MATH 55203–55303

Theory of Functions of a Complex Variable I–II

Fall 2025–Spring 2026

(Fall) MEEG 217, MWF 2:00–2:50 p.m.

Next class day: Monday, August 25, 2025

Contents

1.1. Elementary Properties of the Complex Numbers

[Definition: The complex numbers] The set of complex numbers is $\mathbb{R}^2$, denoted $\mathbb{C}$. (In this class, you may use everything you know about $\mathbb{R}$ and $\mathbb{R}^2$—in particular, that $\mathbb{R}^2$ is an abelian group and a normed vector space.)

[Definition: Real and imaginary parts] If (x, y) is a complex number, then $\operatorname{Re}(x, y) = x$ and $\operatorname{Im}(x, y) = y$.

[Definition: Addition and multiplication] If (x, y) and (ξ, η) are two complex numbers, we define

$$(x, y) + (\xi, \eta) = (x + \xi, y + \eta),$$
$$(x, y) \cdot (\xi, \eta) = (x\xi - y\eta, x\eta + y\xi).$$

(Problem 10) Show that multiplication in the complex numbers is commutative.

Let (x, y) and (ξ, η) be two complex numbers. Then

$$(x, y) \cdot (\xi, \eta) = (x\xi - y\eta, x\eta + y\xi),$$
$$(\xi, \eta) \cdot (x, y) = (\xi x - \eta y, \xi y + \eta x).$$

Because multiplication in the real numbers is commutative, we have that

$$(\xi, \eta) \cdot (x, y) = (x\xi - y\eta, y\xi + x\eta).$$

Because addition in the real numbers is commutative, we have that

$$(\xi, \eta) \cdot (x, y) = (x\xi - y\eta, x\eta + y\xi) = (x, y) \cdot (\xi, \eta)$$

as desired.

(Fact 20) This notion of addition and multiplication makes the complex numbers a ring—thus, multiplication is also associative and distributes over addition.

(Problem 30) What is the multiplicative identity?

(Problem 40) Let r be a real number. Recall that $\mathbb{C} = \mathbb{R}^2$ is a vector space over $\mathbb{R}$, so we can multiply vectors (complex numbers) by scalars (real numbers). Is there a complex number (ξ, η) such that $r(x, y) = (\xi, \eta) \cdot (x, y)$ for all $(x, y) \in \mathbb{C}$?

[Definition: Notation for the complex numbers]

- If $r \in \mathbb{R}$, we identify r with the number $(r, 0) \in \mathbb{C}$.
- We let i denote $(0, 1)$.

(Problem 50) If x, y are real numbers, what complex number is $x + iy$?

(Problem 60) If $z = x + iy$ for x, y real, what are $\operatorname{Re} z$ and $\operatorname{Im} z$?

(Problem 70) If $z \in \mathbb{C}$ and r is real, what are $\operatorname{Re}(zr)$ and $\operatorname{Im}(zr)$?

(Problem 80) If $z, w \in \mathbb{C}$, what are $\operatorname{Re}(z + w)$, $\operatorname{Im}(z + w)$ in terms of $\operatorname{Re} z$, $\operatorname{Re} w$, $\operatorname{Im} z$, and $\operatorname{Im} w$?

(Problem 90) If $z, w \in \mathbb{C}$, what are $\operatorname{Re}(zw)$, $\operatorname{Im}(zw)$ in terms of $\operatorname{Re} z$, $\operatorname{Re} w$, $\operatorname{Im} z$, and $\operatorname{Im} w$?

[Definition: Conjugate] The conjugate to the complex number $x + iy$, where x, y are real, is $\overline{x + iy} = x - iy$.[1]

(Problem 100) If z and w are complex numbers, show that $\bar{z} + \bar{w} = \overline{z + w}$.

(Problem 110) Show that $\bar{z} \cdot \bar{w} = \overline{zw}$.

(Problem 120) Write $\operatorname{Re} z$ and $\operatorname{Im} z$ in terms of z and $\bar{z}$.

