MATH 55003

Theory of Functions of a Real Variable I

Fall 2024

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I: Lebesgue Integration for Functions of a Single Real Variable

Preliminaries on Sets, Mappings, and Relations

Unions and Intersections of Sets

(Problem 10) [De Morgan] If X is a set and $\mathcal{F}$ is a family of sets, show that

$$X \sim \Big(\bigcup_{F\in\mathcal{F}} F\Big) = \bigcap_{F\in\mathcal{F}} (X \sim F).$$

(Problem 20) [De Morgan] If X is a set and $\mathcal{F}$ is a family of sets, show that

$$X \sim \Big(\bigcap_{F\in\mathcal{F}} F\Big) = \bigcup_{F\in\mathcal{F}} (X \sim F).$$

Mappings Between Sets

(Problem 30) If $f : A \to B$ is a function, and E, $F \subseteq B$, show that

$$f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F).$$

(Problem 40) If $f : A \to B$ is a function, and E, $F \subseteq B$, show that

$$f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F).$$

(Problem 50) If $f : A \to B$ is a function, and E, $F \subseteq B$, show that

$$f^{-1}(E \sim F) = f^{-1}(E) \sim f^{-1}(F).$$

(Problem 60) If $f : A \to B$ is a function, and we define $f(E) = \{f(e) : e \in E\}$ for all $E \subseteq A$, are the analogues to the above formulas true?

Equivalence Relations, the Axiom of Choice and Zorn's Lemma

[Definition: Relation] Let X be a set. A subset R of the Cartesian product $X \times X$ is called a relation on X; we write xRy if $(x, y) \in R$.

[Definition: Reflexive relation] A relation R is reflexive if xRx for all $x \in X$.

[Definition: Transitive relation] A relation R is transitive if xRy and yRz implies xRz.

[Definition: Symmetric relation] A relation R is symmetric if xRy if and only if yRx.

[Definition: Equivalence relation] A relation R is an equivalence if it is reflexive, symmetric, and transitive.

[Definition: Partial ordering] A relation R is a partial ordering if it is reflexive, transitive, and as far from symmetric as possible: if xRy and yRx then $x = y$.

[Definition: Totally ordered] Let R be a partial ordering on a set X and let $E \subseteq X$. We say that E is totally ordered if, for every x, $y \in E$, either xRy or yRx.

[Definition: Equivalence class] The equivalence class of an element x of a set X with respect to an equivalence relation R on X is $R_x = \{y \in X : xRy\}$, and $X/R = \{R_x : x \in X\}$.

[Definition: Partition] Let X be a set and let $\mathcal{P}$ be a collection of subsets of X. If, for every $x \in X$, there is exactly one $P \in \mathcal{P}$ that satisfies $x \in P$, we say that $\mathcal{P}$ is a partition of X.

(Problem 70) Let R be an equivalence relation. Show that X/R is a partition of X.

(Problem 80) Let $\mathcal{P}$ be a partition of X. Define $R = \bigcup_{P\in\mathcal{P}} P \times P$. Show that R is an equivalence relation and that xRy if and only if there is a $P \in \mathcal{P}$ with x, $y \in P$.

[Definition: Choice function] Let $\mathcal{F}$ be a nonempty family of nonempty sets and let $X = \bigcup_{F\in\mathcal{F}} F$. (We do not require that $\mathcal{F}$ be a partition of X.) A choice function on $\mathcal{F}$ is a function $f : \mathcal{F} \to X$ such that $f(F) \in F$ for each $F \in \mathcal{F}$.

Zermelo's axiom of choice. If $\mathcal{F}$ is a nonempty collection of nonempty sets, then there exists a choice function on $\mathcal{F}$.

(Problem 90) Suppose that $f : X \to Y$ is a function from one metric space to another. Suppose that $x \in X$ and that f is continuous at x in the sequential sense: for all sequences $\{x_n\}_{n=1}^{\infty} \subset X$ that satisfy $x_n \to x$, we have that $f(x_n) \to f(x)$. Use the axiom of choice to show that f is continuous in the ε-δ sense as well. Be sure to explain where you must use a choice axiom.

[Definition: Upper bound] Let R be a partial ordering on a set X and let $E \subseteq X$. If $x \in X$ and eRx for all $e \in E$, then x is an upper bound on E.

[Definition: Maximal element] Let R be a partial ordering on a set X and let $x \in X$. If $\{y \in X : xRy\} = \{x\}$, then x is said to be maximal.

[Homework 1.1] Let $\mathcal{F}$ be a family of sets.

(a) Show that $\subseteq$ and $\supseteq$ are both relations on $\mathcal{F}$ and that they are partial orderings.
(b) When is $A \in \mathcal{F}$ maximal under the relation $\subseteq$? Under the relation $\supseteq$?
(c) If $\mathcal{E} \subseteq \mathcal{F}$, when is $A \in \mathcal{F}$ an upper bound for $\mathcal{E}$ under the relation $\subseteq$? Under the relation $\supseteq$?

Zorn's lemma. Let X be a nonempty partially ordered set. Assume that, if $E \subseteq X$ is totally ordered, then E has an upper bound in X (not necessarily in E). Then X contains a maximal element.

[Definition: Cartesian product of a parameterized collection of sets] If $\{E_\lambda\}_{\lambda\in\Lambda}$ is a parameterized collection of sets, then the Cartesian product $\prod_{\lambda\in\Lambda} E_\lambda$ is defined to be the set of functions f from Λ to $\bigcup_{\lambda\in\Lambda} E_\lambda$ such that $f(\lambda) \in E_\lambda$ for all $\lambda \in \Lambda$.

(Problem 100) When is the Cartesian product of a parameterized collection of sets equal to the set of choice functions on the family of sets? Can you modify the axiom of choice so that it is equivalent to the statement that the Cartesian product of of a parameterized collection of nonempty sets is nonempty?

1. The Real Numbers: Sets, Sequences and Functions

1.1 The Field, Positivity and Completeness Axioms

[Definition: Field] A field is a set F together with two functions from $F \times F$ to itself (denoted $a + b$ and ab rather than $s((a, b))$ and $p((a, b))$) that satisfy the axioms

- $a + b = b + a$ for all $a, b \in F$,
- $a + (b + c) = (a + b) + c$ for all $a, b, c \in F$,
- There is a $0 \in F$ such that $a + 0 = a$ for all $a \in F$,
- If $a \in F$ then there is a $-a \in F$ such that $a + (-a) = 0$,
- $ab = ba$ for all $a, b \in F$,
- $a(bc) = (ab)c$ for all $a, b, c \in F$,
- There is a $1 \in F$ such that $1a = a$ for all $a \in F$,
- If $a \in F$ and $a \neq 0$ then there is a $1/a \in F$ such that $a(1/a) = 1$,
- $1 \neq 0$,
- $a(b + c) = ab + ac$ for all $a, b, c \in F$,

(Elliott, Problem 110) Show from the axioms that $0a = 0$ for all $a \in F$.

Because 0 is the additive identity, $0 + 0 = 0$ and so $0a = (0 + 0)a$. By the distributivity of multiplication over addition (and commutativity of multiplication), $(0 + 0)a = 0a + 0a$. Thus $0a = 0a + 0a$. We add $-(0a)$ to both sides to see that $0 = 0a$, as desired.

[Definition: Ordered field] A field F is an ordered field if

- F is a field,
- F is totally ordered with respect to some relation on F (which we will call $\leq$),
- If $a \geq 0$ then $(-a) \leq 0$,
- $a - b \geq 0$ if and only if $a \geq b$,
- If $a \geq 0$ and $b \geq 0$ then both $a + b \geq 0$ and $ab \geq 0$.

(Problem 120) Show that $\mathbb{Q}$ and $\mathbb{R}$ (as defined in undergraduate analysis/advanced calculus) are both ordered fields. If you have taken complex analysis, show that $\mathbb{C}$ is not an ordered field.

[Definition: Complete ordered field] An ordered field F is complete if, whenever $E \subset F$ is a nonempty subset with an upper bound, then E has a least upper bound.

[Definition: Absolute value] If x is an element of an ordered field, we define $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x \leq 0$.

The Triangle inequality. If x and y are real numbers (or rational numbers), then $|x + y| \leq |x| + |y|$.

[Definition: Extended real numbers] We introduce the symbols $\infty = +\infty$ and $-\infty$ denoting two objects that are not in $\mathbb{R}$, and let the extended real numbers be $\mathbb{R} \cup \{-\infty, \infty\}$. (The extended real numbers are not a field!)

- We extend the relation $\leq$ by stating $-\infty \leq r \leq \infty$ for all $r \in \mathbb{R}$.
- We extend the operation $+$ by writing

$$\infty + r = \infty, \quad -\infty + r = -\infty, \quad \infty + \infty = \infty, \quad -\infty + (-\infty) = -\infty$$

  for all $r \in \mathbb{R}$. $(\infty + (-\infty)$ and $(-\infty) + \infty$ are not defined.)
- If a, $b \in \mathbb{R} \cup \{-\infty, \infty\}$, then

$$(a, b) = \{r \in \mathbb{R} \cup \{-\infty, \infty\} : a < r < b\}, \quad [a, b) = \{r \in \mathbb{R} \cup \{-\infty, \infty\} : a \leq r < b\},$$
$$(a, b] = \{r \in \mathbb{R} \cup \{-\infty, \infty\} : a < r \leq b\}, \quad [a, b] = \{r \in \mathbb{R} \cup \{-\infty, \infty\} : a \leq r \leq b\}.$$

  We may write $\mathbb{R} \cup \{-\infty, \infty\} = [-\infty, \infty]$.

(Irina, Problem 130) Show that every subset of the extended real numbers has a supremum in the extended real numbers. (A similar argument shows that every subset of the extended real numbers has an infimum in the extended real numbers.)

Let $X \subset [-\infty, \infty]$. There are then four cases:

- $X = \emptyset$ or $X = \{-\infty\}$. Then $-\infty$ is an upper bound for X, because $x \leq -\infty$ for all $x \in X$. Furthermore, if $y \in [-\infty, \infty]$ then $-\infty \leq y$, and so $-\infty \leq y$ for all upper bounds y of X. Thus $-\infty$ is the least upper bound on X.
- For every $r \in \mathbb{R}$ there is an $x \in X$ with $x > r$. In particular there is an $x \in X$ with $x > 0 > -\infty$, and so $-\infty$ is not an upper bound for X. Furthermore, no element of $\mathbb{R}$ is an upper bound for X. However, ∞ is an upper bound for $[-\infty, \infty]$, so is an upper bound for X; as it is the only upper bound
- None of the preceding cases is valid. Thus, there is some $r \in R$ such that $x \leq r$ for all $x \in X$. Furthermore, $X \neq \emptyset$ and $X \neq \{-\infty\}$. In this case $\infty \notin X$ (because $\infty \not< r$). Thus, $X \not\subseteq \{-\infty, \infty\}$, and so $X \cap \mathbb{R} \neq \emptyset$. r is an upper bound for $X \cap \mathbb{R}$, so by the completeness axiom $X \cap \mathbb{R}$ has a (real) least upper bound s. If $X = X \cap \mathbb{R}$ we are done; otherwise, $X \cap \mathbb{R} = X \sim \{-\infty\}$. s is an upper bound for X because $s > -\infty$, and is the least upper bound for X because it is the least upper bound for a subset of X. This completes the proof.

1.2 The Natural and Rational Numbers

[Definition: Inductive set] Suppose that F is a field and that $E \subseteq F$. Suppose furthermore that $1 \in E$ and that, if $x \in E$, then $x + 1 \in E$. Then we say that E is inductive.

[Definition: Natural numbers] If F is a field, we define the natural numbers $\mathbb{N}_F$ in F as the intersection of all inductive subsets of F.

(Zach, Problem 140) If F is an ordered field, show that $1 \in \mathbb{N}_F$ and that $f \notin \mathbb{N}_F$ for all $f < 1$.

Let $S = \{x \in F : x \geq 1\}$. Then $1 \in S$. Furthermore, if $x \in F$ then $x \geq 1$. Because $1 > 0$, we have that $x + 1 > x \geq 1$ and so $x + 1 \in S$. Thus S is inductive, and so $\mathbb{N}_F \subseteq S$. Because $\geq$ is a partial ordering, if $f < 1$ then $f \not\geq 1$ and so $f \notin S$, so $f \notin \mathbb{N}_F$, as desired.

(Problem 141) If n, $m \in \mathbb{N}_F$, show that $n + m \in \mathbb{N}_F$. Phrase your proof in terms of inductive sets rather than in terms of induction as done in undergraduate mathematics.

[Chapter 1, Problem 8] If F is an ordered field and $n \in \mathbb{N}_F$, then $\mathbb{N}_F \cap (n, n + 1) = \emptyset$.

[Chapter 1, Problem 9] If F is an ordered field and n, $m \in \mathbb{N}_F$ with $n > m$, then $n - m \in \mathbb{N}_F$.

Theorem 1.1. Let F be an ordered field and let $E \subseteq \mathbb{N}_F$. Suppose that $E \neq \emptyset$. Then E has a smallest element.

(Juan, Problem 150) Prove Theorem 1.1. For bonus points, prove this in a general ordered field; the proof in your book is only valid in complete ordered fields.

Corollary. Let $\mathbb{Z}_F = \{n - m : n, m \in \mathbb{N}_F\}$. If $S \subseteq \mathbb{Z}_F$ and S is bounded above (respectively, below) then S contains a maximal (respectively, minimal) element.

The Archimedean property. An ordered field F has the Archimedean property if, for every $a \in F$, there is a $n \in \mathbb{N}_F$ with $n > a$.

(Micah, Problem 160) Let F be a complete ordered field. Prove that F has the Archimedean property.

Suppose that F does not have the Archimedian property. Then there is some $a \in F$ fuch that $n \leq a$ for all $n \in \mathbb{N}_F$.

Thus a is an upper bound for $\mathbb{N}_F$. Because F is complete, there is a least upper bound c of $\mathbb{N}_F$. Consider $c - 1$. Then $c - 1 < c$, and so $c - 1$ is not an upper bound for $\mathbb{N}_F$, and so there is some $m \in \mathbb{N}_F$ such that $c - 1 < m$. But then $c < m + 1$, and so c is not an upper bound for $\mathbb{N}_F$. This is a contradiction.

[Potential homework problem] Let F and R be two complete ordered fields. Show that there is a unique bijection $\varphi : F \to R$ that satisfies

- $\varphi(a + b) = \varphi(a) + \varphi(b)$,
- $\varphi(ab) = \varphi(a)\varphi(b)$,
- If $a \leq b$ then $\varphi(a) \leq \varphi(b)$.

Thus (up to isomorphism) there is only one complete ordered field.

[Definition: Dense] Let F be an ordered field. A subset S of F is *dense* if, whenever $a, b \in F$ with $a < b$, there is a $s \in S$ with $a < s < b$.

Theorem 1.2. The rational numbers $\mathbb{Q}$ are dense in the real numbers $\mathbb{R}$.

(Muhammad, Problem 170) Prove Theorem 1.2.

Since $b - a > 0$, we have that $\frac{2}{b-a} > 0$. Thus there is a $q \in \mathbb{N}$ such that $q > \frac{1}{b-a}$ (and thus $1/q < b - a$) by the Archimedean property.

Let $S = \{n \in \mathbb{Z} : n < qb\}$. S is clearly bounded above, so by (the corollary to) Theorem 1.1 we have that S contains a maximal element p.

Thus $p < qb$. Because $q \in \mathbb{N}$, we have that $q > 0$ and so $\frac{p}{q} < b$. Furthermore, $p + 1$ is not in S and so $p + 1 \geq qb$, that is, $\frac{p+1}{q} \geq b$. But then $\frac{p}{q} = \frac{p+1}{q} - \frac{1}{q} \geq b - \frac{1}{q} > b - (b - a) = a$, and so $a < \frac{p}{q} < b$, as desired.

The axiom of dependent choice

The axiom of countable choice. Let X be a set. For each $n \in \mathbb{N}$, let $E_n \subseteq X$. Then there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that, for each $n \in \mathbb{N}$, we have that $x_n \in E_n$.

The axiom of dependent choice. Let X be a set and let R be a relation on X. Suppose that for each $x \in X$, there is at least one $y \in X$ such that xRy.

If $x \in X$, then there is a sequence (a function from $\mathbb{N}$ to X) such that $x_1 = x$ and such that $x_n R x_{n+1}$ for all $n \in \mathbb{N}$.

The Bolzano-Weierstraß theorem. If $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence of points in $\mathbb{R}$, then there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ that converges.

(Bonus Problem 171) Use the axiom of dependent choice to prove the Bolzano-Weierstraß theorem. Can you do this using only the axiom of countable choice?

(Bonus Problem 172) Show that the axiom of dependent choice implies the axiom of countable choice.

(Bonus Problem 180) Use Zorn's lemma to prove the axiom of dependent choice.

(Problem 181) Choose a standard inductive proof from undergraduate analysis and rephrase it in terms of inductive sets.

1.3 Countable and Uncountable Sets

[Definition: Finite] A subset of $\mathbb{N}$ is finite if it is bounded above. An arbitrary set S is finite if there is a bijection f from S to a finite subset of $\mathbb{N}$.

(Memory 190) If S is finite, then there is a $m \in \mathbb{N}$ and a bijection $f : S \to \{k \in \mathbb{N} : k \leq m\}$.

(Memory 200) If S is a set, T is a finite set, and there exists either

- An injection $g : S \to T$,
- A surjection $h : T \to S$,

then S is finite.

[Definition: Countable] A set S is countable if there exists an injection $g : S \to \mathbb{N}$.

(Memory 210) S is countable if and only if there exists a surjection $h : \mathbb{N} \to S$.

(Memory 220) All finite sets are countable.

[Definition: Countably infinite] A set S is countably infinite if it is countable but not finite.

(Memory 230) S is countably infinite if and only if there exists a bijection $f : S \to \mathbb{N}$.

(Memory 240) $\mathbb{Q}$ is countable.

(Memory 250) The countable union of countable sets is countable.

[Definition: Uncountable] A set is uncountable if it is not countable.

(Memory 260) The real numbers are uncountable. In fact, if $I \subseteq \mathbb{R}$ is an interval, then either $I = \emptyset$, $I = \{a\}$ is a single point, or I is uncountable.

(Anjuman, Problem 270) Let S be a set. Let 2^S (or $P(S)$) denote the set of all subsets of S. Show that there does not exist a bijection $f : S \to 2^S$.

1.4 Open Sets and Closed Sets of Real Numbers

[Definition: Open interval] A subset I of $\mathbb{R}$ is an open interval if there exist a, $b \in [-\infty, \infty]$ with $a \leq b$ such that $I = (a, b) = \{r \in \mathbb{R} : a < r < b\}$.

(Problem 280) Is the empty set an open interval?

[Definition: Open set] A subset $\mathcal{O}$ of $\mathbb{R}$ is open if it is the union of open intervals.

Proposition 1.9. In $\mathbb{R}$ (but not in other metric spaces), every open set may be written as a union of *countably* many open intervals that are *pairwise-disjoint*.

[Definition: Closed set] A set in $\mathbb{R}$ is closed if its complement is open.

