Math 2584, Spring 2024

The final exam will occur on Wednesday, May 8, 2024, at 3:00 p.m., in SCEN 407. You are allowed a non-graphing calculator and two double-sided 3 inch by 5 inch cards of notes.

I am including the review sheets for Exams 1–3 to emphasize that the final is cumulative. New material appears in Problems 34, 37, –71, 87, 88, 91, and from Problem 92 onwards.

Please complete the online course evaluation at `course-val.uark.edu` on or before Friday, May 3. If at least 80% of the class completes the course evaluation before the deadline, I will drop your 2 lowest group project scores; otherwise, I will drop your 1 lowest group project score.

Please check your final exam schedule. If you have 3 or more final exams scheduled for the same day, and you need to reschedule the final exam for this class, please let me know by email immediately.

The following theorems will be written on the cover page of the exam:

Picard-Lindelöf theorem. Consider $\frac{dy}{dt} = f(t, y)$. If the functions of two variables f and $\frac{\partial f}{\partial y}$ are continuous near the point (t_0, y_0), we call (t_0, y_0) a "good" point for f. If (t_0, y_0) is a "good" point for f, then exists is a unique solution to the initial value problem $\frac{dy}{dt} = f(y, t)$, $y(t_0) = y_0$.

The solution will continue to exist and be unique until either it approaches a "bad" point, or until the solution becomes unbounded.

Theorem 4.1.1. *Consider the initial-value problem*

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dt} + a_0(x)\, y = g(x),$$
$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad y''(x_0) = y_2, \ldots, y^{(n-1)}(x_0) = y_{n-1}.$$

Suppose that $a_0(x)$, $a_1(x)$, ..., $a_n(x)$ and $g(x)$ are all continuous everywhere, and $a_n(x) \neq 0$ for all real numbers x. Then there is a unique solution to the initial-value problem for all x.

The following Laplace transforms will be written on the last page of the exam:

If α, β, a, and k are real numbers, $n \geq 0$ is an integer, $f(t)$ and $g(t)$ are functions with Laplace transforms, $F(s)$ is the Laplace transform of some function, and y is a differentiable function such that both y and $\frac{dy}{dt}$ have Laplace transforms, then

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s\,\mathcal{L}\{y(t)\} - y(0)$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha\,\mathcal{L}\{f(t)\} + \beta\,\mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad s > 0$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad s > 0$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad s > a$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2} \qquad s > 0$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2} \qquad s > 0$$

$$\mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} = f(t), \quad \mathcal{L}\{\mathcal{L}^{-1}\{F(s)\}\} = F(s)$$

$\mathcal{L}\{f(t)\} = F(s)$ if and only if $\mathcal{L}^{-1}\{F(s)\} = f(t)$

$$\mathcal{L}^{-1}\{F(s)\} = e^{at}\,\mathcal{L}^{-1}\{F(s+a)\}$$

If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$$\mathcal{L}\{\mathcal{U}(t-c)\} = \frac{e^{-cs}}{s} \qquad s > 0, c \geq 0$$

$$\mathcal{L}\{\delta(t-c)\} = e^{-cs} \qquad s > 0, c \geq 0$$

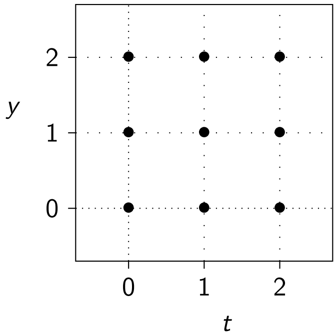
$$\mathcal{L}\{\mathcal{U}(t-c)f(t)\} = e^{-cs}\,\mathcal{L}\{f(t+c)\} \qquad c \geq 0$$

$$\mathcal{L}\{\mathcal{U}(t-c)g(t-c)\} = e^{-cs}\,\mathcal{L}\{g(t)\} \qquad c \geq 0$$

$$\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}\,\mathcal{L}\{f(t)\}$$

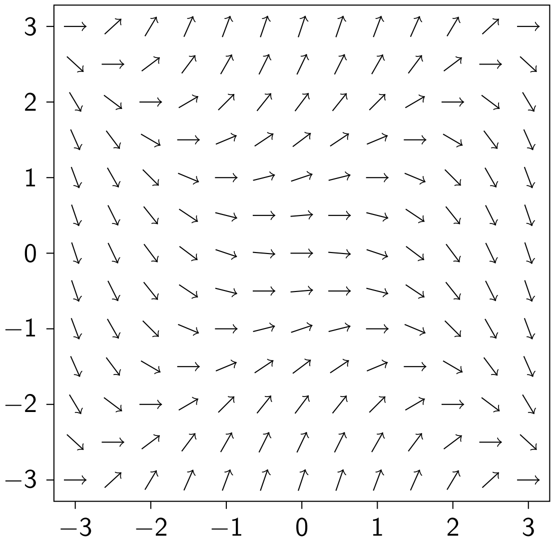
$$\mathcal{L}\left\{\int_0^t f(r)\,g(t-r)\,dr\right\} = \mathcal{L}\left\{\int_0^t f(t-r)\,g(r)\,dr\right\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$$

(AB 1) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.



(AB 2) Consider the differential equation $\frac{dy}{dx} = (y^2 - x^2)/3$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

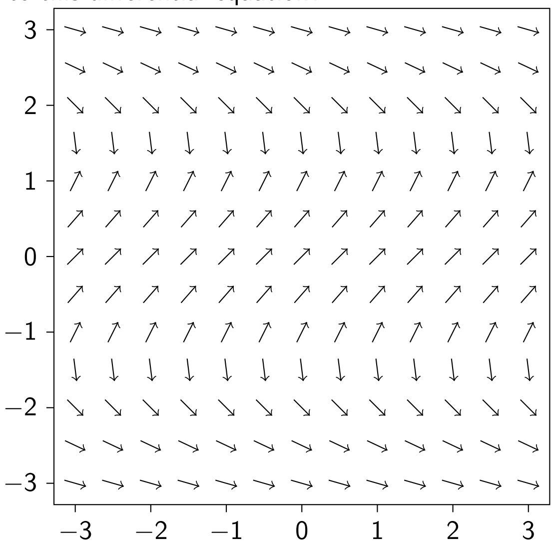
$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-2) = 1.$$



(AB 3) Consider the differential equation $\frac{dy}{dx} = \frac{2}{2-y^2}$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{2}{2-y^2}, \quad y(1) = 0.$$

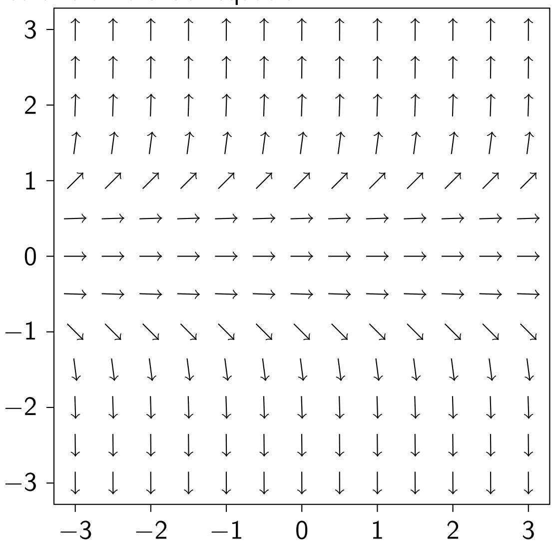
Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 4) Consider the differential equation $\frac{dy}{dx} = y^5$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = y^5, \quad y(1) = -1.$$

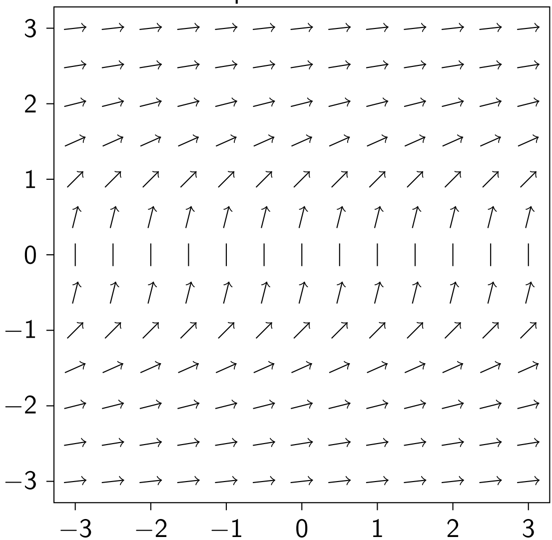
Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 5) Consider the differential equation $\frac{dy}{dx} = 1/y^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{1}{y^2}, \quad y(3) = 2.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 6) A lake initially has a population of 600 trout. The birth rate of trout is proportional to the number of trout living in the lake. Fishermen are allowed to harvest 30 trout/year from the lake. Write an initial value problem for the fish population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 7) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write an initial value problem for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 8) A tank contains nitrogen dioxide. Initially, there are 200 grams of NO_2. The gas decays (converts to molecular oxygen and nitrogen) at a rate proportional to the square of the amount of NO_2 remaining. Write an initial value problem for the amount of NO_2 in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 9) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen consumed. Write an initial value problem for the amount of hydrogen in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 10) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C. Write an initial value problem for the temperature of the cup of coffee. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 11) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Write an initial value problem for the amount of salt in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 12) I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write an initial value problem for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 13) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write an initial value problem for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 14) According to the Stefan-Boltzmann law, a black body in a dark vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 15) According to the Stefan-Boltzmann law, a black body in a vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). If there is some ambient light in the vacuum, then the black body absorbs energy at a rate proportional to the fourth power of the effective temperature of the light. The proportionality constant is the same in both cases.

Suppose that a planet in space is currently at a temperature of 400 K. The effective temperature of its surroundings is 3 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 16) A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 17) A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 18) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y-2)(y+1)^2$. By hand, sketch some typical solutions.

(AB 19) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 20) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = y^2(y-2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 21) Trout live in a lake. Each trout produces, on average, one baby trout every two years. Fishermen are allowed to harvest 100 trout/year from the lake. You may neglect other reasons why trout die.

(a) Formulate a differential equation for the number of trout in the lake.

(b) Find the critical points of this differential equation and classify them as to stability.

(c) What is the real-world meaning of the critical points?

(AB 22) I owe my bank a debt of B dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

(a) Formulate a differential equation for the amount of money I owe.

(b) Find the critical points of this differential equation and classify them as to stability.

(c) What is the real-world meaning of the critical points?

(AB 23) A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

(a) Write the differential equation for the amount of salt in the tank.

(b) Find the critical points of this differential equation and classify them as to stability.

(c) What is the real-world meaning of the critical points?

(AB 24) A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where v is her velocity in meters per second.

(a) Write a differential equation for her velocity. Assume her velocity is always downwards.

(b) Find the critical points of this differential equation and classify them as to stability. Be sure to include units.

(c) What is the real-world meaning of these critical points?

(AB 25) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen.

(a) Write a differential equation for the amount of hydrogen left in the tank.

(b) Find the critical points of this differential equation and classify them as to stability. Be sure to include units.

(c) What is the real-world meaning of these critical points?

(AB 26) Young oak trees grow in a large field with area 1 square kilometer. Initially, there are 30 trees in the field. Each tree shades 20 square meters. The trees produce acorns. Squirrels scatter the acorns across the field at random. Squirrels eat most of the acorns; on average, each tree produces two acorns per year that are not eaten by squirrels. Every acorn not eaten by squirrels and planted in sunlight sprouts.

(a) Write a differential equation for the number of trees in the field.

(b) Find the critical points of this differential equation and classify them as to stability.

(c) What is the real-world meaning of these critical points?

(AB 27) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate of 3 cm^3 per exposed cm2 per hour.

(a) Write a differential equation for the amount of water in the tank.

(b) Find the critical points of this differential equation and classify them as to stability.

(c) What is the real-world meaning of these critical points?

(AB 28) Is $y = e^t$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 29) Is $y = e^{2t}$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 30) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 1$?

(AB 31) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 2$?

(AB 32) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 1$?

(AB 33) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 2$?

(AB 34) For each of the following differential equations, determine whether it is linear, separable, exact, Bernoulli, of the form $\frac{dy}{dt} = f(y/t)$, or of the form $\frac{dy}{dt} = f(At+By+C)$. Then solve the differential equation.

(a) $t + \cos t + (y - \sin y)\frac{dy}{dt} = 0$
(b) $\ln y + y + x + \left(\frac{x}{y} + x\right)\frac{dy}{dx} = 0$
(c) $1 + t^2 - ty + (t^2+1)\frac{dy}{dt} = 0$
(d) $ty - y^2 - t^2 + t^2\frac{dy}{dt} = 0$
(e) $4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5)\frac{dy}{dt} = 0$
(f) $y\tan 2x + y^3\cos 2x + \frac{dy}{dx} = 0$
(g) $3t - 5x + (t+x)\frac{dx}{dt} = 0$
(h) $\frac{dy}{dt} = \frac{1}{3t+2y+7}$.
(i) $y^3\cos(2t) + \frac{dy}{dt} = 0$
(j) $4ty\frac{dy}{dt} = 3y^2 - 2t^2$
(k) $t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$
(l) $\sin^2(x-t)\frac{dx}{dt} = \csc^2(x-t)$.
(m) $\frac{dy}{dt} = 8y - y^8$
(n) $t\frac{dz}{dt} = -\cos t - 3z$
(o) $\frac{dy}{dt} = \cot(y/t) + y/t$
(p) $\frac{dy}{dt} = ty + t^2\sqrt[3]{y}$

(AB 35) For each of the following differential equations, find all the equilibrium solutions or state that no such solutions exist. You do not need to find the nonequilibrium solutions.

(a) $\frac{dy}{dt} = y^3 - yt$
(b) $\frac{dy}{dt} = t^2e^y$
(c) $\frac{dy}{dt} = ty + t^3$
(d) $\frac{dy}{dt} = (y^2 + 3y + 2)\sin(t)$
(e) $\frac{dy}{dt} = \ln(y^t)$

(AB 36) Suppose that $\frac{dy}{dt} = \cos(t)\sin(y)$, $y(0) = 3\pi$. Find $y(2)$.

(AB 37) For each of the following differential equations, determine whether it is linear or separable, exact, Bernoulli, of the form $\frac{dy}{dt} = f(y/t)$, or a function of a linear term. Then solve the given initial-value problem.

(a) $\cos(t+y^3) + 2t + 3y^2\cos(t+y^3)\frac{dy}{dt} = 0$, $y(\pi/2) = 0$

(b) $ty^2 - 4t^3 + 2t^2 y\frac{dy}{dt} = 0$, $y(1) = 3$.

(c) $1 + y^2 + t\frac{dy}{dt} = 0$, $y(1) = 1$

(d) $\frac{dy}{dt} = (2y + 2t - 5)^2$, $y(0) = 3$.

(e) $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$.

(f) $3y + e^{-3t}\sin t + \frac{dy}{dt} = 0$, $y(0) = 2$

(g) $\frac{dy}{dt} = (y^2 - 5y + 4)te^{t^2}$, $y(0) = 4$

(AB 38) Solve the initial-value problem $\frac{dy}{dt} = \frac{t-5}{y}$, $y(0) = 3$. Determine the domain of definition of the solution to the initial value problem. If the solution ceases to exist at a finite point, tell me whether the solution ceases to exist by approaching a "bad" point (in the sense of the Picard-Lindelöf theorem) or by becoming unbounded.

(AB 39) Solve the initial-value problem $\frac{dy}{dt} = y^2$, $y(0) = 1/4$. Determine the domain of definition of the solution to the initial value problem. If the solution ceases to exist at a finite point, tell me whether the solution ceases to exist by approaching a "bad" point (in the sense of the Picard-Lindelöf theorem) or by becoming unbounded.

(AB 40) Find all of the "bad" points (in the sense of the Picard-Lindelöf theorem) for the differential equation $\frac{dy}{dt} = \frac{1}{(t-3)\ln y}$.

(AB 41) Find all of the "bad" points (in the sense of the Picard-Lindelöf theorem) for the differential equation $\frac{dy}{dt} = \sqrt[3]{(y-4)(t-2)}$.

(AB 42)

(a) Find all equilibrium solutions to the differential equation $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$.

(b) Use the method of separation of variables to find a solution to the initial value problem $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$, $y(3) = 3$.

(c) Use the method of separation of variables to find a solution to the initial value problem $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$, $y(3) = 5$.

(d) You have found three solutions to the differential equation $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$. Do your solutions cross at any point?

(e) Find two piecewise-defined solutions to the initial value problem $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$, $y(3) = 3$ that are different from each other and also different from the solution you found in Problem (b).

(AB 43) The function $y_1(t) = e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2} - (1+2t)\frac{dy}{dt} + (t+1)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

(AB 44) The function $y_1(t) = t$ is a solution to the differential equation $t^2\frac{d^2y}{dt^2} - (t^2+2t)\frac{dy}{dt} + (t+2)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

(AB 45) The function $y_1(x) = x^3$ is a solution to the differential equation $x^2\frac{d^2y}{dx^2} + (x^2\tan x - 6x)\frac{dy}{dx} + (12 - 3x\tan x)y = 0$. Find the general solution to this differential equation on the interval $0 < x < \pi/2$.

(AB 46) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + \arctan(t)\, y = e^t$, $y(3) = 7$.

(b) $\frac{1}{1+t^2}\frac{dy}{dt} - t^5 y = \cos(6t)$, $y(2) = -1$, $y'(2) = 3$.

(c) $\frac{d^2y}{dt^2} - 5\sin(t)\, y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.

(d) $e^t\frac{d^2y}{dt^2} + 3(t-4)\frac{dy}{dt} + 4t^6 y = 2$, $y(3) = 1$, $y'(3) = -1$.

(e) $\frac{d^2y}{dt^2} + \cos(t)\frac{dy}{dt} + 3\ln(1+t^2)\, y = 0$, $y(2) = 3$.

(f) $\frac{d^3y}{dt^3} - t^7\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\, y = t^3$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$, $y'''(1) = 3$.

(g) $(1+t^2)\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = t^3$, $y(3) = 9$, $y'(3) = 7$, $y''(3) = 5$.

(h) $\frac{d^3y}{dt^3} - e^t\frac{d^2y}{dt^2} + e^{2t}\frac{dy}{dt} + e^{3t}y = e^{4t}$, $y(-1) = 1$, $y'(-1) = 3$.

(i) $(2+\sin t)\frac{d^3y}{dt^3} + \cos t\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\, y = t^3$, $y(7) = 2$.

(AB 47) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $e^t\frac{dy}{dt} + y = \cos t$, $y(0) = 3$, $y'(0) = -2$.

(b) $(t^2+4)\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + 6y = 7t^3$, $y(2) = 4$, $y'(2) = 4$, $y''(2) = 1$.

(c) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0$, $y(4) = 3$, $y'(4) = -2$, $y''(4) = 0$, $y'''(4) = 3$.

(AB 48) Find the general solution to the following differential equations.

(a) $\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 85x = 0$.

(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.

(c) $\frac{d^4z}{dt^4} + 7\frac{d^2z}{dt^2} - 144z = 0$.

(d) $\frac{d^4w}{dt^4} - 8\frac{d^2w}{dt^2} + 16w = 0$.

(AB 49) Solve the following initial-value problems. Express your answers in terms of real functions.

(*a*) $9\frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 2v = 0$, $v(0) = 3$, $v'(0) = 2$.

(*b*) $\frac{d^2u}{dt^2} + 10\frac{du}{dt} + 25u = 0$, $u(0) = 1$, $u'(0) = 4$.

(*c*) $8\frac{d^2f}{dt^2} - 6\frac{df}{dt} + f = 0$, $f(0) = 3$, $f'(0) = 1$.