(Problem 130) Show that $z\bar{z}$ is always real and nonnegative. If $z\bar{z} = 0$, what can you say about z?

(Problem 140) If z is a complex number with $z \neq 0$, show that there exists another complex number w such that $zw = 1$. Give a formula for w in terms of z. We will write $w = \frac{1}{z}$.

$z\bar{z}$ is a positive real number, and we know from real analysis that positive real numbers have reciprocals. Thus $\frac{1}{z\bar{z}} \in \mathbb{R}$. We can multiply complex numbers by real numbers, so $\frac{1}{z\bar{z}}\bar{z}$ is a complex number and it is the w of the problem statement.

[Definition: Modulus] If z is a complex number, we define its modulus $|z|$ as $|z| = \sqrt{z\bar{z}}$.

(Fact 150) $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$ (where the first $|\cdot|$ denotes the absolute value in the real numbers and the second $|\cdot|$ denotes the modulus in the complex numbers.)

(Problem 160) If z and w are complex numbers, show that $|zw| = |z|\,|w|$.

(Problem 170) Give an example of a non-constant polynomial that has no roots (solutions) that are real numbers. Find a root (solution) to your polynomial that is a complex number.

1.2. Real Analysis

(Fact 180) If $z = x + iy = (x, y)$, then the complex modulus $|z|$ is equal to the vector space norm $\|(x, y)\|$ in $\mathbb{R}^2$.

(Fact 190) $\mathbb{C}$ is complete as a metric space if we use the expected metric $d(z, w) = |z - w|$.

(Bashar, Problem 200) Recall that $(\mathbb{R}^2, d)$ is a metric space, where $d(u, v) = \|u - v\|$. In particular, this metric satisfies the triangle inequality. Write the triangle inequality as a statement about moduli of complex numbers. Simplify your statement as much as possible.

The conclusion is that $|z + w| \leq |z| + |w|$ for all $z, w \in \mathbb{C}$. This is Proposition 1.2.3 in your textbook.

[1] Some authors, especially in physics, write z^* instead of $\bar{z}$ for the complex conjugate of z.

(Memory 210) If $\{a_n\}_{n=1}^{\infty}$ is a sequence of points in $\mathbb{R}^p$, $a \in \mathbb{R}^p$, and we write $a_n = (a_n^1, a_n^2, \dots, a_n^p)$, $a = (a^1, \dots a^p)$, then $a_n \to a$ (in the metric space sense) if and only if $a_n^k \to a^k$ for each $1 \leq k \leq p$.

(Dibyendu, Problem 220) What does this tell you about the complex numbers?

If $\{z_n\}_{n=1}^{\infty}$ is a sequence of points in $\mathbb{C}$ and $z \in \mathbb{C}$, then $z_n \to z$ if and only if both $\operatorname{Re} z_n \to \operatorname{Re} z$ and $\operatorname{Im} z_n \to \operatorname{Im} z$.

[Definition: Maclaurin series] If $f : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable function, then the Maclaurin series for f is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

with the convention that $0^0 = 1$.

(Memory 221) If x is real, then the Maclaurin series for $\exp x$, $\sin x$, or $\cos x$ converges to $\exp x$, $\sin x$, or $\cos x$, respectively.

(Memory 230) The Maclaurin series for the exp function is $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

(Memory 240) The Maclaurin series for the sin function is $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$.

(Memory 250) The Maclaurin series for the cos function is $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$.

(Memory 270) If x and t are real numbers then

$$\sin(x+t) = \sin x \cos t + \sin t \cos x,$$
$$\cos(x+t) = \cos x \cos t - \sin x \sin t.$$

(Memory 280) The Cauchy-Schwarz inequality for real numbers states that if $n \in \mathbb{N}$ is a positive integer, and if for each k with $1 \leq k \leq n$ the numbers x_k, ξ_k are real, then

$$\left(\sum_{k=1}^{n} x_k \, \xi_k\right)^2 \leq \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} \xi_k^2\right).$$

1.2. Further Properties of the Complex Numbers

(Hope, Problem 290) State the Cauchy-Schwarz inequality for complex numbers and prove that it is valid.