[Definition: Closure] If $E \subset \mathbb{R}$, then $\overline{E} = \bigcap_{F : F \text{ closed}, E \subseteq F} F$.

(Memory 290) If $E \subseteq \mathbb{R}$ then $\overline{E}$ is closed. If $E \subseteq \mathbb{R}$ is closed then $E = \overline{E}$.

(Memory 300) If $E \subseteq \mathbb{R}$ then $\overline{E} = \{r \in \mathbb{R} : \text{if } \varepsilon > 0 \text{ then there is an } e \in E \text{ with } |r - e| < \varepsilon\}$.

The nested set theorem. If $\{F_n\}_{n=1}^{\infty}$ is a sequence of sets such that $F_n \supseteq F_{n+1}$, F_n is compact, and $F_n \neq \emptyset$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

[Definition: σ-algebra] Let X be a set and let $\mathcal{A} \subseteq 2^X$. We say that $\mathcal{A}$ is a σ-algebra of subsets of X, or a σ-algebra over X, if

(a) $\emptyset \in \mathcal{A}$.
(b) If $S \in \mathcal{A}$ then $X \sim S \in \mathcal{A}$.
(c) If $\mathcal{B} \subseteq \mathcal{A}$ is countable then $\bigcup_{S \in \mathcal{B}} S$ is in $\mathcal{A}$.

(Problem 310) Show that 2^X and $\{X, \emptyset\}$ are both σ-algebras over X.

(Ashley, Problem 320) Suppose that $\mathcal{A}$ is a σ-algebra and that $\mathcal{B} \subseteq \mathcal{A}$ is countable. Show that $\bigcap_{S \in \mathcal{B}} S$ is in $\mathcal{A}$.

(Problem 321) Let $\mathcal{G}$ be a collection of σ-algebras over X. Let $\mathcal{A} = \bigcap_{\mathcal{B} \in \mathcal{G}} \mathcal{B}$. Show that $\mathcal{A}$ is also a σ-algebra over X.

(Problem 322) Is the same true for unions of σ-algebras?

Proposition 1.13. Let $\mathcal{F} \subseteq 2^X$. Let

$$\mathcal{S} = \{\mathcal{A} \subseteq 2^X : \mathcal{F} \subseteq \mathcal{A},\ \mathcal{A} \text{ is a } \sigma\text{-algebra}\}.$$

Let $\mathcal{A}_\mathcal{F} = \bigcap_{\mathcal{A} \in \mathcal{S}} \mathcal{A}$. Then $\mathcal{A}_\mathcal{F}$ is a σ-algebra.

Furthermore, $\mathcal{F} \subseteq \mathcal{A}_\mathcal{F}$ and if $\mathcal{A}$ is a σ-algebra with $\mathcal{F} \subseteq \mathcal{A}$ then $\mathcal{A}_\mathcal{F} \subseteq \mathcal{A}$.

We call $\mathcal{A}_\mathcal{F}$ the smallest σ-algebra that contains $\mathcal{F}$.

(Bashar, Problem 330) Prove Proposition 1.13.

(Dibyendu, Problem 340) Let $\mathcal{A}$ be a σ-algebra on X and let $\{E_n\}_{n=1}^\infty \subseteq \mathcal{A}$. Let

$$\limsup_{n\to\infty} E_n = \{x \in X : x \in E_n \text{ for infinitely many values of } n\}.$$

Show that $\limsup_{n\to\infty} E_n \in \mathcal{A}$.

The statement that $x \in E_n$ for infinitely many values of n is equivalent to the statement that, for all $m \in \mathbb{N}$, there is a $n \geq m$ such that $x \in E_n$.

The statement that there is a $n \geq m$ such that $x \in E_n$ is equivalent to the statment that $x \in \bigcup_{n=m}^\infty E_n$.

Thus, the statement that $x \in E_n$ for infinitely many values of n is equivalent to the statement that, for all $m \in \mathbb{N}$, $x \in \bigcup_{n=m}^\infty E_n$.

But for any sequence of sets $\{S_m\}_{m=1}^\infty$, $x \in S_m$ for all $m \in \mathbb{N}$ if and only if $x \in \bigcap_{m=1}^\infty S_m$.

Thus,

$$\limsup_{n\to\infty} E_n = \bigcap_{k=1}^\infty \left[\bigcup_{n=k}^\infty E_n\right]$$

and the result that $\limsup_{n\to\infty} E_n \in \mathcal{A}$ follows immediately from the definition of a σ-algebra and from Problem 320.

(Elliott, Problem 350) Let $\mathcal{A}$ be a σ-algebra on X and let $\{E_n\}_{n=1}^\infty \subseteq \mathcal{A}$. Let

$$\liminf_{n\to\infty} E_n = \{x \in X : x \notin E_n \text{ for at most finitely many values of } n\}.$$

Show that $\liminf_{n\to\infty} E_n \in \mathcal{A}$.

A similar argument to that of the previous problem yields that

$$\liminf_{n\to\infty} E_n = \bigcup_{k=1}^\infty \left[\bigcap_{n=k}^\infty E_n\right]$$

and so again the result follows from the definition of a σ-algebra and from Problem 320.

[Definition: Borel sets] The collection $\mathcal{B}$ of Borel subsets of $\mathbb{R}$ is the smallest σ-algebra containing all open subsets of $\mathbb{R}$.

1.5 Sequences of Real Numbers

[We will assume that all students saw all material in this section in advanced calculus or real analysis.]

1.6 Continuous Real-Valued Functions of a Real Variable

[We will assume that all students saw all material in this section in advanced calculus or real analysis.]

2. Lebesgue Measure

2.1 Introduction

[Definition: Characteristic function] Let $E \subset \mathbb{R}$ be a set. The characteristic function of E is defined to be

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \in \mathbb{R} \sim E. \end{cases}$$

[Definition: Jordan content] Let $E \subset \mathbb{R}$ be a bounded set. If χ_E is Riemann integrable, then we say that E is *Jordan measurable* and that its Jordan content is $\mathcal{J}(E) = \int_{-\infty}^{\infty} \chi_E$.

(Irina, Problem 360) Suppose that $\{E_n\}_{n=1}^m$ is a finite collection of pairwise disjoint Jordan measurable sets. Show that $\cup_{n=1}^m E_n$ is also Jordan measurable and that

$$\mathcal{J}\Big(\bigcup_{n=1}^m E_n\Big) = \sum_{n=1}^m \mathcal{J}(E_n).$$

(Zach, Problem 370) Find a bounded set E that is not Jordan measurable, but such that $E = \bigcup_{n=1}^\infty E_n$, where $\{E_n\}_{n=1}^\infty$ is a sequence of pairwise disjoint Jordan measurable sets.

Let $E = \mathbb{Q} \cap [0,1]$. Then E is countable and so is the union of countably many pairwise disjoint singleton sets. It is an elementary real analysis argument to show that singleton sets are Jordan measurable but that E is not.

[Definition: Measure] If X is a set and $\mathcal{M}$ is a σ-algebra of subsets of X, then we call $(X, \mathcal{M})$ a measurable space. A measure on a measurable space $(X, \mathcal{M})$ is a function μ such that:

- $\mu : \mathcal{M} \to [0, \infty]$.
- $\mu(\emptyset) = 0$,
- If $\{E_k\}_{k=1}^\infty \subseteq \mathcal{M}$ and $E_k \cap E_j = \emptyset$ for all $j \neq k$, then

$$\mu\Big(\bigcup_{k=1}^\infty E_k\Big) = \sum_{k=1}^\infty \mu(E_k).$$

(Recall that if $\mathcal{M}$ is a σ-algebra then $\emptyset \in \mathcal{M}$.)

[Chapter 2, Problem 1] If μ is a measure on a σ-algebra $\mathcal{M}$, and if A, $B \in \mathcal{M}$ with $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

[Chapter 2, Problem 2] We may replace the second condition by the condition "$\mu(E) < \infty$ for at least one $E \in \mathcal{M}$".

[Chapter 2, Problem 3] If μ is a measure on a σ-algebra $\mathcal{M}$, and if $\{E_k\}_{k=1}^\infty \subseteq \mathcal{M}$, then

$$\mu\Big(\bigcup_{k=1}^\infty E_k\Big) \leq \sum_{k=1}^\infty \mu(E_k).$$

[Chapter 2, Problem 4] Let $(X, \mathcal{M})$ be a measurable space. If $E \in \mathcal{M}$, let $c(E) = \infty$ if E is infinite and $c(E) = \#E$ if E is finite. Then c is a measure on $(X, \mathcal{M})$.

(Juan, Problem 380) Let $(X, \mathcal{M})$ be a measurable space. There are functions from $\mathcal{M}$ to $\{0, \infty\}$ that are measures. Find two of them. (These are the trivial measures on $\mathcal{M}$.)

[Definition: Length] Let $I \subseteq \mathbb{R}$ be an interval, so $I = (a, b)$, $[a, b]$, $[a, b)$, or $(a, b]$ for some a, $b \in [-\infty, \infty]$ with $a \leq b$. We define $\ell(I) = b - a$.

[Definition: Outer measure] Let $A \subseteq \mathbb{R}$. The Lebesgue outer measure of A is

$$m^*(A) = \inf\Big\{\sum_{k=1}^{\infty} \ell(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k, \text{ each } I_k \text{ is a bounded open interval}\Big\}.$$

(Problem 390) Let $A \subseteq B \subseteq \mathbb{R}$. Show that $m^*(A) \leq m^*(B)$.

(Problem 391) Let $S \subset \mathbb{R}$ be a finite set. Show that $m^*(S) = 0$.

(Micah, Problem 400) Let $S \subset \mathbb{R}$ be a countably infinite set. Show that $m^*(S) = 0$. In particular, $m^*(\mathbb{Q}) = 0$.

Proposition 2.1. The outer measure of an interval in $\mathbb{R}$ is its length.

(Muhammad, Problem 410) Let $I = [a, b]$ be a closed bounded interval (that is, a and b are both real numbers). Show that $m^*(I) = \ell(I) = b - a$.

(Anjuman, Problem 420) Prove Proposition 2.1.

Proposition 2.2. Outer measure is translation invariant: if $E \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, then

$$m^*(E) = m^*\big(\{e + r : e \in E\}\big).$$

(Ashley, Problem 430) Prove Proposition 2.2.

Let $\{I_k\}_{k=1}^{\infty}$ be a countable sequence of bounded open intervals that satisfies $E \subseteq \bigcup_{k=1}^{\infty} I_k$. There are real numbers a_k, b_k with $a_k \leq b_k$ and $I_k = (a_k, b_k)$.

Let $J_k = (a_k + r, b_k + r)$.

If $f \in E_r = \{e + r : e \in E\}$, then $f = e + r$ for some $e \in E$. Because $E \subseteq \bigcup_{k=1}^{\infty} I_k$, we have that $e \in I_k$ (that is, $a_k < e < b_k$) for at least one value of k. Then $a_k + r < e + r < b_k + r$, so $f \in J_k$. Thus $E_r \subset \bigcup_{k=1}^{\infty} J_k$. But $\{J_k : k \in \mathbb{N}\}$ is a countable collection of bounded open intervals, and so by definition of Lebesgue outer measure

$$m^*(E_r) \leq \sum_{k=1}^{\infty} \ell(J_k) = \sum_{k=1}^{\infty} (b_k - a_k) = \sum_{k=1}^{\infty} \ell(I_k).$$

Taking the infimum of both sides over the set of all such $\{I_k\}_{k=1}^{\infty}$, we see that

$$m^*(E_r) \leq m^*(E).$$

But $E = (E_r)_{-r}$, and so by the previous argument $m^*(E) = m^*((E_r)_{-r}) \leq m^*(E_r)$. Thus $m^*(E) = m^*(E_r)$, as desired.

(Problem 431) If $E \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, then

$$|r| m^*(E) = m^*\big(\{re : e \in E\}\big).$$

Proposition 2.3. If for each $k \in \mathbb{N}$ we have a set $E_k \subseteq \mathbb{R}$, then

$$m^*\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

(Bashar, Problem 440) Prove Proposition 2.3.

If $\sum_{k=1}^{\infty} m^*(E_k) = \infty$ we are done because $m^*(S) \in [0,\infty]$ for all $S \subseteq \mathbb{R}$. Therefore we assume that $\sum_{k=1}^{\infty} m^*(E_k) < \infty$. In particular, $m^*(E_k) < \infty$ for all $k \in \mathbb{N}$.

Let $\varepsilon > 0$. By definition of infimum and by definition of m^*, for each k, there is a sequence $\{I_{k,\ell}\}_{\ell=1}^{\infty}$ of bounded open intervals that satisfies

$$E_k \subseteq \bigcup_{\ell=1}^{\infty} I_{k,\ell}, \quad \sum_{\ell=1}^{\infty} \ell(I_{k,\ell}) \leq m^*(E_k) + 2^{-k}\varepsilon.$$

Now, $\mathcal{B} = \{I_{k,\ell} : k, \ell \in \mathbb{N}\}$ is the union of countably many collections of countably many bounded open intervals, and so by Problem 250 $\mathcal{B}$ is a countable set of bounded open intervals.

Furthermore, if $x \in \bigcup_{k=1}^{\infty} E_k$ then $x \in E_k$ for some k, and so $x \in I_{k,\ell}$ for some ℓ. Thus $E_k \subset \bigcup_{I \in \mathcal{B}} I$.

Thus by definition of m^*,

$$m^*\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) \leq \sum_{I \in \mathcal{B}} \ell(I).$$

By definition of $\mathcal{B}$

$$m^*\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) \leq \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \ell(I_{k,\ell}).$$

By definition of $I_{k,\ell}$,

$$m^*\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) \leq \sum_{k=1}^{\infty}\bigl(m^*(E_k) + 2^{-k}\varepsilon\bigr) = \varepsilon + \sum_{k=1}^{\infty} m^*(E_k).$$

Because this is true for all $\varepsilon > 0$, we must have that

$$m^*\Bigl(\bigcup_{k=1}^{\infty} E_k\Bigr) \leq \sum_{k=1}^{\infty} m^*(E_k)$$

as desired.

[Chapter 2, Problem 9] If A, $B \subseteq \mathbb{R}$ and $m^*(A) = 0$, then $m^*(A \cup B) = m^*(A) + m^*(B)$.

[Chapter 2, Problem 10] Let A, $B \subset \mathbb{R}$ be bounded. Suppose that $\inf\{|a-b| : a \in A, b \in B\} > 0$. Show that $m^*(A \cup B) = m^*(A) + m^*(B)$.

2.6 Vitali's Example of a Nonmeasurable Set

[Definition: Rationally equivalent] If r, $s \in \mathbb{R}$, we say that r and s are rationally equivalent if $r - s \in \mathbb{Q}$.

(Dibyendu, Problem 450) Show that rational equivalence is an equivalence relation.

Let R denote rational equivalence.

- Suppose rRs. Then $r - s \in \mathbb{Q}$. Because $\mathbb{Q}$ is a field, $-(r-s) = s - r \in \mathbb{Q}$ and so sRr. Thus R is symmetric.
- Suppose $r \in \mathbb{R}$. Then $r - r = 0 \in \mathbb{Q}$ and so rRr. Thus R is reflexive.
- Suppose rRs and sRt. Then $r - s \in \mathbb{Q}$ and $s - t \in \mathbb{Q}$. Because $\mathbb{Q}$ is a field, $r - t = (r-s) + (s-t) \in \mathbb{Q}$ and so rRt. Thus R is transitive.

Because R is symmetric, reflexive, and transitive, it is an equivalence relation.

(Problem 451) Let R denote rational equivalence. Let $[-1,1]/R$ be the set of equivalence classes in $[-1,1]$ under R. This is a collection of (pairwise disjoint) sets.

If $S \in [-1,1]/R$, is S finite, countably infinite, or uncountable?

Let $x \in S$. Then $S = \{y \in [-1,1] : xRy\} = \{y \in [-1,1] : y = x + q \text{ for some } q \in \mathbb{Q}\}$. S is clearly infinite. Conversely, $S \subset \{x + q : q \in \mathbb{Q}\}$. This set is countable because it has a natural bijection to the countable set $\mathbb{Q}$. Thus S is countably infinite.

(Problem 452) Is the collection of equivalence classes $[-1,1]/R$ finite, countably infinite, or uncountable?

We have that

$$[-1,1] = \bigcup_{S\in[-1,1]/R} S.$$

The right hand side is uncountable but each S on the left hand side is countable. The countable union of countable sets is countable, and so we must have that $[-1,1]/R$ is uncountable.

(Problem 453) Let φ be a choice function on $[-1,1]/R$ and let $V = \varphi([-1,1]/R)$. Is V countable or uncountable?

The elements of $[-1,1]/R$ are pairwise disjoint because R is an equivalence relation. Thus, if $\varphi(S) = \varphi(T)$, then $\varphi(T) \in S\cap T$ because φ is a choice function and so $\varphi(S)\in S$, $\varphi(T)\in T$. Thus $S\cap T\neq\emptyset$ and so $S = T$. Thus φ is a bijection from $[-1,1]/R$ to V. Because $[-1,1]/R$ is uncountable, so is V.

(Elliott, Problem 460) If q is rational, define $V_q = \{v+q : v\in V\}$. Show that if $q, p\in\mathbb{Q}\cap[-2,2]$ then either $q = p$ or $V_q\cap V_p = \emptyset$.

Let $p, q\in\mathbb{Q}\cap[-2,2]$ and suppose that $V_q\cap V_p\neq\emptyset$. Let $x\in V_q\cap V_p$. Then $x = v+q = w+p$ for some $v, w\in V$ by definition of V_q. We must have that $v = \varphi(S)$, $w = \varphi(T)$ for some $S, T\in[-1,1]/R$ by definition of V.

Thus $v-w = p-q\in\mathbb{Q}$ and so vRw. Because φ is a choice function, $v = \varphi(S)\in S$. Because vRw, we must also have $w\in S$. But $w = \varphi(T)\in T$. As in the previous problem this implies $S = T$, and so $v = \varphi(S) = \varphi(T) = w$. Thus $0 = v-w = p-q$ and so $p = q$, as desired.

(Irina, Problem 470) Show that

$$[-1,1]\subseteq\bigcup_{q\in[-2,2]\cap\mathbb{Q}} V_q\subseteq[-3,3].$$

(Zach, Problem 480) Show that $m^*(V) > 0$.

By Proposition 2.1 and Problem 390, we have that

$$2 = m^*([-1,1]) \leq m^*\Bigl(\bigcup_{q\in[-2,2]\cap\mathbb{Q}} V_q\Bigr).$$

By Proposition 2.3, and because the rational numbers are countable, we have that

$$2\leq\sum_{q\in[-2,2]\cap\mathbb{Q}} m^*(V_q).$$

By Proposition 2.2, we have that $m^*(V_p) = m^*(V_q)$ for all $p, q\in\mathbb{Q}$. Thus

$$2\leq\sum_{q\in[-2,2]\cap\mathbb{Q}} m^*(V).$$

The right hand side is either zero (if $m^*(V) = 0$) or ∞ (if $m^*(V) > 0$). The first possibility is precluded by the positive lower bound, and thus we must have that $m^*(V) > 0$.