(AB 50) A spring is suspended vertically. When an object with mass 5 kg is attached, it stretches the spring to a new equilibrium 4 cm lower. The system moves in a medium which damps it with damping constant 16 newton $\cdot$ seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Write the differential equation and initial conditions that describe the position of the object. You may use 9.8 meters/second^2 for the acceleration of gravity.

(AB 51) A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant β. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20$ N $\cdot$ s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30$ N $\cdot$ s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(AB 52) A 3-kg object is attached to a spring with constant k and to a viscous damper with damping constant 42 N·sec/m. The object is set in motion from its equilibrium position with initial velocity 5 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object.

Find the values of k for which the system is underdamped, overdamped, and critically damped. Be sure to include units for k.

(AB 53) A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant β. The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

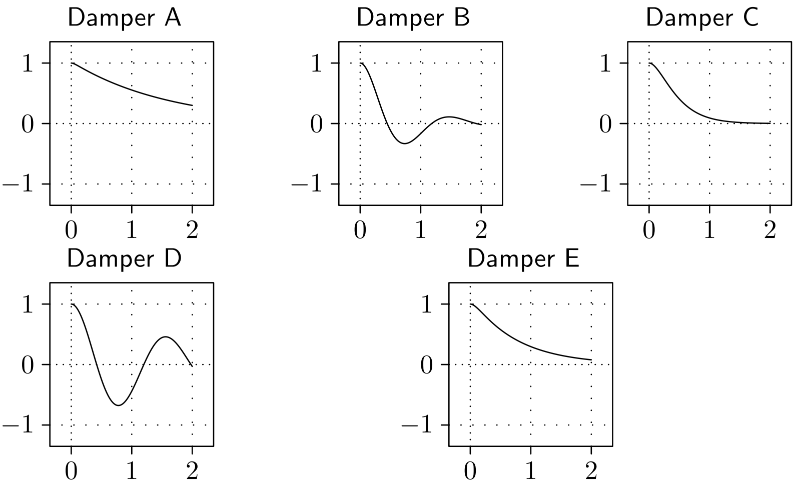
Write the differential equation and initial conditions that describe the position of the object. Then find the values of β for which the system is underdamped, overdamped, and critically damped. Be sure to include units for β.

(AB 54) An object of mass m is attached to a spring with constant 80 N/m and to a viscous damper with damping constant 20 N·s/m. The object is pulled down to 5cm below its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of m for which the system is underdamped, overdamped, or critically damped. Be sure to include units for m.

(AB 55) Five objects, each with mass 3 kg, are attached to five springs, each with constant 48 N/m. Five dampers with unknown constants are attached to the objects. In each case, the object is pulled down to a distance 1 cm below the equilibrium position and released from rest. You are given that the system is critically damped in exactly one of the five cases.

Here are the graphs of the objects' positions with respect to time:



(*a*) For which damper is the system critically damped?

(*b*) For which dampers is the system overdamped?

(*c*) For which dampers is the system underdamped?

(*d*) Which damper has the highest damping constant? Which damper has the lowest damping constant?

(AB 56) An object with mass 5 kg stretches a spring 4 cm. It is attached to a viscous damper with damping constant 16 newton · seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Using only first derivatives, write the differential equations and initial conditions for the object's position. That is, write a first-order system involving the object's position. You may use 9.8 meters/second^2 for the acceleration of gravity.

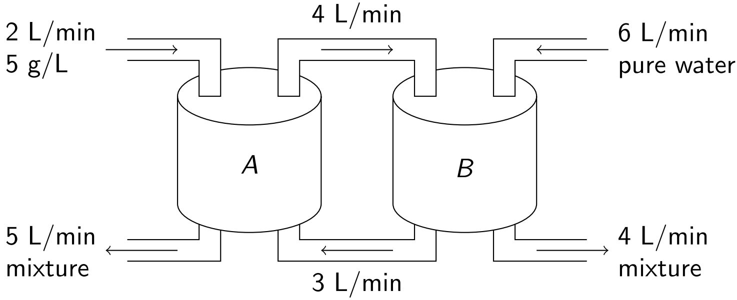
(AB 57) Imperial stormtroopers and Rebel Alliance fighters battle each other on an open plain, where both groups can easily see and aim at all members of the other group. Every minute, each stormtrooper has a 2% chance of killing a rebel, and each rebel has a 5% chance of killing a stormtrooper. There are initially 4000 stormtroopers and 1000 rebels.

Write the initial value problem for the number of stormtroopers and rebels still alive.

(AB 58) Jedi knights and Sith lords battle in a dense forest. Every minute, each Jedi has a 1% chance of finding each Sith lord. If a Jedi finds a Sith lord, they fight; the Jedi has a 60% chance of dying and the Sith lord has a 40% chance of dying. Initially there are 90 Jedi and 50 Sith lords.

Write the initial value problem for the number of Jedi and Sith lords still alive.

(AB 59) Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



Write the differential equations and initial conditions that describe the amount of salt in each tank.

(AB 60) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(AB 61) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(AB 62) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

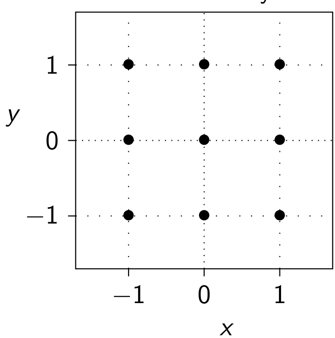
Set up the system of differential equations that describes the number of susceptible, infected, and resistant people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(AB 63) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers.

Each infected person encounters an average of 20 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. If an infected person encounters a recovered person, the recovered person has a 1% chance of contracting the disease again. Each infected person has a 12% chance of recovering from the disease on any given day.

Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(AB 64) Here is a grid. Draw a small phase plane (vector field) with nine arrows for the autonomous system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = x$.

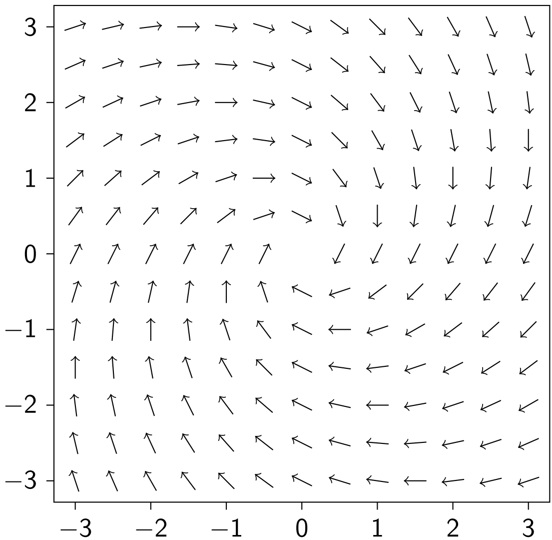


(AB 65) Here is the phase plane for the system

$$\frac{dx}{dt} = 2y - x, \quad \frac{dy}{dt} = -2x - y$$

Sketch the solution to the initial value problem

$$\frac{dx}{dt} = 2y - x, \quad \frac{dy}{dt} = -2x - y, \quad x(0) = 2, \quad y(0) = 1.$$



(AB 66) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Note: On the exam I may ask you to find the general solution instead. However, for a problem like this there are always many ways to write the general solution, not all of which are obviously equivalent; solutions to initial value problems take much more predictable forms and therefore make the answer key much easier to read.

(AB 67) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 68) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 69) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 4x{+}y{-}z, \qquad \frac{dy}{dt} = x{+}4y{-}z, \qquad \frac{dz}{dt} = 4z{-}x{-}y, \quad x(0) = 3, \quad y($$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 4-m & 1 & -1 \\ 1 & 4-m & -1 \\ -1 & -1 & 4-m \end{pmatrix} = -(m-3)^2(m-6).$

(AB 70) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 3x{+}y, \qquad \frac{dy}{dt} = 2x{+}3y{-}2z, \qquad \frac{dz}{dt} = y{+}3z, \qquad x(0) = 3, \quad y(0$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 3-m & 1 & 0 \\ 2 & 3-m & -2 \\ 0 & 1 & 3-m \end{pmatrix} = -(m-3)^3.$

(AB 71) Find the solution to the initial-value problem

$$\frac{dx}{dt} = x{+}y{+}z, \qquad \frac{dy}{dt} = x{+}3y{-}z, \qquad \frac{dz}{dt} = 2y{+}2z, x(0) = 4, \quad y(0) =$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 1-m & 1 & 1 \\ 1 & 3-m & -1 \\ 0 & 2 & 2-m \end{pmatrix} = -(m-2)^3.$

(AB 72) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x+9y-15, \qquad \frac{dy}{dt} = -5x+6y-8, \qquad x(0) = 2, \quad y(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 73) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 74) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y + e^{2t}\cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 5, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 75) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 2x+3y+\sin(e^t), \qquad \frac{dy}{dt} = -4x-5y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 76) Find the general solution to the equation $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3}\ln t$ on the interval $0 < t < \infty$.

(AB 77) Find the general solution to the equation $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$.

(AB 78) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - t\frac{dx}{dt} - 3x = 0$, $t > 0$, is $x(t) = C_1t^3 + C_2t^{-1}$. Find the general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

(AB 79) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - 2x = 0$, $t > 0$, is $x(t) = C_1t^2 + C_2t^{-1}$. Solve the initial-value problem $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(AB 80) You are given that the general solution to the differential equation $(1-t)\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = 0$ on the interval $t < 1$ is $x(t) = C_1t + C_2e^t$. Find the general solution to $(1-t)\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = (1-t)^2e^t$ on the interval $t < 1$.

(AB 81) Find the general solution to the following differential equations.

(a) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 3e^{4t}$.
(b) $16\frac{d^2y}{dt^2} - y = e^{t/4}\sin t$.
(c) $\frac{d^2y}{dt^2} + 49y = 3t\sin 7t$.
(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$.
(e) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = t\sin(3t)$.
(f) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$.
(g) $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 6t^2 + \cos(2t)$.
(h) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = t^2e^{3t}$.
(i) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}$.
(j) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t)$.
(k) $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t)$.
(l) $\frac{d^2y}{dt^2} + 9y = 5\sin(3t)$.

(m) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$.
(n) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$.
(o) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$.
(p) $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$.
(q) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$.

(AB 82) A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 $\text{N}\cdot\text{s/m}$ and is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(20t)$ N upwards. You may take the acceleration of gravity to be 9.8 meters/second^2.

Write the differential equation and initial conditions that describe the position of the object.

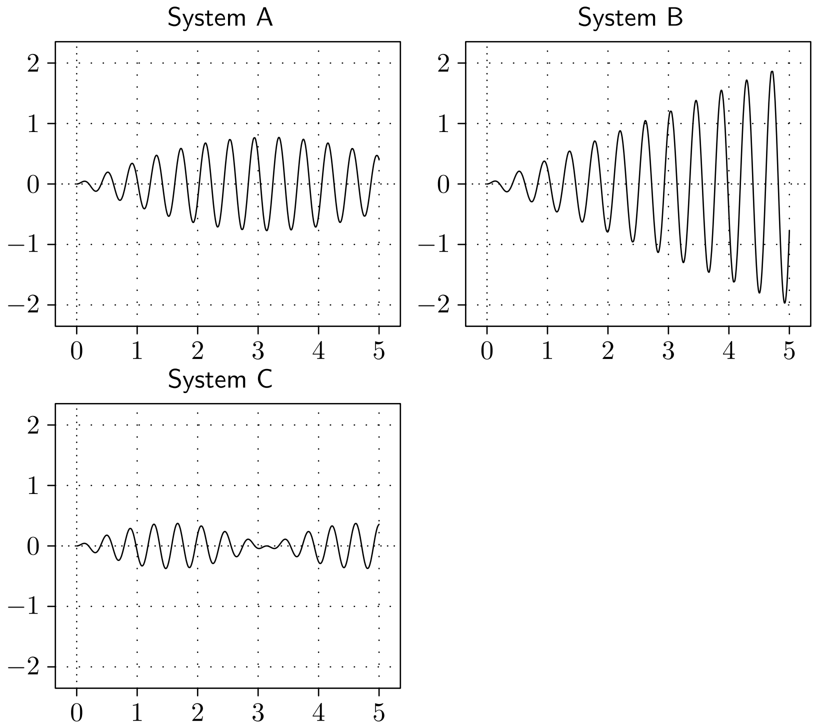
(AB 83) An object weighing 4 lbs stretches a spring 2 inches. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(\omega t)$ pounds, directed upwards. There is no damping. You may take the acceleration of gravity to be 32 feet/second^2.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of ω for which resonance occurs. Be sure to include units for ω.

(AB 84) A 4-kg object is suspended from a spring. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $7\sin(\omega t)$ newtons, directed upwards.

(a) Write the differential equation and initial conditions that describe the position of the object.
(b) It is observed that resonance occurs when $\omega = 20$ rad/sec. What is the constant of the spring? Be sure to include units.

(AB 85) A 1-kg object is suspended from a spring with constant 225 N/m. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $15\cos(\omega t)$ newtons, directed upwards. Illustrated are the object's position as a function of time for three different values of ω. You are given that the three values are $\omega = 15$ radians/second, $\omega = 16$ radians/second, and $\omega = 17$ radians/second. Determine the value of ω that will produce each image.



(AB 86) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ (**not** the table on the cover page of this exam), find the Laplace transforms of the following functions.

(a) $f(t) = e^{-11t}$

(b) $f(t) = t$

(c) $f(t) = \begin{cases} 3e^t, & 0 < t < 4, \\ 0, & 4 \leq t. \end{cases}$

(AB 87) Find the Laplace transforms of the following functions. You may use the table in your notes, on Blackboard, or in your book.

(a) $f(t) = t^4 + 5t^2 + 4$

(b) $f(t) = (t+2)^3$

(c) $f(t) = 9e^{4t+7}$

(d) $f(t) = -e^{3(t-2)}$

(e) $f(t) = (e^t + 1)^2$

(f) $f(t) = 8\sin(3t) - 4\cos(3t)$

(g) $f(t) = t^2 e^{5t}$

(h) $f(t) = 7e^{3t}\cos 4t$

(i) $f(t) = 4e^{-t}\sin 5t$

(j) $f(t) = t\, e^t \sin t$

(k) $f(t) = t^2 \sin 5t$

(l) $f(t) = \int_0^t e^{-4r}\sin(3r)\, dr$

(m) You are given that $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$. Find $\mathcal{L}\{t\, J_0(t)\}$. (The function J_0 is called a Bessel function and is important in the theory of partial differential equations in polar coordinates.)

(n) You are given that $\mathcal{L}\{\frac{1}{\sqrt{t}} e^{-1/t}\} = \frac{\sqrt{\pi}}{\sqrt{s}} e^{-2\sqrt{s}}$. Find $\mathcal{L}\{\sqrt{t}\, e^{-1/t}\}$.

(o) $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$

(p) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$

(q) $f(t) = \begin{cases} 0, & t < 1, \\ t - 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$

(r) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

(s) $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

(t) $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi \end{cases}$

(u) $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi \end{cases}$

(AB 88) For each of the following problems, find y.

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$

(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$

(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$

(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$

(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$

(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$

(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$

(h) $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$. *Hint*: Start by finding $\mathcal{L}\{t \sin 3t\}$ and $\mathcal{L}\{t \cos 3t\}$.

(i) $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$. You may express your answer as a definite integral.

(j) $\mathcal{L}\{y\} = \frac{s}{s^2-9} \mathcal{L}\{\sqrt{t}\}$. You may express your answer as a definite integral.

(k) $\mathcal{L}\{y\} = \frac{s}{s^4(s^2+36)}$. You may express your answer as a definite integral.

(l) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$

(m) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$

(AB 89) If a and b are constants, find $\mathcal{L}\{a \sin(4t) + b\,t \cos(4t)\}$. Then find values of a and b such that $\mathcal{L}\{a \sin(4t) + b\,t \cos(4t)\} = \frac{1}{(s^2+16)^2}$.

(AB 90) Sketch the graph of $y = t^2 - t^2\mathcal{U}(t-1) + (2-t)\mathcal{U}(t-1)$.

(AB 91) Solve the following initial-value problems using the Laplace transform. Do you expect the graphs of y, $\frac{dy}{dt}$, or $\frac{d^2y}{dt^2}$ to show any corners or jump discontinuities? If so, at what values of t?

(a) $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$

(b) $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$

(c) $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

(e) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

(f) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$

(g) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$

(h) $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

(i) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$

(j) $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

(k) $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$

(l) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$

(m) $y(t) + \int_0^t r\,y(t-r)\,dr = t$.

(n) $y(t) = te^t + \int_0^t (t-r)\,y(r)\,dr$.

(o) $\frac{dy}{dt} = 1 - \sin t - \int_0^t y(r)\,dr$, $y(0) = 0$.

(p) $\frac{dy}{dt} + 2y + 10\int_0^t e^{4r}y(t-r)\,dr = 0$, $y(0) = 7$.

(q) $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \leq t < 4, \\ 0, & 4 \leq t \end{cases}$, $y(0) = 2$, $y'(0) = 0$

(r) $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \leq t < 2\pi, \\ 0, & 2\pi \leq t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

(s) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \leq t < 10, \\ 0, & 10 \leq t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

(t) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \leq t < 2, \\ 4, & 2 \leq t \end{cases}$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = 2$.

(u) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$

(v) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$.

(w) $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$.

(x) $\frac{d^2y}{dt^2} + 4y = -2\delta(t-4\pi)$, $y(0) = 1/2$, $y'(0) = 0$

(y) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + \mathcal{U}(t-2)$, $y(0) = 1$, $y'(0) = 0$

(z) $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 5\delta(t-4)$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 2$.

(AB 92) Consider the initial value problem $\frac{dx}{dt} = x\cos y$, $\frac{dy}{dt} = x^2 \sin y$, $x(0) = 4$, $y(0) = \pi/2$. Use the phase plane method to find a nondifferential equation relating x and y.

(AB 93) Consider the system of differential equations $\frac{dx}{dt} = 3x - 4y$, $\frac{dy}{dt} = 4x - 3y$. Use the phase plane method to find a nondifferential equation relating x and y.

(AB 94) Consider the system of differential equations $\frac{dx}{dt} = 3y - 2xy$, $\frac{dy}{dt} = 4xy - 3y$. Use the phase plane method to find a nondifferential equation relating x and y.

(AB 95) Consider the system of differential equations $\frac{dx}{dt} = 3x - 4xy$, $\frac{dy}{dt} = 5xy - 2y$. Use the phase plane method to find a nondifferential equation relating x and y.

(AB 96) Imperial stormtroopers and Rebel Alliance fighters battle each other on an open plain, where both groups can easily see and aim at all members of the other group. Every minute, each stormtrooper has a 2% chance of killing a rebel, and each rebel has a 5% chance of killing a stormtrooper. There are initially 4000 stormtroopers and 1000 rebels.

Find a nondifferential equation relating the number of surviving rebels and the number of surviving stormtroopers.

(AB 97) Jedi knights and Sith lords battle in a dense forest. Every minute, each Jedi has a 1% chance of finding each Sith lord. If a Jedi finds a Sith lord, they fight; the Jedi has a 60% chance of dying and the Sith lord has a 40% chance of dying. Initially there are 90 Jedi and 50 Sith lords.

Find a nondifferential equation relating the number of Jedi and Sith lords still alive.

(AB 98) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(b) Use the phase plane method to find a nondifferential equation relating the number of susceptible and infected people.

(c) Use the phase plane method to find a nondifferential equation relating the number of resistant and susceptible people.

(d) What is the maximum number of people that are infected at any one time?

(AB 99) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(AB 100) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

(*a*) Set up the system of differential equations that describes the number of susceptible, infected, and resistant people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(*b*) Assume that there are initially no recovered or vaccinated people and 7 infected people. Use the phase plane method to find a nondifferential equation relating S and I.

(*c*) Find a nondifferential equation involving the maximum number of people that are infected with the virus at any one time.

(AB 101) A small town has a population of 16,000 people. It is expected that soon, one of them will be infected with a contagious disease.