This is Proposition 1.2.4 in your book. If $n \in \mathbb{N}$, and if $z_1, z_2, \dots, z_n$ and $w_1, w_2, \dots, w_n$ are complex numbers, then

$$\left|\sum_{k=1}^{n} z_k \, w_k\right|^2 \leq \sum_{k=1}^{n} |z_k|^2 \sum_{k} |w_k|^2.$$

We can prove this as follows. By the triangle inequality, $|z_1 w_1 + z_2 w_2| \leq |z_1 w_1| + |z_2 w_2| = |z_1||w_1| + |z_2||w_2|$. A straightforward induction argument yields that

$$\left|\sum_{k=1}^{n} z_k \, w_k\right| \leq \sum_{k=1}^{n} |z_k||w_k|.$$

Applying the real Cauchy-Schwarz inequality with $x_k = |z_k|$ and $\xi_k = |w_k|$ completes the proof.

(James, Problem 300) Let $z \in \mathbb{C}$. Consider the series $\sum_{k=0}^{\infty} \frac{z^k}{k!}$, that is, the sequence of complex numbers $\left\{\sum_{k=0}^{n} \frac{z^k}{k!}\right\}_{n=0}^{\infty}$. Show that this sequence is a Cauchy sequence.

(Problem 310) Since $\mathbb{C}$ is complete, the series converges. If $z = x$ is a real number, to what number does the series converge?

It converges to e^x.

(Micah, Problem 320) If $z = iy$ is purely imaginary (that is, if $y \in \mathbb{R}$), show that $\sum_{k=0}^{\infty} \frac{(iy)^k}{k!}$ converges to $\cos y + i \sin y$.

An induction argument establishes that

$$\operatorname{Re} i^k = \begin{cases} 0, & k \text{ is odd,} \\ 1, & k \text{ is even and a multiple of 4,} \\ -1, & k \text{ is even and not a multiple of 4,} \end{cases}$$

and

$$\operatorname{Im} i^k = \begin{cases} 0, & k \text{ is even,} \\ 1, & k \text{ is odd and one more than a multiple of 4,} \\ -1, & k \text{ is even and one less than a multiple of 4.} \end{cases}$$

We then see that we may write the Maclaurin series for cos and sin as

$$\cos(y) = \sum_{k=0}^{\infty} \operatorname{Re} i^k \frac{y^k}{k!}, \qquad \sin(y) = \sum_{k=0}^{\infty} \operatorname{Im} i^k \frac{y^k}{k!}.$$

We then have that

$$\operatorname{Re}\Big(\sum_{k=0}^{n} \frac{(iy)^k}{k!}\Big) = \sum_{k=0}^{n} \operatorname{Re}\left(\frac{(iy)^k}{k!}\right) = \sum_{k=0}^{n} \operatorname{Re} i^k \frac{y^k}{k!}$$

which converges to $\cos y$ as $n \to \infty$. Similarly

$$\operatorname{Im}\Big(\sum_{k=0}^{n} \frac{(iy)^k}{k!}\Big) = \sum_{k=0}^{n} \operatorname{Im}\left(\frac{(iy)^k}{k!}\right) = \sum_{k=0}^{n} \operatorname{Im} i^k \frac{y^k}{k!}$$

converges to $\sin y$ as $n \to \infty$. Thus the series $\sum_{k=0}^{\infty} \frac{(iy)^k}{k!}$ converges to $\cos y + i \sin y$, as desired.

(Bonus Problem 330) If $z = x + iy$, show that $\sum_{j=0}^{\infty} \frac{z^j}{j!}$ converges to the product $\big(\sum_{j=0}^{\infty} \frac{x^j}{j!}\big)\big(\sum_{j=0}^{\infty} \frac{(iy)^j}{j!}\big)$.