(Juan, Problem 490) Let $\{q_k\}_{k=1}^\infty$ be a sequence that contains each rational number in $[-2,2]$ exactly once and contains no other numbers. Show that $\sum_{k=1}^\infty m^*(V_{q_k}) \neq m^*\Bigl(\bigcup_{k=1}^\infty V_{q_k}\Bigr)$.

(Micah, Problem 500) Show that there exist two disjoint sets A and B such that $m^*(A\cup B)\neq m^*(A)+m^*(B)$.

[Definition: Measurable set] Let $E \subseteq \mathbb{R}$. We say that E is Lebesgue measurable (or just measurable) if, for all $A \subseteq \mathbb{R}$, we have that

$$m^*(A) = m^*(A \cap E) + m^*(A \sim E).$$

(Problem 501) Show that $E \subseteq \mathbb{R}$ is measurable if and only if, for all $A \subseteq \mathbb{R}$, we have that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \sim E).$$

This is an easy consequence of Proposition 2.3.

(Problem 510) Show that the complement of a measurable set is measurable.

(Problem 520) Show that the empty set is measurable.

Proposition 2.10. Let $E \subseteq \mathbb{R}$ be a measurable set. Let $y \in \mathbb{R}$. Define $E + y = \{e + y : e \in E\}$. Show that $E + y$ is also measurable.

(Muhammad, Problem 530) Prove Proposition 2.10.

(Problem 531) Let $E \subseteq \mathbb{R}$ be a measurable set. Let $r \in \mathbb{R}$. Define $rE = \{re : e \in E\}$. Show that rE is also measurable.

Proposition 2.4. Let $E \subset \mathbb{R}$. Suppose that $m^*(E) = 0$. Then E is measurable.

(Anjuman, Problem 540) Prove Proposition 2.4.

Proposition 2.5. The union of finitely many measurable sets is measurable.

(Ashley, Problem 550) Prove Proposition 2.5.

Let $S = \{n \in \mathbb{N}$:the union of n measurable sets is measurable$\}$. It is trivially true that $1 \in S$. We will now show that S is inductive and thus $S = \mathbb{N}$; this will complete the proof.

Suppose that $n \in S$. Let $\{E_k\}_{k=1}^{n+1}$ be a set of $n+1$ measurable sets. Let $F = \bigcup_{k=1}^{n} E_k$; because $n \in S$ we have that F is measurable. Let $G = E_{n+1}$; by assumption G is measurable. We need only show that $F \cup G = \bigcup_{k=1}^{n+1}$ is measurable; because $\{E_k\}_{k=1}^{n+1}$ was arbitrary this will imply that $n+1 \in S$, which will show that S is inductive and thus complete the proof.

Let $A \subseteq \mathbb{R}$. Applying the measurability of F yields that

$$m^*(A) = m^*(A \cap F) + m^*(A \cap F^C)$$

where the superscript C denotes the complement. Applying the measurability of G yields that

$$m^*(A) = m^*(A \cap F) + m^*(G \cap (A \cap F^C)) + m^*((A \cap F^C) \cap G^C).$$

Set intersection is associative and so $(A \cap F^C) \cap G^C = A \cap (F^C \cap G^C)$. Recall that $F^C = \mathbb{R} \sim F$. By De Morgan's laws (Problem 10),

$$F^C \cap G^C = (\mathbb{R} \sim F) \cap (\mathbb{R} \sim G) \quad = \mathbb{R} \sim (F \cup G) = (F \cup G)^C.$$

Thus

$$m^*(A) = m^*(A \cap F) + m^*(G \cap (A \cap F^C)) + m^*(A \cap (F \cup G)^C).$$

By Proposition 2.2,

$$m^*(A \cap F) + m^*(G \cap (A \cap F^C)) \geq m^*((A \cap F) \cup (G \cap (A \cap F^C))).$$

Again using associativity of intersection,

$$G \cap (A \cap F^C) = A \cap (G \cap F^C).$$

Because unions distribute over intersections,

$$(A \cap F) \cup (A \cap (G \cap F^C)) = A \cap (F \cup (G \cap F^C)).$$

But $F \cup (G \cap F^C) = F \cup G$ and so

$$m^*(A) \geq m^*(A \cap (F \cup G)) + m^*(A \cap (F \cup G)^C).$$

By Proposition 2.2, $m^*(A) \leq m^*(A \cap (F \cup G)) + m^*(A \cap (F \cup G)^C)$, and so we must have that

$$m^*(A) = m^*(A \cap (F \cup G)) + m^*(A \cap (F \cup G)^C)$$

for any set A. Thus $F \cup G$ is measurable, as desired.

Proposition 2.6. If $\{E_k\}_{k=1}^n$ is a collection of finitely many measurable pairwise disjoint sets, then

$$m^*\Bigl(\bigcup_{k=1}^n E_k\Bigr) = \sum_{k=1}^n m^*(E_k).$$

More generally, if $A \subseteq \mathbb{R}$ then

$$m^*\Bigl(\bigcup_{k=1}^n (A \cap E_k)\Bigr) = \sum_{k=1}^n m^*(A \cap E_k).$$

(Bashar, Problem 560) Prove Proposition 2.6.

Let $A \subseteq \mathbb{R}$. We prove by induction. The base case $n = 1$ is trivially true.

Suppose that the proposition is true for some n, that is, that

$$m^*\Bigl(\bigcup_{k=1}^n (A \cap E_k)\Bigr) = \sum_{k=1}^n m^*(A \cap E_k)$$

whenever $\{E_k\}_{k=1}^n$ is a collection of n measurable sets. Let $\{E_k\}_{k=1}^{n+1}$ be a collection of $n+1$ measurable sets.

Because E_{n+1} is measurable, we have that

$$m^*(A \cap \bigcup_{k=1}^{n+1} E_k) = m^*\Bigl(\Bigl(A \cap \bigcup_{k=1}^{n+1} E_k\Bigr) \cap E_{n+1}\Bigr) + m^*\Bigl(\Bigl(A \cap \bigcup_{k=1}^{n+1} E_k\Bigr) \sim E_{n+1}\Bigr).$$

Now,

$$\Bigl(A \cap \bigcup_{k=1}^{n+1} E_k\Bigr) \cap E_{n+1} = A \cap \Bigl(\bigcup_{k=1}^{n+1} E_k \cap E_{n+1}\Bigr) = A \cap E_{n+1}$$

because the E_ks are pairwise disjoint.

Similarly,

$$\Bigl(A \cap \bigcup_{k=1}^{n+1} E_k\Bigr) \sim E_{n+1} = \Bigl(A \cap \bigcup_{k=1}^{n+1} E_k\Bigr) \cap E_{n+1}^C = A \cap \Bigl(\bigcup_{k=1}^{n+1} E_k \cap E_{n+1}^C\Bigr) = A \cap \Bigl(\bigcup_{k=1}^{n} E_k\Bigr).$$

Thus

$$m^*(A \cap \bigcup_{k=1}^{n+1} E_k) = m^*(A \cap E_{n+1}) + m^*\Bigl(A \cap \Bigl(\bigcup_{k=1}^{n} E_k\Bigr)\Bigr).$$

By our induction hypothesis, the second term is equal to

$$\sum_{k=1}^n m^*(A \cap E_k)$$

and so

$$m^*(A \cap \bigcup_{k=1}^{n+1} E_k) = \sum_{k=1}^{n+1} m^*(A \cap E_k).$$

A standard inductive argument completes the proof.

(Problem 561) Show that the intersection of finitely many measurable sets is measurable.

Let $\{E_k\}_{k=1}^n$ be a collection of finitely many measurable sets.
By Problem 510, each $E_k^C = \mathbb{R} \sim E_k$ is measurable.
By Proposition 2.4, $\bigcup_{k=1}^n E_k^C$ is measurable.
Again by Problem 510, $\mathbb{R} \sim \left(\bigcup_{k=1}^n E_k^C\right)$ is measurable.
Finally, by Problem 10,

$$\mathbb{R} \sim \left(\bigcup_{k=1}^n E_k^C\right) = \bigcap_{k=1}^n (\mathbb{R} \sim E_k^C) = \bigcap_{k=1}^n E_k$$

and so $\bigcap_{k=1}^n E_k$ is measurable.

Proposition 2.13. Let $\{E_k\}_{k=1}^\infty$ be a sequence of pairwise disjoint measurable sets. Then

$$m^*\Bigl(\bigcup_{k=1}^\infty E_k\Bigr) = \sum_{k=1}^\infty m^*(E_k).$$

[Chapter 2, Problem 26]If $A \subseteq \mathbb{R}$ and $\{E_k\}_{k=1}^\infty$ is as in Proposition 2.13, then

$$m^*\Bigl(\bigcup_{k=1}^\infty (A \cap E_k)\Bigr) = \sum_{k=1}^\infty m^*(A \cap E_k).$$

The inequality

$$m^*\Bigl(\bigcup_{k=1}^\infty (A \cap E_k)\Bigr) \leq \sum_{k=1}^\infty m^*(A \cap E_k)$$

is Proposition 2.3.
By Problem 390, if $n \in \mathbb{N}$ then

$$m^*\Bigl(\bigcup_{k=1}^\infty (A \cap E_k)\Bigr) \geq m^*\Bigl(\bigcup_{k=1}^n (A \cap E_k)\Bigr).$$

By Proposition 2.6 we have that

$$m^*\Bigl(\bigcup_{k=1}^\infty (A \cap E_k)\Bigr) \geq \sum_{k=1}^n m^*(A \cap E_k).$$

Taking the supremum over n yields that

$$m^*\Bigl(\bigcup_{k=1}^\infty (A \cap E_k)\Bigr) \geq \sum_{k=1}^\infty m^*(A \cap E_k)$$

as desired.

(Dibyendu, Problem 570) Let $\{E_k\}_{k=1}^\infty$ be a sequence of measurable sets. For each n, let

$$F_n = E_n \sim \bigcup_{k=1}^{n-1} E_k, \qquad G_n = \bigcup_{k=1}^n E_k.$$

Show that

- Each F_n is measurable.
- If $m \in \mathbb{N}$ then $G_m = \bigcup_{n=1}^m F_n$.

By Proposition 2.5, $\bigcup_{k=1}^{n-1} E_k$ is measurable. By Problem 510, the complement $\mathbb{R} \sim \bigcup_{k=1}^{n-1} E_k$ is measurable. By Problem 561, $E_n \cap \left(\mathbb{R} \sim \bigcup_{k=1}^{n-1} E_k\right)$ is measurable. It is straightforward to establish that this final expression is equal to F_n.

Because $F_n \subseteq E_n \subseteq G_m$ whenever $n \leq m$, we have that $G_m \supseteq \bigcup_{n=1}^{m} F_n$. Conversely, let $e \in G_m$. Then $e \in E_k$ for some $k \leq m$. Let n be the smallest natural number with $e \in E_n$. Then $e \notin \bigcup_{k=1}^{n-1} E_k$, and so $e \in F_n$. Thus $G_m \subseteq \bigcup_{n=1}^{m} F_n$. This completes the proof.

(Problem 580) Show that

- Each G_n is measurable.
- If the E_ns are pairwise disjoint then $E_n = F_n$.
- If $n < m$ then $F_n \cap F_m = \emptyset$.
- If $n < m$ then $G_n \subseteq G_m$.
- $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} G_n$.

Proposition 2.7. The union of countably many measurable sets is measurable.

(Elliott, Problem 590) Prove Proposition 2.7.

Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of measurable sets and let F_n, G_n be as in the previous problem. It suffices to show that $E = \bigcup_{n=1}^{\infty} F_n$ is measurable.

Let $A \subseteq \mathbb{R}$. By Problem 501, it suffices to show that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \sim E)$$

for all such A.

If $m^*(A) = \infty$ there is nothing to prove, so assume that $m^*(A) < \infty$. By Problem 2.26,

$$m^*(A \cap E) = m^*\left(A \cap \bigcup_{k=1}^{\infty} F_k\right) = \sum_{k=1}^{\infty} m^*(A \cap F_k).$$

The left hand side is at most $m^*(A) < \infty$, and the right hand side is a sum of nonnegative real numbers, and so the sum must converge. In particular, we must have that $\sum_{k=n+1}^{\infty} m^*(A \cap F_k) \to 0$ as $n \to \infty$. Thus

$$m^*(A \cap E) = \lim_{n\to\infty} \sum_{k=1}^{n} m^*(A \cap F_k).$$

By Proposition 2.6 and by the previous two problems,

$$m^*(A \cap E) = \lim_{n\to\infty} m^*(A \cap G_n).$$

Because G_n is measurable and $m^*(A) < \infty$,

$$m^*(A \cap E) = \lim_{n\to\infty} m^*(A) - m^*(A \sim G_n).$$

But $G_n \subseteq E$ and so $A \sim G_n \supseteq A \sim E$, and so

$$m^*(A \cap E) \leq m^*(A) - m^*(A \sim E).$$

Rearranging completes the proof.

[Homework 2.1] The collection of Borel sets $\mathcal{B}$ is the smallest σ-algebra that contains $\{(-\infty, a) : a \in \mathbb{R}\}$.

[Homework 3.1] If A, $B \subseteq \mathbb{R}$ and $\sup A \leq \inf B$, then $m^*(A \cup B) = m^*(A) + m^*(B)$.

(Problem 591) If $a \in \mathbb{R}$ then $(-\infty, a)$ is measurable. (Note that we use Homework 3.1 to prove this, and so you may not use this fact to do Homework 3.1.)

Theorem 2.9. The collection $\mathcal{M}$ of Lebesgue measurable subsets of $\mathbb{R}$ is a σ-algebra on $\mathbb{R}$ and contains the Borel sets.

(Problem 600) Show that the collection of Lebesgue measurable subsets of $\mathbb{R}$ is a σ-algebra on $\mathbb{R}$.

(Irina, Problem 610) Prove Theorem 2.9.

(Problem 611) Let $\mathcal{M}$ denote the collection of Lebesgue measurable subsets of $\mathbb{R}$. Then $(\mathbb{R},\mathcal{M})$ is a measurable space, and $m^*\big|_{\mathcal{M}}$ is a measure on $(\mathbb{R},\mathcal{M})$.

[Definition: Lebesgue measure] We let $m = m^*\big|_{\mathcal{M}}$ and refer to m as the Lebesgue measure.

2.4 Finer Properties of Measurable Sets

(Zach, Problem 620) Let $E \subseteq \mathbb{R}$ be measurable. Show that $E = \bigcup_{n=1}^\infty F_n$, where each F_n is bounded and measurable and where $F_n \cap F_m = \emptyset$ if $n \neq m$.

Let $E_1 = (-1,1)$ and let $E_n = (-n,n) \sim (1-n, n-1)$ for all $n \geq 2$. Because the measurable sets form a σ-algebra and contain the Borel sets, we have that each E_n is measurable. Clearly each E_n is bounded. Furthermore, the E_ns are pairwise disjoint. Let $F_n = E_n \cap E$. Then $F_n \subseteq E_n$, and so F_n is bounded and the F_ns are pairwise disjoint. The fact that $E = \bigcup_{n=1}^\infty F_n$ follows from the fact that $R = \bigcup_{n=1}^\infty E_n$, and the fact that the F_ns are measurable follows from the measurability of E and from Problem 561.

[Definition: G_δ-set] A set $G \subseteq \mathbb{R}$ is a G_δ-set if there is a sequence $\{\mathcal{O}_n\}_{n=1}^\infty$ of countably many open sets such that $G = \bigcap_{n=1}^\infty \mathcal{O}_n$.

[Definition: F_σ-set] A set $F \subseteq \mathbb{R}$ is a F_σ-set if there is a sequence $\{\mathcal{C}_n\}_{n=1}^\infty$ of countably many closed sets such that $F = \bigcup_{n=1}^\infty \mathcal{C}_n$.

(Problem 630) Show that all F_σ sets and all G_δ sets are measurable.

Theorem 2.11. Let $E \subseteq \mathbb{R}$ and let $E^C = \mathbb{R} \sim E$. The following statements are equivalent:

(i) If $\varepsilon > 0$, then there is an open set $\mathcal{O}$ containing E for which $m^*(\mathcal{O} \sim E) < \varepsilon$.
(ii) There is a G_δ-set G with $E \subseteq G$ and with $m^*(G \sim E) = 0$.
(iii) If $\varepsilon > 0$, then there is a closed set $\mathcal{C}$ with $\mathcal{C} \subseteq E^C$ and with $m^*(E^C \sim \mathcal{C}) < \varepsilon$.
(iv) There is a F_σ-set F with $E^C \supseteq F$ and with $m^*(E^C \sim F) = 0$.
(v) E is Lebesgue measurable.
(vi) E^C is Lebesgue measurable.

Recall [Problem Problem 510]: (v) and (vi) are equivalent.

(Juan, Problem 640) Show that (ii) and (iv) are equivalent.

(Problem 650) Show that (i) and (iii) are equivalent.

(Problem 651) Show that (v) implies (i) in the special case where $m^*(E) < \infty$.

(Micah, Problem 660) Show that (v) implies (i) in general.

(Muhammad, Problem 670) Show that (iii) implies (iv).

(Anjuman, Problem 680) Show that (iv) implies (vi).

Theorem 2.12. Let $E \subset \mathbb{R}$ be a measurable set with finite measure. Let $\varepsilon > 0$. Show that there is a collection of finitely many pairwise disjoint open intervals $\{I_k\}_{k=1}^n$ such that

$$m\Big(E \sim \bigcup_{k=1}^n I_k\Big) + m\Big(\bigcup_{k=1}^n I_k \sim E\Big) < \varepsilon.$$

(Ashley, Problem 690) Prove Theorem 2.12.

Let $\mathcal{O}$ be as in Theorem 2.11 with ε replaced by $\varepsilon/3$. By Proposition 1.9, there is a sequence of pairwise disjoint open intervals with $\mathcal{O} = \bigcup_{k=1}^\infty I_k$.

Observe that by Problem 390, Proposition 2.13, and Proposition 2.1,

$$m(E) \leq m(\mathcal{O}) = \sum_{k=1}^\infty m(I_k) = \sum_{k=1}^\infty \ell(I_k) < m(E) + \varepsilon/2.$$

In particular each I_k is bounded. There is also an $n \in \mathbb{N}$ such that $\sum_{k=1}^n \ell(I_k) > m(E) - \varepsilon/3$.

Then

$$E \sim \bigcup_{k=1}^{n} I_k \subseteq \mathcal{O} \sim \bigcup_{k=1}^{n} I_k = \bigcup_{k=n+1}^{\infty} I_k$$

and so

$$m\Big(E \sim \bigcup_{k=1}^{n} I_k\Big) \leq \sum_{k=n+1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \ell(I_k) - \sum_{k=1}^{n} \ell(I_k) < m(E) + \varepsilon/3 - (m(E) - \varepsilon/3) = 2\varepsilon/3.$$

Conversely,

$$\bigcup_{k=1}^{n} I_k \sim E \subseteq \mathcal{O} \sim E$$

and so

$$m\Big(\bigcup_{k=1}^{n} I_k \sim E\Big) \leq m(\mathcal{O} \sim E) < \varepsilon/3.$$

This completes the proof.

(Problem 700) Why do we need the assumption that E has finite outer measure?