Epidemiologists observe the town and expect each infected person to encounter 10 people per day, who are distributed among susceptible, vaccinated, recovered and infected people according to their proportion of the total population. Each time an infected person encounters a susceptible person, there is a 6% chance that the susceptible person becomes infected. Each infected person has a 25% chance per day of recovering. A recovered person can never contract the disease again.

A (not very effective) vaccine can be distributed before the epidemic starts. Each time an infected person encounters a vaccinated person, there is a 2% chance they become infected. (Once a vaccinated person becomes infected, they are identical to an infected person who was never vaccinated.)

The mayor of the town would like to be sure that no more than 2,000 people are ever infected.

(*a*) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(*b*) Use the phase plane method to find a nondifferential equation relating the number of susceptible and recovered people.

(*c*) Use the phase plane method to find a nondifferential equation relating the number of recovered people to the number of vaccinated (and never-infected) people.

(*d*) Find a nondifferential equation relating the number of infected people to the number of recovered people.

(*e*) How many people must be vaccinated in order for there to be at most 2000 previously-infected (and thus resistant) people when the epidemic ends?

(AB 102) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers.

Each infected person encounters an average of 20 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. If an infected person encounters a recovered person, the recovered person has a 1% chance of contracting the disease again. Each infected person has a 12% chance of recovering from the disease on any given day.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(b) Use the phase plane method to find a nondifferential equation relating the number of susceptible and recovered people. Solve for the number of recovered people.

(c) Write a nondifferential equation relating the number of infected people to the number of susceptible people.

(d) Does the number of infected people ever approach zero? If so, how many susceptible people are left at that time?

(e) Does the number of susceptible people ever approach zero? If so, how many infected people and how many resistant people are there at that time?

(AB 103) Suppose that $\frac{d^2x}{dt^2} = 18x^3$, $x(0) = 1$, $x'(0) = 3$. Let $v = \frac{dx}{dt}$. Find a formula for v in terms of x. Then find a formula for x in terms of t.

(AB 104) Suppose that a rocket of mass $m = 1000$ kg is launched straight up from the surface of the planet Gallifrey with initial velocity 10 km/sec. The radius of Gallifrey is 3,000 km. When the rocket is r meters from the center of Gallifrey, it experiences a force due to gravity of magnitude GMm/r^2, where $GM = 6 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the rocket's position.

(b) Find the velocity of the rocket as a function of position.

(c) How far away from the earth is the rocket when it stops moving and starts to fall back?

(AB 105) A 5-kg toolbox is dropped (from rest) out of a spaceship at an altitude of 9,000 km above the surface of Gallifrey. The radius of Gallifrey is 3,000 km. When the toolbox is r meters from the center of the earth, it experiences a force due to gravity of magnitude $5GM/r^2$, where $GM = 6 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the toolbox's position.

(b) Find the velocity of the toolbox as a function of position.

(c) How fast is the toolbox moving when it strikes the surface of Gallifrey?

(AB 106) A particle of mass $m = 3$ kg a distance r from an infinitely long string experiences a force due to gravity of magnitude Gm/r, where $G = 2000$ meters^2/second^2, directed directly toward the string. Suppose that the particle is initially 1000 meters from the string and takes off with initial velocity 200 meters/second directly away from the string.

(a) Formulate the initial value problem for the particle's position.

(b) Find the velocity of the particle as a function of position.

(c) How far away from the string is the particle when it stops moving and starts to fall back?

(AB 107) A charged particle of mass $m = 20$ g a distance r meters from an electric dipole experiences a force of $3/r^3$ newtons, directed directly toward the dipole. Suppose that the particle is initially 3 meters from the dipole and is set in motion with initial velocity 5 meters/second away from the dipole.

(a) Formulate the initial value problem for the particle's position.

(b) Find the velocity of the particle as a function of position.

(c) What is the limiting velocity of the particle?

(AB 108) Suppose that a bob of mass 300g hangs from a pendulum of length 15cm. The pendulum is set in motion from its equilibrium point with an initial velocity of 3 meters/second.

If θ denotes the angle between the pendulum and the vertical, then the pendulum satisfies the equation of motion $m\frac{d^2\theta}{dt^2} = -\frac{mg}{\ell}\sin\theta$, where m is the mass of the pendulum bob, ℓ is the length of the pendulum and g is the acceleration of gravity (which you may take to be 9.8 meters/second^2).

Find a formula for $\omega = \frac{d\theta}{dt}$ in terms of θ.

(AB 109) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

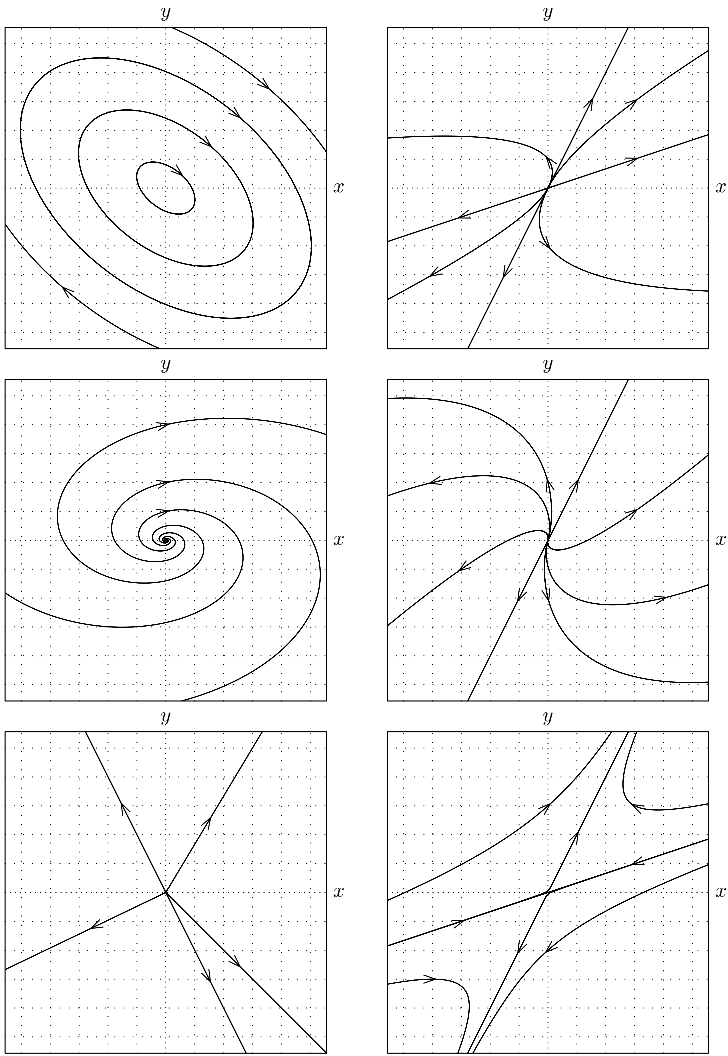
(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2.6 & 1.8 \\ -1.2 & 1.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4/3 & -1/6 \\ 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} t+3 \\ 2t \end{pmatrix}$$

(d) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4.5 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 3 \sin 3t \\ 2 \cos 3t \end{pmatrix} + C_2 e^{t} \begin{pmatrix} -3 \cos 3t \\ 2 \sin 3t \end{pmatrix}$$

(e) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 5 \sin 4t \\ 4 \cos 4t - 2 \sin 4t \end{pmatrix} + C_2 \begin{pmatrix} 5 \cos 4t \\ -4 \sin 4t - 2 \cos 4t \end{pmatrix}$$

(f) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$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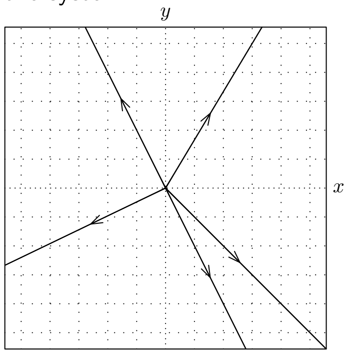
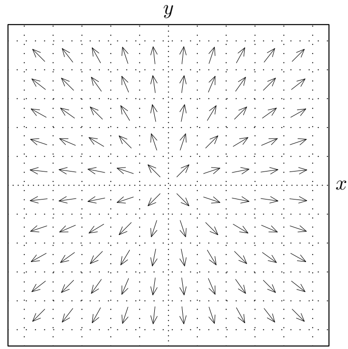
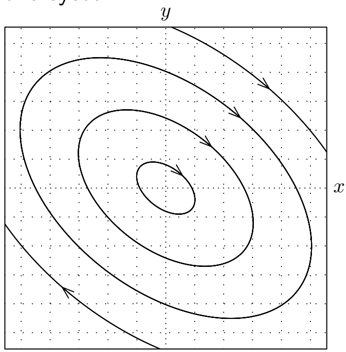
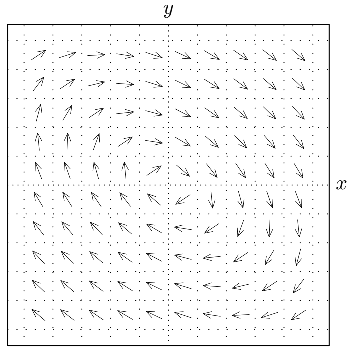
(AB 110) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



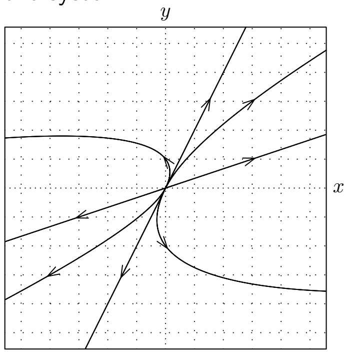


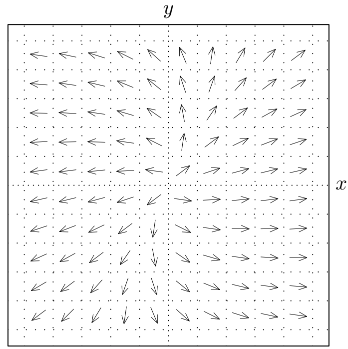
(AB 111) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



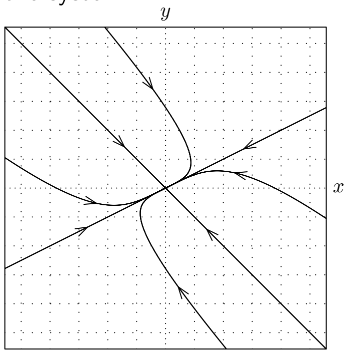


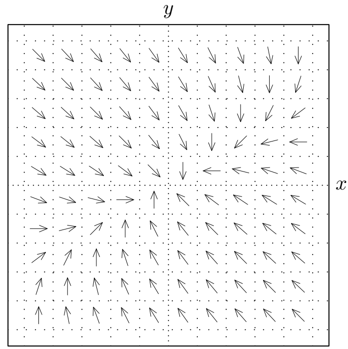
(AB 112) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



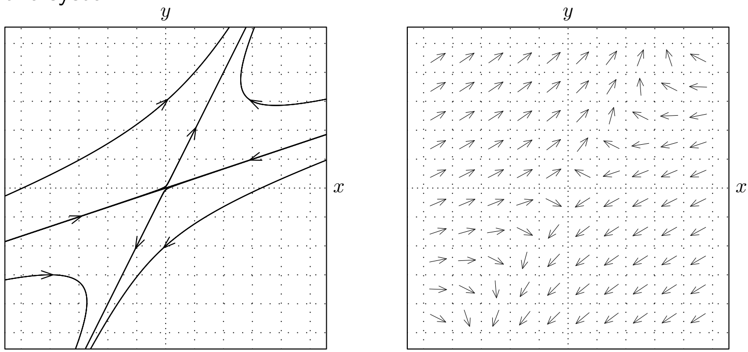


(AB 113) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?

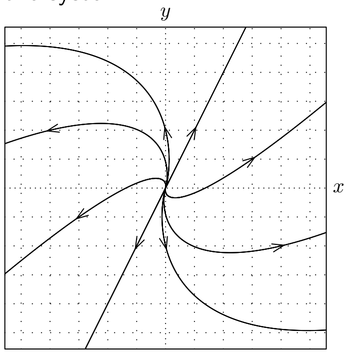


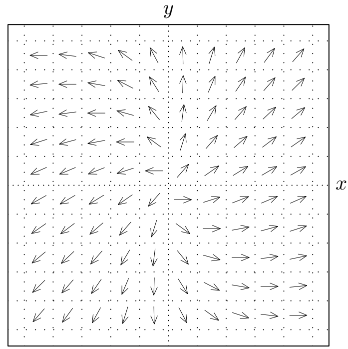


(AB 114) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?

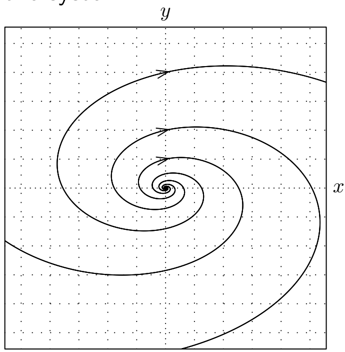


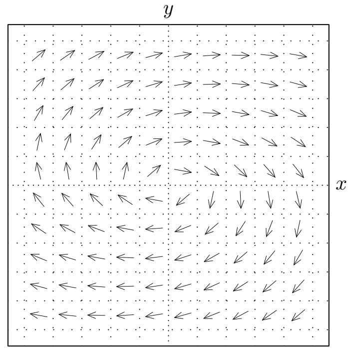
(AB 115) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?





(AB 116) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?

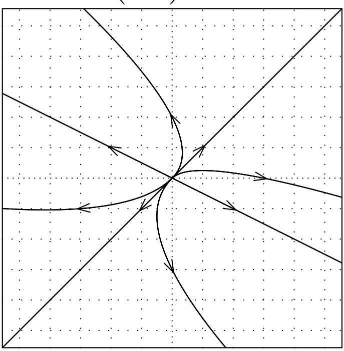


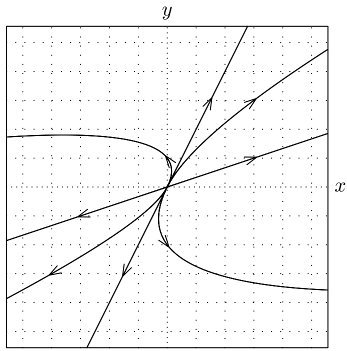


(AB 117) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

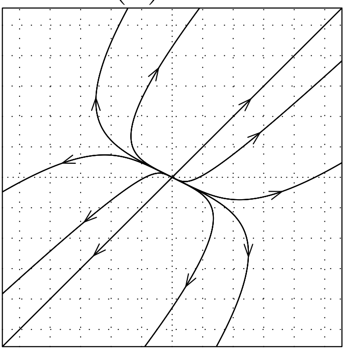


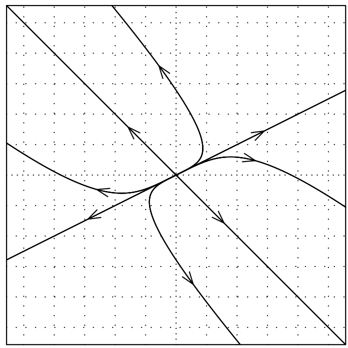


(AB 118) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

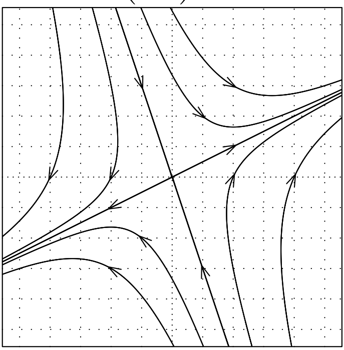


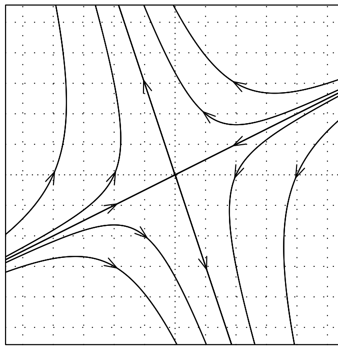


(AB 119) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 9 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-14t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -4 & -6 \\ -9 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{14t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

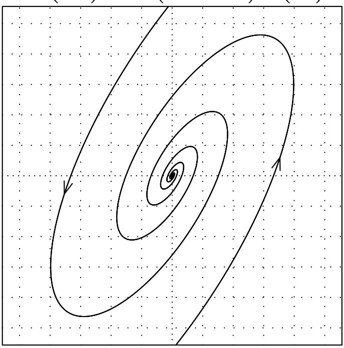


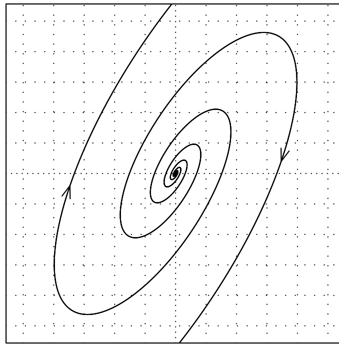


(AB 120) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -8 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, eigenvalues $r = 1 \pm 4i$

(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 8 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, eigenvalues $r = -1 \pm 4i$

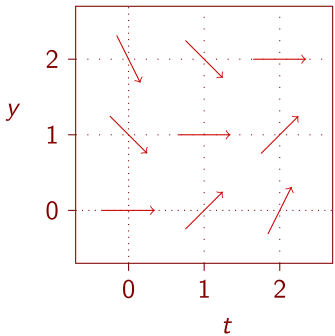




Answer key

(AB 1) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.

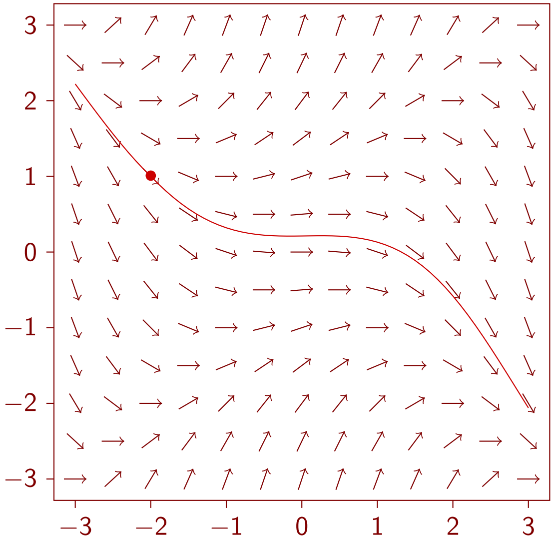
(Answer 1) Here is the direction field for the differential equation $\frac{dy}{dt} = t - y$.



(AB 2) Consider the differential equation $\frac{dy}{dx} = (y^2 - x^2)/3$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-2) = 1.$$

(Answer 2)

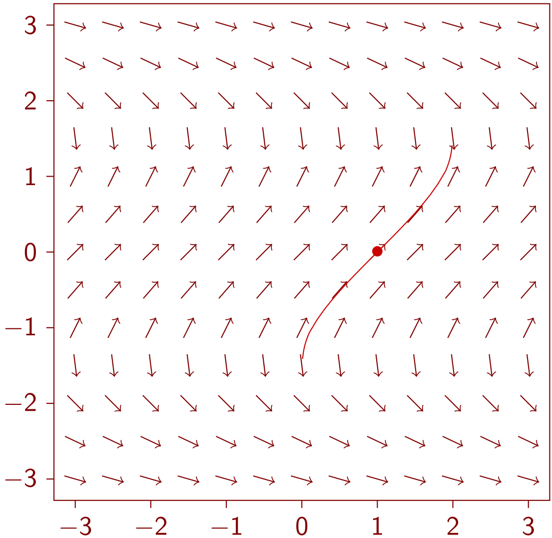


(AB 3) Consider the differential equation $\frac{dy}{dx} = \frac{2}{2-y^2}$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{2}{2-y^2}, \quad y(1) = 0.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?

