\Psi_j = 0$ in $2^{j+1}Q\times(-2^j\ell(Q),\infty)$. Thus, $L(\Pi^L(\mathbf{1}_-\boldsymbol{A}\nabla^m\Psi_j)) = 0$ in this set. If m is large enough, then by the bound (58),

$$|\partial_t^{m+1} u_f(x,t)| \leq \sum_{j=1}^\infty C \fint_{2^{j+1/3}Q} \fint_{-2^{j-2/3}\ell(Q)}^{2^{j-2/3}\ell(Q)} |\partial_t^{m+1}\Pi^L(\mathbf{1}_-\boldsymbol{A}\nabla^m\Psi_j)(y,s)|\,ds\,dy.$$

By Hölder's inequality and the Caccioppoli inequality,

$$|\partial_t^{m+1} u_f(x,t)| \leq \sum_{j=1}^\infty \frac{C}{2^j\ell(Q)}\Big(\fint_{2^{j+2/3}Q} \fint_{-2^{j-1/3}\ell(Q)}^{2^{j-1/3}\ell(Q)} |\partial_t^m \Pi^L(\mathbf{1}_-\boldsymbol{A}\nabla^m\Psi_j)(y,s)|^2\,ds\,dy\Big)^{1/2}$$

and by Theorem 3.2, if $q>0$ then

$$|\partial_t^{m+1} u_f(x,t)| \leq \sum_{j=1}^\infty \frac{C}{2^j\ell(Q)}\Big(\fint_{2^{j+1}Q} \fint_{-2^j\ell(Q)}^{2^j\ell(Q)} |\partial_t^m \Pi^L(\mathbf{1}_-\boldsymbol{A}\nabla^m\Psi_j)|^q\Big)^{1/q}.$$

If $p^-_{0,L^*} < q < p^+_{0,L}$, then again by boundedness of $\nabla^m\Pi^L$ and our bounds on $\nabla^m\Psi_j$,

$$|\partial_t^{m+1} u_f(x,t)| \leq \sum_{j=1}^\infty \frac{C}{(2^j\ell(Q))^{n/q+1}} \|\nabla_\parallel \dot{\boldsymbol{\varphi}}\|_{L^q(2^{j+4}Q)} \leq \frac{C}{\ell(Q)}\mathcal{M}(|\nabla_\parallel\dot{\boldsymbol{\varphi}}|^q)(x_0)^{1/q}.$$

Thus, by Lemma 3.5 and formula (94),

$$\widetilde{N}_+(\nabla^m \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}})(x_0) \leq C\mathcal{M}(|\nabla_\parallel\dot{\boldsymbol{\varphi}}|^q)(x_0)^{1/q} + \mathcal{M}(\dot{\mathbf{T}}\mathbf{r}^+_m \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}})$$

for any q sufficiently close to 2. Choosing $q < p$, we have that by boundedness of the maximal operator $\mathcal{M}$ and by Lemma 5.1,

$$\|\widetilde{N}_+(\nabla^m \mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}})\|_{L^p(\mathbb{R}^n)} \leq C\|\nabla_\parallel \dot{\boldsymbol{\varphi}}\|_{L^p(\mathbb{R}^n)}$$

as desired.

5.3. **Reduction to operators of high order.** We must now extend to the case of operators of lower order. Recall formulas (84) and (85). Our goal is to establish an analogous formula for $\mathcal{D}^{\boldsymbol{A}}$. That is, we wish to find an operator $\mathcal{O}$ such that

$$\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}} = \Delta^M \mathcal{D}^{\widetilde{\boldsymbol{A}}}(\mathcal{O}\dot{\boldsymbol{\varphi}}),$$

where $\widetilde{\boldsymbol{A}}$ is given by formula (83). We remark that we will need to take somewhat more care in this case, as the natural domain of $\mathcal{D}^{\widetilde{\boldsymbol{A}}}$ is not $\dot{B}^{2,2}_{1/2}$ but instead a closed proper subset $\dot{W}A^2_{m-1,1/2}$.

Let $m \geq 1$ and $M \geq 1$ be integers. Let $\dot{\boldsymbol{\varphi}}$ be an array indexed by multiindices of length $m-1$. We define $\mathcal{O}\dot{\boldsymbol{\varphi}}$ as follows.

If $\delta \in (\mathbb{N}_0)^{n+1}$ is a multiindex with $|\delta| = m + 2M - 1$, then there is some nonnegative integer $\ell = \delta_{n+1}$ and some multiindex $\xi \in (\mathbb{N}_0)^n$ with $|\xi| = m + 2M - 1 - \ell$ such that $\delta = (\xi, \ell)$. We define

$$(\mathcal{O}\dot{\boldsymbol{\varphi}})_{(\xi,\ell)} = 0 \qquad \text{if} \qquad 0 \leq \ell < 2M,$$

$$(\mathcal{O}\dot{\boldsymbol{\varphi}})_{(\xi,\ell)} = \varphi_{(\xi,\ell-2M)} - \sum_{k=1}^{M}\sum_{|\zeta|=k} \tilde{\kappa}^M_\zeta (\mathcal{O}\dot{\boldsymbol{\varphi}})_{(\xi+2\zeta,\ell-2k)} \quad \text{if} \quad 2M \leq \ell \leq 2M+m-1$$

where $\tilde{\kappa}^M_\zeta = \kappa_{(\zeta,M-|\zeta|)} = M!/\zeta!(M-|\zeta|)!$, and where κ_ξ and $\zeta!$ are as in Section 4.4.