(Bashar, Problem 710) Give an example of a measurable set $E \subset \mathbb{R}$ with finite measure and a $\varepsilon > 0$ such that there is no collection $\{I_k\}_{k=1}^n$ of finitely many pairwise disjoint open intervals with $\bigcup_{k=1}^n I_k \subseteq E$ and with $m\Big(E \sim \bigcup_{k=1}^n I_k\Big) < \varepsilon$. *Note*: The empty set is an open interval, and the empty collection is a finite collection of open intervals.

Let $E = [0,1] \sim \mathbb{Q}$. There are no nonempty open intervals that are subsets of E, and so $m\Big(E \sim \bigcup_{k=1}^n I_k\Big) = m(E) = 1$ for all finite collections $\{I_k\}_{k=1}^n$ of open intervals contained in E.

(Dibyendu, Problem 720) Give an example of a measurable set $E \subset \mathbb{R}$ with finite measure and a $\varepsilon > 0$ such that there is no collection $\{I_k\}_{k=1}^n$ of finitely many pairwise disjoint open intervals with $\bigcup_{k=1}^n I_k \supseteq E$ and with $m^*\Big(\bigcup_{k=1}^n I_k \sim E\Big) < \varepsilon$.

Let $E = [0,1] \cap \mathbb{Q}$. Suppose $E \subseteq \bigcup_{k=1}^n I_k$ where each I_k is an open interval. Then $\overline{E} \subseteq \overline{\bigcup_{k=1}^n I_k} = \bigcup_{k=1}^n \overline{I_k}$, where $\overline{E}$ denotes the closure of E. But $\overline{E} = [0,1]$ and $\overline{I_k} \sim I_k$ contains exactly two points, so $\bigcup_{k=1}^n I_k$ contains all but finitely many points of $[0,1]$. Thus $\sum_{k=1}^n \ell(I_k) \geq m^*([0,1]) = 1$. Because E has measure zero and is measurable, $m^*\Big(\bigcup_{k=1}^n I_k \sim E\Big) = m^*\Big(\bigcup_{k=1}^n I_k\Big) \geq 1$, and so no such collection can have measure less than ε for any $\varepsilon < 1$.

[Chapter 2, Problem 19] Suppose that E is not measurable but does have finite outer measure. Show that there is an open set $\mathcal{O}$ containing E such that $E \subset \mathcal{O}$ but such that $m^*(\mathcal{O} \sim E) \neq m^*(\mathcal{O}) - m^*(E)$.

[Chapter 2, Problem 20] In the definition of measurable set, it suffices to check for sets A that are bounded open intervals; that is, $E \subseteq \mathbb{R}$ is measurable if and only if, for every $a < b$, we have that

$$b - a = m^*((a,b) \cap E) + m^*((a,b) \sim E).$$

2.7 Undergraduate analysis

(Memory 721) Let X, Y be two metric spaces. Let $f : X \to Y$ and let $f_n : X \to Y$ for each $n \in \mathbb{N}$. Suppose that each f_n is continuous and that $f_n \to f$ uniformly on X. Then f is continuous.

(Memory 722) A sequence of functions $f_n : X \to Y$ is uniformly Cauchy if, for every $\varepsilon > 0$, there is a $K \in \mathbb{N}$ such that if $m, n \in \mathbb{N}$ with $m \geq n \geq K$, then $d(f_n(x), f_m(x)) < \varepsilon$ for all $x \in X$. Suppose that $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy and that Y is complete. Then $f_n \to f$ uniformly for some function $f : X \to Y$.

(Memory 723) (The intermediate value theorem.) Suppose that $a < b$ and that $f : [a, b] \to \mathbb{R}$ is continuous. If $f(a) \leq y \leq f(b)$, then there is an $x \in [a, b]$ such that $f(x) = y$.

(Memory 724) (Definition of interval) A set $I \subseteq \mathbb{R}$ is an interval if, whenever $a < b < c$ and $a, c \in I$, we also have that $b \in I$. Then $\{I \subseteq \mathbb{R} : I \text{ is an interval}\}$ is the union of the following ten collections of sets:

- $\{\emptyset\}$
- $\{\mathbb{R}\}$
- $\{(a, b) : a < b \text{ and } a, b \in \mathbb{R}\}$
- $\{[a, b : a < b \text{ and } a, b \in \mathbb{R}\}$
- $\{[a, b) : a < b \text{ and } a, b \in \mathbb{R}\}$
- $\{(a, b] : a < b \text{ and } a, b \in \mathbb{R}\}$
- $\{(a, \infty) : a \in \mathbb{R}\}$
- $\{[a, \infty) : a \in \mathbb{R}\}$
- $\{(-\infty, b) : b \in \mathbb{R}\}$
- $\{(-\infty, b] : b \in \mathbb{R}\}$

[Definition: Relatively open] Let (X, d) be a metric space and let $Y \subset X$. Then $(Y, d\big|_{Y \times Y})$ is also a metric space. If $G \subseteq Y$ is open in $(Y, d\big|_{Y \times Y})$, then we say that G is relatively open in Y.

(Memory 725) If (X, d) is a metric space and $G \subseteq Y \subseteq X$, then G is relatively open in Y if and only if $G = Y \cap U$ for some $U \subseteq X$ that is open in (X, d).

[Definition: Connected] Let (X, d) be a metric space. We say that (X, d) is disconnected if there exist two sets $A, B \subseteq X$ such that

- A and B are both open in (X, d),
- $X = A \cup B$,
- $\emptyset = A \cap B$,
- $A \neq \emptyset \neq B$.

If no such A and B exist then we say that (X, d) is connected.

(Memory 726) Let (X, d) be a metric space and let $Y \subseteq X$. Then the metric space $(Y, d\big|_{Y \times Y})$ is disconnected if and only if there exist two sets $A, B \subseteq X$ such that

- A and B are both open in (X, d),
- $Y \subseteq A \cup B$,
- $\emptyset = A \cap B$,
- $A \cap Y \neq \emptyset \neq B \cap Y$.

(Memory 727) A subset of $\mathbb{R}$ is connected if and only if it is an interval.

(Memory 728) If (X, d) is a connected metric space and $f : X \to Y$ is continuous, then $f(X)$ is also connected.

2.7 An uncountable set of measure zero

(Memory 729) By Problem 400 and Proposition 2.4, if $E \subset \mathbb{R}$ is countable, then E is measurable and has measure zero.

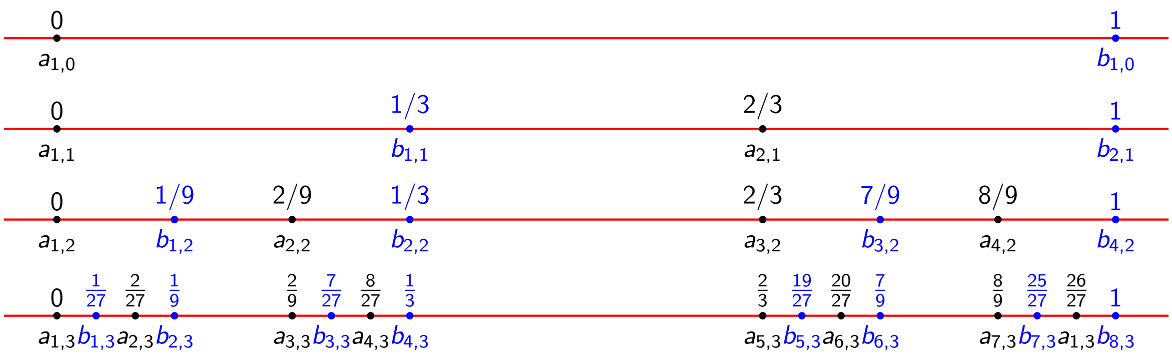
Proposition 2.19. There is a set of measure zero that is uncountable.

(Elliott, Problem 730) In this problem we begin the construction of an uncountable set of measure zero. We define the points $a_{k,n}$ and $b_{k,n}$, for $n \in \mathbb{N}_0$ and for $1 \leq k \leq 2^n$, as follows.

$$a_{1,0} = 0, \qquad b_{1,0} = 1,$$
$$a_{2\ell-1,n} = a_{\ell,n-1}, \qquad b_{2\ell-1,n} = \frac{2}{3}a_{\ell,n-1} + \frac{1}{3}b_{\ell,n-1},$$
$$a_{2\ell,n} = \frac{1}{3}a_{\ell,n-1} + \frac{2}{3}b_{\ell,n-1}, \qquad b_{2\ell,n} = b_{\ell,n-1}.$$

Show that:

(a) if $1 \leq k \leq 2^n$, then $b_{k,n} - a_{k,n} = 3^{-n}$.
(b) if $1 \leq k < k+1 \leq 2^n$ and k is even then $a_{k+1,n} - b_{k,n} > 3^{-n}$.
(c) if $1 \leq k < k+1 \leq 2^n$ and k is odd then $a_{k+1,n} - b_{k,n} = 3^{-n}$.



We prove by induction. Our base case is $n = 0$. In this case $2^n = 1$, so in (a) k must also be 1 and so (a) is true by inspection. (b) and (c) are vacuously true as there are no k that satisfy $1 \leq k < k+1 \leq 2^0 = 1$.

Now suppose that the statements are all true for some $n - 1 \geq 0$. Let $1 \leq k \leq 2^n$.

If k is even, then $k = 2\ell$ for some $1 \leq \ell \leq 2^{n-1}$, and so

$$b_{k,n} - a_{k,n} = b_{2\ell,n} - a_{2\ell,n} = b_{\ell,n-1} - \left(\frac{1}{3}a_{\ell,n-1} + \frac{2}{3}b_{\ell,n-1}\right) = \frac{1}{3}(b_{\ell,n-1} - a_{\ell,n-1})$$

and thus (a) holds by the induction hypothesis. If in addition $k+1 \leq 2^n$, then $\ell < 2^{n-1}$ and so $\ell+1 \leq 2^{n-1}$, and so

$$a_{k+1,n} - b_{k,n} = a_{2(\ell+1)-1,n} - b_{2\ell,n} = a_{\ell+1,n-1} - b_{\ell,n-1}.$$

By assumption (b) and (c) hold for $n-1$, and so

$$a_{k+1,n} - b_{k,n} = a_{\ell+1,n-1} - b_{\ell,n-1} \geq 3^{1-n} > 3^{-n}.$$

If k is odd, then $k = 2\ell - 1$ for some $1 \leq \ell \leq 2^{n-1}$. Then

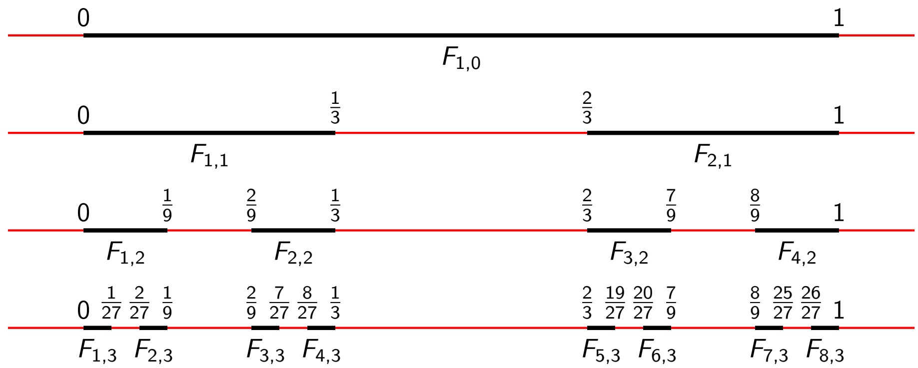
$$b_{k,n} - a_{k,n} = b_{2\ell-1,n} - a_{2\ell-1,n} = \left(\frac{2}{3}a_{\ell,n-1} + \frac{1}{3}b_{\ell,n-1}\right) - a_{\ell,n-1} = \frac{1}{3}(b_{\ell,n-1} - a_{\ell,n-1})$$

and thus (a) holds by the induction hypothesis. If in addition $k+1 \leq 2^n$, then $\ell \leq 2^{n-1}$ and so

$$a_{k+1,n} - b_{k,n} = a_{2\ell,n} - b_{2\ell-1,n} = \left(\frac{1}{3}a_{\ell,n-1} + \frac{2}{3}b_{\ell,n-1}\right) - \left(\frac{2}{3}a_{\ell,n-1} + \frac{1}{3}b_{\ell,n-1}\right) = \frac{1}{3}(b_{\ell,n-1} - a_{\ell,-1}).$$

By assumption (a) holds for $n-1$, and so this must equal 3^{-n}. This completes the proof.

(Problem 731) If $n \in \mathbb{N}_0$ and $1 \leq k \leq 2^n$, let $F_{k,n} = [a_{k,n}, b_{k,n}]$ be the closed interval of length 3^{-n} with endpoints at $a_{k,n}$ and $b_{k,n}$. Show that $F_{j,n} \cap F_{k,n} = \emptyset$ if $j \neq k$.



(Irina, Problem 740) Define $F_n = \bigcup_{k=1}^{2^n} [a_{k,n}, b_{k,n}] = \bigcup_{k=1}^{2^n} F_{k,n}$. Show that $F_n \subsetneq F_{n-1}$ for all $n \in \mathbb{N}$, and that $F_n \sim F_{n-1}$ may be written as the union of 2^{n-1} open intervals each of length 3^{-n}.

Let $n \geq 1$. Observe that

$$\begin{aligned} F_{2\ell-1,n} \cup F_{2\ell,n} &= [a_{2\ell-1,n}, b_{2\ell-1,n}] \cup [a_{2\ell,n}, b_{2\ell,n}] \\ &= [a_{\ell,n-1}, b_{2\ell-1,n}] \cup [a_{2\ell,n}, b_{\ell,n-1}] \subset [a_{\ell,n-1}, b_{\ell,n-1}] = F_{\ell,n-1}. \end{aligned}$$

Thus, if $1 \leq \ell \leq 2^{n-1}$, then there are at least two distinct values of k, namely $k = 2\ell - 1$ and $k = 2\ell$, that satisfy $1 \leq k \leq 2^n$ and $F_{k,n} \subset F_{\ell,n-1}$.

Because there are only $2^n = 2 \cdot 2^{n-1}$ such values of k, and there are 2^{n-1} such values of ℓ, we must have that each $F_{k,n}$ is contained in a $F_{\ell,n-1}$. (This may also be seen by choosing $\ell = k/2$ if k is even and $\ell = (k+1)/2$ if k is odd, and observing that $F_{k,n} \subset F_{\ell,n-1}$ by the above analysis.) Thus $F_n = \bigcup_{k=1}^{2^n} F_{k,n} \subset \bigcup_{\ell=1}^{2^{n-1}} F_{\ell,n-1} = F_{n-1}$, as desired.

Again because there are only $2^n = 2 \cdot 2^{n-1}$ such values of k, and there are 2^{n-1} such values of ℓ, we must have that each $F_{\ell,n-1}$ contains $F_{k,n}$ for exactly two values of k, namely $k = 2\ell$ and $k = 2\ell - 1$. (This may also be seen by observing that if $1 \leq j \leq 2^n$ and $j \notin \{2\ell - 1, 2\ell\}$, then either $j > 2\ell$ or $j < 2\ell - 1$. In the first case $a_{j,n} > b_{2\ell,n} = b_{\ell,n-1}$, while in the second case $b_{j,n} < a_{2\ell-1,n} = a_{\ell,n-1}$; in either case $[a_{j,n}, b_{j,n}]$ is clearly disjoint from $[a_{\ell,n-1}, b_{\ell,n-1}]$.)

Thus $F_{\ell,n-1} \sim F_n = F_{\ell,n-1} \sim (F_{2\ell-1,n} \cup F_{2\ell,n})$, which by the above analysis is equal to the interval $(b_{2\ell-1,n}, a_{2\ell,n})$. There are 2^{n-1} such intervals (one for each ℓ) and by the above analysis, because $2\ell - 1$ is odd we have that the interval is of length 3^{-n}.

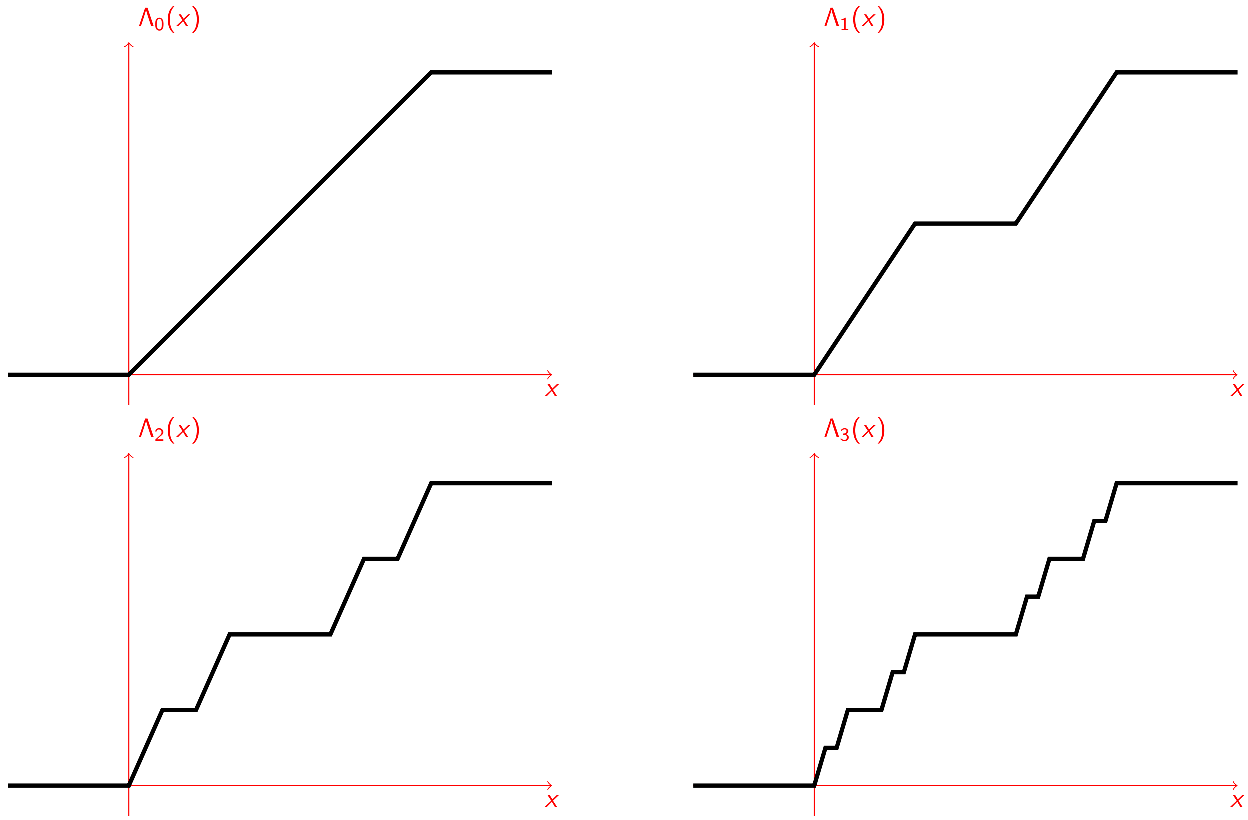
(Zach, Problem 750) Let $C = \bigcap_{n=0}^{\infty} F_n$. The set C is called the Cantor set. Show that $m(F_n) = (2/3)^n$ for all $n \in \mathbb{N}_0$ and that $m(C) = 0$.