(Answer 3)



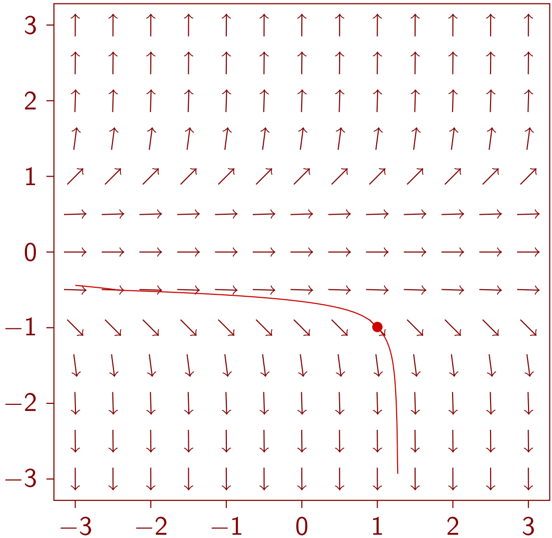
The domain of definition of the solution appears to be $0 < t < 2$.

(AB 4) Consider the differential equation $\frac{dy}{dx} = y^5$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = y^5, \quad y(1) = -1.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?

(Answer 4)



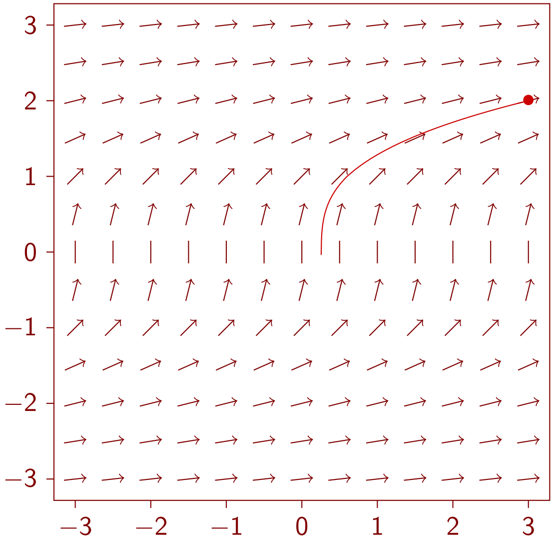
The domain of definition of the solution appears to be approximately $t < 1.3$.

(AB 5) Consider the differential equation $\frac{dy}{dx} = 1/y^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{1}{y^2}, \quad y(3) = 2.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?

(Answer 5)



The domain of definition of the solution appears to be $\frac{1}{4} < t$.

(AB 6) A lake initially has a population of 600 trout. The birth rate of trout is proportional to the number of trout living in the lake. Fishermen are allowed to harvest 30 trout/year from the lake. Write an initial value problem for the fish population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 6) Independent variable: t = time (in years).

Dependent variable: P = Number of trout in the lake

Initial condition: $P(0) = 600$.

Parameters: α = birth rate (in 1/years).

Differential equation: $\frac{dP}{dt} = \alpha P - 30$.

(AB 7) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write an initial value problem for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 7) Independent variable: t = time (in years).

Dependent variable: P = Number of birds on the island.

Parameters: α = birth rate parameter (in 1/years); β = death rate parameter (in 1/(bird·years)).

Initial condition: $P(0) = 5$.

Differential equation: $\frac{dP}{dt} = \alpha P - \beta P^2$.

(AB 8) A tank contains nitrogen dioxide. Initially, there are 200 grams of NO_2. The gas decays (converts to molecular oxygen and nitrogen) at a rate proportional to the square of the amount of NO_2 remaining. Write an initial value problem for the amount of NO_2 in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 8) Independent variable: t = time (in minutes).

Dependent variable: M = Amount of NO_2 in the tank (in grams).

Parameters: α = reaction rate parameter (in 1/second·grams).

Initial condition: $M(0) = 200$.

Differential equation: $\frac{dM}{dt} = -\alpha M^2$.

(AB 9) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen consumed. Write an initial value problem for the amount of hydrogen in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 9) Independent variable: $t =$ time (in minutes).

Dependent variable: $H =$ Amount of hydrogen in the tank (in grams).

Parameters: $\alpha =$ reaction rate parameter (in 1/minute·grams).

Initial condition: $H(0) = 50$.

Differential equation: $\frac{dH}{dt} = -\alpha H(126.9H - 3345)$.

(AB 10) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C. Write an initial value problem for the temperature of the cup of coffee. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 10) Independent variable: $t =$ time (in seconds).

Dependent variable: $T =$ Temperature of the cup (in degrees Celsius)

Initial condition: $T(0) = 95$.

Differential equation: $\frac{dT}{dt} = -\alpha(T - 25)$, where α is a positive parameter (constant of proportionality) with units of 1/seconds.

(AB 11) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Write an initial value problem for the amount of salt in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 11) Independent variable: $t =$ time (in minutes).
Dependent variable: $Q =$ amount of dissolved salt (in kilograms).
Initial condition: $Q(0) = 3$.
Differential equation: $\frac{dQ}{dt} = 0.01 - \frac{3Q}{600-t}$.

(AB 12) I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write an initial value problem for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 12) Independent variable: $t =$ time (in years).
Dependent variable: $B =$ balance of my loan (in dollars).
Initial condition: $B(0) = 300{,}000$.
Differential equation: $\frac{dB}{dt} = 0.04B - 12 \cdot 1600$

(AB 13) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write an initial value problem for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 13) Independent variable: t = time (in minutes).

Dependent variables:

h = depth of water in the hole (in centimeters)

V = volume of water in the hole (in cubic centimeters); notice that $V = \frac{1}{3}\pi(5h)^2 h$

Initial condition: $V(0) = 0$.

Differential equation: $\frac{dV}{dt} = 1 - \alpha\pi(5h)^2 = 1 - 25\alpha\pi(3V/25\pi)^{2/3}$, where α is a proportionality constant with units of cm/s.

(AB 14) According to the Stefan-Boltzmann law, a black body in a dark vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 14) Independent variable: t = time (in seconds).

Dependent variable: T = object's temperature (in kelvins)

Parameter: σ = proportionality constant (in $1/(\text{seconds}\cdot\text{kelvin}^3)$)

Initial condition: $T(0) = 400$.

Differential equation: $\frac{dT}{dt} = -\sigma T^4$.

(AB 15) According to the Stefan-Boltzmann law, a black body in a vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). If there is some ambient light in the vacuum, then the black body absorbs energy at a rate proportional to the fourth power of the effective temperature of the light. The proportionality constant is the same in both cases.

Suppose that a planet in space is currently at a temperature of 400 K. The effective temperature of its surroundings is 3 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 15) Independent variable: $t =$ time (in seconds).

Dependent variable: $T =$ object's temperature (in kelvins)

Parameter: $\sigma =$ proportionality constant (in 1/(seconds·kelvin^3))

Initial condition: $T(0) = 400$.

Differential equation: $\frac{dT}{dt} = -\sigma T^4 + 81\sigma$.

(AB 16) A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 16) Independent variable: $t =$ time (in seconds).

Dependent variable: $v =$ velocity of the ball (in meters/second, where a positive velocity denotes upward motion).

Parameters: $\alpha =$ proportionality constant of the drag force (in newton·seconds/meter)

$m =$ mass of the ball (in kilograms)

Initial condition: $v(0) = 10$.

Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/m)v$.

(AB 17) A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 17) Independent variable: $t =$ time (in seconds).

Dependent variable: $v =$ velocity of the ball (in meters/second, where a positive velocity denotes upward motion).

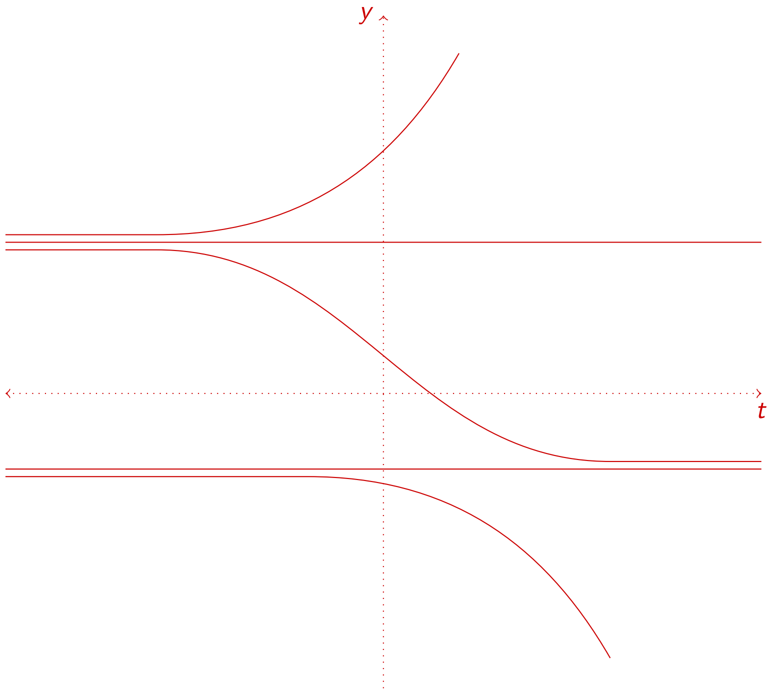
Parameter: $\alpha =$ proportionality constant of the drag force (in newton·seconds/meter).

Initial condition: $v(0) = 20$.

Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/0.3)v|v|$, or $\frac{dv}{dt} = \begin{cases} -9.8 - (\alpha/0.3)v^2, & v \geq 0, \\ -9.8 + (\alpha/0.3)v^2, & v < 0. \end{cases}$

(AB 18) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y-2)(y+1)^2$. By hand, sketch some typical solutions.

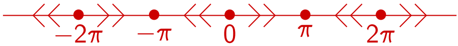
(Answer 18)



(AB 19) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(Answer 19)

Critical points: $y = k\pi$ for any integer k.

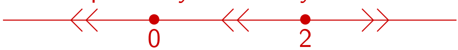


If k is even then $y = k\pi$ is unstable.
If k is odd then $y = k\pi$ is stable.

(AB 20) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = y^2(y-2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(Answer 20)

Critical points: $y = 0$ and $y = 2$.



$y = 0$ is semistable. $y = 2$ is unstable.

(AB 21) Trout live in a lake. Each trout produces, on average, one baby trout every two years. Fishermen are allowed to harvest 100 trout/year from the lake. You may neglect other reasons why trout die.

(a) Formulate a differential equation for the number of trout in the lake.
$\frac{dP}{dt} = \frac{1}{2}P - 100$, where t denotes time in years and P denotes the number of trout in the lake.

(b) Find the critical points of this differential equation and classify them as to stability.
The critical point is $P = 200$. It is unstable.

(c) What is the real-world meaning of the critical points?
If there are initially fewer than 200 trout in the lake, then eventually the trout will go extinct. If there are initially more than 200 trout in the lake, the trout population will grow without limit (or at least, until the present model stops being applicable).

(AB 22) I owe my bank a debt of B dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

(a) Formulate a differential equation for the amount of money I owe.

$\frac{dB}{dt} = 0.05B - 19200$, where t denotes time in years.

(b) Find the critical points of this differential equation and classify them as to stability.

The critical point is $B = \$384{,}000$. It is unstable.

(c) What is the real-world meaning of the critical points?

If my initial debt is less than \$384,000, then I will eventually pay it off (have a debt of zero dollars), but if my initial debt is greater than \$384,000, my debt will grow exponentially. The critical point $B = 384{,}000$ corresponds to the balance that will allow me to make interest-only payments on my debt.

(AB 23) A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

(a) Write the differential equation for the amount of salt in the tank.

$\frac{dQ}{dt} = 10 - Q/300$, where Q denotes the amount of salt in grams and t denotes time in minutes.

(b) Find the critical points of this differential equation and classify them as to stability.

The critical point is $Q = 3000$. It is stable.

(c) What is the real-world meaning of the critical points?

No matter the initial amount of salt in the tank, as time passes the amount of salt will approach 3000 g or 3 kg.

(AB 24) A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where v is her velocity in meters per second.

(a) Write a differential equation for her velocity. Assume her velocity is always downwards.
If $v \leq 0$ then $70\frac{dv}{dt} = -70 * 9.8 + 2v^2$. (If $v > 0$ then $70\frac{dv}{dt} = -70 * 9.8 - 2v^2$.)

(b) Find the critical points of this differential equation and classify them as to stability. Be sure to include units.
$v = -\sqrt{343}$ meters/second is a stable critical point.

(c) What is the real-world meaning of these critical points?
As $t \to \infty$, her velocity will approach the stable critical point of $v = -\sqrt{343}$ meters/second.

(AB 25) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen.

(a) Write a differential equation for the amount of hydrogen left in the tank.
$\frac{dH}{dt} = -\alpha H(126.9H - 3345)$, where H is the amount of hydrogen remaining (in grams), t denotes time in minutes, and α is a positive parameter.

(b) Find the critical points of this differential equation and classify them as to stability. Be sure to include units.
$H = 0$ grams (unstable) and $H = 26.36$ grams (stable).

(c) What is the real-world meaning of these critical points?
As $t \to \infty$, the amount of hydrogen in the tank will tend towards 26.36 grams. The $H = 0$ critical point is not physically meaningful, as if $H = 0$ grams then the model predicts -3345 grams of iodine in the tank.

(AB 26) Young oak trees grow in a large field with area 1 square kilometer. Initially, there are 30 trees in the field. Each tree shades 20 square meters. The trees produce acorns. Squirrels scatter the acorns across the field at random. Squirrels eat most of the acorns; on average, each tree produces two acorns per year that are not eaten by squirrels. Every acorn not eaten by squirrels and planted in sunlight sprouts.

(*a*) Write a differential equation for the number of trees in the field.

$\frac{dP}{dt} = 2P(1 - 20P/1{,}000{,}000)$, where P is the number of trees in the field and t denotes time in years.

(*b*) Find the critical points of this differential equation and classify them as to stability.
$P = 0$ (unstable) and $P = 50{,}000$ trees (stable).

(*c*) What is the real-world meaning of these critical points?
If there are initially no trees in the field, then the number of trees will remain at the $P = 0$ equilibrium. However, if there are initially any trees in the field, then as $t \to \infty$, the number of trees will approach 50,000.

(AB 27) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate of 3 cm^3 per exposed cm2 per hour.

(*a*) Write a differential equation for the amount of water in the tank.

$\frac{dV}{dt} = 1 - 75\pi(3V/25\pi)^{2/3}$, where t denotes time in hours and V is the volume of water in the hole in cm^3.

(*b*) Find the critical points of this differential equation and classify them as to stability.
$V = (25/3)(1/75\pi)^{3/2}$. The critical point is stable.

(*c*) What is the real-world meaning of these critical points?
As $t \to \infty$, the amount of water in the hole will approach $(25/3)(1/75\pi)^{3/2}$ cubic centimeters.

(AB 28) Is $y = e^t$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(Answer 28) No, $y = e^t$ is not a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(AB 29) Is $y = e^{2t}$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(Answer 29) Yes, $y = e^{2t}$ is a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(AB 30) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 1$?

(Answer 30) No.

(AB 31) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 2$?

(Answer 31) No.

(AB 32) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 1$?

(Answer 32) Yes.

(AB 33) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 2$?

(Answer 33) No.

(AB 34) For each of the following differential equations, determine whether it is linear, separable, exact, Bernoulli, of the form $\frac{dy}{dt} = f(y/t)$, or of the form $\frac{dy}{dt} = f(At + By + C)$. Then solve the differential equation.

(a) $t + \cos t + (y - \sin y)\frac{dy}{dt} = 0$

$t + \cos t + (y - \sin y)\frac{dy}{dt} = 0$ is separable (and also exact) and has solution $\frac{1}{2}y^2 + \cos y = -\frac{1}{2}t^2 - \sin t + C$.

(b) $\ln y + y + x + \left(\frac{x}{y} + x\right)\frac{dy}{dx} = 0$

$\ln y + y + x + \left(\frac{x}{y} + x\right)\frac{dy}{dx} = 0$ is exact and has solution $x \ln y + xy + \frac{1}{2}x^2 = C$.

(c) $1 + t^2 - ty + (t^2 + 1)\frac{dy}{dt} = 0$

$1 + t^2 - ty + (t^2 + 1)\frac{dy}{dt} = 0$ is linear and has solution $y = -\sqrt{t^2+1}\ln(t + \sqrt{t^2+1}) + C\sqrt{t^2+1}$.

(d) $ty - y^2 - t^2 + t^2\frac{dy}{dt} = 0$

$ty - y^2 - t^2 + t^2\frac{dy}{dt} = 0$ is of the form $\frac{dy}{dt} = f(y/t)$ and has solution $y = \frac{t}{C - \ln|t|} + t$.

(e) $4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5)\frac{dy}{dt} = 0$

$4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5)\frac{dy}{dt} = 0$ is exact and has solution $2t^2y^3 + \frac{3}{2}t^4 + 3y + \frac{1}{6}y^6 = C$.

(f) $y \tan 2x + y^3 \cos 2x + \frac{dy}{dx} = 0$

$y \tan 2x + y^3 \cos 2x + \frac{dy}{dx} = 0$ is Bernoulli. Let $v = y^{-2}$. Then $\frac{dv}{dx} = 2v\tan 2x + 2\cos 2x$, so $v = \frac{1}{2}\sin(2x) + x\sec(2x) + C\sec(2x)$ and $y = \frac{1}{\sqrt{\frac{1}{2}\sin(2x) + x\sec(2x) + C\sec(2x)}}$.

(g) $3t - 5x + (t + x)\frac{dx}{dt} = 0$

$3t - 5x + (t + x)\frac{dx}{dt} = 0$ is of the form $\frac{dy}{dt} = f(y/t)$ and has solution $(x - 3t)^2 = C(x - t)$.

(h) $\frac{dy}{dt} = \frac{1}{3t + 2y + 7}$.

If $\frac{dy}{dt} = \frac{1}{3t+2y+7}$, then $\frac{3t+2y+7}{3} - \frac{2}{9}\ln|3t + 2y + 7 + 2/3| = \ln|t| + C$.

(i) $y^3\cos(2t) + \frac{dy}{dt} = 0$

$y^3\cos(2t) + \frac{dy}{dt} = 0$ is separable and has solution $y = \pm\frac{1}{\sqrt{\sin(2t)+C}}$ or $y = 0$.

(*j*) $4ty\frac{dy}{dt} = 3y^2 - 2t^2$

$4ty\frac{dy}{dt} = 3y^2 - 2t^2$ is of the form $\frac{dy}{dt} = f(y/t)$ (and also Bernoulli) and has solution $2\ln(y^2/t^2 + 2) = -\ln|t| + C$.

(*k*) $t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$

$t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$ is a Bernoulli equation. Let $v = y^6$. Then $t\frac{dv}{dt} = 18v - 6t^2$, so $v = Ct^{18} + \frac{3}{8}t^2$ and $y = \sqrt[6]{Ct^{18} + \frac{3}{8}t^2}$.

(*l*) $\sin^2(x-t)\frac{dx}{dt} = \csc^2(x-t)$.

If $\frac{dx}{dt} = \csc^2(x-t)$, then $\tan(x-t) - x = C$.

(*m*) $\frac{dy}{dt} = 8y - y^8$

$\frac{dy}{dt} = 8y - y^8$ is Bernoulli (and also separable, but separating variables results in an impossible integral). Make the substitution $v = y^{-7}$. Then $\frac{dv}{dt} = -56v + 7$, so $v = Ce^{-56t} + \frac{1}{8}$ and $y = \frac{1}{\sqrt[7]{Ce^{-56t}+1/8}}$.

(*n*) $t\frac{dz}{dt} = -\cos t - 3z$

$t\frac{dz}{dt} = -\cos t - 3z$ is linear and has solution $z = \frac{1}{t}\sin t + \frac{2}{t^2}\cos t - \frac{2}{t^3}\sin t + \frac{C}{t^3}$.