There are then constants $\mu_{\gamma,\delta}$ depending only on γ, δ, m, M and the dimension $n+1$ such that

$$(\mathcal{O}\dot{\boldsymbol{\varphi}})_\delta = \sum_{|\gamma|=m-1} \mu_{\gamma,\delta}\varphi_\gamma$$

for all $|\delta| = m + 2M - 1$. As such, $\mathcal{O}$ is bounded on $L^p(\mathbb{R}^n)$ and $\dot{W}^p_1(\mathbb{R}^n)$ for any $1 \leq p \leq \infty$.

We now show that if $\dot{\boldsymbol{\varphi}}$ is in the domain of $\mathcal{D}^{\boldsymbol{A}}$, then $\mathcal{O}\dot{\boldsymbol{\varphi}}$ is in the domain of $\mathcal{D}^{\widetilde{\boldsymbol{A}}}$.

Lemma 5.3. *Let $m \geq 1$ and let $M \geq 1$.*

If $\dot{\boldsymbol{\varphi}} = \dot{\mathbf{Tr}}_{m-1} F$ for some $F \in C^\infty_0(\mathbb{R}^{n+1})$, then $\mathcal{O}\dot{\boldsymbol{\varphi}} = \dot{\mathbf{Tr}}_{m+2M-1} H$ for some $H \in C^\infty_0(\mathbb{R}^{n+1})$.

If $\dot{\boldsymbol{\varphi}} = \dot{\mathbf{Tr}}_{m-1} F$ for some $F \in \dot{W}^2_m(\mathbb{R}^{n+1})$, then $\mathcal{O}\dot{\boldsymbol{\varphi}} = \dot{\mathbf{Tr}}_{m+2M-1} H$ for some $H \in \dot{W}^2_{m+2M}(\mathbb{R}^{n+1})$.

Proof. Let $F_j = \operatorname{Tr}^+ \partial^j_{n+1} F$. If $F \in C^\infty_0(\mathbb{R}^{n+1})$ then $F_j \in C^\infty_0(\mathbb{R}^n)$. If $F \in \dot{W}^2_m(\mathbb{R}^{n+1})$, then $\partial^j_{n+1} F \in \dot{W}^2_{m-j}(\mathbb{R}^{n+1})$ for all $0 \leq j \leq m-1$, and so $F_j = \operatorname{Tr}^+ \partial^j_{n+1} F$ lies in the space $\dot{B}^{2,2}_{m-j-1/2}(\mathbb{R}^n)$.

Observe that if $|\gamma| = m-1$ and $\gamma = (\xi, j)$ for some $0 \leq j \leq m-1$ and some $\xi \in (\mathbb{N}_0)^n$, then $\varphi_\gamma = \partial^\xi_\parallel F_j$.

Claim. There exist functions Φ_ℓ, in either $C^\infty_0(\mathbb{R}^n)$ or $\dot{B}^{2,2}_{m+2M-\ell-1/2}(\mathbb{R}^n)$, such that

$$(\mathcal{O}\dot{\boldsymbol{\varphi}})_{(\xi,\ell)} = \partial^\xi_\parallel \Phi_\ell. \tag{96}$$

We will prove this by induction on ℓ.

If $\ell < 2M$ (and in particular if $\ell = 0$ or $\ell = 1$), let $\Phi_\ell = 0$. By definition of $\mathcal{O}$, formula (96) is valid for all $\ell < 2M$.

If $\ell \geq 2M$, then by the induction hypothesis

$$(\mathcal{O}\dot{\boldsymbol{\varphi}})_{(\xi,\ell)} = \partial_\parallel^\xi F_{\ell-2M} - \sum_{k=1}^{M} \sum_{|\zeta|=k} \widetilde{\kappa}_\zeta^M \partial^{\xi+2\zeta} \Phi_{\ell-2k}.$$

Recall that if $|\zeta| = k$, then

$$\widetilde{\kappa}_\zeta^M = \frac{M!}{\zeta!(M-k)!} = \frac{k!}{\zeta!}\binom{M}{k}.$$

By the multinomial theorem,

$$\sum_{|\zeta|=k} \frac{k!}{\zeta!}\partial^{2\zeta} = \Delta_\parallel^k$$

and so

$$(\mathcal{O}\dot{\boldsymbol{\varphi}})_{(\xi,\ell)} = \partial_\parallel^\xi F_{\ell-2M} - \sum_{k=1}^{M} \binom{M}{k} \partial^\xi \Delta_\parallel^k \Phi_{\ell-2k}.$$

Taking $\Phi_\ell = F_{\ell-2M} - \sum_{k=1}^{M} \binom{M}{k} \Delta_\parallel^k \Phi_{\ell-2k}$, we see that the claim is valid.