Each $F_{k,n}$ is a closed interval, and so is closed. The union of finitely many closed sets is closed, and so each F_n is closed. The intersection of an arbitrary collection of closed sets is closed, and so C must be closed.

For each n, $C \subset F_n$, and so $m^*(C) \leq m^*(F_n)$. But each $F_{k,n}$ has length (thus measure) 3^{-n}, the $F_{k,n}$s are disjoint for distinct k, and there are 2^n intervals $F_{k,n}$ in F_n; thus $m(F_n) = \sum_{k=1}^{n} m(F_{k,n}) = (2/3)^n$. Recalling from undergraduate analysis that $(2/3)^n \to 0$ as $n \to \infty$, we have that $m^*(C) = 0$. Sets of measure zero are measurable by Proposition 2.4, and so $m(C)$ exists and equals zero.

[Homework 3.1b] If $E \subseteq \mathbb{R}$ and $m^*(E) < \infty$, and if we define f by $f(x) = m(E \cap (-\infty, x))$, then $f : \mathbb{R} \to \mathbb{R}$ is continuous.

(Problem 751) Let $\Lambda_k(x) = \frac{m(F_k \cap (-\infty, x))}{m(F_k)}$. Then Λ_k is continuous and nondecreasing. Sketch the graphs of Λ_0, Λ_1, and Λ_2.



(Juan, Problem 760) Suppose that $x \notin F_n$. Show that $\Lambda(x) = 2^{-n}|\{k \in \{1, 2, \dots, 2^n\} : b_{k,n} < x\}|$.

(Micah, Problem 770) Show that $\Lambda_n(\mathbb{R} \sim F_n) = \{i2^{-n} : 0 \leq i \leq 2^n, i \in \mathbb{Z}\}$ and that, if $m \geq n$, then $\Lambda_n(x) = \Lambda_m(x)$ for all $x \notin F_n$.

(Muhammad, Problem 780) Show that $\{\Lambda_k\}_{k=1}^\infty$ is uniformly Cauchy.

[Definition: The Cantor function] Let $\Lambda(x) = \lim_{k\to\infty} \Lambda_k(x)$.

(Ashley, Problem 790) Show that Λ exists and is continuous, nondecreasing, and surjective $\Lambda : [0, 1] \to [0, 1]$.

We have that $\Lambda_k : \mathbb{R} \to \mathbb{R}$ is uniformly Cauchy and $\mathbb{R}$ is complete. Thus by Problem 722, the sequence $\{\Lambda_k\}_{k=1}^\infty$ converges uniformly to some function $\Lambda : \mathbb{R} \to \mathbb{R}$. Thus Λ exists.

By Homework 3.1b, each Λ_k is continuous, so by Problem 721, Λ must also be continuous.

Suppose $x < y$. Clearly $\Lambda_k(x) \leq \Lambda_k(y)$ for all $k \in \mathbb{N}$, and so we must have that $\Lambda_k(x) \leq \Lambda_k(y)$ as well.

Finally, observe that $\Lambda_k(0) = 0$ and $\Lambda_k(1) = 1$ for all $k \in \mathbb{N}$. Thus $\Lambda(0) = 0$ and $\Lambda(1) = 1$. By the intermediate value theorem (Problem 723), if $0 < y < 1$ then $y = \Lambda(x)$ for some $x \in (0, 1)$, and so Λ is surjective $[0, 1] \to [0, 1]$.

[Definition: Almost everywhere] Suppose that $E \subseteq \mathbb{R}$ is a set. If a property P is true for every $x \in E \sim E_0$, where $m^*(E_0) = 0$, we say that P is true almost everywhere on E.

(Bashar, Problem 800) Show that $\Lambda'(x) = 0$ for every $x \in \mathbb{R} \sim C$, and thus for almost every $x \in \mathbb{R}$.

(Dibyendu, Problem 810) Show that $\Lambda([0, 1] \sim C)$ is countable.

Note that $0, 1 \in C$, and so $[0, 1] \sim C = (0, 1) \sim C$.

Because C is closed, we must have that $(0, 1) \sim C$ is open. Thus by Proposition 1.9, $(0, 1) \sim C = \bigcup_{k=1}^\infty I_k$, where the I_ks are (possibly empty) open intervals. Because $I_k \subset (0, 1) \sim C \subset \mathbb{R} \sim C$, we have that $\Lambda' = 0$ on I_k and so Λ is constant on each I_k. Because each $\Lambda(I_k)$ is empty or a single point, and there are countably many I_ks, we must have that $\Lambda((0,1) \sim C) = \bigcup_{k=1}^{\infty} \Lambda(I_k)$ is countable.

Alternatively, observe that if $x \notin C$ then $x \notin F_n$ for some n. Then $\Lambda_m(x) = \Lambda_n(x)$ for all $m \geq n$, and so $\Lambda(x) = \lim_{m\to\infty} \Lambda_m(x) = \Lambda_n(x) \in \Lambda_n(\mathbb{R} \sim F_n) = \{i2^{-n} : 0 \leq i \leq 2^n, i \in \mathbb{Z}\}$. In particular $\Lambda(x)$ is rational. Thus $\Lambda((0,1) \sim \mathbb{C}) \subset \mathbb{Q}$ and so must be countable.

(Elliott, Problem 820) Show that C is uncountable.

We know that $\Lambda([0,1]) = [0,1]$ is uncountable. But $\Lambda([0,1] \sim C)$ is countable, and $[0,1] = \Lambda([0,1]) = \Lambda(C) \cup \Lambda([0,1] \sim C)$. Thus $\Lambda(C)$ must be uncountable, for if it were countable then $\Lambda(C) \cup \Lambda([0,1] \sim C)$ would be countable, which is a contradiction. But if C were countable then $\Lambda(C)$ would also be countable, so C must be uncountable.

2.6 Vitali's Example of a Nonmeasurable Set

Theorem 2.17. If $E \subseteq \mathbb{R}$ has positive outer measure, then there is an $A \subseteq E$ that is not measurable.

(Irina, Problem 830) In this problem we begin the proof of Theorem 2.17. Let $F \subset [-1,1]$ and let V_{q_k} be as in Problem 490. Show that either $F \cap V_{q_k}$ is not measurable or $m^*(F \cap V_{q_k}) = 0$.

(Zach, Problem 840) Suppose that $m^*(F) > 0$. Show that we must have that $m^*(F \cap V_{q_k}) > 0$ for at least one value of k.

Recall that $[-1,1] \subseteq \bigcup_{k=1}^{\infty} V_{q_k}$. Thus $F = F \cap [-1,1] = F \cap \bigcup_{k=1}^{\infty} V_{q_k} = \bigcup_{k=1}^{\infty} F \cap V_{q_k}$.

Thus by Proposition 2.3, we must have that $m^*(F) \leq \sum_{k=1}^{\infty} m^*(F \cap V_{q_k})$. Because the left hand side is positive, at least one of the summands on the right hand side must be positive.

(Problem 841) Prove Theorem 2.17.

2.7 A non-measurable set that is not a Borel set

Proposition 2.22. There is a measurable set that is not a Borel set.

(Juan, Problem 850) In this problem we begin the proof of Proposition 2.22. Let Λ be the Cantor-Lebesgue function and let $f(x) = x + \Lambda(x)$. Show that f is continuous, strictly increasing, and surjective $\mathbb{R} \to \mathbb{R}$.

[Chapter 2, Problem 47] If $f : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing, and if B is a Borel set, then $f(B)$ is also a Borel set.

(Micah, Problem 860) Show that $f(C)$ has positive measure.

(Muhammad, Problem 870) Prove Proposition 2.22.

Because $m(f(C)) > 0$, by Theorem 2.17 there is an $A \subseteq f(C)$ that is not measurable.

Let $B = f^{-1}(A) \cap C$. Then $B \subseteq C$, and $m^*(C) = 0$, so $m^*(B) = 0$; thus B is measurable by Proposition 2.4.

We claim that $f(B) = A$. Because $B \subseteq f^{-1}(A)$, by definition $f(B) \subseteq A$. Conversely, suppose that $a \in A$. Then $a \in f(C)$ because $A \subseteq f(C)$, so $a = f(b)$ for some $b \in C$. Then $b \in f^{-1}(A)$ by definition, so $b \in f^{-1}(A) \cap C = B$. Thus $b \in B$ and so $a = f(b) \in f(B)$; because a was arbitrary we have that $A \subseteq f(B)$. Thus $A = f(B)$.

If B were Borel, then by Problem 2.47 we would have that $f(B)$ was Borel and therefore measurable. But $f(B) = A$ is not measurable, and so B is not Borel. We have seen that B is measurable, and so B must be a set that is measurable but not Borel.

Theorem 2.15.

(i) Let $\{A_k\}_{k=1}^\infty$ be such that A_k is measurable and $A_k \subseteq A_{k+1}$ for all $k \in \mathbb{N}$. Then

$$m\Big(\bigcup_{k=1}^\infty A_k\Big) = \lim_{n\to\infty} m(A_n).$$

(ii) Let $\{B_k\}_{k=1}^\infty$ be such that B_k is measurable and $B_k \supseteq B_{k+1}$ for all $k \in \mathbb{N}$. If $m(B_\ell) < \infty$ for some $\ell \in \mathbb{N}$, then

$$m\Big(\bigcap_{k=1}^\infty B_k\Big) = \lim_{n\to\infty} m(B_n).$$

(Ashley, Problem 880) Prove part (ii) of Theorem 2.15 without using part (i).

(Problem 881) Prove part (i).

(Bashar, Problem 890) Find a sequence $\{B_k\}_{k=1}^\infty$ such that B_k is measurable and $B_k \supseteq B_{k+1}$ for all $k \in \mathbb{N}$, but such that

$$m\Big(\bigcap_{k=1}^\infty B_k\Big) \neq \lim_{n\to\infty} m(B_n).$$

Take $B_k = (k, \infty)$. Then $m(B_n) = \infty$ for all k and so $\lim_{n\to\infty} m(B_n) = \infty$, but $\bigcap_{k=1}^\infty B_k = \emptyset$ and so $m(\bigcap_{k=1}^\infty B_k) = 0$.

The Borel-Cantelli Lemma. Let $\{E_k\}_{k=1}^\infty$ be a sequence of measurable sets. Suppose that $\sum_{k=1}^\infty m(E_k) < \infty$. Then $|\{k \in \mathbb{N} : x \in E_k\}| < \infty$ for almost every $x \in \mathbb{R}$.

(Dibyendu, Problem 900) Prove the Borel-Cantelli lemma. Start by writing a formula for the set of $x \in \mathbb{R}$ such that $|\{k \in \mathbb{N} : x \in E_k\}| = \infty$ using unions and intersections. Explain carefully why your formula is true.

We claim that $|\{k \in \mathbb{N} : x \in E_k\}| = \infty$ if and only if, for all $n \in \mathbb{N}$, there is a $k \in \mathbb{N}$ with $k \geq n$ such that $x \in E_k$.

To prove the claim, we will prove that the two negations are equivalent. Suppose first that $|\{k \in \mathbb{N} : x \in E_k\}| \neq \infty$. Then the set $\{k \in \mathbb{N} : x \in E_k\}$ is finite, and so it contains a largest element K. Then $n = K + 1 \in \mathbb{N}$, but there is no $k \in \mathbb{N}$ with $k \geq n$ such that $x \in E_k$.

Suppose to the contrary that for some n, there does not exist a $k \geq n$ with $x \in E_k$. Then $x \notin E_k$ for all $k \geq n$. Thus $x \in E_k$ for at most $n-1$ possible values of k, and so $x \in E_k$ for only finitely many k.

Let

$$\begin{aligned} A &= \{x \in \mathbb{R} : |\{k \in \mathbb{N} : x \in E_k\}| = \infty\} \\ &= \{x \in \mathbb{R} : \text{for all } n \in \mathbb{N} \text{ there exists a } k \geq n \text{ such that } x \in E_k\}. \end{aligned}$$

For any fixed $n \in \mathbb{N}$, the set

$$\{x \in \mathbb{R} : \text{there exists a } k \geq n \text{ such that } x \in E_k\} = \bigcup_{k=n}^\infty E_k.$$

Thus

$$A = \Big\{x \in \mathbb{R} : \text{for all } n \in \mathbb{N},\ x \in \bigcup_{k=n}^\infty E_k\Big\} = \bigcap_{n=1}^\infty \Big(\bigcup_{k=n}^\infty E_k\Big).$$

Let $B_n = \bigcup_{k=n}^\infty E_k$. Then $B_n = E_n \cup B_{n+1} \supseteq B_{n+1}$ for all n. Furthermore, $m(B_1) \leq \sum_{k=1}^\infty m(E_k) < \infty$, and so by Theorem 2.15ii,

$$m(A) = m\Big(\bigcap_{n=1}^\infty B_n\Big) = \lim_{n\to\infty} m(B_n).$$

But

$$m(B_n) \leq \sum_{k=n}^{\infty} m(E_k).$$

Because $\sum_{k=1}^{\infty} m(E_k) < \infty$, and each $m(E_k) \geq 0$, we have that the series converges absolutely and so $\lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0$. Thus $m(A) = 0$, as desired.

3. Lebesgue Measurable Functions

[Definition: Measurable function] Let $E \subseteq \mathbb{R}$ be measurable and let $f : E \to [-\infty, \infty]$. Suppose that for every $c \in \mathbb{R}$ the set

$$\{x \in E : f(x) > c\} = f^{-1}((c, \infty])$$

is measurable. Then we say that f is a measurable function (or that f is measurable on E).

Proposition 3.3. Let $E \subseteq \mathbb{R}$ be measurable and let $f : E \to \mathbb{R}$ be continuous. Then f is measurable.

(Elliott, Problem 910) Prove Proposition 3.3.

(Irina, Problem 920) Let $f : \mathbb{R} \to [-\infty, \infty]$. Suppose that $\lim_{y\to x} f(y) = f(x)$ for all $x \in \mathbb{R}$. Show that f is measurable.

[Chapter 3, Problem 24] A monotonic function defined on a measurable set is measurable.

Proposition 3.1. Let $E \subseteq \mathbb{R}$ be measurable and let $f : E \to [-\infty, \infty]$. The following statements are equivalent.

(i) If $c \in \mathbb{R}$, then $\{x \in E : f(x) > c\} = f^{-1}((c, \infty])$ is measurable. (That is, f is a measurable function.)
(ii) If $c \in \mathbb{R}$, then $\{x \in E : f(x) \geq c\} = f^{-1}([c, \infty])$ is measurable.
(iii) If $c \in \mathbb{R}$, then $\{x \in E : f(x) < c\} = f^{-1}([-\infty, c))$ is measurable.
(iv) If $c \in \mathbb{R}$, then $\{x \in E : f(x) \leq c\} = f^{-1}([-\infty, c])$ is measurable.

Furthermore, if any of these conditions is true, then $f^{-1}(\{c\})$ is measurable for all $c \in [-\infty, \infty]$.

(Zach, Problem 930) Prove Proposition 3.1.

[Chapter 3, Problem 4] If $f^{-1}(\{c\})$ is measurable for all $c \in [-\infty, \infty]$, is it necessarily the case that f is measurable?

(Problem 931) Let $E \subseteq \mathbb{R}$ be measurable and let $f : E \to [-\infty, \infty]$. Suppose that, for all $c \in \mathbb{R}$, the set $\{x \in E : f(x) > c\} = f^{-1}((c, \infty))$ is measurable. Is f necessarily measurable? If not, what additional assumptions must be imposed to show that f is measurable?

f need not be measurable. Let A be a non-measurable set; such sets exist by Theorem 2.17. Let

$$f(x) = \begin{cases} \infty, & x \in A, \\ -\infty, & x \notin A. \end{cases}$$

Then $f^{-1}((c, \infty)) = \emptyset$ is measurable for all $c \in \mathbb{R}$, but $f^{-1}((0, \infty]) = f^{-1}(\{\infty\}) = A$ is not measurable.

However, if for all $c \in \mathbb{R}$, the set $\{x \in E : f(x) > c\} = f^{-1}((c, \infty))$ is measurable, and if in addition the set $f^{-1}(\{\infty\})$ is measurable, then f is measurable.

(Problem 932) Let $\mathcal{A}$ be a σ-algebra over a set X, let $Y \in \mathcal{A}$, and define $\mathcal{S} = \{S \cap Y : S \in \mathcal{A}\}$. Show that $\mathcal{S}$ is a σ-algebra over Y.

(Juan, Problem 940) Let $(X, \mathcal{A})$ be a measurable space (that is, $\mathcal{A}$ is a σ-algebra over X). Let $f : X \to Y$ be a function and let $\mathcal{F} = \{S \subseteq Y : f^{-1}(S) \in \mathcal{A}\}$. Show that $\mathcal{F}$ is a σ-algebra over Y.

[Homework 2.1] The collection $\mathcal{B}$ of Borel sets is the smallest σ-algebra containing $\{(-\infty, a) : a \in \mathbb{R}$.

Proposition 3.2. If f is measurable, then $f^{-1}(\mathcal{O})$ is measurable for all open sets $\mathcal{O}$.

(Micah, Problem 950) Prove that in fact, if f is measurable, then then $f^{-1}(B)$ is measurable for all Borel sets B.

(Muhammad, Problem 960) If g is measurable, is it true that $g^{-1}(E)$ is measurable for all *measurable* sets E?

(Ashley, Problem 970) If h is measurable, is it true that $h^{-1}(B)$ is Borel for all Borel sets B?

No. Let A be a set that is measurable but not Borel; such an A must exist by Proposition 2.22. Let

$$h(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

If $E \subseteq [-\infty, \infty]$, then $h^{-1}(E)$ is either $\emptyset$ (if $0, 1 \notin E$), A (if $1 \in E$, $0 \notin E$), $\mathbb{R} \sim A$ (if $0 \in E$, $1 \notin E$), or $\mathbb{R}$ (if $0, 1 \in E$). In any case $h^{-1}(E)$ is measurable, and so h is a measurable function. However, $h^{-1}(\{1\}) = A$ is not Borel.

Proposition 3.5. Let $E \subseteq \mathbb{R}$ be measurable and let $f : E \to [-\infty, \infty]$.

(i) Suppose that $g : E \to [-\infty, \infty]$ satisfies $f = g$ almost everywhere on E and that g is measurable. Then f is also measurable.

(ii) Suppose that $D \subseteq E$, that $f\big|_D$ is measurable, and that $f\big|_{E\sim D}$ is measurable. Then f is measurable on E.

(Problem 971) Prove Proposition 3.5, part (ii).

(Bashar, Problem 980) Prove Proposition 3.5, part (i).

Let $S = \{x \in E : f(x) \neq g(x)\}$. By assumption $m^*(S) = 0$, and so by Proposition 2.4 S and all of its subsets are measurable.