(*o*) $\frac{dy}{dt} = \cot(y/t) + y/t$

$\frac{dy}{dt} = \cot(y/t) + y/t$ is of the form $\frac{dy}{dt} = f(y/t)$ and has solution $\sec(y/t) = Ct$.

(*p*) $\frac{dy}{dt} = ty + t^2\sqrt[3]{y}$

$\frac{dy}{dt} = ty + t^2\sqrt[3]{y}$ is Bernoulli. Make the substitution $v = y^{2/3}$. Then $v = -t - \frac{3}{2} + Ce^{t^2/3}$ and so $y = \sqrt{(-t - \frac{3}{2} + Ce^{t^2/3})^3}$.

(AB 35) For each of the following differential equations, find all the equilibrium solutions or state that no such solutions exist. You do not need to find the nonequilibrium solutions.

(a) $\frac{dy}{dt} = y^3 - yt$
$y = 0$.

(b) $\frac{dy}{dt} = t^2 e^y$
There are no equilibrium solutions.

(c) $\frac{dy}{dt} = ty + t^3$
There are no equilibrium solutions.

(d) $\frac{dy}{dt} = (y^2 + 3y + 2)\sin(t)$
$y = -1$ and $y = -2$.

(e) $\frac{dy}{dt} = \ln(y^t)$
$y = 1$.

(AB 36) Suppose that $\frac{dy}{dt} = \cos(t)\sin(y)$, $y(0) = 3\pi$. Find $y(2)$.

(Answer 36) $y(t) = 3\pi$ is an equilibrium solution to the differential equation and satisfies $y(0) = 3\pi$, so we must have that $y(t) = 3\pi$ for all t. In particular, $y(2) = 3\pi$.

(AB 37) For each of the following differential equations, determine whether it is linear or separable, exact, Bernoulli, of the form $\frac{dy}{dt} = f(y/t)$, or a function of a linear term. Then solve the given initial-value problem.

(a) $\cos(t+y^3)+2t+3y^2\cos(t+y^3)\frac{dy}{dt}=0$, $y(\pi/2)=0$

If $\cos(t+y^3)+2t+3y^2\cos(t+y^3)\frac{dy}{dt}=0$, $y(\pi/2)=0$, then $\sin(t+y^3)+t^2=1+\pi^2/4$.

(b) $ty^2-4t^3+2t^2y\frac{dy}{dt}=0$, $y(1)=3$.

If $ty^2-4t^3+2t^2y\frac{dy}{dt}=0$, $y(1)=3$, then $y=\sqrt{\frac{4}{3}t^2+\frac{23}{3t}}$.

(c) $1+y^2+t\frac{dy}{dt}=0$, $y(1)=1$

If $1+y^2+t\frac{dy}{dt}=0$, $y(1)=1$, then $y=\tan(\pi/4-\ln t)$.

(d) $\frac{dy}{dt}=(2y+2t-5)^2$, $y(0)=3$.

If $\frac{dy}{dt}=(2y+2t-5)^2$, $y(0)=3$, then $y=\frac{1}{2}\tan(2t+\pi/4)+\frac{5}{2}-t$.

(e) $\frac{dy}{dt}=2y-\frac{6}{y^2}$, $y(0)=7$.

If $\frac{dy}{dt}=2y-\frac{6}{y^2}$, $y(0)=7$, then $y=\sqrt[3]{340e^{6t}+3}$.

(f) $3y+e^{-3t}\sin t+\frac{dy}{dt}=0$, $y(0)=2$

If $3y+e^{-3t}\sin t+\frac{dy}{dt}=0$, $y(0)=2$, then $y=e^{-3t}\cos t+e^{-3t}$.

(g) $\frac{dy}{dt}=(y^2-5y+4)te^{t^2}$, $y(0)=4$

If $\frac{dy}{dt}=(y^2-5y+4)te^{t^2}$, $y(0)=4$, then $y=4$ for all t.

(AB 38) Solve the initial-value problem $\frac{dy}{dt}=\frac{t-5}{y}$, $y(0)=3$. Determine the domain of definition of the solution to the initial value problem. If the solution ceases to exist at a finite point, tell me whether the solution ceases to exist by approaching a "bad" point (in the sense of the Picard-Lindelöf theorem) or by becoming unbounded.

(Answer 38) $y=\sqrt{(t-5)^2-16}=\sqrt{(t-1)(t-9)}=\sqrt{t^2-10t+9}$. The solution is valid for all $t<1$. The solution ceases to exist by approaching a "bad" point.

(AB 39) Solve the initial-value problem $\frac{dy}{dt} = y^2$, $y(0) = 1/4$. Determine the domain of definition of the solution to the initial value problem. If the solution ceases to exist at a finite point, tell me whether the solution ceases to exist by approaching a "bad" point (in the sense of the Picard-Lindelöf theorem) or by becoming unbounded.

(Answer 39) $y = \frac{1}{4-t}$. The solution is valid for all $t < 4$. The solution ceases to exist by becoming unbounded.

(AB 40) Find all of the "bad" points (in the sense of the Picard-Lindelöf theorem) for the differential equation $\frac{dy}{dt} = \frac{1}{(t-3)\ln y}$.

(Answer 40) The "bad" points occur when either $t = 3$ or $y = 1$.

(AB 41) Find all of the "bad" points (in the sense of the Picard-Lindelöf theorem) for the differential equation $\frac{dy}{dt} = \sqrt[3]{(y-4)(t-2)}$.

(Answer 41) The "bad" points occur when $y = 4$.

(AB 42)

(a) Find all equilibrium solutions to the differential equation $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$.

The equilibrium solution is $y = 4$.

(b) Use the method of separation of variables to find a solution to the initial value problem $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$, $y(3) = 3$.

One such solution is $y = 4 - (t-2)^2 = 4t - t^2$.

(c) Use the method of separation of variables to find a solution to the initial value problem $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$, $y(3) = 5$.

One such solution is $y = 4 + (t-2)^2$.

(d) You have found three solutions to the differential equation $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$. Do your solutions cross at any point?

Yes, all three solutions cross at the point $(2, 4)$.

(e) Find two piecewise-defined solutions to the initial value problem $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$, $y(3) = 3$ that are different from each other and also different from the solution you found in Problem (b).

One such solution is

$$y(t) = \begin{cases} 4, & t < 2, \\ 4 - (t-2)^2, & 2 \leq t. \end{cases}$$

Another such solution is

$$y(t) = \begin{cases} 4 + (t-2)^2, & t < 2, \\ 4 - (t-2)^2, & 2 \leq t. \end{cases}$$

(AB 43) The function $y_1(t) = e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2} - (1+2t)\frac{dy}{dt} + (t+1)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

(Answer 43) $y(t) = C_1\, e^t + C_2\, t^2 e^t$.

(AB 44) The function $y_1(t) = t$ is a solution to the differential equation $t^2 \frac{d^2y}{dt^2} - (t^2 + 2t)\frac{dy}{dt} + (t + 2)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

(Answer 44) $y(t) = C_1\, t + C_2\, te^t$.

(AB 45) The function $y_1(x) = x^3$ is a solution to the differential equation $x^2 \frac{d^2y}{dx^2} + (x^2 \tan x - 6x)\frac{dy}{dx} + (12 - 3\,x \tan x)y = 0$. Find the general solution to this differential equation on the interval $0 < x < \pi/2$.

(Answer 45) $y(x) = C_1\, x^3 + C_2\, x^3 \sin x$.

(AB 46) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + \arctan(t)\, y = e^t$, $y(3) = 7$.
We expect a unique solution.

(b) $\frac{1}{1+t^2}\frac{dy}{dt} - t^5 y = \cos(6t)$, $y(2) = -1$, $y'(2) = 3$.
We do not expect any solutions.

(c) $\frac{d^2y}{dt^2} - 5\sin(t)\, y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.
We do not expect any solutions.

(d) $e^t \frac{d^2y}{dt^2} + 3(t-4)\frac{dy}{dt} + 4t^6 y = 2$, $y(3) = 1$, $y'(3) = -1$.
We expect a unique solution.

(e) $\frac{d^2y}{dt^2} + \cos(t)\frac{dy}{dt} + 3\ln(1+t^2)\, y = 0$, $y(2) = 3$.
We expect an infinite family of solutions.

(f) $\frac{d^3y}{dt^3} - t^7\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\, y = t^3$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$, $y'''(1) = 3$.
We do not expect any solutions.

(g) $(1+t^2)\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = t^3$, $y(3) = 9$, $y'(3) = 7$, $y''(3) = 5$.
We expect a unique solution.

(h) $\frac{d^3y}{dt^3} - e^t\frac{d^2y}{dt^2} + e^{2t}\frac{dy}{dt} + e^{3t} y = e^{4t}$, $y(-1) = 1$, $y'(-1) = 3$.
We expect an infinite family of solutions.

(i) $(2+\sin t)\frac{d^3y}{dt^3} + \cos t\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\, y = t^3$, $y(7) = 2$.
We expect an infinite family of solutions.

(AB 47) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $e^t \frac{dy}{dt} + y = \cos t$, $y(0) = 3$, $y'(0) = -2$.
We expect a unique solution.

(b) $(t^2+4)\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + 6y = 7t^3$, $y(2) = 4$, $y'(2) = 4$, $y''(2) = 1$.
We expect a unique solution.

(c) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0$, $y(4) = 3$, $y'(4) = -2$, $y''(4) = 0$, $y'''(4) = 3$.
We expect a unique solution.

(AB 48) Find the general solution to the following differential equations.

(a) $\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 85x = 0$.
If $\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 85x = 0$, then $x = C_1 e^{-6t}\cos(7t) + C_2 e^{-6t}\sin(7t)$.

(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.
If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$, then $y = C_1 e^{(-2+\sqrt{2})t} + C_2 e^{(-2-\sqrt{2})t}$.

(c) $\frac{d^4z}{dt^4} + 7\frac{d^2z}{dt^2} - 144z = 0$.
If $\frac{d^4z}{dt^4} + 7\frac{d^2z}{dt^2} - 144z = 0$, then $z = C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos 4t + C_4 \sin 4t$.

(d) $\frac{d^4w}{dt^4} - 8\frac{d^2w}{dt^2} + 16w = 0$.
If $\frac{d^4w}{dt^4} - 8\frac{d^2w}{dt^2} + 16w = 0$, then $w = C_1 e^{2t} + C_2 t e^{2t} + C_3 e^{-2t} + C_4 t e^{-2t}$.

(AB 49) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $9\frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 2v = 0$, $v(0) = 3$, $v'(0) = 2$.
If $9\frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 2v = 0$, $v(0) = 3$, $v'(0) = 2$, then $v = 3e^{-t/3}\cos(t/3) + 9e^{-t/3}\sin(t/3)$.

(b) $\frac{d^2u}{dt^2} + 10\frac{du}{dt} + 25u = 0$, $u(0) = 1$, $u'(0) = 4$.
If $\frac{d^2u}{dt^2} + 10\frac{du}{dt} + 25u = 0$, $u(0) = 1$, $u'(0) = 4$, then $u = e^{-5t} + 9te^{-5t}$.

(c) $8\frac{d^2f}{dt^2} - 6\frac{df}{dt} + f = 0$, $f(0) = 3$, $f'(0) = 1$.
If $8\frac{d^2f}{dt^2} - 6\frac{df}{dt} + f = 0$, then $f = e^{t/2} + 2e^{t/4}$.

(AB 50) A spring is suspended vertically. When an object with mass 5 kg is attached, it stretches the spring to a new equilibrium 4 cm lower. The system moves in a medium which damps it with damping constant 16 newton · seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Write the differential equation and initial conditions that describe the position of the object. You may use 9.8 meters/second^2 for the acceleration of gravity.

(Answer 50) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$5\frac{d^2x}{dt^2} + 16\frac{dx}{dt} + 1225x = 0, \qquad x(0) = -0.04, \quad x'(0) = 3.$$

(AB 51) A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant β. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20$ N $\cdot$ s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30$ N $\cdot$ s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(Answer 51) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$2\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.1, \quad x'(0) = 0.$$

If $\beta = 20$ N $\cdot$ s/m, then the system is underdamped, and we do expect to see decaying oscillations.

If $\beta = 30$ N $\cdot$ s/m, then the system overdamped, and we do not expect to see decaying oscillations.

(AB 52) A 3-kg object is attached to a spring with constant k and to a viscous damper with damping constant 42 N·sec/m. The object is set in motion from its equilibrium position with initial velocity 5 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object.

Find the values of k for which the system is underdamped, overdamped, and critically damped. Be sure to include units for k.

(Answer 52) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$3\frac{d^2x}{dt^2} + 42\frac{dx}{dt} + kx = 0, \qquad x(0) = 0, \quad x'(0) = -5.$$

The system is critically damped if $k = 147$ newtons/meter. It is underdamped if $k > 147$ newtons/meter and overdamped if $0 < k < 147$ newtons/meter.

(AB 53) A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant β. The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the values of β for which the system is underdamped, overdamped, and critically damped. Be sure to include units for β.

(Answer 53) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$4\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 70x = 0, \qquad x(0) = 0.05, \quad x'(0) = -3.$$

Critical damping occurs when $\beta = 4\sqrt{70}$ $\mathrm{N\cdot s/m}$. The system is underdamped if $0 < \beta < 4\sqrt{70}$ $\mathrm{N\cdot s/m}$ and is overdamped if $\beta > 4\sqrt{70}$ $\mathrm{N\cdot s/m}$.

(AB 54) An object of mass m is attached to a spring with constant 80 N/m and to a viscous damper with damping constant 20 N·s/m. The object is pulled down to 5cm below its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of m for which the system is underdamped, overdamped, or critically damped. Be sure to include units for m.

(Answer 54) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

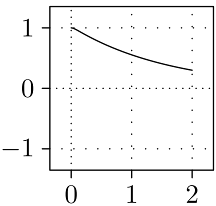
$$m\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.05, \quad x'(0) = -3.$$

Critical damping occurs when $m = \frac{5}{4}$ kg. The system is underdamped if $m > \frac{5}{4}$ kg and overdamped if $0 < m < \frac{5}{4}$ kg.

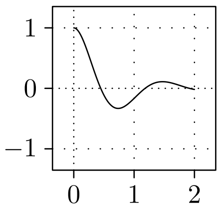
(AB 55) Five objects, each with mass 3 kg, are attached to five springs, each with constant 48 N/m. Five dampers with unknown constants are attached to the objects. In each case, the object is pulled down to a distance 1 cm below the equilibrium position and released from rest. You are given that the system is critically damped in exactly one of the five cases.

Here are the graphs of the objects' positions with respect to time:

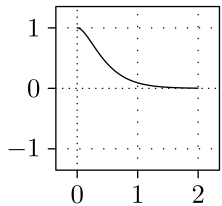
System D



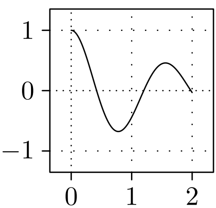
System E



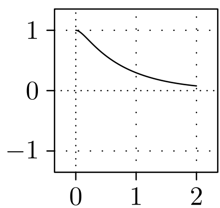
System F



System G



System H



(*a*) For which damper is the system critically damped?
The system is critically damped for Damper F.

(*b*) For which dampers is the system overdamped?
The system overdamped for Dampers H and D.

(*c*) For which dampers is the system underdamped?
The system underdamped for Dampers E and G.

(*d*) Which damper has the highest damping constant? Which damper has the lowest damping constant?
Damper D has the highest damping constant. Damper G has the lowest damping constant.

(AB 56) An object with mass 5 kg stretches a spring 4 cm. It is attached to a viscous damper with damping constant 16 newton · seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Using only first derivatives, write the differential equations and initial conditions for the object's position. That is, write a first-order system involving the object's position. You may use 9.8 meters/second^2 for the acceleration of gravity.

(Answer 56) Let t denote time (in seconds), let x denote the object's displacement above equilibrium (in meters), and let v denote the object's velocity (in meters per second). Then

$$5\frac{dv}{dt} + 16v + 1225x = 0, \qquad \frac{dx}{dt} = v, \qquad x(0) = -0.04, \quad v(0) = 3.$$

(AB 57) Imperial stormtroopers and Rebel Alliance fighters battle each other on an open plain, where both groups can easily see and aim at all members of the other group. Every minute, each stormtrooper has a 2% chance of killing a rebel, and each rebel has a 5% chance of killing a stormtrooper. There are initially 4000 stormtroopers and 1000 rebels.

Write the initial value problem for the number of stormtroopers and rebels still alive.

(Answer 57) Let t denote time (in minutes), let S denote the number of stormtroopers, and let R denote the number of rebels.

Then

$$\frac{dR}{dt} = -0.02S, \quad \frac{dS}{dt} = -0.05R, \quad R(0) = 1000, \quad S(0) = 4000.$$

(AB 58) Jedi knights and Sith lords battle in a dense forest. Every minute, each Jedi has a 1% chance of finding each Sith lord. If a Jedi finds a Sith lord, they fight; the Jedi has a 60% chance of dying and the Sith lord has a 40% chance of dying. Initially there are 90 Jedi and 50 Sith lords.

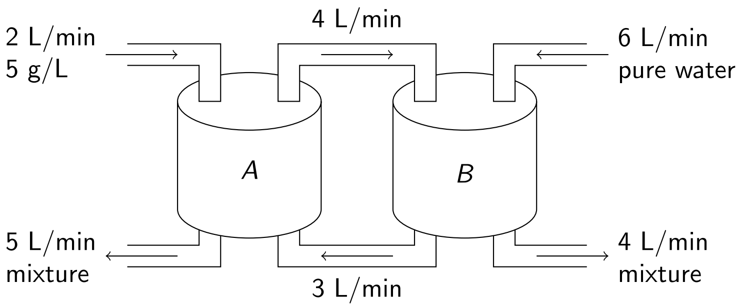
Write the initial value problem for the number of Jedi and Sith lords still alive.

(Answer 58) Let t denote time (in minutes), let S denote the number of Sith lords, and let J denote the number of Jedi.

Then

$$\frac{dJ}{dt} = -0.006JS, \quad \frac{dS}{dt} = -0.004JS, \quad J(0) = 90, \quad S(0) = 50.$$

(AB 59) Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



Write the differential equations and initial conditions that describe the amount of salt in each tank.

(Answer 59) Let t denote time (in minutes).

Let x denote the amount of salt (in grams) in tank A.

Let y denote the amount of salt (in grams) in tank B.

Then $x(0) = 3000$ and $y(0) = 2000$.

If $t < 50$, then

$$\frac{dx}{dt} = 10 - \frac{9x}{200-4t} + \frac{3y}{300+3t}, \qquad \frac{dy}{dt} = \frac{4x}{200-4t} - \frac{7y}{300+3t}.$$

(AB 60) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(Answer 60) Let t denote time (in days), let S denote the number of susceptible people (who have never had the disease), let I denote the number of infected people (who currently have the disease), and let R denote the number of recovered people (who now are resistant, that is, cannot get the disease again). Then

$\boxed{\frac{dS}{dt} = -\frac{1}{22500}SI,\ \frac{dI}{dt} = \frac{1}{22500}SI - 0.17I,\ \frac{dR}{dt} = 0.17I,\ S(0) = 8997,\ I(0) =}$

(AB 61) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(Answer 61) $\frac{dS}{dt} = -\frac{0.2}{5000}SI - 15$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I + 15$.

(AB 62) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

Set up the system of differential equations that describes the number of susceptible, infected, and resistant people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(Answer 62) $\boxed{\frac{dS}{dt} = -\frac{0.2}{5000}SI - 15\frac{S}{5000},\ \frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I,\ \frac{dR}{dt} = 0.1I +}$

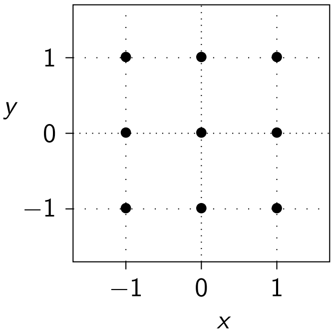
(AB 63) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers.