We now must assemble the function H from the functions Φ_ℓ.

If $F \in C_0^\infty(\mathbb{R}^{n+1})$, let η be a smooth cutoff function, and let

$$H(x,t) = \sum_{\ell=0}^{m+2M-1} \frac{1}{\ell!} t^\ell \, \eta(t) \, \Phi_\ell(x).$$

If $F \in \dot{W}_m^2(\mathbb{R}^{n+1})$, and so $\Phi_\ell \in \dot{B}_{m+2M-\ell-1/2}^{2,2}(\mathbb{R}^n)$ for all $0 \leq \ell \leq m+2M-1$, it is well known that there is a function $H \in \dot{W}_{m+2M}^2(\mathbb{R}^{n+1})$ such that $\partial_{n+1}^\ell H(x,0) = \Phi_\ell(x)$ for all such ℓ. For example, as in Lemma 5.2 we may let

$$H(x,t) = \sum_{\ell=0}^{m+2M-1} \frac{1}{\ell!} t^\ell \, \Phi_\ell * \rho_t(x)$$

where $\rho_t(x) = t^{-n}\rho(x/t)$ for some function ρ that is smooth, compactly supported, and satisfies $\int_{\mathbb{R}^n} \rho = 1$, $\int_{\mathbb{R}^n} x^\gamma \rho(x)\, dx = 0$ for all γ with $1 \leq |\gamma| \leq m+2M-1$. An elementary argument involving the Fourier transform completes the proof. □

We have now shown that $\mathcal{O}\dot{\boldsymbol{\varphi}}$ is the trace of some function H. We now make explicit the relationship between $\dot{\boldsymbol{\varphi}}$ and H.

Lemma 5.4. *Let $m \geq 1$ and let $M \geq 1$. Let $\dot{\boldsymbol{\varphi}}$ and H be as in the previous lemma. Then $\dot{\boldsymbol{\varphi}} = \dot{\mathbf{T}}\mathbf{r}_{m-1}\, \Delta^M H$.*

Proof. Let $\gamma = (\xi, j)$ for some $0 \leq j \leq m-1$ and some $|\xi| = m-1-j$. Recall that $\widetilde{\kappa}_{\vec{0}}^M = \kappa_{(\vec{0},M)} = 1$. We then have that

$$\varphi_\gamma(x) = \varphi_{(\xi,j)}(x) = \sum_{k=0}^{M} \sum_{|\zeta|=k} \widetilde{\kappa}_\zeta^M (\mathcal{O}\dot{\boldsymbol{\varphi}})_{(\xi+2\zeta, j+2M-2k)}(x).$$

By definition of H,

$$\varphi_\gamma(x) = \sum_{k=0}^{M} \sum_{|\zeta|=k} \widetilde{\kappa}_\zeta^M \partial_\parallel^\xi \partial_{n+1}^j \partial_\parallel^{2\zeta} \partial_{n+1}^{2M-2k} H(x,0)$$

and by definition of $\widetilde{\kappa}_\zeta^M$,

$$\varphi_\gamma(x) = \partial_\parallel^\xi \partial_{n+1}^j \Delta^M H(x,0) = \partial^\gamma \Delta^M H(x,0)$$

as desired. □

Finally, we establish the analogue to formulas (84) and (85) for the double layer potential.

Lemma 5.5. *Let L be an operator of the form* (1) *of order $2m$ associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44)*. Let $M \geq 1$ and let $\widetilde{\boldsymbol{A}}$ be as in formula 83.*

Let $H \in \dot{W}^2_{m+2M}(\mathbb{R}^{n+1})$. Then

$$\mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}_{m-1}\, \Delta^M H) = \Delta^M \mathcal{D}^{\widetilde{\boldsymbol{A}}}(\dot{\mathbf{Tr}}_{m+2M-1}\, H).$$

In particular, by Lemma 5.4, if $\dot{\boldsymbol{\varphi}} \in \dot{W}A^2_{m-1,1/2}(\mathbb{R}^n)$, then

$$\mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}} = \Delta^M \mathcal{D}^{\widetilde{\boldsymbol{A}}}(\mathcal{O}\dot{\boldsymbol{\varphi}}).$$

Proof. Recall that by formula (53) for the double layer potential

$$\mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}_{m-1}\, \Delta^M H) = \mathbf{1}_- \Delta^M H - \Pi^L(\mathbf{1}_- \boldsymbol{A} \nabla^m \Delta^M H)$$

and

$$\Delta^M \mathcal{D}^{\widetilde{\boldsymbol{A}}}(\dot{\mathbf{Tr}}_{m+2M-1}\, H) = \mathbf{1}_- \Delta^M H - \Delta^M \Pi^{\widetilde{L}}(\mathbf{1}_- \widetilde{\boldsymbol{A}} \nabla^{m+2M} H).$$

Thus, we need only show that $\Delta^M \Pi^{\widetilde{L}}(\mathbf{1}_- \widetilde{\boldsymbol{A}} \nabla^{m+2M} H) = \Pi^L(\mathbf{1}_- \boldsymbol{A} \nabla^m \Delta^M H)$.