[Definition: The complex exponential] If x is real, we define

$$\exp(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} \quad \text{and} \quad \exp(ix) = \sum_{j=0}^{\infty} \frac{(ix)^j}{j!}.$$

If $z = x + iy$ is a complex number, we define

$$\exp(z) = \exp(x) \cdot \exp(iy).$$

(Muhammad, Problem 340) If y, η are real, show that $\exp(iy + i\eta) = \exp(iy) \cdot \exp(i\eta)$.

Using the sum angle identities for sine and cosine, we compute

$$\begin{aligned} \exp(iy + i\eta) &= \exp(i(y + \eta)) = \cos(y + \eta) + i \sin(y + \eta) \\ &= \cos y \cos \eta - \sin y \sin \eta + i \sin y \cos \eta + i \cos y \sin \eta \end{aligned}$$

and

$$\begin{aligned} \exp(iy) \exp(i\eta) &= (\cos y + i \sin y)(\cos \eta + i \sin \eta) \\ &= \cos y \cos \eta - \sin y \sin \eta + i \sin y \cos \eta + i \cos y \sin \eta \end{aligned}$$

and observe that they are equal.

(Robert, Problem 350) If z, w are any complex numbers, show that $\exp(z+w) = \exp(z)\cdot\exp(w)$.

There are real numbers x, y, ξ, η such that $z = x+iy$ and $w = \xi + i\eta$.
By definition

$$\exp(z) = \exp(x)\exp(iy), \qquad \exp(w) = \exp(\xi)\exp(i\eta).$$

Because multiplication in the complex numbers is associative and commutative,

$$\exp(z)\exp(w) = [\exp(x)\exp(iy)][\exp(\xi)\exp(i\eta)] = [\exp(x)\exp(\xi)][\exp(iy)\exp(i\eta)].$$

By properties of exponentials in the real numbers and by the previous problem, we see that

$$\exp(z)\exp(w) = \exp(x+\xi)\exp(iy+i\eta).$$

By definition of the complex exponential,

$$\exp(z)\exp(w) = \exp((x+\xi)+i(y+\eta)) = \exp(z+w)$$

as desired.

(Sam, Problem 360) Suppose that z is a complex number and that $|z| = 1$. Show that there is a number $\theta \in \mathbb{R}$ with $\exp(i\theta) = z$. How many such numbers θ exist? (Use only undergraduate real analysis and methods established so far in this course.)

We know from real analysis that, if (x, y) lies on the unit circle, then $(x, y) = (\cos\theta, \sin\theta)$ for some real number θ. By definition of complex modulus, if $|z| = 1$ and $z = x+iy$ then (x, y) lies on the unit circle. Thus $z = \cos\theta + i\sin\theta = \exp(i\theta)$ for some $\theta \in \mathbb{R}$.

Infinitely many such numbers θ exist.

[Chapter 1, Problem 25] If θ, $\varpi \in \mathbb{R}$, then $e^{i\theta} = e^{i\varpi}$ if and only if $(\theta - \varpi)/(2\pi)$ is an integer.

(William, Problem 370) Suppose that z is a complex number. Show that there exist numbers $r \in [0,\infty)$ and $\theta \in \mathbb{R}$ such that $z = r\exp(i\theta)$. How many possible values of r exist? How many possible values of θ exist? (Use only undergraduate real analysis and methods established so far in this course.)

Observe that $|re^{i\theta}| = r|e^{i\theta}|$ because $r \geq 0$ and because the modulus distributes over products. But $|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$, and so the only choice for r is $r = |z|$.

If $z = 0$ then we must have that $r = 0$ and can take any real number for θ.

If $z \neq 0$, let $r = |z|$. Then $w = \frac{1}{r}z$ is a complex number with $|z| = 1$, and so there exist infinitely many values θ with $e^{i\theta} = w$ and thus $z = re^{i\theta}$.

(Wilson, Problem 380) Find all solutions to the equation $z^6 = i$. Use only undergraduate real analysis and methods established so far in this course.

Suppose that $z = re^{i\theta}$ for some $r \geq 0$, $\theta \in \mathbb{R}$.