Each infected person encounters an average of 20 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. If an infected person encounters a recovered person, the recovered person has a 1% chance of contracting the disease again. Each infected person has a 12% chance of recovering from the disease on any given day.

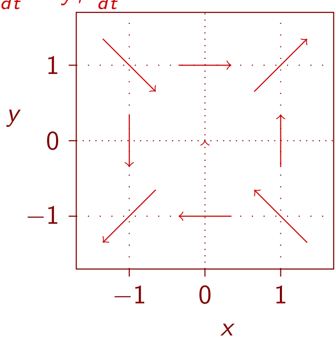
Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(Answer 63) Let t denote time (in days), let S denote the number of susceptible people (who have never had the disease), let I denote the number of infected people (who currently have the disease), and let R denote the number of recovered people (who now are resistant, that is, are less likely to get the disease again). Then $\frac{dS}{dt} = -\frac{1}{9000}SI$, $\frac{dI}{dt} = \frac{1}{9000}SI + \frac{1}{45000}RI - 0.12I$, $\frac{dR}{dt} = 0.12I - \frac{1}{45000}RI$, $S(0) = 8997$, $I(0) = 3$, $R(0) = 0$.

(AB 64) Here is a grid. Draw a small phase plane (vector field) with nine arrows for the autonomous system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = x$.



(Answer 64) Here is the direction field for the differential equation system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = x$.

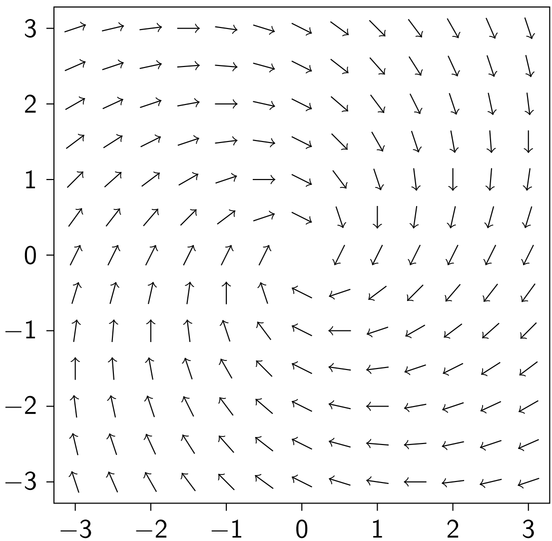


(AB 65) Here is the phase plane for the system

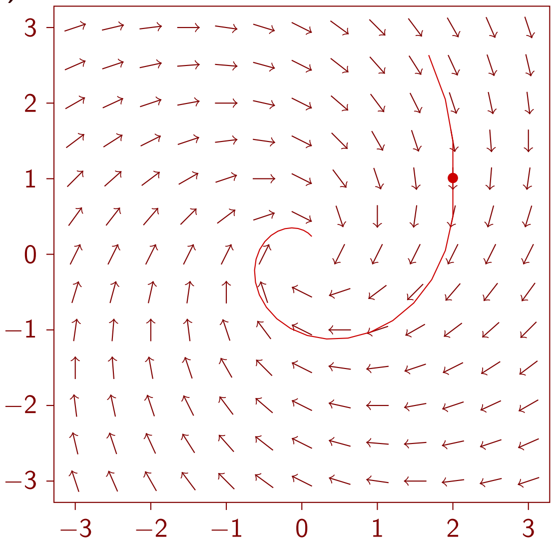
$$\frac{dx}{dt} = 2y - x, \quad \frac{dy}{dt} = -2x - y$$

Sketch the solution to the initial value problem

$$\frac{dx}{dt} = 2y - x, \quad \frac{dy}{dt} = -2x - y, \quad x(0) = 2, \quad y(0) = 1.$$



(Answer 65)



(AB 66) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 66) If

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} -3\cos t + 16\sin t \\ 2\cos t + 7\sin t \end{pmatrix}.$$

(AB 67) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 67) If

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 5\cos 12t + \sin 12t \\ 2\cos 12t + 2\sin 12t \end{pmatrix}.$$

(AB 68) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 68) If

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos 3t - 11 \sin 3t \\ -3 \cos 3t - (23/3) \sin 3t \end{pmatrix}.$$

(AB 69) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 4x+y-z, \qquad \frac{dy}{dt} = x+4y-z, \qquad \frac{dz}{dt} = 4z-x-y, \quad x(0) = 3, \quad y($$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 4-m & 1 & -1 \\ 1 & 4-m & -1 \\ -1 & -1 & 4-m \end{pmatrix} = -(m-3)^2(m-6).$

(Answer 69) If

$$\frac{dx}{dt} = 4x+y-z, \qquad \frac{dy}{dt} = x+4y-z, \qquad \frac{dz}{dt} = 4z-x-y, \quad x(0) = 3, \quad y($$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{3t} \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} + e^{6t} \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}.$$

(AB 70) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 3x+y, \qquad \frac{dy}{dt} = 2x+3y-2z, \qquad \frac{dz}{dt} = y+3z, \qquad x(0) = 3, \quad y(0$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 3-m & 1 & 0 \\ 2 & 3-m & -2 \\ 0 & 1 & 3-m \end{pmatrix} = -(m-3)^3.$

(Answer 70) If

$$\frac{dx}{dt} = 3x+y, \qquad \frac{dy}{dt} = 2x+3y-2z, \qquad \frac{dz}{dt} = y+3z, \qquad x(0) = 3, \quad y(0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{3t} \begin{pmatrix} 2t^2+2t+3 \\ 4t+2 \\ 2t^2+2t+1 \end{pmatrix}.$$

(AB 71) Find the solution to the initial-value problem

$$\frac{dx}{dt} = x+y+z, \qquad \frac{dy}{dt} = x+3y-z, \qquad \frac{dz}{dt} = 2y+2z, x(0) = 4, \quad y(0) =$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 1-m & 1 & 1 \\ 1 & 3-m & -1 \\ 0 & 2 & 2-m \end{pmatrix} = -(m-2)^3.$

(Answer 71) If

$$\frac{dx}{dt} = x+y+z, \qquad \frac{dy}{dt} = x+3y-z, \qquad \frac{dz}{dt} = 2y+2z, \qquad x(0) = 4, \quad y($$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 4+4t+2t^2 \\ 2t+3 \\ 5+6t+2t^2 \end{pmatrix}.$$

(AB 72) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x+9y-15, \qquad \frac{dy}{dt} = -5x+6y-8, \qquad x(0) = 2, \quad y(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 72) If

$$\frac{dx}{dt} = -6x+9y-15, \qquad \frac{dy}{dt} = -5x+6y-8, \qquad x(0) = 2, \quad y(0) = 5$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 6\sin 3t \\ 4\sin 3t + 2\cos 3t \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

(AB 73) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x+8y+t^8e^{2t}, \qquad \frac{dy}{dt} = -2x-2y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 73) If

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 1, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} (2/45)t^{10} + (1/9)t^9 + 4t + 1 \\ -(1/45)t^{10} - 2t \end{pmatrix}.$$

(AB 74) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y + e^{2t} \cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 5, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 74) If

$$\frac{dx}{dt} = 5y + e^{2t} \cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 5, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} -t \sin t + (1/2)t \cos t + 5 \cos t - (19/2) \sin t \\ -(1/2)t \sin t - 5 \sin t \end{pmatrix}.$$

(AB 75) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 75) If

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t}\begin{pmatrix} 4 - 4\cos(e^t) \\ 4\cos(e^t) - 4 \end{pmatrix} + e^{-2t}\begin{pmatrix} 3\sin(e^t) - 3e^t\cos(e^t) - 3\sin 1 + \\ 4e^t\cos(e^t) - 4\sin(e^t) - 4\cos 1 + \end{pmatrix}$$

(AB 76) Find the general solution to the equation $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3}\ln t$ on the interval $0 < t < \infty$.

(Answer 76) If $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3}\ln t$, then $y(t) = c_1\, e^{t/3} + c_2\, t\, e^{t/3} + \frac{1}{2}t^2e^{t/3}\ln t - \frac{3}{4}t^2e^{t/3}$ for all $t > 0$.

(AB 77) Find the general solution to the equation $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$.

(Answer 77) If $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$, then $y = c_1e^{-t} + c_2e^{-t/2} - 2e^{-t}\sin(e^{t/2})$.

(AB 78) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - t\frac{dx}{dt} - 3x = 0$, $t > 0$, is $x(t) = C_1t^3 + C_2t^{-1}$. Find the general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

(Answer 78) If $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$, then $y(t) = -\frac{3}{2}t^{-1}\ln t + C_1t^3 + C_2t^{-1}$ for all $t > 0$.

(AB 79) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - 2x = 0$, $t > 0$, is $x(t) = C_1t^2 + C_2t^{-1}$. Solve the initial-value problem $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(Answer 79) If $t^2 \frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$, then $y(t) = -4\sqrt{t} + 3t^2 + \frac{2}{t}$ for all $0 < t < \infty$.

(AB 80) You are given that the general solution to the differential equation $(1-t)\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = 0$ on the interval $t < 1$ is $x(t) = C_1 t + C_2 e^t$. Find the general solution to $(1-t)\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = (1-t)^2 e^t$ on the interval $t < 1$.

(Answer 80) If $(1-t)\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = (1-t)^2 e^t$, then $y(t) = te^t - \frac{1}{2}t^2 e^t + C_1 t + C_2 e^t$.

(AB 81) Find the general solution to the following differential equations.

(*a*) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 3e^{4t}$.

The general solution to $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 0$ is $y_g = C_1 e^{-t/2} + C_2 e^{-t/3}$. To solve $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 2e^{4t}$ we make the guess $y_p = Ae^{4t}$. The solution is $y = C_1 e^{-t/2} + C_2 e^{-t/3} + \frac{1}{39}e^{4t}$.

(*b*) $16\frac{d^2y}{dt^2} - y = e^{t/4}\sin t$.

The general solution to $16\frac{d^2y}{dt^2} - y = 0$ is $y_g = C_1 e^{t/4} + C_2 e^{-t/4}$. To solve $16\frac{d^2y}{dt^2} - y = e^{t/4}\sin t$ we make the guess $y_p = Ae^{t/4}\sin t + Be^{t/4}\cos t$. The solution is $y = C_1 e^{t/4} + C_2 e^{-t/4} - (1/20)e^{t/4}\sin t - (1/40)e^{t/4}\cos t$.

(*c*) $\frac{d^2y}{dt^2} + 49y = 3t\sin 7t$.

The general solution to $\frac{d^2y}{dt^2} + 49y = 0$ is $y_g = C_1 \sin 7t + C_2 \cos 7t$. To solve $\frac{d^2y}{dt^2} + 49y = 3t\sin 7t$ we make the guess $y_p = At^2\sin 7t + Bt\sin 7t + Ct^2\cos 7t + Dt\cos 7t$. The solution is $y = C_1\sin 7t + C_2\cos 7t - (3/28)t^2\cos 7t + (3/4)t\sin 7t$.

(*d*) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$.

The general solution to $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 0$ is $y_g = C_1 + C_2 e^{4t}$. To solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$ we make the guess $y_p = Ae^{3t}$. The solution is $y = C_1 + C_2 e^{4t} - (4/3)e^{3t}$.

(*e*) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = t\sin(3t)$.

The general solution to $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$ is $y_g = C_1 e^{-6t}\cos(7t) + C_2 e^{-6t}\sin(7t)$. To solve $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = t\sin(3t)$ we make the guess $y_p = At\sin(3t) + Bt\cos(3t) + C\sin(3t) + D\cos(3t)$. The solution is $y = C_1 e^{-6t}\cos(7t) + C_2 e^{-6t}\sin(7t) + \frac{19}{1768}t\sin(3t) - \frac{9}{1768}t\cos(3t) - \frac{1353}{781456}\sin(3t) + \frac{606}{781456}\cos(3t)$.

(*f*) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$.

The general solution to $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 0$ is $y_g = C_1 e^{2t} +$

C_2e^{-5t}. To solve $\frac{d^2y}{dt^2}+3\frac{dy}{dt}-10y=7e^{-5t}$ we make the guess $y_p=Ate^{-5t}$. The solution is $y=C_1e^{2t}+C_2e^{-5t}-(1/7)te^{-5t}$.

(g) $16\frac{d^2y}{dt^2}-24\frac{dy}{dt}+9y=6t^2+\cos(2t)$.
The general solution to $16\frac{d^2y}{dt^2}-24\frac{dy}{dt}+9y=0$ is $y_g=C_1e^{(3/4)t}+C_2te^{(3/4)t}$. To solve $16\frac{d^2y}{dt^2}-24\frac{dy}{dt}+9y=6t^2+\cos(2t)$ we make the guess $y_p=At^2+Bt+C+D\cos(2t)+E\sin(2t)$. The solution is $y=C_1e^{(3/4)t}+C_2te^{(3/4)t}+(2/3)t^2+(32/9)t+64/9-(48/5329)\cos(2t)-(55/5329)\sin(2t)$.

(h) $\frac{d^2y}{dt^2}-6\frac{dy}{dt}+25y=t^2e^{3t}$.
The general solution to $\frac{d^2y}{dt^2}-6\frac{dy}{dt}+25y=0$ is $y_g=C_1e^{3t}\cos(4t)+C_2e^{3t}\sin(4t)$. To solve $\frac{d^2y}{dt^2}-6\frac{dy}{dt}+25y=t^2e^{3t}$ we make the guess $y_p=At^2e^{3t}+Bte^{3t}+Ce^{3t}$. The solution is $y=C_1e^{3t}\cos(4t)+C_2e^{3t}\sin(4t)+(1/16)t^2e^{3t}-(1/128)e^{3t}$.

(i) $\frac{d^2y}{dt^2}+10\frac{dy}{dt}+25y=3e^{-5t}$.
The general solution to $\frac{d^2y}{dt^2}+10\frac{dy}{dt}+25y=0$ is $y_g=C_1e^{-5t}+C_2te^{-5t}$. To solve $\frac{d^2y}{dt^2}+10\frac{dy}{dt}+25y=3e^{-5t}$ we make the guess $y_p=At^2e^{-5t}$. The solution is $y=C_1e^{-5t}+C_2te^{-5t}+(3/2)t^2e^{-5t}$.

(j) $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=3\cos(2t)$.
The general solution to $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=0$ is $y_g=C_1e^{-2t}+C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=3\cos(2t)$, we make the guess $y_p=A\cos(2t)+B\sin(2t)$. The solution is $y=C_1e^{-2t}+C_2e^{-3t}+\frac{15}{52}\sin(2t)+\frac{3}{52}\cos(2t)$.

(k) $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=5\sin(4t)$.
The general solution to $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=0$ is $y_g=C_1e^{-3t}+C_2te^{-3t}$. To solve $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=5\sin(4t)$, we make the guess $y_p=A\cos(4t)+B\sin(4t)$. The solution is $y=C_1e^{-3t}+C_2te^{-3t}-\frac{7}{125}\sin(4t)-\frac{24}{125}\cos(4t)$.

(l) $\frac{d^2y}{dt^2}+9y=5\sin(3t)$.
The general solution to $\frac{d^2y}{dt^2}+9y=0$ is $y_g=C_1\cos(3t)+$

$C_2 \sin(3t)$. To solve $\frac{d^2y}{dt^2} + 9y = 5\sin(3t)$, we make the guess $y_p = C_1 t\cos(3t) + C_2 t\sin(3t)$. The solution is $y = C_1\cos(3t) + C_2\sin(3t) - \frac{5}{6}t\cos(3t)$.

(m) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$.

The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ is $y_g = C_1e^{-t} + C_2te^{-t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$, we make the guess $y_p = At^2 + Bt + C$. The solution is $y = C_1e^{-t} + C_2te^{-t} + 2t^2 - 8t + 12$.

(n) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$.

The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 0$ is $y_g = C_1 + C_2e^{-2t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$, we make the guess $y_p = At^2 + Bt$. The solution is $y = C_1 + C_2e^{-2t} + \frac{3}{4}t^2 - \frac{3}{4}t$.

(o) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$.

The general solution to $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 0$ is $y_g = C_1e^{3t} + C_2e^{4t}$. To solve $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$, we make the guess $y_p = At + B + C\cos(2t) + D\sin(2t)$. The solution is $y = C_1e^{3t} + C_2e^{4t} + \frac{5}{12}t + \frac{35}{144} + \frac{16}{177}\cos 2t - \frac{28}{177}\sin 2t$.

(p) $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$.

The general solution to $\frac{d^2y}{dt^2} - 9y = 0$ is $y_g = C_1e^{3t} + C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$, we make the guess $y_p = Ae^t + Be^{-t} + Ct + D$. The solution is $y = C_1e^{3t} + C_2e^{-3t} - \frac{1}{4}e^t - \frac{1}{8}e^{-t} - \frac{5}{9}t - \frac{2}{9}$.

(q) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$.

The general solution to $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$ is $y_g = C_1e^{2t} + C_2te^{2t}$. To solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$, we make the guess $y_p = At^2e^{2t} + B\cos t + C\sin t$. The solution is $y = C_1e^{2t} + C_2te^{2t} + \frac{3}{2}t^2e^{2t} + \frac{3}{5}\cos t - \frac{4}{5}\sin t$.

(AB 82) A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 $\mathrm{N \cdot s/m}$ and is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(20t)$ N upwards. You may take the acceleration of gravity to be 9.8 meters/second^2.

Write the differential equation and initial conditions that describe the position of the object.

(Answer 82) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters).

Then

$$3\frac{d^2x}{dt^2} + 27\frac{dx}{dt} + 588x = 3\cos(20t), \qquad u(0) = 0, \quad u'(0) = 0.$$

(AB 83) An object weighing 4 lbs stretches a spring 2 inches. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(\omega t)$ pounds, directed upwards. There is no damping. You may take the acceleration of gravity to be 32 feet/second^2.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of ω for which resonance occurs. Be sure to include units for ω.

(Answer 83) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then

$$\frac{4}{32}\frac{d^2x}{dt^2} + 24x = 3\cos(\omega t), \qquad x(0) = 0, \quad x'(0) = 0.$$

Resonance occurs when $\omega = 8\sqrt{3}\ \mathrm{s}^{-1}$.

(AB 84) A 4-kg object is suspended from a spring. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $7\sin(\omega t)$ newtons, directed upwards.

(a) Write the differential equation and initial conditions that describe the position of the object.
Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Let k denote the constant of the spring (in N·s/m). Then

$$4\frac{d^2x}{dt^2} + kx = 7\sin(\omega t), \qquad x(0) = 0, \quad x'(0) = 0.$$

(b) It is observed that resonance occurs when $\omega = 20$ rad/sec. What is the constant of the spring? Be sure to include units.
The spring constant is $k = 1600$ N·s/m.

(AB 85) A 1-kg object is suspended from a spring with constant 225 N/m. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $15\cos(\omega t)$ newtons, directed upwards. Illustrated are the object's position as a function of time for three different values of ω. You are given that the three values are $\omega = 15$ radians/second, $\omega = 16$ radians/second, and $\omega = 17$ radians/second. Determine the value of ω that will produce each image.

(Answer 85) $\omega = 15$ radians/second in Picture B. $\omega = 16$ radians/second in Picture A. $\omega = 17$ radians/second in Picture C.

(AB 86) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ (**not** the table on the cover page of this exam), find the Laplace transforms of the following functions.

(a) $f(t) = e^{-11t}$

(b) $f(t) = t$

(c) $f(t) = \begin{cases} 3e^t, & 0 < t < 4, \\ 0, & 4 \leq t. \end{cases}$

(Answer 86)

(a) $\mathcal{L}\{t\} = \frac{1}{s^2}$.