By the definition (51) of the Newton potential, we have that

$$u = \Pi^L(\mathbf{1}_- \boldsymbol{A} \nabla^m \Delta^M H)$$

is the unique element of $\dot{W}^2_m(\mathbb{R}^{n+1})$ that satisfies

$$\langle \nabla^m \varphi, \boldsymbol{A} \nabla^m u \rangle_{\mathbb{R}^{n+1}} = \langle \nabla^m \varphi, \boldsymbol{A} \nabla^m \Delta^M H \rangle_{\mathbb{R}^{n+1}_-}$$

for all $\varphi \in \dot{W}^2_m(\mathbb{R}^{n+1})$.

Choose some such φ. Then there is a $\Phi \in \dot{W}^2_{m+2M}(\mathbb{R}^{n+1})$ such that $\Delta^M \Phi = \varphi$. Let $v = \Delta^M \Pi^{\widetilde{L}}(\mathbf{1}_- \widetilde{\boldsymbol{A}} \nabla^{m+2M} H)$. Then

$$\langle \nabla^m \varphi, \boldsymbol{A} \nabla^m v \rangle_{\mathbb{R}^{n+1}} = \langle \nabla^m \Delta^M \Phi, \boldsymbol{A} \nabla^m \Delta^M \Pi^{\widetilde{L}}(\mathbf{1}_- \widetilde{\boldsymbol{A}} \nabla^{m+2M} H) \rangle_{\mathbb{R}^{n+1}}.$$

It is clear from the definition of $\widetilde{L}$ in Section 4.4 that

$$\langle \nabla^m \varphi, \boldsymbol{A} \nabla^m v \rangle_{\mathbb{R}^{n+1}} = \langle \nabla^{m+2M} \Phi, \widetilde{\boldsymbol{A}} \nabla^{m+2M} \Pi^{\widetilde{L}}(\mathbf{1}_- \widetilde{\boldsymbol{A}} \nabla^{m+2M} H) \rangle_{\mathbb{R}^{n+1}}.$$

Again by formula (51),

$$\langle \nabla^m \varphi, \boldsymbol{A} \nabla^m v \rangle_{\mathbb{R}^{n+1}} = \langle \nabla^{m+2M} \Phi, \widetilde{\boldsymbol{A}} \nabla^{m+2M} H \rangle_{\mathbb{R}^{n+1}_-}$$

and again by the definition of $\widetilde{L}$,

$$\langle \nabla^m \varphi, \boldsymbol{A} \nabla^m v \rangle_{\mathbb{R}^{n+1}} = \langle \nabla^m \Delta^M \Phi, \boldsymbol{A} \nabla^m \Delta^M H \rangle_{\mathbb{R}^{n+1}_-} = \langle \nabla^m \varphi, \boldsymbol{A} \nabla^m \Delta^M H \rangle_{\mathbb{R}^{n+1}_-}.$$

This equation is valid for all $\varphi \in \dot{W}^2_m(\mathbb{R}^{n+1})$, and so $u = v$, as desired. □

5.4. **Nontangential and area integral estimates.** By Lemmas 5.2 and 5.5, we have that the bound (29) is valid; that is, if L and $\boldsymbol{A}$ are as in Theorem 1.2, then there is some $\varepsilon > 0$ such that if $2-\varepsilon < p < 2+\varepsilon$, then

$$\|\widetilde{N}_+(\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}})\|_{L^p(\mathbb{R}^n)} \leq C_p \|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)} \tag{97}$$

for any $\dot{\boldsymbol{\varphi}}$ that satisfies $\dot{\boldsymbol{\varphi}} = \dot{\mathbf{Tr}}_{m-1}\,\Phi$ for some Φ smooth and compactly supported. By density, we may extend $\mathcal{D}^{\boldsymbol{A}}$ to an operator from $\dot{W}A^p_{m-1,1}(\mathbb{R}^n)$ to $\dot{W}^2_{m,loc}(\mathbb{R}^{n+1}_+)$ that satisfies this bound.

Using this bound, it is straightforward to establish the bounds (30), (38) and (39).

Corollary 98. *Let L and $\boldsymbol{A}$ be as in Theorem 1.2. Then the bound* (30) *is valid; that is, there is some $\varepsilon > 0$ such that*

$$\|\widetilde{N}_+(\nabla^{m-1} \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}})\|_{L^p(\mathbb{R}^n)} \leq C_p \|\dot{\boldsymbol{f}}\|_{\dot{W}A^p_{m-1,0}(\mathbb{R}^n)} \qquad \text{if } 2-\varepsilon \leq p < 2+\varepsilon, \tag{99}$$

whenever $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1}\,F$ for some $F \in C_0^\infty(\mathbb{R}^{n+1})$.