Let $c \in \mathbb{R}$. Then

$$\{x \in E : f(x) > c\} = \big(\{x \in E : g(x) > c\} \cup \{x \in E : f(x) > c \geq g(x)\}\big) \sim \{x \in E : g(x) > c \geq f(x)\}.$$

The first set $S_1 = \{x \in E : g(x) > c\}$ is measurable by assumption. The second set $S_2 = \{x \in E : f(x) > c \geq g(x)\} \subseteq \{x \in E : f(x) \neq g(x)\} = S$ has outer measure zero and thus is measurable. Thus $S_1 \cup S_2$ is measurable by Proposition 2.5. The third set $S_3 = \{x \in E : g(x) > c \geq f(x)\} \subseteq \{x \in E : g(x) \neq f(x)\} = S$ also has outer measure zero and thus is measurable, and so $(S_1 \cup S_2) \sim S_3$ is measurable by Problem 510 and Problem 561. Thus $\{x \in E : f(x) > c\}$ is measurable for all $c \in \mathbb{R}$, and so f is a measurable function.

(Problem 981) Let $E \subseteq \mathbb{R}$. Show that E is measurable if and only if the characteristic function χ_E is measurable.

[Chapter 3, Problem 6] Let $E \subseteq \mathbb{R}$ be measurable. Let $f : E \to [-\infty, \infty]$. Show that f is measurable on E if and only if the function

$$g(x) = \begin{cases} f(x), & x \in E, \\ 0, & x \notin E \end{cases}$$

is measurable.

(Problem 982) Did we need the condition that E was measurable?

Theorem 3.6. Let $E \subseteq \mathbb{R}$ be measureable, and let $f, g : E \to [-\infty, \infty]$ be measurable functions that are finite almost everywhere in E

If $\alpha, \beta \in \mathbb{R}$, then fg and $\alpha f + \beta g$ are defined almost everywhere on E and are measurable on E in the sense of Proposition 3.5, that is, in the sense that any of the extensions of fg and $\alpha f + \beta g$ to E are measurable.

(Problem 983) If f is measurable and $\alpha \in \mathbb{R}$, then αf is measurable.

(Dibyendu, Problem 990) Suppose that f and g are measurable and finite almost everywhere. Show that $f + g$ is measurable.

First observe that $D = \{x \in E : f(x) + g(x) \text{ does not exist}\} = \{x \in E : f(x) = \infty, g(x) = -\infty\} \cup \{x \in E : f(x) = -\infty, g(x) = \infty\}$ and so has measure zero.

Let $c \in \mathbb{R}$. Observe that if $f(x) + g(x)$ exists and is greater than c, then neither $f(x)$ nor $g(x)$ can equal $-\infty$. Thus $E = \bigcup_{q \in \mathbb{Q}} \{x \in E : f(x) > q\}$.

Now, if $f(x) > q$ and $g(x) > c - q$, then $f(x) + g(x) > c$. Conversely, if $f(x) + g(x) > c$, then $f(x) > c - g(x)$, the left hand side is not $-\infty$, and the right hand side is not $+\infty$. By density of the rationals (if both sides are finite) or by the Archimedean property (if $f(x) = \infty$ or $g(x) = \infty$ or both), there is a $q \in \mathbb{Q}$ with $f(x) > q > c - g(x)$ and so $g(x) > c - q$. We thus have that

$$\{x \in E : f(x) + g(x) > c\} = \bigcup_{q \in \mathbb{Q}} \{x \in E : f(x) > q\} \cap \{x \in E : g(x) > c - q\}.$$

Because f and g are measurable, the two sets $\{x \in E : f(x) > q\}$ and $\{x \in E : g(x) > c - q\}$ are measurable. Thus their intersection is measurable. The rationals are countable, and the union of countably many measurable sets is measurable. Thus $\{x \in E : f(x) + g(x) > c\}$ is measurable for all $c \in \mathbb{R}$, as desired.

(Elliott, Problem 1000) Suppose that f is measurable. Show that f^2 is measurable. Then prove Proposition 3.5.

If f is measurable and $c \in \mathbb{R}$, then

$$\{x \in E : f(x)^2 \leq c\} = \{x \in E : -\sqrt{c} \leq f(x) \leq \sqrt{c}\} = \{x \in E : f(x) \leq \sqrt{c}\} \cap \{x \in E : f(x) \geq -\sqrt{c}\}$$

is the intersection of two measurable sets, and so is measurable. Thus f^2 is measurable by Proposition 3.1.

Now, observe that

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2}.$$

Each of the three elements of the numerator is measurable by the previous argument, while their sum is measurable by Problem Problem 990 and so fg is measurable by Problem Problem 983.

(Irina, Problem 1010) Give an example a measurable function h and a continuous function g such that $h \circ g$ is not measurable.

Proposition 3.7. If $D, E \subseteq \mathbb{R}$ are measurable, if $h : E \to D$ is measurable, and if $g : D \to \mathbb{R}$ is continuous, then $g \circ h$ is measurable.

[Chapter 3, Problem 8iv] More generally, if $E \subseteq \mathbb{R}$ is measurable, if $D \subseteq \mathbb{R}$ is a Borel set, if $h : E \to D$ is measurable, and if $g : D \to \mathbb{R}$ is such that $\{x \in D : g(x) > c\}$ is Borel for all $c \in \mathbb{R}$, then $g \circ h$ is measurable.

(Zach, Problem 1020) Prove Proposition 3.7.

(Problem 1021) If f is measurable, show that $|f|$ is measurable.

(Problem 1022) Define

$$f^+(x) = \begin{cases} f(x), & f(x) \geq 0, \\ 0, & f(x) \leq 0, \end{cases} \qquad f^-(x) = \begin{cases} 0, & f(x) \geq 0, \\ f(x), & f(x) \leq 0. \end{cases}$$

If f is measurable, show that f^+ and f^- are measurable.

Proposition 3.8. Let $E \subseteq \mathbb{R}$ be measurable and let $f_1, f_2, \dots, f_k : E \to [-\infty, \infty]$ be finitely many measurable functions. Then $f(x) = \max\{f_1(x), \dots, f_k(x)\}$ is also measurable.

(Juan, Problem 1030) Prove Proposition 3.8.

3.2 Sequential Pointwise Limits

[Definition: Pointwise convergence] Let E be a set and let $f_n, f : E \to [-\infty, \infty]$. If $f_n(x) \to f(x)$ for all $x \in E$, then we say that $f_n \to f$ pointwise on E.

[Definition: Almost everywhere convergence] Let $E \subseteq \mathbb{R}$ be a set and let $f_n, f : E \to [-\infty, \infty]$. If $f_n(x) \to f(x)$ for all $x \in E \sim D$, where $m(D) = 0$, then we say that $f_n \to f$ almost everywhere on E.

[Definition: Uniform convergence] Let $E \subseteq \mathbb{R}$ be a set and let $f_n, f : E \to \mathbb{R}$. Suppose that for all $\varepsilon > 0$ there is a $m \in \mathbb{N}$ such that, if $n \geq m$, then $|f_n(x) - f(x)| < \varepsilon$. Then we say that $f_n \to f$ uniformly on E.

(Problem 1031) Show that uniform convergence implies pointwise convergence and that pointwise convergence implies almost everywhere convergence.

(Problem 1032) Show that none of the reverse implications hold.

Proposition 3.9. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on a measurable set E. Suppose that $f_n \to f$ almost everywhere on E for some $f : E \to [-\infty, \infty]$. Then f is measurable.

(Micah, Problem 1040) Prove Proposition 3.9 by showing that $\{x \in E : c \leq f(x)\}$ is measurable for all $c \in \mathbb{R}$. Be sure to explain why your proof works even if f_n, f are allowed to be infinite.

3.2 Simple Approximation

[Definition: Characteristic function] If $A \subseteq \mathbb{R}$, then the characteristic function of A, denoted χ_A, is defined by

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

[Definition: Simple function] A function φ is simple if its domain $E \subseteq \mathbb{R}$ is measurable, if φ is measurable on E, and if $\{\varphi(x) : x \in E\}$ is a set of of finitely many real numbers.

[Chapter 3, Problem 6] If E is measurable and $f : E \to \mathbb{R}$, then f is measurable on E if and only if

$$g(x) = \begin{cases} f(x), & x \in E, \\ 0, & x \notin E \end{cases}$$

is measurable on $\mathbb{R}$.

(Problem 1041) A function φ with domain $E \subseteq \mathbb{R}$ is simple if and only if $\varphi = \psi\big|_E$ for some simple function $\psi : \mathbb{R} \to \mathbb{R}$.

(Problem 1042) The set of simple functions contains all of the characteristic functions and is closed under taking finite linear combinations.

(Muhammad, Problem 1050) Suppose that $\varphi : \mathbb{R} \to \mathbb{R}$ is simple. Show that there is a unique list of numbers $c_1 < c_2 < \cdots < c_n$ and a unique list of nonempty measurable sets $E_1, E_2, \ldots, E_n$ such that $\varphi = \sum_{j=1}^{n} c_j \chi_{E_j}$. Furthermore, show that $\mathbb{R} = \bigcup_{j=1}^{n} E_j$ and $E_j \cap E_k = \emptyset$ for all $j \neq k$.

We may write $\varphi(\mathbb{R}) = \{c_1, \ldots, c_n\}$ because φ takes on finitely many values. As $\varphi(\mathbb{R})$ is a set, we may require the c_ks to be distinct. Any finite set can be ordered, so we may require $c_1 < c_2 < \cdots < c_n$. Let $E_k = \varphi^{-1}(\{c_k\})$; the given properties are straightforward to check.

(Problem 1051) If φ and ψ are simple functions with the same domain, show that $\max(\varphi, \psi)$ is also simple.

The simple approximation lemma. Let $f : E \to \mathbb{R}$ be measurable and bounded. Let $\varepsilon > 0$. Then there are two simple functions φ_ε and ψ_ε with

$$\psi_\varepsilon(x) - \varepsilon \leq \varphi_\varepsilon(x) \leq f(x) \leq \psi_\varepsilon(x) \leq \varphi_\varepsilon(x) + \varepsilon$$

for all $x \in E$.

(Ashley, Problem 1060) Prove the simple approximation lemma.

Let M be such that $-M \leq f(x) \leq M$ for all $x \in E$; such an M exists by definition of bounded function. Let $N \in \mathbb{N}$ be such that $N\varepsilon > M$; such an N exists by the Archimedean property.

For each $k \in \mathbb{Z}$, let $D_k = f^{-1}([k\varepsilon, (k+1)\varepsilon))$ and let $E_k = f^{-1}(((k-1)\varepsilon, k\varepsilon])$. If $|k| \geq N+1$, then D_k and E_k are empty. Furthermore, $E = \bigcup_{k=-N}^{N} D_k = \bigcup_{k=-N}^{N} E_k$ and $D_k \cap D_j = \emptyset = E_k \cap E_j$ if $j \neq k$.

Let $\varphi_\varepsilon = \sum_{k=-N}^{N} k\varepsilon\chi_{D_k}$ and let $\psi_\varepsilon = \sum_{k=-N}^{N} k\varepsilon\chi_{E_k}$. These functions are simple by construction.

If $f(x) = k\varepsilon$ for some $k \in \mathbb{Z}$, then $x \in D_k \cap E_k$ and so $\varphi_\varepsilon(x) = k\varepsilon = f(x) = \psi_\varepsilon(x)$, and thus the desired inequalities hold.

Otherwise, $k\varepsilon < f(x) < (k+1)\varepsilon$ for some $k \in \mathbb{Z}$, and so $x \in D_k \cap E_{k+1}$. Thus $\varphi(x) = k\varepsilon < f(x) < (k+1)\varepsilon = \psi_\varepsilon(x)$, and $\psi_\varepsilon(x) = \varphi_\varepsilon(x) + \varepsilon$, and so the desired inequalities are again satisfied.

The simple approximation theorem. Let $f : E \to [-\infty, \infty]$. Then f is measurable if and only if there is a sequence $\{\varphi_n\}_{n=1}^{\infty}$ such that

(a) $\varphi_n \to f$ pointwise,

(b) Each φ_n is simple,
(c) $\{|\varphi_n(x)|\}_{n=1}^\infty$ is nondecreasing for all $x \in E$.

(Problem 1061) Prove the easy direction; that is, suppose that such a sequence $\{\varphi_n\}_{n=1}^\infty$ exists and show that f is measurable.

(Bashar, Problem 1070) Prove the simple approximation theorem.

For each $n \in \mathbb{N}$, $k \in \mathbb{N}$, let $E_{n,k} = \{x \in E : |f(x)| \geq k/2^n\}$. Define

$$\psi_n(x) = \sum_{k=1}^{n2^n} \frac{1}{2^n} \chi_{E_{n,k}}.$$

Then ψ is simple and nonnegative.

If $x \in E_{n,k}$ and $k \leq n2^n$, then $2k \leq n2^{n+1} \leq (n+1)2^{n+1}$, $x \in E_{n+1,2k}$, and $x \in E_{n+1,2k-1}$. Thus $\sum_{k=1}^{(n+1)2^{n+1}} \frac{1}{2^{n+1}} \chi_{E_{n+1,k}}(x)$ has at least twice as many nonzero terms as $\sum_{k=1}^{n2^n} \frac{1}{2^n} \chi_{E_{n,k}}(x)$, so $\{|\psi_n(x)|\}_{n=1}^\infty$ is nondecreasing.

Finally, for any x and n, either $\psi_n(x) = n$ if $|f(x)| \geq n$, or $|f(x)| - 1/2^n \leq \psi_n(x) \leq |f(x)|$ if $|f(x)| \leq n$. It is clear that $\psi_n(x) \to |f(x)|$ pointwise.

Now, define $\varphi_n(x) = \psi(x) \operatorname{sgn}(f(x))$, that is,

$$\varphi_n(x) = \begin{cases} \psi(x), & f(x) \geq 0, \\ -\psi(x), & f(x) < 0. \end{cases}$$

Then φ_n is also simple, $|\varphi_n| = |\psi_n|$ so $\{|\varphi_n(x)|\}_{n=1}^\infty$ is nondecreasing for all $x \in E$, and $\varphi_n \to f$ pointwise.

(Dibyendu, Problem 1080) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a simple function and let c_j, E_j be as in Problem 1050. What do you expect $\int_\mathbb{R} \varphi \, dm$ to equal?

3.3. Undergraduate analysis

(Memory 1081) (Tietze's Extension Theorem in $\mathbb{R}$). Let $F \subseteq \mathbb{R}$ be closed and let $f : F \to \mathbb{R}$ be continuous. Then there is a function $g : \mathbb{R} \to \mathbb{R}$ that is continuous on all of $\mathbb{R}$ and satisfies $g = f$ on F.

(Memory 1082) Let F and D be two disjoint closed sets and let $f : F \cup D \to \mathbb{R}$ be a function. Suppose that $f\big|_F$ and $f\big|_D$ are continuous on F and D, respectively. Then f is continuous on $F \cup D$.

3.3 Egoroff's Theorem and Lusin's Theorem

Egoroff's theorem. Let $E \subseteq \mathbb{R}$ be measurable with $m(E) < \infty$. Let $\{f_n\}_{n=1}^\infty$ be a sequence of measurable functions on E that converges pointwise almost everywhere to some function f that is finite almost everywhere.

Then for every $\varepsilon > 0$ there is a closed set $F \subseteq E$ with $m(E \sim F) < \varepsilon$ and such that $f_n \to f$ uniformly on F.

Lemma 3.10. Under the conditions of Egoroff's theorem, if $\mu > 0$ and $\delta > 0$, then there is a measurable set $A \subseteq E$ and a $k \in \mathbb{N}$ such that $m(E \sim A) < \delta$ and such that $|f_n(x) - f(x)| < \mu$ for all $x \in A$ and all $n \geq k$.

(Elliott, Problem 1090) Prove Lemma 3.10. (Note that we will use Lemma 3.10 to prove Egoroff's theorem, and so you may not use Egoroff's theorem to prove Lemma 3.10.)

(Zach, Problem 1100) Use Lemma 3.10 to prove Egoroff's theorem.

Let $E_0 = \{x \in E : |f(x)| < \infty, f_n(x) \to f(x)\}$. By assumption $\{x \in E : |f(x)| = \infty\}$ and $\{x \in E : f_n(x) \not\to f(x)\}$ have measure zero, so E_0 is measurable and $m(E_0) = m(E)$.

If $\ell \in \mathbb{N}$, let $\mu = 1/\ell$ and let $\delta = \varepsilon/2^{\ell+1}$, and let $A = A_\ell$ and $k = k_\ell$ be as in Lemma 3.10. Then $A_\ell \subseteq E$, $m(E \sim A_\ell) < \varepsilon/2^{\ell+1}$, and $|f_n(x) - f(x)| < 1/\ell$ for all $x \in A_\ell$ and all $n \geq k_\ell$.

Let $B = \bigcap_{\ell=1}^\infty A_\ell$. Then B is measurable because the sets A_ℓ are measurable. Then $B \subseteq E$ and

$$m(E \sim B) = m\Big(E \sim \bigcap_{\ell=1}^\infty A_\ell\Big) = m\Big(\bigcup_{\ell=1}^\infty E \sim A_\ell\Big) \leq \sum_{\ell=1}^\infty m(E \sim A_\ell) < \frac{\varepsilon}{2}.$$

We claim that $f_n \to f$ uniformly on B. Let $\eta > 0$. There is a $\ell \in \mathbb{N}$ with $1/\ell < \eta$. If $n > k_\ell$, then $|f_n(x) - f(x)| < 1/\ell < \eta$ for all $x \in A_\ell$, and thus all $x \in B$ because $B \subseteq A_\ell$. Thus $f_n \to f$ uniformly on B.

Finally, by Theorem 2.11, there is a closed set F with $F \subseteq B$ and $m(B \sim F) < \varepsilon/2$. Then $F \subseteq B \subseteq E$ so $F \subseteq E$, $m(E \sim F) = m(E \sim B) + m(B \sim F) < \varepsilon$, and $f_n \to f$ uniformly on F because $F \subseteq B$ and $f_n \to f$ uniformly on B. This completes the proof.

(Irina, Problem 1110) Give an example of a sequence of measurable functions on an *unbounded* measurable set E that converges pointwise almost everywhere to some function f that is finite almost everywhere, but such that the conclusion of Egoroff's theorem fails.

[Chapter 3, Problem 16] Let $I \subseteq \mathbb{R}$ be a closed, bounded interval and let $E \subseteq I$ be measurable. Show that, for each $\varepsilon > 0$, there exists a step function $h : I \to \mathbb{R}$ and a measurable set $F \subseteq I$ such that $h = \chi_E$ on F and such that $m(I \sim F) < \varepsilon$.

Proposition 3.11. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a simple function and let $\varepsilon > 0$. Then there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $m^*(\{x \in \mathbb{R} : g(x) \neq \varphi(x)\}) < \varepsilon$.

(Juan, Problem 1120) Prove Proposition 3.11.

Lusin's theorem. Let $E \subseteq \mathbb{R}$ be a measurable set and let $f : E \to [-\infty, \infty]$ be measurable and finite almost everywhere. If $\varepsilon > 0$, then there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ and a closed set $F \subseteq E$ such that $f = g$ on F and such that $m(E \sim F) < \varepsilon$.

(Micah, Problem 1130) Prove Lusin's theorem in the case $m(E) < \infty$.