(b) $\mathcal{L}\{e^{-11t}\} = \frac{1}{s+11}$.

(c) $\mathcal{L}\{f(t)\} = \frac{3-3e^{4-4s}}{s-1}$.

(AB 87) Find the Laplace transforms of the following functions. You may use the table in your notes, on Blackboard, or in your book.

(a) $f(t) = t^4 + 5t^2 + 4$
$\mathcal{L}\{t^4 + 5t^2 + 4\} = \frac{96}{s^5} + \frac{10}{s^3} + \frac{4}{s}$.

(b) $f(t) = (t+2)^3$
$\mathcal{L}\{(t+2)^3\} = \mathcal{L}\{t^3 + 6t^2 + 12t + 8\} = \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{8}{s}$.

(c) $f(t) = 9e^{4t+7}$
$\mathcal{L}\{9e^{4t+7}\} = \mathcal{L}\{9e^7 e^{4t}\} = \frac{9e^7}{s-4}$.

(d) $f(t) = -e^{3(t-2)}$
$\mathcal{L}\{-e^{3(t-2)}\} = \mathcal{L}\{-e^{-6}e^{3t}\} = -\frac{e^{-6}}{s-3}$.

(e) $f(t) = (e^t + 1)^2$
$\mathcal{L}\{(e^t+1)^2\} = \mathcal{L}\{e^{2t} + 2e^t + 1\} = \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s}$.

(f) $f(t) = 8\sin(3t) - 4\cos(3t)$
$\mathcal{L}\{8\sin(3t) - 4\cos(3t)\} = \frac{24}{s^2+9} - \frac{4s}{s^2+9}$.

(g) $f(t) = t^2 e^{5t}$
$\mathcal{L}\{t^2 e^{5t}\} = \frac{2}{(s-5)^3}$.

(h) $f(t) = 7e^{3t}\cos 4t$
$\mathcal{L}\{7e^{3t}\cos 4t\} = \frac{7s-21}{(s-3)^2+16}$.

(i) $f(t) = 4e^{-t}\sin 5t$
$\mathcal{L}\{4e^{-t}\sin 5t\} = \frac{20}{(s+1)^2+25}$.

(j) $f(t) = t\,e^t \sin t$
$\mathcal{L}\{t\,e^t\sin t\} = \frac{2s-2}{(s^2-2s+2)^2}$.

(k) $f(t) = t^2 \sin 5t$
$\mathcal{L}\{t^2\sin 5t\} = \frac{30s^2-250}{(s^2+25)^3}$.

(l) $f(t) = \int_0^t e^{-4r}\sin(3r)\,dr$
$\mathcal{L}\left\{\int_0^t e^{-4r}\sin(3r)\,dr\right\} = \frac{3}{s((s+4)^2+9)}$.

(m) You are given that $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$. Find $\mathcal{L}\{t\,J_0(t)\}$. (The function J_0 is called a Bessel function and is important in the theory of partial differential equations in polar coordinates.)
$\mathcal{L}\{t\,J_0(t)\} = \frac{s}{(s^2+1)^{3/2}}$.

(n) You are given that $\mathcal{L}\{\frac{1}{\sqrt{t}}e^{-1/t}\} = \frac{\sqrt{\pi}}{\sqrt{s}}e^{-2\sqrt{s}}$. Find $\mathcal{L}\{\sqrt{t}\, e^{-1/t}\}$.

$\mathcal{L}\{\sqrt{t}\, e^{-1/t}\} = \frac{\sqrt{\pi}(1+2\sqrt{s})}{2s\sqrt{s}}e^{-2\sqrt{s}}$

(o) $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$

If $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-3s}e^3}{s-1}$.

(p) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$

If $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$ then $\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{1}{s} + \frac{2}{s^3}\right)$.

(q) $f(t) = \begin{cases} 0, & t < 1, \\ t - 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$

If $f(t) = \begin{cases} 0, & t < 1, \\ t - 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2}$.

(r) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

If $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{5}{s-2} - \frac{5e^6e^{-3s}}{s-2}$.

(s) $f(t) = \begin{cases} 7t^2e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

If $f(t) = \begin{cases} 7t^2e^{-t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{14}{(s+1)^3} - \frac{14e^{-3s-3}}{(s+1)^3} - \frac{42e^{-3s-3}}{(s+1)^2} - \frac{63e^{-3s-3}}{s+1}$.

(t) $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi \end{cases}$

If $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{s}{s^2+9} + e^{-\pi s}\frac{s}{s^2+9} - e^{-\pi s}\frac{3}{s^2+9}$.

(u) $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \le t < \pi, \\ 0, & t \ge \pi \end{cases}$

If $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \le t < \pi, \\ 0, & t \ge \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s/2}\frac{1}{s^2+1} + e^{-\pi s}\frac{s}{s^2+1}$.

(AB 88) For each of the following problems, find y.

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$
If $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$, then $y = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$.

(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$
If $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$, then $y = \frac{5}{2}t^2 - \frac{7}{6}t^3$.

(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$
If $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$, then $y = \frac{1}{2}t^2e^{-t} + \frac{1}{6}t^3e^{-t}$.

(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$
If $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$, then $y = (1/4)e^{2t} + (7/4)e^{-2t}$.

(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$
If $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$, then $y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{1}{9}t\,e^{3t}$.

(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$
If $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$, then $y = \frac{1}{2} - \frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t)$.

(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$
If $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$, then $y = \frac{1}{8}\cos t - \frac{1}{8}\cos 3t$.

(h) $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$. *Hint*: Start by finding $\mathcal{L}\{t\sin 3t\}$ and $\mathcal{L}\{t\cos 3t\}$.
If $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$, then $y = \frac{1}{6}t\sin(3t)$.

(i) $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$. You may express your answer as a definite integral.
If $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$, then $y = e^{-2t}\int_0^t \sin(2r)\sin(2t-2r)\,dr = \frac{1}{4}e^{-2t}\sin(2t) - \frac{1}{2}e^{-2t}t\cos(2t)$.

(j) $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$. You may express your answer as a definite integral.
If $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$, then $y = \int_0^t \frac{e^{3r}+e^{-3r}}{2}\sqrt{t-r}\,dr$.

(k) $\mathcal{L}\{y\} = \frac{s}{s^4(s^2+36)}$. You may express your answer as a definite integral.
$y = \int_0^t \cos(6r)\frac{1}{6}(t-r)^3\,dr$ or $y = \int_0^t \cos(6(t-r))\frac{1}{6}r^3\,dr$.

(l) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$

If $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$, then $y = 2\,\mathcal{U}(t-2)\,e^{t-2}\cos(t-2) + \mathcal{U}(t-2)\,e^{t-2}\sin(t-2)$.

(m) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$

If $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$, then $y = \frac{1}{2}\,\mathcal{U}(t-1)\,e^{3(t-1)} + \frac{1}{2}\,\mathcal{U}(t-1)\,e^{t-1}$.

(AB 89) If a and b are constants, find $\mathcal{L}\{a\sin(4t) + b\,t\cos(4t)\}$. Then find values of a and b such that $\mathcal{L}\{a\sin(4t) + b\,t\cos(4t)\} = \frac{1}{(s^2+16)^2}$.

(Answer 89)

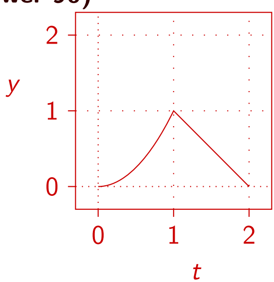
$$\mathcal{L}\{a\sin(4t)+b\,t\cos(4t)\} = \frac{4a}{s^2+16} + \frac{bs^2-16b}{(s^2+16)^2} = \frac{4as^2+bs^2+64a-16}{(s^2+16)^2}$$

Thus

$$\mathcal{L}\left\{\frac{1}{128}\sin(4t) - \frac{1}{32}\,t\cos(4t)\right\} = \frac{1}{(s^2+16)^2}.$$

(AB 90) Sketch the graph of $y = t^2 - t^2\,\mathcal{U}(t-1) + (2-t)\,\mathcal{U}(t-1)$.

(Answer 90)



(AB 91) Solve the following initial-value problems using the Laplace transform. Do you expect the graphs of y, $\frac{dy}{dt}$, or $\frac{d^2y}{dt^2}$ to show any corners or jump discontinuities? If so, at what values of t?

(a) $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$

If $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$, then $y(t) = -\frac{1}{30}\sin 3t - \frac{1}{90}\cos 3t + \frac{31}{30}e^{9t}$.

(b) $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$

If $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$, then $y = 3te^{2t} + 2e^{2t}$.

(c) $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$

If $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$, then $y = \frac{1}{5}t^3 - \frac{3}{25}t^2 + \frac{6}{125}t - \frac{6}{625} + \frac{1881}{625}e^{-5t}$.

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$, then $y(t) = e^{2t} - te^{2t}$.

(e) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$, then $y(t) = \frac{1}{5}(e^{-t} - e^t\cos t + 7e^t\sin t)$.

(f) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$

If $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$, then $y(t) = 4e^{3t} - 2e^{4t} - te^{3t}$.

(g) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$

If $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$, then $y(t) = -\frac{15}{8}e^{4t} + \frac{39}{8}e^{2t} - \frac{1}{4}te^{2t} - \frac{1}{2}t^2e^{2t} - t^3e^{2t}$.

(h) $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{40}e^{2t} - \frac{1}{72}e^{-2t} - \frac{1}{90}e^t\cos(3t) - \frac{1}{45}e^t\sin(3t)$.

(i) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$

If $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$, then $y(t) = \frac{1}{5}\cos 2t + \cos 3t + \frac{5}{3}\sin 3t$.

(j) $\frac{d^2y}{dt^2}+9y=t\sin(3t),\ y(0)=0,\ y'(0)=0$

If $\frac{d^2y}{dt^2}+9y=t\sin(3t),\ y(0)=0,\ y'(0)=0$, then $y(t)=\frac{1}{3}\int_0^t r\sin 3r\sin(3t-3r)\,dr$.

(k) $\frac{d^2y}{dt^2}+9y=\sin(3t),\ y(0)=0,\ y'(0)=0$

If $\frac{d^2y}{dt^2}+9y=\sin(3t),\ y(0)=0,\ y'(0)=0$, then $y(t)=\frac{1}{3}\int_0^t \sin 3r\sin(3t-3r)\,dr=\frac{1}{6}\sin 3t-\frac{1}{2}t\cos 3t$.

(l) $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=\sqrt{t+1},\ y(0)=2,\ y'(0)=1$

If $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=\sqrt{t+1},\ y(0)=2,\ y'(0)=1$, then $y(t)=7e^{-2t}-5e^{-3t}+\int_0^t(e^{-2r}-e^{-3r})\sqrt{t-r+1}\,dr$.

(m) $y(t)+\int_0^t r\,y(t-r)\,dr=t$.

If $y(t)+\int_0^t y(r)(t-r)\,dr=t$, then $y(t)=\sin t$.

(n) $y(t)=te^t+\int_0^t(t-r)\,y(r)\,dr$.

If $y(t)=te^t+\int_0^t(t-r)\,y(r)\,dr$, then $y(t)=-\frac{1}{8}e^{-t}+\frac{1}{8}e^t+\frac{3}{4}te^t+\frac{1}{4}t^2e^t$.

(o) $\frac{dy}{dt}=1-\sin t-\int_0^t y(r)\,dr,\ y(0)=0$.

If $\frac{dy}{dt}=1-\sin t-\int_0^t y(r)\,dr,\ y(0)=0$, then $y(t)=\sin t-\frac{1}{2}t\sin t$.

(p) $\frac{dy}{dt}+2y+10\int_0^t e^{4r}y(t-r)\,dr=0,\ y(0)=7$.

If $\frac{dy}{dt}+2y+10\int_0^t e^{4r}y(t-r)\,dr=0,\ y(0)=7$, then $y=7e^t\cos t-21e^t\sin t$.

(q) $\frac{dy}{dt}+3y=\begin{cases}2, & 0\le t<4,\\ 0, & 4\le t\end{cases},\ y(0)=2,\ y'(0)=0$

If $\frac{dy}{dt}+3y=\begin{cases}2, & 0\le t<4,\\ 0, & 4\le t\end{cases},\ y(0)=2$, then

$$y(t)=\frac{2}{3}+\frac{4}{3}e^{-3t}-\frac{2}{3}\mathcal{U}(t-4)+\frac{2}{3}\mathcal{U}(t-4)e^{12-3t}.$$

The graph of y has a corner at $t=4$. The graph of $\frac{dy}{dt}$ has a jump at $t=4$.

(r) $\frac{d^2y}{dt^2}+4y=\begin{cases}\sin t, & 0\le t<2\pi,\\ 0, & 2\pi\le t\end{cases},\ y(0)=0,\ y'(0)=0$

If $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = (1/6)(1 - \mathcal{U}(t - 2\pi))(2\sin t - \sin 2t).$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2\pi$.

(s) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - \mathcal{U}(t-10)\left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right].$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 10$. The graph of $\frac{d^2y}{dt^2}$ has a jump discontinuity at $t = 10$.

(t) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \le t < 2, \\ 4, & 2 \le t \end{cases}$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = 2$.

If $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \le t < 2, \\ 4, & 2 \le t \end{cases}$, $y(0) = 3$, $y'(0) = 2$, $y''(0) = 1$, then

$$y(t) = 3e^{-t} + 5te^{-t} + 4t^2e^{-t} + \mathcal{U}(t-2)(4 - 4e^{2-t} - 4te^{2-t} - 2(t-2)^2e$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2$. The graph of $\frac{d^3y}{dt^3}$ has a jump discontinuity at $t = 2$.

(u) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$, then

$$y(t) = 2e^{-2t} + 5te^{-2t} + \frac{3}{4}u_2(t) - \frac{3}{4}e^{-2t+4}u_2(t) - \frac{3}{2}(t-2)e^{-2t+4}u_2(t)$$

The graph of $\frac{dy}{dt}$ has a corner at $t=2$. The graph of $\frac{d^2y}{dt^2}$ has a jump at $t=2$.

(v) $6\frac{d^2y}{dt^2}+5\frac{dy}{dt}+y=4\mathcal{U}(t-2)$, $y(0)=0$, $y'(0)=1$.

If $6\frac{d^2y}{dt^2}+5\frac{dy}{dt}+y=4\mathcal{U}(t-2)$, $y(0)=0$, $y'(0)=1$, then

$$y=6e^{-t/3}-6e^{-t/2}+4\mathcal{U}(t-2)-12\mathcal{U}(t-2)e^{-(t-2)/3}+8\mathcal{U}(t-2)e^{-(t-$$

The graph of $y'(t)$ has a corner at $t=2$, and the graph of $y''(t)$ has a jump at $t=2$.

(w) $\frac{dy}{dt}+9y=7\delta(t-2)$, $y(0)=3$.

If $\frac{dy}{dt}+9y=7\delta(t-2)$, $y(0)=3$, then $y(t)=3e^{-9t}+7\mathcal{U}(t-2)e^{-9t+18}$. The graph of $y(t)$ has a jump at $t=2$.

(x) $\frac{d^2y}{dt^2}+4y=-2\delta(t-4\pi)$, $y(0)=1/2$, $y'(0)=0$

If $\frac{d^2y}{dt^2}+4y=\delta(t-4\pi)$, $y(0)=1/2$, $y'(0)=0$, then

$$y=\frac{1}{2}\cos(2t)-\mathcal{U}(t-4\pi)\sin(2t).$$

The graph of $y(t)$ has a corner at $t=4\pi$, and graph of $y'(t)$ has a jump at $t=4\pi$.

(y) $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+3y=2\delta(t-1)+\mathcal{U}(t-2)$, $y(0)=1$, $y'(0)=0$

If $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+3y=2\delta(t-1)+\mathcal{U}(t-2)$, $y(0)=1$, $y'(0)=0$, then

$$y=\frac{3}{2}e^{-t}-\frac{1}{2}e^{-3t}+\frac{1}{2}\mathcal{U}(t-1)e^{-t+1}-\frac{1}{2}\mathcal{U}(t-1)e^{-3t+3}+\frac{1}{3}\mathcal{U}(t-2)-$$

The graph of $y(t)$ has a corner at $t=1$. The graph of $y'(t)$ has a corner at $t=2$, and a jump at $t=1$. $y''(t)$ has an impulse at $t=1$, and a jump at $t=2$.

(z) $\frac{d^3y}{dt^3}-2\frac{d^2y}{dt^2}+\frac{dy}{dt}-2y=5\delta(t-4)$, $y(0)=1$, $y'(0)=0$, $y''(0)=2$.

If $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - \frac{dy}{dt} + 2y = 5\delta(t-4)$, $y(0) = 3$, $y'(0) = 0$, $y''(0) = 0$, then

$$y = \frac{3}{5}e^{2t} + \frac{4}{5}\cos t - \frac{2}{5}\sin t + \mathcal{U}(t-4)(e^{2t} - \cos t - 2\sin t)$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 4$, and the graph of $\frac{d^2y}{dt^2}$ has a jump at $t = 4$.

(AB 92) Consider the initial value problem $\frac{dx}{dt} = x\cos y$, $\frac{dy}{dt} = x^2 \sin y$, $x(0) = 4$, $y(0) = \pi/2$. Use the phase plane method to find a nondifferential equation relating x and y.

(Answer 92) We compute that $\frac{dy}{dx} = x\tan y$, so $\ln|\sin y| = \frac{1}{2}x^2 - 8$.

(AB 93) Consider the system of differential equations $\frac{dx}{dt} = 3x - 4y$, $\frac{dy}{dt} = 4x - 3y$. Use the phase plane method to find a nondifferential equation relating x and y.

(Answer 93) The trajectories of solutions to $\frac{dx}{dt} = 3x - 4y$, $\frac{dy}{dt} = 4x - 3y$ satisfy $4y^2 - 6xy + 4x^2 = C$ for constants C.

(AB 94) Consider the system of differential equations $\frac{dx}{dt} = 3y - 2xy$, $\frac{dy}{dt} = 4xy - 3y$. Use the phase plane method to find a nondifferential equation relating x and y.

(Answer 94) If $\frac{dx}{dt} = 3y - 2xy$, $\frac{dy}{dt} = 4xy - 3y$, then $y = -2x - \frac{3}{2}\ln|x - 3/2| + C$.

(AB 95) Consider the system of differential equations $\frac{dx}{dt} = 3x - 4xy$, $\frac{dy}{dt} = 5xy - 2y$. Use the phase plane method to find a nondifferential equation relating x and y.

(Answer 95) If $\frac{dx}{dt} = 3x - 4xy$, $\frac{dy}{dt} = 5xy - 2y$, then $3\ln|y| + 2\ln|x| - 4y - 5x = C$.

(AB 96) Imperial stormtroopers and Rebel Alliance fighters battle each other on an open plain, where both groups can easily see and aim at all members of the other group. Every minute, each stormtrooper has a 2% chance of killing a rebel, and each rebel has a 5% chance of killing a stormtrooper. There are initially 4000 stormtroopers and 1000 rebels.

Find a nondifferential equation relating the number of surviving rebels and the number of surviving stormtroopers.

(Answer 96) Let t denote time (in minutes), let S denote the number of stormtroopers, and let R denote the number of rebels.

Then

$$\frac{dR}{dt} = -0.02S, \quad \frac{dS}{dt} = -0.05R, \quad R(0) = 1000, \quad S(0) = 4000.$$

The phase plane method tells us that

$$\frac{dR}{dS} = \frac{2S}{5R}.$$

This is a separable equation and we compute

$$5R^2 = 2S^2 - 27{,}000{,}000.$$

(AB 97) Jedi knights and Sith lords battle in a dense forest. Every minute, each Jedi has a 1% chance of finding each Sith lord. If a Jedi finds a Sith lord, they fight; the Jedi has a 60% chance of dying and the Sith lord has a 40% chance of dying. Initially there are 90 Jedi and 50 Sith lords.