Then $z^6 = r^6 e^{6i\theta}$. If $z^6 = i$, then $1 = |i| = |z^6| = r^6$ and so $r = 1$ because $r \geq 0$. We must then have that $i = e^{6i\theta}$. Observe that $i = e^{i\pi/2}$. By Homework 1.25, we must have that $6\theta = \pi/2 + 2\pi n$ for some $n \in \mathbb{Z}$, and so $(e^{i\theta})^6 = i$ if and only if $\theta = \pi/12 + n\pi/3$. Thus the solutions are

$$e^{\pi/12}, \quad e^{5\pi/12}, \quad e^{9\pi/12}, \quad e^{13\pi/12}, \quad e^{17\pi/12}, \quad e^{21\pi/12}.$$

Any other solution is of the form $e^{i\theta}$, where θ differs from one of the listed numbers by 2π.

(Problem 390) Give an example of a function that can be written in two different ways.

[Definition: Real polynomial] Let $p:\mathbb{R}\to\mathbb{R}$ be a function. We say that p is a (real) polynomial in one (real) variable if there is a $n\in\mathbb{N}_0$ and constants a_0, $a_1,\dots,a_n\in\mathbb{R}$ such that $p(x)=\sum_{k=0}^n a_k x^k$ for all $x\in\mathbb{R}$.

[Definition: Real polynomial in two variables] Let $p:\mathbb{R}^2\to\mathbb{R}$ be a function. We say that p is a (real) polynomial in two (real) variables if there is a $n\in\mathbb{N}_0$ and constants $a_{k,\ell}\in\mathbb{R}$ such that $p(x,y)=\sum_{k=0}^n\sum_{\ell=0}^n a_{k,\ell}x^k y^\ell$ for all x, $y\in\mathbb{R}$.

(Adam, Problem 400) Let $p(x)=\sum_{k=0}^n a_k\,x^k$ and let $q(x)=\sum_{k=0}^n b_k\,x^k$ be two polynomials in one variable, with a_k, $b_k\in\mathbb{R}$. Show that if $p(x)=q(x)$ for all $x\in\mathbb{R}$ then $a_k=b_k$ for all $k\in\mathbb{N}_0$.

p and q are infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, and because $p(x)=q(x)$ for all $x\in\mathbb{R}$, we must have that $p'=q'$, $p''=q'',\dots,p^{(k)}=q^{(k)}$ for all $k\in\mathbb{N}$.

We compute $p^{(k)}(0)=k!a_k$ and $q^{(k)}(0)=k!b_k$. Setting them equal we see that $a_k=b_k$.

[Definition: Degree] If $p(z)=\sum_{k=0}^n a_k\,z^k$, then the degree of p is the largest nonnegative integer m such that $a_m\neq 0$. (The degree of the zero polynomial $p(z)=0$ is either undefined, -1, or $-\infty$.)

(Amani, Problem 410) Let p be a polynomial. Suppose that $x_0\in\mathbb{R}$ and that $p(x_0)=0$. Show that there exists a polynomial q such that $p(x)=(x-x_0)q(x)$ for all $x\in\mathbb{R}$. Further show that, if p is a polynomial of degree $m\geq 0$, then q is a polynomial of degree $m-1$. *Hint*: Use induction.

(Bashar, Problem 420) Let $p(x)=\sum_{k=0}^n a_k\,x^k$ and let $q(x)=\sum_{k=0}^n b_k\,x^k$ be two polynomials of degree at most n, with a_k, $b_k\in\mathbb{R}$ and $n\in\mathbb{N}_0$. Suppose that there are $n+1$ distinct numbers $x_0,x_1,\dots,x_n\in\mathbb{R}$ such that $p(x_j)=q(x_j)$ for all $0\leq j\leq n$. Show that $a_k=b_k$ for all $k\in\mathbb{N}_0$. *Hint*: Consider the polynomial $r(x)=p(x)-q(x)$.