Furthermore, there is some $\varepsilon > 0$ such that the bounds (38) *and* (39) *are valid; that is,*

$$\|\mathcal{A}_2^+(t\nabla^m \partial_t \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{\varphi}})\|_{L^p(\mathbb{R}^n)} \leq C_p \|\dot{\boldsymbol{\varphi}}\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)} \qquad \text{if } 2 \leq p < 2+\varepsilon, \tag{100}$$

$$\|\mathcal{A}_2^+(t\nabla^m \mathcal{D}^{\boldsymbol{A}} \dot{\boldsymbol{f}})\|_{L^p(\mathbb{R}^n)} \leq C_p \|\dot{\boldsymbol{f}}\|_{\dot{W}A^p_{m-1,0}(\mathbb{R}^n)} \qquad \text{if } 2 \leq p < 2+\varepsilon, \tag{101}$$

whenever $\dot{\boldsymbol{f}} = \dot{\mathbf{Tr}}_{m-1}\,F$ and $\dot{\boldsymbol{\varphi}} = \dot{\mathbf{Tr}}_{m-1}\,\Phi$ for some F, $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$.

Proof. We will use Lemma 3.7 to establish the bound (100). Let $u = \partial_{n+1}\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}$ and $u_Q = \partial_{n+1}\mathcal{D}^{\boldsymbol{A}}\dot{\boldsymbol{\varphi}}_{3Q}$, where $\dot{\boldsymbol{\varphi}}_R$ is as in Definition 95, and let $\psi = C|\dot{\boldsymbol{\varphi}}|$. By the bounds (97) and (31), the conditions of Lemma 3.7 are satisfied, and so the bound (100) is valid.

By formula (93) (or [20, formula (6.3)]), if F, $\Phi \in C_0^\infty(\mathbb{R}^n)$ and $\dot{\mathbf{Tr}}_{m-1}\,F = \dot{\mathbf{Tr}}_{m-1}\,\partial_{n+1}\Phi$, then

$$\mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}_{m-1}\,F) = \partial_t \mathcal{D}^{\boldsymbol{A}}(\dot{\mathbf{Tr}}_{m-1}\,\Phi) - \mathcal{S}^L_\nabla(\boldsymbol{A}\,\dot{\mathbf{Tr}}_m\,\Phi).$$

As in the proof of [20, Theorem 6.1], given $\dot{\mathbf{Tr}}_{m-1}\,F$ we may find an appropriate Φ such that

$$\|\boldsymbol{A}\,\dot{\mathbf{Tr}}_m\,\Phi\|_{L^p(\mathbb{R}^n)} + \|\dot{\mathbf{Tr}}_{m-1}\,\Phi\|_{\dot{W}A^p_{m-1,1}(\mathbb{R}^n)} \leq \|\dot{\mathbf{Tr}}_{m-1}\,F\|_{\dot{W}A^p_{m-1,0}(\mathbb{R}^n)}.$$

Thus, the bound (99) follows from Lemma 4.6 and the bound (97), and the bound (101) follows from Lemma 4.7 and the bound (100). □

This concludes the proof of Theorem 1.2 (and with it Theorem 1.1) and of Theorem 1.3, that is, of the main results of this paper.

Appendix A. Neumann boundary values in the half space

In this appendix, we will prove the following lemma. This lemma will allow us to characterize the Neumann boundary values $\dot{\mathbf{M}}^+_{\boldsymbol{A}}\,u$ of a solution u to $Lu = 0$ in terms of the boundary values of u and its derivatives. This generalizes the second order formula $\mathrm{M}^+_{\boldsymbol{A}}\,u = -\vec{e}_{n+1} \cdot \boldsymbol{A}\nabla u\big|_{\partial\mathbb{R}^{n+1}_+}$, and shows that formula (19) in the introduction is valid.

Lemma A.1. *Let L be an operator of the form* (1) *of order $2m$ associated to t-independent coefficients $\boldsymbol{A}$ that satisfy the bounds* (43) *and* (44).

Let $u \in \dot{W}^2_{m,loc}(\mathbb{R}^{n+1}_+) \cap \dot{W}^1_{m,loc}(\overline{\mathbb{R}^{n+1}_+})$ satisfy $Lu = 0$ in $\mathbb{R}^{n+1}_+$. Suppose that for any integer ℓ with $0 \le \ell \le m-1$, we have that

$$\sum_{\substack{|\zeta|=\ell \\ \zeta\in(\mathbb{N}_0)^{n+1}}} \partial^\zeta\Big(\big(\boldsymbol{A}\nabla^m u(\,\cdot\,,t)\big)_{\zeta+(m-|\zeta|)\vec{e}_{n+1}}\Big) = \sum_{\substack{|\xi|\le\ell \\ \xi\in(\mathbb{N}_0)^{n}}} \partial_\parallel^\xi\Big(\big(\boldsymbol{A}\nabla^m \partial_t^{\ell-|\xi|} u(\,\cdot\,,t)\big)_{(\xi,m-|\xi|)}\Big)$$