Let $\{\varphi_n\}_{n=1}^\infty$ be as in the simple approximation theorem, so $\varphi_n \to f$ pointwise. For each n, apply Proposition 3.11 to obtain a continuous function $g_n : \mathbb{R} \to \mathbb{R}$ such that $m(\{x \in \mathbb{R} : g_n \neq \varphi_n\}) < \frac{\varepsilon}{2^{n+1}}$.

Let $A_n = \{x \in \mathbb{R} : g_n \neq \varphi_n\}$. Then $m(\bigcup_{n=1}^\infty A_n) < \varepsilon/2$, and $g_n = \varphi_n$ on $E \sim \bigcup_{n=1}^\infty A_n$. Thus $g_n \to f$ on $E \sim \bigcup_{n=1}^\infty A_n$.

By Egoroff's theorem, there is a closed set $F \subseteq E \sim \bigcup_{n=1}^\infty A_n$ such that $g_n \to f$ uniformly on F and such that $m((E \sim \bigcup_{n=1}^\infty A_n) \sim F) < \varepsilon/2$. Thus, $F \subseteq E$ is closed and $m(E \sim F) < \varepsilon$, and because f is the uniform limit of a sequence of continuous functions, f is continuous on F.

By Tietze's extension theorem, there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ on F. This completes the proof.

(Problem 1131) Prove Lusin's theorem.

By the previous result, for each $n \in \mathbb{Z}$, there is a closed set $F_n \subset (n, n+1) \cap E$ such that f is continuous on F_n and such that $m(E \cap (n, n+1) \sim F_n) < \varepsilon/2^{|n|+2}$.

It is elementary to show that f is continuous on $\bigcup_{n \in \mathbb{Z}} F_n$, and $m(E \sim \bigcup_{n \in \mathbb{Z}} F_n) < \varepsilon$. The conclusion follows from Tietze's theorem.

4. Lebesgue Integration

4.1 Comments on the Riemann Integral

[Definition: Step function] If $[a, b] \subset \mathbb{R}$ is a closed and bounded interval, we say that $\varphi : [a, b] \to \mathbb{R}$ is a step function if there are finitely many points $a = x_0 < x_1 < \cdots < x_n = b$ such that φ is constant on each of the intervals (x_{k-1}, x_k) for all $1 \leq k \leq n$.

[Definition: Integral of a step function] If $\varphi : [a, b] \to \mathbb{R}$ is a step function and $x_0, x_1, \ldots, x_n$ are the numbers in the definition of step function, we define

$$\int_a^b \varphi = \sum_{k=1}^n (x_k - x_{k-1})\, \varphi\left(\frac{x_{k-1} + x_k}{2}\right).$$

[Definition: Riemann integrable] Let $[a, b] \subset \mathbb{R}$ be a closed and bounded interval and let $f : [a, b] \to \mathbb{R}$ be bounded. We say that f is Riemann integrable on $[a, b]$ if

$$\sup\left\{\int_a^b \varphi \;\middle|\; \varphi : [a, b] \to \mathbb{R} \text{ is a step function and } \varphi(x) \leq f(x) \text{ for all } x \in [a, b]\right\}$$
$$= \inf\left\{\int_a^b \psi \;\middle|\; \psi : [a, b] \to \mathbb{R} \text{ is a step function and } \psi(x) \geq f(x) \text{ for all } x \in [a, b]\right\}.$$

If f is Riemann integrable we define

$$\int_a^b f = \sup\left\{\int_a^b \varphi \;\middle|\; \varphi : [a, b] \to \mathbb{R} \text{ is a step function and } \varphi \leq f\right\}.$$

4.2 The Integral of a Bounded, Finitely Supported, Measurable Function

[Definition: Integral of a simple function] Let $E \subseteq \mathbb{R}$ be measurable with $m(E) < \infty$ and let $\varphi : E \to \mathbb{R}$ be simple. Let $\varphi(E) = \{c_1, c_2, \dots, c_n\}$; as in Problem 1050, $\varphi = \sum_{k=1}^n c_k \chi_{E_k}$, where $E_k = \varphi^{-1}(\{c_k\})$. We define

$$\int_E \varphi = \sum_{k=1}^n c_k \, m(E_k).$$

(Problem 1132) Let E be a measurable set. Let $\{D_1, \dots, D_\ell\}$ be a *partition* of E: $E = \bigcup_{j=1}^\ell D_j$ and $D_j \cap D_k = \emptyset$ if $j \neq k$. Suppose furthermore that each D_j is measurable. Let $\varphi : E \to \mathbb{R}$ and suppose that φ is constant on each D_j. Then φ takes on at most ℓ values, so is simple. Let b_j be such that $\varphi(x) = b_j$ for all $x \in D_j$. Show that

$$\int_E \varphi = \sum_{j=1}^\ell b_j \, m(D_j)$$

even if the D_js are not as in Problem 1050.

Let n, c_k, and E_k be as in Problem 1050. If $1 \leq j \leq \ell$, then either $D_j = \emptyset$ or D_j contains at least one point $x \in E$. But then $x \in E_k$ for some k, and so $b_j = \varphi(x) = c_k$ because $x \in D_j$ and $x \in E_k$. Thus $b_j = c_k$ and so $D_j \subseteq E_k$. Thus if D_j is not empty then $D_j \subseteq E_k$ for some k. Conversely, the E_ks are pairwise disjoint and so if $D_j \subseteq E_k$ then $D_j \cap E_r \subseteq E_k \cap E_r = \emptyset$ if $k \neq r$.

Thus we may write

$$\sum_{j=1}^\ell b_j \, m(D_j) = \sum_{\substack{1 \leq j \leq \ell \\ D_j = \emptyset}} b_j \, m(D_j) + \sum_{k=1}^n \sum_{\substack{1 \leq j \leq \ell \\ D_j \neq \emptyset \\ D_j \subseteq E_k}} b_j \, m(D_j).$$

If $D_j = \emptyset$ then $m(D_j) = 0$. Thus

$$\sum_{j=1}^\ell b_j \, m(D_j) = \sum_{k=1}^n \sum_{\substack{1 \leq j \leq \ell \\ D_j \subseteq E_k}} b_j \, m(D_j).$$

But if $D_j \subseteq E_k$ then $b_j = c_k$. So

$$\sum_{j=1}^\ell b_j \, m(D_j) = \sum_{k=1}^n c_k \sum_{\substack{1 \leq j \leq \ell \\ D_j \subseteq E_k}} m(D_j).$$

Because the D_js are pairwise disjoint and measurable,

$$\sum_{\substack{1 \leq j \leq \ell \\ D_j \subseteq E_k}} m(D_j) = m\Big(\bigcup_{\substack{1 \leq j \leq \ell \\ D_j \subseteq E_k}} D_j\Big).$$

By assumption $\bigcup_{\substack{1\leq j\leq \ell\\ D_j\subseteq E_k}} D_j \subseteq E_k$, while $E_k \subseteq E = \bigcup_{j=1}^{\ell} D_j$ and so $E_k = \bigcup_{j=1}^{\ell}(D_j \cap E_k)$. But either $D_j \cap E_k = \emptyset$ or $D_j \subseteq E_k$, and so $\bigcup_{\substack{1\leq j\leq \ell\\ D_j\subseteq E_k}} D_j = E_k$. Thus

$$\sum_{j=1}^{\ell} b_j\, m(D_j) = \sum_{k=1}^{n} c_k m(E_k)$$

as desired.

(Muhammad, Problem 1140) Suppose that $E = [a, b]$ and that φ is a step function. Show that φ is also a simple function and that $\int_E \varphi = \int_a^b \varphi$.

Lemma 4.1. If $E_1, E_2, \dots, E_n$ are measurable, $c_1, c_2, \dots, c_n \in \mathbb{R}$, $\varphi = \sum_{k=1}^n c_k\, \chi_{E_k}$, and $\bigcup_{k=1}^n E_k \subseteq E$ for some measurable set E, then $\int_E \varphi = \sum_{k=1}^n c_k\, m(E_k)$ even if the E_ks and c_ks are not as in Problem 1050.

(Ashley, Problem 1150) Prove Lemma 4.1.

Let $S = \{1, 2, \dots, n\}$. Recall that 2^S is the set of all subsets of S. Observe that 2^S is also a finite set. If $x \in E$, let $\sigma(x) = \{k \in S : x \in E_k\}$; then σ is a well defined function.

For each $A \subseteq S$, let

$$D_A = \Big(\bigcap_{k\in A} E_k\Big) \cap \Big(\bigcap_{k\in S\sim A} E \sim E_k\Big).$$

Then each D_A is measurable and $x \in D_{\sigma(x)}$ for each $x \in E$.

Furthermore, we claim that $A \neq B$ then $D_A \cap D_B = \emptyset$. To see this, observe that if $k \in A$ and $k \notin B$, then $D_A \subseteq E_k$ and $D_B \subseteq E \sim E_k$, and so $D_A \cap D_B = \emptyset$. Similarly if $k \in B$ and $k \notin A$ then $D_A \cap D_B = \emptyset$. If $A \neq B$ there is at least one such k.

Thus $\{D_A : A \in 2^S\}$ is a partition of E. Furthermore, observe that

$$\varphi(x) = \sum_{k=1}^{n} c_k\, \chi_{E_k}(x) = \sum_{k\in\sigma(x)} c_k$$

and so $\varphi = \sum_{k\in A} c_k$ on D_A, and in particular is constant on D_A.

Thus by Problem 1132,

$$\int_E \varphi = \sum_{A\in 2^S} \sum_{k\in A} c_k m(D_A).$$

Changing the order of summation, we see that

$$\int_E \varphi = \sum_{k=1}^{n} c_k \sum_{\substack{A\in 2^S\\ A\ni k}} m(D_A).$$

The D_As are a partition of E, and each D_A is either a subset of E_k or a subset of $E \sim E_k$ (that is, disjoint from E_k); thus $E_k = \bigcup_{\substack{A\in 2^S\\ D_A\subseteq E_k}} D_A$ and $m(E_k) = \sum_{\substack{A\in 2^S\\ A\ni k}} m(D_A)$. Thus

$$\int_E \varphi = \sum_{k=1}^{n} c_k m(E_k)$$

as desired.

Proposition 4.2. Let φ, ψ be simple functions defined on a set of finite measure E.

(i) If $\alpha, \beta \in \mathbb{R}$, then $\int_E(\alpha\varphi + \beta\psi) = \alpha \int_E \varphi + \beta \int_E \psi$.
(ii) If $\varphi \leq \psi$ on E, then $\int_E \varphi \leq \int_E \psi$.

(Bashar, Problem 1160) Prove Proposition 4.2, part (i).

Because φ and ψ are simple, we may write $\varphi = \sum_{k=1}^{n} a_k \chi_{D_k}$ and $\psi = \sum_{k=1}^{\ell} b_k \chi_{S_k}$ for some real numbers a_k, b_k and some measurable sets D_k, $S_k \subseteq E$.

Define

$$\widetilde{a}_k = \begin{cases} a_k, & 1 \leq k \leq n, \\ 0, & n+1 \leq k \leq n+\ell, \end{cases} \qquad \widetilde{b}_k = \begin{cases} 0, & 1 \leq k \leq n, \\ b_{k-n}, & n+1 \leq k \leq n+\ell, \end{cases} \qquad E_k = \begin{cases} D_k, & 1 \leq k \leq n, \\ S_{k-n}, & n+1 \leq k \leq n+\ell. \end{cases}$$

Then $\varphi = \sum_{k=1}^{n+\ell} \widetilde{a}_k \chi_{E_k}$, $\psi = \sum_{k=1}^{n+\ell} \widetilde{b}_k \chi_{E_k}$, and $\alpha\varphi + \beta\psi = \sum_{k=1}^{n+\ell} (\alpha\widetilde{a}_k + \beta\widetilde{b}_k) \chi_{E_k}$, and so by Lemma 4.1

$$\alpha \int_E \varphi + \beta \int_E \psi = \alpha \sum_{k=1}^{n+\ell} \widetilde{a}_k \, m(E_k) + \beta \sum_{k=1}^{n+\ell} \widetilde{b}_k \, m(E_k) = \sum_{k=1}^{n+\ell} (\alpha\widetilde{a}_k + \beta\widetilde{b}_k) \, m(E_k) = \int_E (\alpha\varphi + \beta\psi)$$

as desired.

(Dibyendu, Problem 1170) Prove Proposition 4.2, part (ii).

Let $\eta = \psi - \varphi$. Then $\eta \geq 0$ on E. Furthermore, η is simple. Let $\{c_1, \dots, c_n\} = \eta(E)$ and let $E_k = \eta^{-1}(\{c_k\})$. Then by part (i) and by definition

$$\int_E \psi - \int_E \varphi = \int_E (\psi - \varphi) = \int_E \eta = \sum_{k=1}^{n} c_k \, m(E_k).$$

But each $m(E_k) \geq 0$ by definition of measure, and each $c_k \geq 0$ because $\eta \geq 0$ and so $\eta(E) \subset [0, \infty)$. Thus $\sum_{k=1}^{n} c_k \, m(E_k) \geq 0$, so $\int_E \psi \geq \int_E \varphi$, as desired.

[Definition: Integral of a bounded function over a bounded set] Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$ and let $f : E \to [-M, M]$ be a bounded function. We say that f is Lebesgue integrable over E if

$$\sup\left\{ \int_E \varphi \;\middle|\; \varphi : [a, b] \to \mathbb{R} \text{ is a simple function and } \varphi(x) \leq f(x) \text{ for all } x \in E \right\}$$
$$= \inf\left\{ \int_E \psi \;\middle|\; \psi : [a, b] \to \mathbb{R} \text{ is a simple function and } \psi(x) \geq f(x) \text{ for all } x \in E \right\}.$$

If f is Lebesgue integrable we define

$$\int_E f = \sup\left\{ \int_E \varphi \;\middle|\; \varphi : [a, b] \to \mathbb{R} \text{ is a simple function and } \varphi(x) \leq f(x) \text{ for all } x \in E \right\}.$$

(Problem 1171) Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$ and let $\theta : E \to \mathbb{R}$ be simple. Show that

$$\int_E \theta = \sup\left\{ \int_E \varphi \;\middle|\; \varphi : [a, b] \to \mathbb{R} \text{ is a simple function and } \varphi(x) \leq \theta(x) \text{ for all } x \in E \right\}$$
$$= \inf\left\{ \int_E \psi \;\middle|\; \psi : [a, b] \to \mathbb{R} \text{ is a simple function and } \psi(x) \geq \theta(x) \text{ for all } x \in E \right\}.$$

Thus all simple functions with domains of bounded measure are integrable and there is no ambiguity in using $\int_E \theta$ to denote both the integral of a simple function and of an arbitrary Lebesgue integrable function.

Theorem 4.3. If f is Riemann integrable on $[a, b]$, then f is Lebesgue integrable over $[a, b]$ and $\int_a^b f = \int_{[a,b]} f$.

(Elliott, Problem 1180) Prove Theorem 4.3 and give an example of a bounded measurable function defined on an interval $[a, b]$ that is Lebesgue integrable over $[a, b]$ but is not Riemann integrable.

Assume that $f : [a, b] \to \mathbb{R}$ is bounded and Riemann integrable. By Problem 1140, if φ is a step function on $[a, b]$, then φ is a simple function and $\int_a^b \varphi = \int_{[a,b]} \varphi$. Thus,

$$\sup\left\{\int_a^b \varphi \,\middle|\, \varphi : [a, b] \to \mathbb{R} \text{ is a step function and } \varphi(x) \le f(x) \text{ for all } x \in [a, b]\right\}$$
$$= \sup\left\{\int_{[a,b]} \varphi \,\middle|\, \varphi : [a, b] \to \mathbb{R} \text{ is a step function and } \varphi(x) \le f(x) \text{ for all } x \in [a, b]\right\}.$$

Because all step functions are simple functions, by definition of supremum

$$\sup\left\{\int_{[a,b]} \varphi \,\middle|\, \varphi : [a, b] \to \mathbb{R} \text{ is a step function and } \varphi(x) \le f(x) \text{ for all } x \in [a, b]\right\}$$
$$\le \sup\left\{\int_{[a,b]} \varphi \,\middle|\, \varphi : [a, b] \to \mathbb{R} \text{ is a simple function and } \varphi(x) \le f(x) \text{ for all } x \in [a, b]\right\}.$$

Now, if φ and ψ are simple functions and $\varphi \le f \le \psi$, then $\varphi \le \psi$ and so by Proposition 4.2 $\int_{[a,b]} \varphi \le \int_{[a,b]} \psi$. Thus, again by definition of supremum and infimum,

$$\sup\left\{\int_{[a,b]} \varphi \,\middle|\, \varphi : [a, b] \to \mathbb{R} \text{ is a simple function and } \varphi(x) \le f(x) \text{ for all } x \in [a, b]\right\}$$
$$\le \inf\left\{\int_{[a,b]} \psi \,\middle|\, \psi : [a, b] \to \mathbb{R} \text{ is a simple function and } \psi(x) \ge f(x) \text{ for all } x \in [a, b]\right\}.$$

Again by Problem 1140 and because all step functions are simple functions,

$$\inf\left\{\int_{[a,b]} \psi \,\middle|\, \psi : [a, b] \to \mathbb{R} \text{ is a simple function and } \psi(x) \ge f(x) \text{ for all } x \in [a, b]\right\}$$
$$\le \inf\left\{\int_a^b \psi \,\middle|\, \psi : [a, b] \to \mathbb{R} \text{ is a step function and } \psi(x) \ge f(x) \text{ for all } x \in [a, b]\right\}.$$

But because f is Riemann integrable, we have that

$$\inf\left\{\int_a^b \psi \,\middle|\, \psi : [a, b] \to \mathbb{R} \text{ is a step function and } \psi(x) \ge f(x) \text{ for all } x \in [a, b]\right\}$$
$$= \sup\left\{\int_a^b \varphi \,\middle|\, \varphi : [a, b] \to \mathbb{R} \text{ is a step function and } \varphi(x) \le f(x) \text{ for all } x \in [a, b]\right\}.$$

Thus the chain of inequalities collapses and we must have that all of the above quantities are equal. In particular,

$$\sup\left\{\int_{[a,b]} \varphi \,\middle|\, \varphi : [a, b] \to \mathbb{R} \text{ is a simple function and } \varphi(x) \le f(x) \text{ for all } x \in [a, b]\right\}$$
$$= \inf\left\{\int_{[a,b]} \psi \,\middle|\, \psi : [a, b] \to \mathbb{R} \text{ is a simple function and } \psi(x) \ge f(x) \text{ for all } x \in [a, b]\right\}$$

This is the definition of Lebesgue integrability. This completes the proof.

Now let $f = \chi_{\mathbb{Q}\cap[0,1]}$ be the characteristic function of the rationals restricted to $[0, 1]$. By Problem 370, f is not Riemann integrable. However, f is simple (the set $\mathbb{Q}$ is measurable because it has outer measure zero), and so is Lebesgue integrable by Problem 1171.

Theorem 4.4. Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$ and let $f : E \to [-M, M]$ be bounded and measurable. Then f is Lebesgue integrable.

(Irina, Problem 1190) Prove Theorem 4.4.