(Dibyendu, Problem 430) Let $p(x,y)=\sum_{j=0}^n\sum_{k=0}^n a_{j,k}\,x^j\,y^k$ and let $q(x,y)=\sum_{j=0}^n\sum_{k=0}^n b_{j,k}\,x^j\,y^k$ be two polynomials of two variables, with $a_{j,k}$, $b_{j,k}\in\mathbb{R}$. Show that if $p(x,y)=q(x,y)$ for all $(x,y)\in\mathbb{R}^2$ then $a_{j,k}=b_{j,k}$ for all j, $k\in\mathbb{N}_0$.

(Memory 440) If $\Omega\subseteq\mathbb{R}^2$ is both open and connected, then Ω is path connected: for every z, $w\in\Omega$ there is a continuous function $\gamma:[0,1]\to\Omega$ such that $\gamma(0)=z$ and $\gamma(1)=w$.

(Memory 450) If $\Omega\subseteq\mathbb{R}^2$ is open and connected, we may require the paths in the definition of path connectedness to be C^1.

(Memory 460) If $\Omega\subseteq\mathbb{R}^2$ is open and connected, we may require the paths in the definition of path connectedness to consist of finitely many horizontal or vertical line segments.

Definition 1.3.1 (part 1). Let $\Omega\subseteq\mathbb{R}^2$ be open. Suppose that $f:\Omega\to\mathbb{R}$. We say that f is continuously differentiable, or $f\in C^1(\Omega)$, if the two partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere in Ω and f, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ are all continuous on Ω.

(Hope, Problem 470) Let $B=B(z,r)$ be a ball in $\mathbb{R}^2$. Let $f\in C^1(B)$ and suppose that $\frac{\partial f}{\partial y}=\frac{\partial f}{\partial x}=0$ everywhere in B. Show that f is a constant.

(James, Problem 480) Suppose that $\Omega\subseteq\mathbb{R}^2$ is open and connected. Let $f\in C^1(\Omega)$ and suppose that $\frac{\partial f}{\partial y}=\frac{\partial f}{\partial x}=0$ everywhere in Ω. Show that f is a constant.

1.3. Complex Polynomials

[Definition: Complex polynomials in one variable] Let $p:\mathbb{C}\to\mathbb{C}$ be a function. We say that p is a polynomial in one (complex) variable if there is a $n\in\mathbb{N}_0$ and constants a_0, $a_1,\dots,a_n\in\mathbb{C}$ such that $p(z)=\sum_{k=0}^n a_k z^k$ for all $z\in\mathbb{C}$.

[Definition: Complex polynomial in two variables] Let $p:\mathbb{C}\to\mathbb{C}$ be a function. We say that p is a polynomial in two real variables if there is a $n\in\mathbb{N}_0$ and constants $a_{k,\ell}\in\mathbb{C}$ such that $p(x+iy)=\sum_{k=0}^n\sum_{\ell=0}^n a_{k,\ell}x^k y^\ell$ for all x, $y\in\mathbb{R}$.

(Micah, Problem 490) Show that p is a polynomial in two real variables if and only if there are constants $b_{k,\ell} \in \mathbb{C}$ such that $p(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{n} b_{k,\ell} z^k \overline{z}^\ell$ for all $z \in \mathbb{C}$.

(Fact 500) Problem 430 is true for complex polynomials of two real variables; that is, if $p(x+iy) = \sum_{k,\ell=0}^{n} a_{k,\ell} x^k y^\ell$, $q(x+iy) = \sum_{k,\ell=0}^{n} c_{k,\ell} x^k y^\ell$, and $p(z) = q(z)$ for all $z \in \mathbb{C}$, then $a_{k,\ell} = c_{k,\ell}$ for all k and ℓ.

(Fact 510) Problem 400, Problem 410, and Problem 420 are valid for complex polynomials of one complex variable.

(Muhammad, Problem 520) Give an example to show that Problem 420 is *not* true for complex polynomials of two real variables; that is, give an example of a complex polynomial q of two real variables of degree n such that $q(z_k) = 0$ for at least $n+1$ values of k, but such that q is not the zero polynomial.