converges weakly in the distributional sense on $\mathbb{R}^n$ as $t \to 0^+$. That is, suppose that there are some distributions Γ_ℓ such that, for every $\psi \in C_0^\infty(\mathbb{R}^n)$, we have that

$$\lim_{t\to 0^+} \sum_{\substack{|\xi|\le\ell \\ \xi\in(\mathbb{N}_0)^{n}}} (-1)^{|\xi|}\Big\langle \partial_\parallel^\xi\psi, \big(\boldsymbol{A}\nabla^m \partial_t^{\ell-|\xi|} u(\,\cdot\,,t)\big)_{(\xi,m-|\xi|)}\Big\rangle_{\mathbb{R}^n} = \langle \psi, \Gamma_\ell\rangle_{\mathbb{R}^n}. \tag{102}$$

Let $\vec{\mathrm{M}}^+_{\boldsymbol{A}} u$, $\ddot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$ and $\dot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$ be given by formulas (12), (13) *and* (14) *with $\Omega = \mathbb{R}^{n+1}_+$. Then*

$$(\vec{\mathrm{M}}^+_{\boldsymbol{A}} u)_\ell = (-1)^m \Gamma_{m-1-\ell}, \quad 0 \le \ell \le m-1,$$

and an array of distributions $\dot{\boldsymbol{M}}$ or $\ddot{\boldsymbol{M}}$ satisfies $\dot{\boldsymbol{M}} \in \dot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$ or $\ddot{\boldsymbol{M}} \in \ddot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$ if and only if

$$\sum_{\substack{|\zeta|=m-1-\ell \\ \zeta\in(\mathbb{N}_0)^{n}}} \partial_\parallel^\zeta(\dot{\boldsymbol{M}})_{(\zeta,\ell)} = -\Gamma_{m-\ell-1} \quad \text{or} \quad \sum_{\substack{|\zeta|\le m-1-\ell \\ \zeta\in(\mathbb{N}_0)^{n}}} (-1)^{m-\ell-|\zeta|}\partial_\parallel^\zeta(\ddot{\boldsymbol{M}})_{(\zeta,\ell)} = \Gamma_{m-\ell-1}$$

for each integer ℓ with $0 \le \ell \le m-1$.

Proof. By formulas (12), (13) and (14), if $\nabla^m u$ is locally integrable up to the boundary, then $\vec{M} = \vec{\mathrm{M}}^+_{\boldsymbol{A}} u$, $\dot{\boldsymbol{M}} \in \dot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$ and $\ddot{\boldsymbol{M}} \in \ddot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$ if and only if

$$\begin{aligned}\sum_{j=0}^{m-1} \langle (-1)^j \operatorname{Tr} \partial_{n+1}^j \varphi, (\vec{M})_j\rangle_{\mathbb{R}^n} &= \sum_{|\gamma|=m-1} \langle \operatorname{Tr} \partial^\gamma \varphi, (\dot{\boldsymbol{M}})_\gamma\rangle_{\mathbb{R}^n} \\ &= \sum_{|\xi|\le m-1} \langle \operatorname{Tr} \partial^\xi \varphi, (\ddot{\boldsymbol{M}})_\xi\rangle_{\mathbb{R}^n} = \langle \nabla^m\varphi, \boldsymbol{A}\nabla^m u\rangle_{\mathbb{R}^{n+1}_+}\end{aligned} \tag{103}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^{n+1})$. If $\nabla^m u$ is locally integrable up to the boundary, then by formula (42) the right hand side depends only on $\dot{\mathbf{Tr}}_{m-1}\,\varphi$. Thus, $\vec{M} = \vec{\mathrm{M}}^+_{\boldsymbol{A}} u$, $\dot{\boldsymbol{M}} \in \dot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$, and $\ddot{\boldsymbol{M}} \in \ddot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$ if and only if formula (103) is valid for all φ of the form

$$\varphi(x,t) = \varphi_\ell(x,t) = \frac{t^\ell}{\ell!}\,\psi(x)\,\eta(t)$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$, where $0 \le \ell \le m-1$, and where $\eta \in C^\infty(\mathbb{R})$ satisfies, for some positive numbers ε and δ, the conditions that $\eta(t) = 1$ for all $|t| < \varepsilon$ and $\eta(t) = 0$ for all $|t| > \varepsilon + \delta$.