Let

$$L = \sup\left\{ \int_E \varphi \;\middle|\; \varphi : E \to \mathbb{R} \text{ is a simple function and } \varphi(x) \leq f(x) \text{ for all } x \in E \right\},$$
$$U = \inf\left\{ \int_E \psi \;\middle|\; \psi : E \to \mathbb{R} \text{ is a simple function and } \psi(x) \geq f(x) \text{ for all } x \in E \right\}.$$

As before, by Proposition 4.2, $L \leq U$.

By the simple approximation lemma, if $\varepsilon > 0$, then there exist simple functions φ_ε, $\psi_\varepsilon : E \to \mathbb{R}$ such that

$$\psi_\varepsilon(x) - \varepsilon \leq \varphi_\varepsilon(x) \leq f(x) \leq \psi_\varepsilon(x) \leq \varphi_\varepsilon(x) + \varepsilon$$

for all $x \in E$.

Thus, if $\varepsilon > 0$, then

$$L \geq \int_E \varphi_\varepsilon$$

and

$$U \leq \int_E \psi_\varepsilon.$$

Thus

$$0 \leq U - L \leq \int_E \psi_\varepsilon - \int_E \varphi_\varepsilon$$

and so by Proposition 4.2

$$0 \leq U - L \leq \int_E (\psi_\varepsilon - \varphi_\varepsilon) \leq \int_E \varepsilon = \varepsilon\, m(E)$$

for all $\varepsilon > 0$. Thus $U - L \leq 0$ and so $U - L = 0$. By definition of U, L, and Lebesgue integrablility, we have that f is Lebesgue integrable, as desired.

Theorem 5.7. Let $E \subset \mathbb{R}$ be measurable with $m(E) < \infty$ and let $f : E \to [-M, M]$ be bounded. Then f is measurable if and only if it is Lebesgue integrable. (You may not use this result until we prove it in Chapter 5, but you may find it interesting at this point.)

Theorem 4.5. Let f and g be bounded measurable functions defined on a set of finite measure E.

(i) If α, $\beta \in \mathbb{R}$, then $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$.

(ii) If $f \leq g$ on E, then $\int_E f \leq \int_E g$.

(Zach, Problem 1200) Prove Theorem 4.5, part (i).

(Juan, Problem 1210) Prove Theorem 4.5, part (ii).

Because f and g are measurable, we have that $g - f$ is measurable by Theorem 3.6, and so $g - f$ is integrable by Theorem 4.4. By part (i),

$$\int_E g - \int_E f = \int_E (g - f).$$

But $\varphi = 0$ is a simple function with $\varphi \leq g - f$ on E, so

$$0 = \int_E \varphi \leq \sup\left\{ \int_E \varphi \;\middle|\; \varphi : E \to \mathbb{R} \text{ is a simple function and } \varphi(x) \leq g(x) - f(x) \text{ for all } x \in E \right\} = \int_E (g - f) = \int_E g - \int_E f$$

and so

$$\int_E f \leq \int_E g$$

as desired.

Corollary 4.6. If A and B are two disjoint measurable sets of finite measure and $f : A \cup B \to \mathbb{R}$ is bounded and measurable, then $\int_{A \cup B} f = \int_A f + \int_B f$.

(Micah, Problem 1220) Prove Corollary 4.6.

Corollary 4.7. If $E \subset \mathbb{R}$ is measurable and has finite measure, and if $f : E \to \mathbb{R}$ is bounded and measurable, then

$$\left|\int_E f\right| \leq \int_E |f|.$$

Proposition 4.8. If $E \subset \mathbb{R}$ is measurable and has finite measure, if $f_n : E \to \mathbb{R}$ is bounded and measurable for each n, and if $f_n \to f$ uniformly on E, then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

(Muhammad, Problem 1230) Prove Proposition 4.8.

(Ashley, Problem 1240) Give an example of a sequence of measurable functions $\{f_n\}_{n=1}^{\infty}$, each of which is bounded, defined on a common measurable domain E of finite measure, such that $f_n \to f$ pointwise on E for some bounded measurable function $f : E \to \mathbb{R}$, but such that

$$\int_E f_n \not\to \int_E f.$$

(The failure can be either because $\lim_{n\to\infty} \int_E f_n$ does not exist, or because it exists but is not equal to $\int_E f$.)

The bounded convergence theorem. If $E \subset \mathbb{R}$ is measurable and has finite measure, if $f_n : E \to \mathbb{R}$ is measurable for each n, if there is a M such that $|f_n(x)| < M$ for all $x \in E$ and all $n \in \mathbb{N}$, and if $f_n \to f$ pointwise on E, then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

(Bashar, Problem 1250) Prove the Bounded Convergence Theorem. *Hint*: Use Egoroff's theorem.

Choose some $\varepsilon > 0$.

Let $F \subseteq E$ be as in Egoroff's theorem, so F is closed, $m(E \sim F) < \varepsilon$, and $f_n \to f$ uniformly on F. Thus, there is a $N \in \mathbb{N}$ such that, if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon$ for all $x \in F$.

If $n \geq N$, then by Theorem 4.5(i), Corollary 4.7, and Corollary 4.6,

$$\left|\int_E f_n - \int_E f\right| = \left|\int_E (f_n - f)\right| \leq \int_E |f_n - f| = \int_F |f_n - f| + \int_{E\sim F} |f_n - f|.$$

Then by Theorem 4.5(ii),

$$\int_F |f_n - f| + \int_{E\sim F} |f_n - f| \leq \int_F \varepsilon + \int_{E\sim F} 2M = m(F)\varepsilon + 2Mm(E \sim F) \leq m(E)\varepsilon + 2M\varepsilon.$$

Thus, if $n \geq N$ then

$$\left|\int_E f_n - \int_E f\right| \leq m(E)\varepsilon + 2M\varepsilon.$$

This suffices to show that $\int_E f_n \to \int_E f$.

4.3 The Integral of a Non-Negative Measurable Function

[Definition: Finite support] Let $E \subseteq \mathbb{R}$ be measurable and let $h : E \to \mathbb{R}$. Suppose that there is a measurable set $E_0 \subseteq E$ with $m(E_0) < \infty$ and such that $h(x) = 0$ for all $x \in E \sim E_0$. Then we say that h has finite support; if h is also bounded then we define $\int_E h = \int_{E_0} h$.

[Definition: Integral of a nonnegative function] Let $E \subseteq \mathbb{R}$ be measurable and let $f : E \to [0, \infty]$ be measurable. We define

$$\int_E f = \sup\left\{\int_E h \,\middle|\, h \text{ is bounded, measurable, of finite support, and } 0 \leq h \leq f \text{ on } E\right\}.$$

(Problem 1251) Show that if $m(E) < \infty$ and $f : E \to [0, M]$ is measurable, nonnegative, and bounded, then the above definition coincides with that in Section 4.2.

Chebychev's inequality. Let f be a nonnegative measurable function on a measurable set E. Let $\lambda > 0$. Then

$$m(\{x \in E : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_E f.$$

(Dibyendu, Problem 1260) Prove Chebychev's Inequality.

Let $E_{\lambda,N} = \{x \in E \cap [-N, N] : f(x) \geq \lambda\}$. Because f is measurable, so is $E_{\lambda,N}$. Let $h_N = \lambda \chi_{E_{\lambda,N}}$. Because f is nonnegative, $h_N(x) = 0 \leq f(x)$ for all $x \in E \sim E_{\lambda,N}$; by definition of $E_{\lambda,N}$, $h_N(x) = \lambda \leq f(x)$ for all $x \in E_{\lambda,N}$.

Furthermore, h_N is clearly a set of finite support.

Thus by definition of $\int_E f$, $\int_E f \geq \int_E h_N = \lambda m(E_{\lambda,N})$ and so $m(E_{\lambda,N}) \leq \frac{1}{\lambda} \int_E f$.

Observe that $\{x \in E : f(x) \geq \lambda\} = \bigcup_{N=1}^{\infty} E_{\lambda,N}$. Thus

$$m(\{x \in E : f(x) \geq \lambda\}) = m\Big(\bigcup_{N=1}^{\infty} E_{\lambda,N}\Big).$$

The sets $E_{\lambda,N}$ are nondecreasing in N, so by Theorem 2.15,

$$m(\{x \in E : f(x) \geq \lambda\}) = m\Big(\bigcup_{N=1}^{\infty} E_{\lambda,N}\Big) = \lim_{N\to\infty} m(E_{\lambda,N}).$$

Because the sets $E_{\lambda,N}$ are nondecreasing in N, the right hand side is the limit of a nondecreasing sequence of real numbers, and so

$$m(\{x \in E : f(x) \geq \lambda\}) = \sup_N m(E_{\lambda,N}).$$

But $m(E_{\lambda,N}) \leq \frac{1}{\lambda} \int_E f$ for each N, and so $\sup_N m(E_{\lambda,N}) \leq \frac{1}{\lambda} \int_E f$, as desired.

Proposition 4.9. Let f be a nonnegative measurable function on a measurable set E. Then $\int_E f = 0$ if and only if $f(x) = 0$ for almost every $x \in E$.

(Elliott, Problem 1270) Prove Proposition 4.9.

Suppose that $f \geq 0$ and $\int_E f = 0$. Let $E_n = \{x \in E : f(x) \geq \frac{1}{n}\}$. Then $\{x \in E : f(x) > 0\} = \bigcup_{n\in\mathbb{N}} E_n$ by the Archimedean property of the real numebrs. By Chebychev's inequality, $m(E_n) \leq n \int_E f = 0$ for each $n \in \mathbb{N}$, and so by the subadditivity of Lebesgue measure (Proposition 2.3), $m(\{x \in E : f(x) > 0\}) \leq \sum_{n=1}^{\infty} m(E_n) = 0$, as desired.

We now come to the converse. If φ is a simple function that is nonpositive almost everywhere, then $\int_E \varphi \leq 0$ by definition of integral of a simple function.

If h is a bounded measurable function of finite support, then h is Lebesgue integrable by Theorem 4.4 and so

$$\int_E h = \sup\Big\{\int_E \varphi \,\Big|\, \varphi : E \to \mathbb{R} \text{ is simple, } \varphi \leq h\Big\}.$$

If $h \leq 0$ almost everywhere, then $\varphi \leq 0$ almost everywhere for all such φ, and so $\int_E h \leq 0$.

Finally, if $f \geq 0$ is measurable and $f = 0$ almost everywhere, then

$$\int_E f = \sup\Big\{\int_E h \,\Big|\, h : E \to \mathbb{R} \text{ is bounded measurable and finitely supported, } h \leq f\Big\}.$$

All of the terms on the right hand side are nonnegative, so $\int_E f \leq 0$. But $h \equiv 0$ is a bounded measurable finitely supported function, and so $\int_E f \geq \int_E 0 = 0$; thus $\int_E f = 0$, as desired.

Theorem 4.10. Let f and g be nonnegative measurable functions defined on a measurable set E.

(i) If $\alpha > 0$ and $\beta > 0$, then $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$.

(ii) If $f \leq g$ on E, then $\int_E f \leq \int_E g$.

(Irina, Problem 1280) Prove Theorem 4.10, part (i).

(Problem 1281) Prove Theorem 4.10, part (ii).

Theorem 4.11. If A and B are two disjoint measurable sets and $f : A \cup B \to \mathbb{R}$ is measurable and nonnegative, then $\int_{A\cup B} f = \int_A f + \int_B f$. In particular, if $m(E_0) = 0$ and $E_0 \subseteq E$ for a measurable set E, then $\int_E f = \int_{E\sim E_0} f$ for every nonnegative measurable function $f : E \to [0, \infty]$.

(Juan, Problem 1290) Prove Theorem 4.11.

Fatou's lemma. Let $E \subseteq \mathbb{R}$ be measurable and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions $f_n : E \to [0, \infty]$. Then

$$\int_E \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_E f_n.$$

(Zach, Problem 1300) Prove Fatou's lemma.

The monotone convergence theorem. Let $E \subseteq \mathbb{R}$ be measurable and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions $f_n : E \to [0, \infty]$. Suppose in addition that $f_n(x) \leq f_{n+1}(x)$ for all $x \in E$. Then

$$\int_E \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_E f_n.$$

(Micah, Problem 1310) Prove the monotone convergence theorem.

Corollary 4.12. Let $E \subseteq \mathbb{R}$ be measurable and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of nonnegative measurable functions $f_n : E \to [0, \infty]$. Then

$$\int_E \sum_{n=1}^{\infty} f_n = sum_{n=1}^{\infty} \int_E f_n.$$

(Problem 1311) Prove Corollary 4.12.

[Definition: Integrable function] A nonnegative measurable function f on a measurable set E is said to be integrable, integrable over E, or in $L^1(E)$, if

$$\int_E f < \infty.$$

Proposition 4.13. If f is integrable then f is finite almost everywhere.

(Muhammad, Problem 1320) Prove Proposition 4.13.

4.4 The General Lebesgue Integral

Proposition 4.14. Let $E \subseteq \mathbb{R}$ be measurable and let $f : E \to [-\infty, \infty]$ be measurable. Then $|f|$ is integrable (that is, $\int_E |f| < \infty$) if and only if both f^+ and f^- are integrable.

[Definition: General Lebesgue integral] Suppose that $f : E \to [-\infty, \infty]$ is measurable and that $|f|$ is integrable. Then we say that f is integrable and that

$$\int_E f = \int_E f^+ - \int_E f^-.$$

(Problem 1321) Show that if f is integrable over E and nonnegative, then the above definition of $\int_E f$ coincides with that in Section 4.3.

Proposition 4.15. If f is integrable over E, then f is finite almost everywhere on E and $\int_E f = \int_{E\sim E_0} f$ whenever $m(E_0) = 0$.

(Problem 1322) Prove Proposition 4.15.

Proposition 4.16. (The integral comparison test.) Suppose that g is nonnegative and integrable over E and that $|f| \leq g$ on E. If f is measurable, then f is also integrable and $\left|\int_E f\right| \leq \int_E |f| \leq \int_E g$.

(Ashley, Problem 1330) Prove Proposition 4.16.

Theorem 4.17. Let f and g be functions integrable over a measurable set E.

(i) If $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$, then $\alpha f + \beta g$ is integrable over E and $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$.
(ii) If $f \leq g$ on E, then $\int_E f \leq \int_E g$.

(Bashar, Problem 1340) Prove Theorem 4.17, part (i).

(Elliott, Problem 1350) Prove Theorem 4.17, part (ii).

Corollary 4.18. If A and B are two disjoint measurable sets and $f : A \cup B \to \mathbb{R}$ is integrable over $A \cup B$, then $\int_{A \cup B} f = \int_A f + \int_B f$.

(Dibyendu, Problem 1360) Prove Corollary 4.18.

The Lebesgue dominated convergence theorem. Let $E \subseteq \mathbb{R}$ be measurable and let f, f_n, and g be measurable functions with domain E. Suppose that g is nonnegative and integrable, that $|f_n(x)| \leq g(x)$ for all $n \in \mathbb{N}$ and all $x \in E$, and that $f_n \to f$ pointwise almost everywhere on E. Then $\int_E f_n \to \int_E f$.

(Zach, Problem 1370) Prove the Lebesgue dominated convergence theorem.

4.5 Countable Additivity and Continuity of Integration

Theorem 4.20. Let $\{E_n\}_{n=1}^{\infty}$ be a countable sequence of pairwise disjoint measurable sets. Let $E = \bigcup_{n=1}^{\infty} E_n$. If $f : E \to [-\infty, \infty]$ is integrable (that is, measurable and $\int_E |f| < \infty$), then

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f.$$

(Irina, Problem 1380) Prove Theorem 4.20.

Theorem 4.21. Let $\{E_n\}_{n=1}^{\infty}$ be a countable sequence of measurable sets, let $E = \bigcup_{n=1}^{\infty} E_n$, and suppose that $f : E \to [-\infty, \infty]$ is integrable (that is, measurable and $\int_E |f| < \infty$).

Suppose that either:

- $E_n \subseteq E_{n+1}$ for all n and $D = E = \bigcup_{n=1}^{\infty} E_n$.
- $E_n \supseteq E_{n+1}$ for all n and $D = \bigcap_{n=1}^{\infty} E_n$.

Then

$$\int_D f = \lim_{n \to \infty} \int_{E_n} f.$$

[Chapter 4, Problem 39] Prove Theorem 4.21.

4.6 Uniform Integrability: the Vitali Convergence Theorem

Lemma 4.22. Let $E \subset \mathbb{R}$ be measurable and suppose $m(E) < \infty$. Let $\delta > 0$. Then there is a $n \in \mathbb{N}$ and a list of pairwise disjoint sets $E_1, E_2, \dots, E_n$ such that $m(E_k) < \delta$ for all k and such that $E = \bigcup_{k=1}^{n} E_k$.

(Micah, Problem 1390) Prove Lemma 4.22.

Proposition 4.23. Let $E \subseteq \mathbb{R}$ be measurable and let f be a measurable function on E.

(a) Suppose that $\int_E |f| < \infty$ (that is, f is integrable) and that $\varepsilon > 0$. Then there is a $\delta > 0$ such that, if $A \subseteq E$ is measurable and $m(A) < \delta$, then $\int_A |f| < \varepsilon$.
(b) Suppose that $m(E) < \infty$ and that, for at least one $\varepsilon > 0$, there is a $\delta > 0$ such that, if $A \subseteq E$ is measurable and $m(A) < \delta$, then $\int_A |f| < \varepsilon$. Then $\int_E |f| < \infty$.

(Muhammad, Problem 1400) Prove Proposition 4.23, part (a).

(Juan, Problem 1410) Prove Proposition 4.23, part (b).

[Definition: Uniformly integrable] Let $E \subseteq \mathbb{R}$ be measurable and let $\mathcal{F}$ be a family of measurable functions on E. We say that $\mathcal{F}$ is uniformly integrable over E if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that, for all $f \in \mathcal{F}$, we have that if $A \subseteq E$ is measurable and $m(A) < \delta$, then $\int_A |f| < \varepsilon$.

(Ashley, Problem 1420) Let $E \subseteq \mathbb{R}$ be measurable and let g be integrable over E. Show that $\mathcal{F} = \{f | f : E \to [-\infty, \infty]$ is measurable and $|f(x)| \leq |g(x)|$ for all $x \in E\}$ is a uniformly integrable family.

Proposition 4.24. Any finite collection of integrable functions over a common domain E is uniformly integrable.

(Bashar, Problem 1430) Prove Proposition 4.24.

Proposition 4.25. Let $E \subset \mathbb{R}$ be measurable and assume $m(E) < \infty$. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on E and suppose that $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ is uniformly integrable. Suppose that $f_n \to f$ pointwise almost everywhere on E for some f. Then f is measurable.

(Elliott, Problem 1440) Prove Proposition 4.25.

(Dibyendu, Problem 1450)

(a) Provide a counterexample to show that the condition that $m(E) < \infty$ is a necessary condition; that is, give a sequence of uniformly integrable functions that converge pointwise on a set of infinite measure to a function that is not integrable.

(b) Provide a counterexample to show that the condition that $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ be uniformly integrable is a necessary condition; that is, give a sequence of integrable functions that converge pointwise to a function that is not integrable.