1.3. The complex derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$

Definition 1.3.1 (part 2). Let $\Omega \subseteq \mathbb{C}$ be an open set. Recall $\mathbb{C} = \mathbb{R}^2$. Let $f : \Omega \to \mathbb{C}$ be a function. Then $f \in C^1(\Omega)$ if $\operatorname{Re} f$, $\operatorname{Im} f \in C^1(\Omega)$.

[Definition: Derivative of a complex function] Let $f \in C^1(\Omega)$. Let $u(z) = \operatorname{Re} f(z)$ and let $v(z) = \operatorname{Im} f(z)$. Then

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}.$$

(Nisa, Problem 530) Establish the Leibniz rules

$$\frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x} g + f\frac{\partial g}{\partial x}, \qquad \frac{\partial}{\partial y}(fg) = \frac{\partial f}{\partial y} g + f\frac{\partial g}{\partial y}$$

for f, $g \in C^1(\Omega)$.

[Definition: Complex derivative] Let $f \in C^1(\Omega)$. Then

$$\frac{\partial f}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}.$$

(Robert, Problem 540) Let $f(z) = z$ and let $g(z) = \overline{z}$. Show that $\frac{\partial f}{\partial z} = 1$, $\frac{\partial f}{\partial \overline{z}} = 0$, $\frac{\partial g}{\partial z} = 0$, $\frac{\partial g}{\partial \overline{z}} = 1$.

(Fact 550) $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ are linear operators.

(Fact 560) Show that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ commute in the sense that, if $\Omega \subseteq \mathbb{C}$ is open and $f \in C^2(\Omega)$, then $\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \overline{z}} f\right) = \frac{\partial}{\partial \overline{z}}\left(\frac{\partial}{\partial z} f\right)$.

(Fact 570) The following Leibniz rules are valid:

$$\frac{\partial}{\partial z}(fg) = \frac{\partial f}{\partial z} g + f\frac{\partial g}{\partial z}, \qquad \frac{\partial}{\partial \overline{z}}(fg) = \frac{\partial f}{\partial \overline{z}} g + f\frac{\partial g}{\partial \overline{z}}.$$

(Sam, Problem 580) Show that $\frac{\partial}{\partial z}(z^\ell \overline{z}^m) = \ell z^{\ell-1} \overline{z}^m$ and $\frac{\partial}{\partial \overline{z}}(z^\ell \overline{z}^m) = m z^\ell \overline{z}^{m-1}$ for all nonnegative integers m and ℓ.

(William, Problem 590) Let p be a complex polynomial in two real variables. Show that p is a complex polynomial in one complex variable if and only if $\frac{\partial p}{\partial \overline{z}} = 0$ everywhere in $\mathbb{C}$.

Definition 1.4.1. Let $\Omega \subseteq \mathbb{C}$ be open and let $f \in C^1(\Omega)$. We say that f is holomorphic in Ω if

$$\frac{\partial f}{\partial \overline{z}} = 0$$

everywhere in Ω.

(Fact 600) A polynomial in two real variables is a polynomial in one complex variable if and only if it is holomorphic.

(Wilson, Problem 610) Suppose that $\Omega \subseteq \mathbb{C}$ is open and connected, that $f \in C^1(\Omega)$, and that $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \bar z} = 0$ in Ω. Show that f is constant in Ω.

[Chapter 1, Problem 34]Suppose that $\Omega \subseteq \mathbb{C}$ is open and that $f \in C^1(\Omega)$. Show that

$$\frac{\partial f}{\partial z} = \overline{\left(\frac{\partial \bar f}{\partial \bar z}\right)}.$$

(Adam, Problem 620) Show that $\frac{\partial}{\partial z}\frac{1}{z} = -\frac{1}{z^2}$ and $\frac{\partial}{\partial \bar z}\frac{1}{z} = 0$ if $z \neq 0$. Then compute $\frac{\partial}{\partial z}\frac{1}{z^n}$ and $\frac{\partial}{\partial \bar z}\frac{1}{z^n}$ for any positive integer n.