Choose some such ψ and ℓ. Observe that if $\xi = (\zeta, j)$, then $\operatorname{Tr}\partial^\xi\varphi_\ell = \partial_\parallel^\zeta\psi$ if $j = \ell$ and that $\operatorname{Tr}\partial^\xi\varphi = 0$ otherwise. Thus, $\vec{M} = \vec{\mathrm{M}}^+_{\boldsymbol{A}} u$, $\dot{\boldsymbol{M}} \in \dot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$, and

$\ddot{\boldsymbol{M}} \in \ddot{\mathbf{M}}^{+}_{\boldsymbol{A}}\, u$ if and only if

$$(-1)^{\ell}\langle \psi, (\vec{M})_{\ell}\rangle_{\mathbb{R}^n} = \sum_{\substack{|\zeta|=m-1-\ell\\ \zeta\in(\mathbb{N}_0)^n}} \langle \partial_{\|}^{\zeta}\psi, (\dot{\boldsymbol{M}})_{(\zeta,\ell)}\rangle_{\mathbb{R}^n} = \sum_{\substack{|\zeta|\leq m-1-\ell\\ \zeta\in(\mathbb{N}_0)^n}} \langle \partial_{\|}^{\zeta}\psi, (\ddot{\boldsymbol{M}})_{(\zeta,\ell)}\rangle_{\mathbb{R}^n} = \langle \nabla^m\varphi_\ell, \boldsymbol{A}\nabla^m u\rangle_{\mathbb{R}^{n+1}_+} \tag{104}$$

for all $\psi \in C_0^\infty(\mathbb{R}^n)$ and all $0 \leq \ell \leq m-1$.

We now consider the right hand side. Recalling the definition of φ_ℓ, we see that

$$\langle \nabla^m\varphi_\ell, \boldsymbol{A}\nabla^m u\rangle_{\mathbb{R}^{n+1}_+} = \int_0^\infty \sum_{i=0}^{m} \sum_{\substack{|\xi|=m-i\\ \xi\in(\mathbb{N}_0)^n}} \frac{d^i}{dt^i}\left(\frac{t^\ell}{\ell!}\eta(t)\right)\Big\langle \partial_{\|}^{\xi}\psi, \big(\boldsymbol{A}\nabla^m u(\,\cdot\,,t)\big)_{(\xi,i)}\Big\rangle_{\mathbb{R}^n}\, dt.$$

We define

$$U_i(t) = \sum_{\substack{|\xi|=m-i\\ \xi\in(\mathbb{N}_0)^n}} \Big\langle \partial_{\|}^{\xi}\psi, \big(\boldsymbol{A}\nabla^m u(\,\cdot\,,t)\big)_{(\xi,i)}\Big\rangle_{\mathbb{R}^n}.$$

Observe that if $\boldsymbol{A}$ is t-independent, then by Lemma 3.3, U_i is smooth on $(0,\infty)$. Recall that $\eta(t) = 1$ for all $t < \varepsilon$ and $\eta(t) = 0$ for all $t > \varepsilon + \delta$. Thus,

$$\langle \nabla^m\varphi_\ell, \boldsymbol{A}\nabla^m u\rangle_{\mathbb{R}^{n+1}_+} = \sum_{i=0}^{\ell}\int_0^\varepsilon \frac{t^{\ell-i}}{(\ell-i)!}U_i(t)\,dt + \sum_{i=0}^{m}\int_\varepsilon^{\varepsilon+\delta}\frac{d^i}{dt^i}\left(\frac{t^\ell}{\ell!}\eta(t)\right)U_i(t)\,dt.$$

Observe that $t^\ell\eta(t) = 0$ for all $t > \varepsilon+\delta$ and that $t^\ell\eta(t) = t^\ell$ for all $t < \varepsilon$. Thus, integrating by parts in the second integral yields that

$$\begin{aligned}\langle \nabla^m\varphi_\ell, \boldsymbol{A}\nabla^m u\rangle_{\mathbb{R}^{n+1}_+} &= \sum_{i=0}^{\ell}\int_0^\varepsilon \frac{t^{\ell-i}}{(\ell-i)!}U_i(t)\,dt + \sum_{i=0}^{m}(-1)^i\int_\varepsilon^{\varepsilon+\delta}\frac{t^\ell}{\ell!}\eta(t)\frac{d^i}{dt^i}U_i(t)\,dt\\ &\quad - \sum_{i=1}^{m}\sum_{k=1}^{i}(-1)^{i-k}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{t^\ell}{\ell!}\right)\bigg|_{t=\varepsilon}\frac{d^{i-k}}{d\varepsilon^{i-k}}U_i(\varepsilon).\end{aligned}$$

Recall that the left hand side is independent of η. We require that $0 \leq \eta(t) \leq 1$ for all $t \in \mathbb{R}$. By Lemma 3.3, for any fixed $\varepsilon > 0$, we have that the middle term converges to zero as $\delta \to 0^+$. If $\nabla^m u$ is locally integrable up to the boundary, then the first term converges to zero as $\varepsilon \to 0^+$ for any fixed ψ. Taking the limit as $\delta \to 0^+$ and then the limit as $\varepsilon \to 0^+$, we have that

$$\langle \nabla^m\varphi_\ell, \boldsymbol{A}\nabla^m u\rangle_{\mathbb{R}^{n+1}_+} = -\lim_{\varepsilon\to 0^+}\sum_{i=1}^{m}\sum_{k=1}^{i}(-1)^{i-k}\frac{d^{k-1}}{dt^{k-1}}\left(\frac{t^\ell}{\ell!}\right)\bigg|_{t=\varepsilon}\frac{d^{i-k}}{d\varepsilon^{i-k}}U_i(\varepsilon).$$

Changing the order of summation and evaluating the derivative, we have that

$$\langle \nabla^m\varphi_\ell, \boldsymbol{A}\nabla^m u\rangle_{\mathbb{R}^{n+1}_+} = -\lim_{\varepsilon\to 0^+}\sum_{k=1}^{\ell+1}\sum_{i=k}^{m}(-1)^{i-k}\frac{\varepsilon^{\ell+1-k}}{(\ell+1-k)!}\frac{d^{i-k}}{d\varepsilon^{i-k}}U_i(\varepsilon).$$

Recalling the definition of U_i, we see that if $\boldsymbol{A}$ is t-independent then

$$\begin{aligned}
&\langle \nabla^m \varphi_\ell, \boldsymbol{A}\nabla^m u\rangle_{\mathbb{R}^{n+1}_+} \\
&= -\lim_{\varepsilon\to 0^+} \sum_{k=1}^{\ell+1}\sum_{i=k}^{m} (-1)^{i-k} \frac{\varepsilon^{\ell+1-k}}{(\ell+1-k)!} \sum_{\substack{\xi\in(\mathbb{N}_0)^n\\ |\xi|=m-i}} \Big\langle \partial_\parallel^\xi \psi, \big(\boldsymbol{A}\nabla^m \partial_\varepsilon^{i-k} u(\,\cdot\,,\varepsilon)\big)_{(\xi,i)}\Big\rangle_{\mathbb{R}^n} \\
&= -\lim_{\varepsilon\to 0^+} \sum_{k=1}^{\ell+1} (-1)^{m-k} \frac{\varepsilon^{\ell+1-k}}{(\ell+1-k)!} \sum_{\substack{\xi\in(\mathbb{N}_0)^n\\ |\xi|\le m-k}} (-1)^{|\xi|} \Big\langle \partial_\parallel^\xi \psi, \big(\boldsymbol{A}\nabla^m \partial_\varepsilon^{m-|\xi|-k} u(\,\cdot\,,\varepsilon)\big)_{(\xi,m-|\xi|)}\Big\rangle_{\mathbb{R}^n}.
\end{aligned}$$

By formula (102),

$$\lim_{\varepsilon\to 0^+} \sum_{\substack{|\xi|\le m-k\\ \xi\in(\mathbb{N}_0)^n}} (-1)^{|\xi|} \Big\langle \partial_\parallel^\xi \psi, \big(\boldsymbol{A}\nabla^m \partial_\varepsilon^{m-|\xi|-k} u(\,\cdot\,,\varepsilon)\big)_{(\xi,m-|\xi|)}\Big\rangle_{\mathbb{R}^n} = \langle \psi, \Gamma_{m-k}\rangle_{\mathbb{R}^n} < \infty$$

and so

$$\begin{aligned}
\langle \nabla^m \varphi_\ell, \boldsymbol{A}\nabla^m u\rangle_{\mathbb{R}^{n+1}_+} &= -\lim_{\varepsilon\to 0^+} \sum_{k=1}^{\ell+1} (-1)^{m-k} \frac{\varepsilon^{\ell+1-k}}{(\ell+1-k)!} \langle \psi, \Gamma_{m-k}\rangle_{\mathbb{R}^n} \\
&= (-1)^{m-\ell} \langle \psi, \Gamma_{m-1-\ell}\rangle_{\mathbb{R}^n}.
\end{aligned}$$

By formula (104), $\vec{M} = \vec{\mathrm{M}}^+_{\boldsymbol{A}} u$, $\dot{\boldsymbol{M}} \in \dot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$, and $\ddot{\boldsymbol{M}} \in \ddot{\mathbf{M}}^+_{\boldsymbol{A}}\, u$ if and only if

$$\begin{aligned}
(-1)^\ell \langle \psi, (\vec{M})_\ell\rangle_{\mathbb{R}^n} &= \sum_{\substack{|\zeta|=m-1-\ell\\ \zeta\in(\mathbb{N}_0)^n}} \langle \partial_\parallel^\zeta \psi, (\dot{\boldsymbol{M}})_{(\zeta,\ell)}\rangle_{\mathbb{R}^n} \\
&= \sum_{\substack{|\zeta|\le m-1-\ell\\ \zeta\in(\mathbb{N}_0)^n}} \langle \partial_\parallel^\zeta \psi, (\ddot{\boldsymbol{M}})_{(\zeta,\ell)}\rangle_{\mathbb{R}^n} = (-1)^{m-\ell} \langle \psi, \Gamma_{m-\ell-1}\rangle_{\mathbb{R}^n}
\end{aligned}$$

for all $\psi \in C_0^\infty(\mathbb{R}^n)$ and all $0 \le \ell \le m-1$. Invoking the distributional definition of derivative and canceling powers of -1 completes the proof. □

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