[Chapter 1, Problem 49] Let Ω, $W \subseteq \mathbb{C}$ be open and let $g : \Omega \to W$, $f : W \to \mathbb{C}$ be two C^1 functions. The following chain rules are valid:

$$\frac{\partial}{\partial z}(f \circ g) = \frac{\partial f}{\partial g}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar g}\frac{\partial \bar g}{\partial z},$$
$$\frac{\partial}{\partial \bar z}(f \circ g) = \frac{\partial f}{\partial g}\frac{\partial g}{\partial \bar z} + \frac{\partial f}{\partial \bar g}\frac{\partial \bar g}{\partial \bar z},$$

where $\frac{\partial f}{\partial g} = \frac{\partial f}{\partial z}\big|_{z \to g(z)}$, $\frac{\partial f}{\partial \bar g} = \frac{\partial f}{\partial \bar z}\big|_{z \to g(z)}$.

In particular, if f and g are both holomorphic then so is $f \circ g$.

1.4. Holomorphic Functions, the Cauchy-Riemann Equations, and Harmonic Functions

Lemma 1.4.2. Let $f \in C^1(\Omega)$, let $u = \operatorname{Re} f$, and let $v = \operatorname{Im} f$. Then f is holomorphic in Ω if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

everywhere in Ω. (These equations are called the Cauchy-Riemann equations.)

(Amani, Problem 630) Prove the "only if" direction of Lemma 1.4.2: show that if f is holomorphic in Ω, $\Omega \subseteq \mathbb{C}$ open, then the Cauchy-Riemann equations hold for $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$.

(Bashar, Problem 640) Prove the "if" direction of Lemma 1.4.2: show that if $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are C^1 in Ω and satisfy the Cauchy-Riemann equations, then f is holomorphic in Ω.

Proposition 1.4.3. [Slight generalization.] Let $f \in C^1(\Omega)$. Then f is holomorphic at $p \in \Omega$ if and only if $\frac{\partial f}{\partial x}(p) = \frac{1}{i}\frac{\partial f}{\partial y}(p)$ and that in this case

$$\frac{\partial f}{\partial z}(p) = \frac{\partial f}{\partial x}(p) = \frac{1}{i}\frac{\partial f}{\partial y}(p).$$

(Dibyendu, Problem 650) Begin the proof of Proposition 1.4.3 by showing that if f is holomorphic then $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$.

(Hope, Problem 660) Complete the proof of Proposition 1.4.3 by showing that if $f \in C^1(\Omega)$ and $\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$, then f is holomorphic.

Definition 1.4.4. We let $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. If $\Omega \subseteq \mathbb{C}$ is open and $u \in C^2(\Omega)$, then u is harmonic if

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

everywhere in Ω.

(Problem 670) Show that if $f \in C^1(\Omega)$ then $\triangle f = 4\frac{\partial}{\partial z}\frac{\partial f}{\partial \bar z} = 4\frac{\partial}{\partial \bar z}\frac{\partial f}{\partial z}$.

(Micah, Problem 680) Suppose that f is holomorphic and C^2 in an open set Ω and that $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. Compute $\triangle u$ and $\triangle v$.

(Muhammad, Problem 690) Let f be a holomorphic polynomial. Show that there is a holomorphic polynomial F such that $\frac{\partial F}{\partial z} = f$. How many such polynomials are there?

(Nisa, Problem 700) Show that if u is a harmonic polynomial (of two real variables) then $u(z) = p(z) + q(\bar z)$ for some polynomials p, q of one complex variable.

Lemma 1.4.5. Let u be harmonic and real valued in $\mathbb{C}$. Suppose in addition that u is a polynomial of two real variables. Then there is a holomorphic polynomial f such that $u(x, y) = \operatorname{Re} f(x + iy)$.

(Robert, Problem 710) Prove Lemma 1.4.5. *Hint*: Start by computing $\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}$ and $\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$.