Find a nondifferential equation relating the number of Jedi and Sith lords still alive.

(Answer 97) Let t denote time (in minutes), let S denote the number of Sith lords, and let J denote the number of Jedi.

Then

$$\frac{dJ}{dt} = -0.006JS, \quad \frac{dS}{dt} = -0.004JS, \quad J(0) = 90, \quad S(0) = 50.$$

Then $\frac{dS}{dJ} = \frac{4}{6}$ and so $S = \frac{2}{3}J - 10$.

(AB 98) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

(*a*) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

Let t denote time (in days), let S denote the number of susceptible people (who have never had the disease), let I denote the number of infected people (who currently have the disease), and let R denote the number of recovered people (who now are resistant, that is, cannot get the disease again). Then

$\boxed{\frac{dS}{dt} = -\frac{1}{22500}SI,\ \frac{dI}{dt} = \frac{1}{22500}SI - 0.17I,\ \frac{dR}{dt} = 0.17I,\ S(0) = 8997,}$

(*b*) Use the phase plane method to find a nondifferential equation relating the number of susceptible and infected people.

$\frac{dI}{dS} = \frac{3825}{S} - 1$, so $\boxed{I = 9000 - S - 3825\ln\frac{8997}{S}.}$

(*c*) Use the phase plane method to find a nondifferential equation relating the number of resistant and susceptible people.

$\frac{dR}{dS} = -\frac{3825}{S}$, so $\boxed{R = 3825\ln\frac{8997}{S}.}$

(*d*) What is the maximum number of people that are infected at any one time?

When I is maximized, $0 = \frac{dI}{dt} = \frac{1}{22500}SI - 0.17I$, and so $S = 3825$. Thus, the maximum value of I is

$\boxed{I = 9000 - 3825 - 3825\ln\frac{62979}{33750}.}$

(AB 99) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(Answer 99) $\frac{dS}{dt} = -\frac{0.2}{5000}SI - 15$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I + 15$.

(AB 100) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

(a) Set up the system of differential equations that describes the number of susceptible, infected, and resistant people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

$\boxed{\frac{dS}{dt} = -\frac{0.2}{5000}SI - 15\frac{S}{5000},\ \frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I,\ \frac{dR}{dt} = 0.1I + 15\frac{S}{5000}}$

(b) Assume that there are initially no recovered or vaccinated people and 7 infected people. Use the phase plane method to find a nondifferential equation relating S and I.

We compute that $\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{0.2SI-500I}{-0.2SI-15S} = \frac{0.2S-500}{S}\frac{I}{-0.2I-15}$. This is a separable differential equation, which we solve to see that $-0.2I - 15\ln I = 0.2S - 500\ln S + C$. Applying the initial conditions $I(0) = 7$, $S(0) = 4993$, we see that

$\boxed{-0.2(I-7) - 15\ln(I/7) = 0.2(S-4993) - 500\ln(S/4993).}$

(c) Find a nondifferential equation involving the maximum number of people that are infected with the virus at any one time.

The maximum I value occurs when $0 = \frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, or $S = 2500$. The maximum I value then satisfies

$\boxed{-0.2(I-7) - 15\ln(I/7) = 0.2(2500-4993) - 500\ln(2500/4993).}$

(AB 101) A small town has a population of 16,000 people. It is expected that soon, one of them will be infected with a contagious disease.

Epidemiologists observe the town and expect each infected person to encounter 10 people per day, who are distributed among susceptible, vaccinated, recovered and infected people according to their proportion of the total population. Each time an infected person encounters a susceptible person, there is a 6% chance that the susceptible person becomes infected. Each infected person has a 25% chance per day of recovering. A recovered person can never contract the disease again.

A (not very effective) vaccine can be distributed before the epidemic starts. Each time an infected person encounters a vaccinated person, there is a 2% chance they become infected. (Once a vaccinated person becomes infected, they are identical to an infected person who was never vaccinated.)

The mayor of the town would like to be sure that no more than 2,000 people are ever infected.

(*a*) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

Let t denote time (in days). Let V denote the number of never-infected vaccinated people, S denote the number of never-infected susceptible people, I denote the number of infected people, and R denote the number of recovered, disease-resistant people. We have a parameter, v, for the number of people vaccinated at the start of the epidemic. Then

$$\frac{dS}{dt} = -\frac{0.6}{16000}IS, \quad \frac{dV}{dt} = -\frac{0.2}{16000}IV, \quad \frac{dI}{dt} = \frac{0.6}{16000}IS + \frac{0.2}{16000}IV$$

and

$$S(0) = 15999 - v, \quad V(0) = v, \quad I(0) = 1, \quad R(0) = 0.$$

(*b*) Use the phase plane method to find a nondifferential equation relating the number of susceptible and recovered people.

We compute that $\frac{dS}{dR} = \frac{dS/dt}{dR/dt} = \frac{-\frac{0.6}{16000}IS}{0.25I} = \frac{-6}{40000}S$. This is a separable differential equation, which we solve to see that $S = Ce^{-6R/40000}$. Applying the initial conditions $S(0) = 15999 - v$, $R(0) = 0$, we see that $\boxed{S = (15999 - v)e^{-6R/40000}.}$

(c) Use the phase plane method to find a nondifferential equation relating the number of recovered people to the number of vaccinated (and never-infected) people.

We compute that $\frac{dV}{dR} = \frac{dV/dt}{dR/dt} = \frac{-\frac{0.2}{16000}IV}{0.25I} = \frac{-2}{40000}V$. This is a separable differential equation, which we solve to see that $V = Ce^{-2R/40000}$. Applying the initial conditions $V(0) = v$, $R(0) = 0$, we see that $\boxed{V = ve^{-2R/40000}.}$

(d) Find a nondifferential equation relating the number of infected people to the number of recovered people.

$I + S + V + R = 16{,}000$, so $I = 16000 - R - S - V = 16000 - R - ve^{-2R/40000} - (15999 - v)e^{-6R/40000}$.

(e) How many people must be vaccinated in order for there to be at most 2000 previously-infected (and thus resistant) people when the epidemic ends?

If $I = 0$ and $R = 2000$, then $0 = 14000 - ve^{-1/10} - (15999 - v)e^{-3/10}$. Solving, we see that $\boxed{v = \frac{14000-(15999)e^{-3/10}}{e^{-1/10}+e^{-3/10}}.}$ Thus, at least this many people must be vaccinated.

(AB 102) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers.

Each infected person encounters an average of 20 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. If an infected person encounters a recovered person, the recovered person has a 1% chance of contracting the disease again. Each infected person has a 12% chance of recovering from the disease on any given day.

(*a*) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

Let t denote time (in days), let S denote the number of susceptible people (who have never had the disease), let I denote the number of infected people (who currently have the disease), and let R denote the number of recovered people (who now are resistant, that is, are less likely to get the disease again). Then $\frac{dS}{dt} = -\frac{1}{9000}SI$, $\frac{dI}{dt} = \frac{1}{9000}SI + \frac{1}{45000}RI - 0.12I$, $\frac{dR}{dt} = 0.12I - \frac{1}{45000}RI$, $S(0) = 8997$, $I(0) = 3$, $R(0) = 0$.

(*b*) Use the phase plane method to find a nondifferential equation relating the number of susceptible and recovered people. Solve for the number of recovered people.

$\frac{dR}{dS} = \frac{dR/dt}{dS/dt} = \frac{R-5400}{5S}$, so $\ln|R - 5400| = \frac{1}{5}\ln S + C$. Using our initial conditions, we see that $\ln\frac{5400-R}{5400} = \frac{1}{5}\ln\frac{S}{8997}$, or

$$\boxed{R = 5400 - 5400\left(\frac{S}{8997}\right)^{1/5}.}$$

(*c*) Write a nondifferential equation relating the number of infected people to the number of susceptible people.

$I + R + S = 9000$, so $I = 9000 - R - S = 3600 + 5400\left(\frac{S}{8997}\right)^{1/5}$.

(d) Does the number of infected people ever approach zero? If so, how many susceptible people are left at that time?
No. Observe that if $S \geq 0$ then $I \geq 3600$.

(e) Does the number of susceptible people ever approach zero? If so, how many infected people and how many resistant people are there at that time?
As $S \to 0$, we see that $I \to 3600$ and $R \to 5400$.

(AB 103) Suppose that $\frac{d^2x}{dt^2} = 18x^3$, $x(0) = 1$, $x'(0) = 3$. Let $v = \frac{dx}{dt}$. Find a formula for v in terms of x. Then find a formula for x in terms of t.

(Answer 103) We have that $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ and also $\frac{dv}{dt} = \frac{d^2x}{dt^2}$. Thus $v\frac{dv}{dx} = 18x^3$, $v(1) = 3$. Solving, we see that $v = 3x^2$.

But then $\frac{dx}{dt} = 3x^2$, $x(0) = 1$, and so $x = \frac{1}{1-3t}$.

(AB 104) Suppose that a rocket of mass $m = 1000$ kg is launched straight up from the surface of the planet Gallifrey with initial velocity 10 km/sec. The radius of Gallifrey is 3,000 km. When the rocket is r meters from the center of Gallifrey, it experiences a force due to gravity of magnitude GMm/r^2, where $GM = 6 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the rocket's position.
The initial value problem is

$$1000\frac{d^2r}{dt^2} = -\frac{6 \times 10^{17}}{r^2}, \quad r(0) = 3{,}000{,}000, \quad r'(0) = 10000$$

where r denotes the distance to the center of Gallifrey in meters and t denotes time in seconds.

(b) Find the velocity of the rocket as a function of position.
Let v be the rocket's velocity in meters/second. We have that

$$1000v\frac{dv}{dr} = -\frac{6 \times 10^{17}}{r^2}, \quad v(3{,}000{,}000) = 10000$$

and so

$$500v^2 = \frac{6 \times 10^{17}}{r} - 1.5 \times 10^{11}.$$

(c) How far away from the earth is the rocket when it stops moving and starts to fall back?
$v = 0$ when $r = 4 \times 10^6$ meters.

(AB 105) A 5-kg toolbox is dropped (from rest) out of a spaceship at an altitude of 9,000 km above the surface of Gallifrey. The radius of Gallifrey is 3,000 km. When the toolbox is r meters from the center of the earth, it experiences a force due to gravity of magnitude $5GM/r^2$, where $GM = 6 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the toolbox's position.
The initial value problem is

$$5\frac{d^2r}{dt^2} = -\frac{3 \times 10^{15}}{r^2}, \quad r(0) = 12{,}000{,}000, \quad r'(0) = 0$$

where r denotes the distance to the center of Gallifrey in meters and t denotes time in seconds.

(b) Find the velocity of the toolbox as a function of position.
Let v be the toolbox's velocity in meters/second. We have that

$$5v\frac{dv}{dr} = -\frac{3 \times 10^{15}}{r^2}, \quad v(12{,}000{,}000) = 0$$

and so

$$\frac{5}{2}v^2 = \frac{3 \times 10^{15}}{r} - 2.5 \times 10^8.$$

(c) How fast is the toolbox moving when it strikes the surface of Gallifrey?
When $r = 3{,}000{,}000$, $v = -10000\sqrt{3}$ meters/second.

(AB 106) A particle of mass $m = 3$ kg a distance r from an infinitely long string experiences a force due to gravity of magnitude Gm/r, where $G = 2000$ meters^2/second^2, directed directly toward the string. Suppose that the particle is initially 1000 meters from the string and takes off with initial velocity 200 meters/second directly away from the string.

(*a*) Formulate the initial value problem for the particle's position.
The initial value problem is

$$3\frac{d^2r}{dt^2} = -\frac{6000}{r}, \quad r(0) = 1000, \quad r'(0) = 200$$

where r denotes the distance to the string in meters and t denotes time in seconds.

(*b*) Find the velocity of the particle as a function of position.
Let v be the particle's velocity in meters/second. We have that

$$3v\frac{dv}{dr} = -\frac{6000}{r}, \quad v(1000) = 200$$

and so

$$v^2 = -4000 \ln r + 4000 \ln 1000 + 40000.$$

(*c*) How far away from the string is the particle when it stops moving and starts to fall back?
$v = 0$ when $r = 1000e^{10}$ meters.

(AB 107) A charged particle of mass $m = 20$ g a distance r meters from an electric dipole experiences a force of $3/r^3$ newtons, directed directly toward the dipole. Suppose that the particle is initially 3 meters from the dipole and is set in motion with initial velocity 5 meters/second away from the dipole.

(a) Formulate the initial value problem for the particle's position.
The initial value problem is

$$0.02\frac{d^2r}{dt^2} = -\frac{3}{r^3}, \quad r(0) = 3, \quad r'(0) = 5$$

where r denotes the distance to the dipole in meters and t denotes time in seconds.

(b) Find the velocity of the particle as a function of position.
Let v be the particle's velocity in meters/second. We have that

$$0.02v\frac{dv}{dr} = -\frac{3}{r^3}, \quad v(3) = 5$$

and so

$$v^2 = \frac{150}{r^2} + \frac{25}{3}.$$

(c) What is the limiting velocity of the particle?
As $r \to \infty$, we see that v approaches $\frac{5}{\sqrt{3}}$ meters/second.

(AB 108) Suppose that a bob of mass 300g hangs from a pendulum of length 15cm. The pendulum is set in motion from its equilibrium point with an initial velocity of 3 meters/second.

If θ denotes the angle between the pendulum and the vertical, then the pendulum satisfies the equation of motion $m\frac{d^2\theta}{dt^2} = -\frac{mg}{\ell}\sin\theta$, where m is the mass of the pendulum bob, ℓ is the length of the pendulum and g is the acceleration of gravity (which you may take to be 9.8 meters/second^2).

Find a formula for $\omega = \frac{d\theta}{dt}$ in terms of θ.

(Answer 108) Let θ be the angle between the pendulum and a vertical line (in radians), and let t denote time in seconds. Then

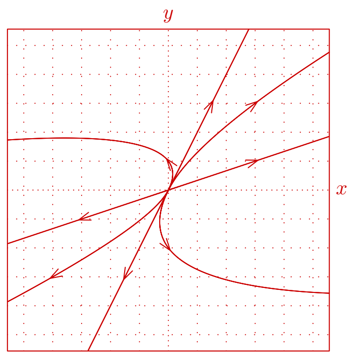
$$0.3\frac{d^2\theta}{dt^2} = -\frac{9.8}{0.5}\sin\theta, \quad \theta(0) = 0, \quad \theta'(0) = 20.$$

Let $\omega = \frac{d\theta}{dt}$ be the pendulum's angular velocity. We have that $\frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}$ and also $\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$. Thus $\omega\frac{d\omega}{d\theta} = -\frac{196}{3}\sin\theta$ and $\omega(0) = 20$. Solving, we see that $\frac{1}{2}\omega^2 = \frac{196}{3}\cos\theta + \frac{404}{3}$.

(AB 109) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

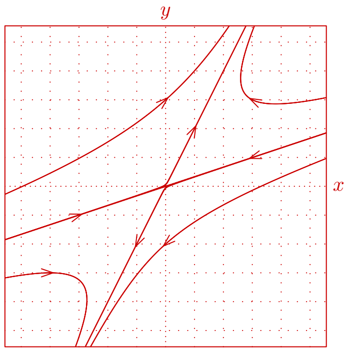
This system has two real eigenvectors, so we expect two pairs of straight line solutions. The eigenvalues are both positive and real, so we expect an unstable node. Thus the phase portrait must be



(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2.6 & 1.8 \\ -1.2 & 1.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution

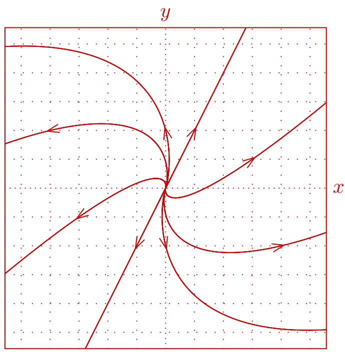
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

This system has two real eigenvectors, so we expect two pairs of straight line solutions. This system has one positive eigenvalue and one negative eigenvalue, so we expect a saddle point. Thus the phase portrait must be



(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4/3 & -1/6 \\ 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution
$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^t \begin{pmatrix} t+3 \\ 2t \end{pmatrix}$

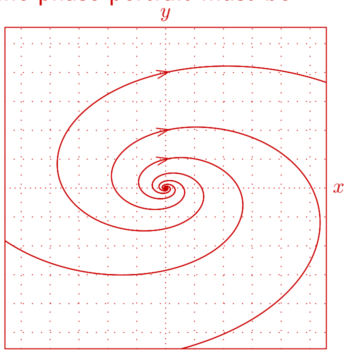
This system has one real eigenvectors, so we expect one pair of straight line solutions. This system has a repeated eigenvalue, and so we expect a degenerate node. Thus the phase portrait must be



(d) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4.5 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 3\sin 3t \\ 2\cos 3t \end{pmatrix} + C_2 e^t \begin{pmatrix} -3\cos 3t \\ 2\sin 3t \end{pmatrix}$$

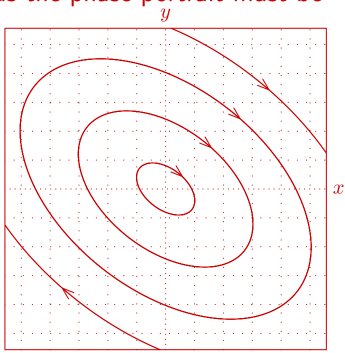
This system has no real eigenvectors, so we don't expect any straight line solutions. The eigenvectors have nonzero real part (we see an e^t, not only sines and cosines), so we expect a spiral point. Thus the phase portrait must be



(e) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 5\sin 4t \\ 4\cos 4t - 2\sin 4t \end{pmatrix} + C_2 \begin{pmatrix} 5\cos 4t \\ -4\sin 4t - 2\cos 4t \end{pmatrix}$$

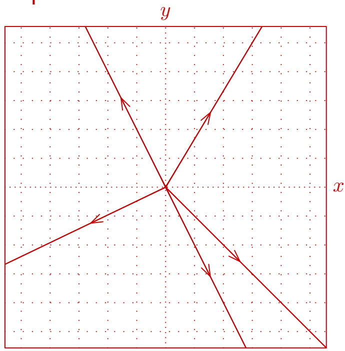
This system has no real eigenvectors, so we don't expect any straight line solutions. The eigenvectors have zero real part (we don't see any exponentials, just sines and cosines), so we expect a center. Thus the phase portrait must be



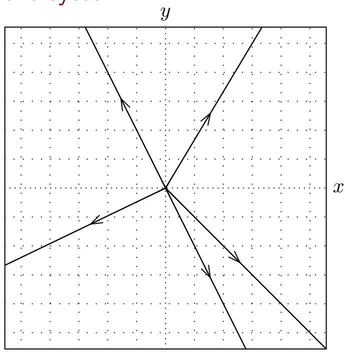
(f) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution

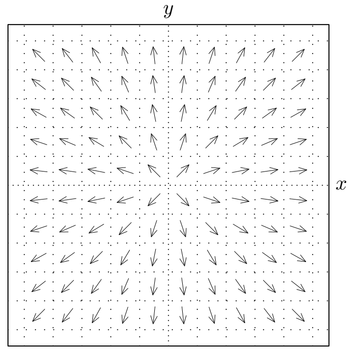
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This system has two real eigenvectors with the same eigenvalue, so we expect infinitely many pairs of straight line solutions. Thus the phase portrait must be



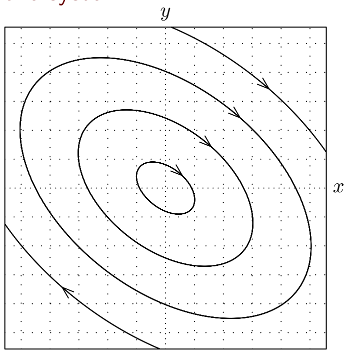
(AB 110) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?

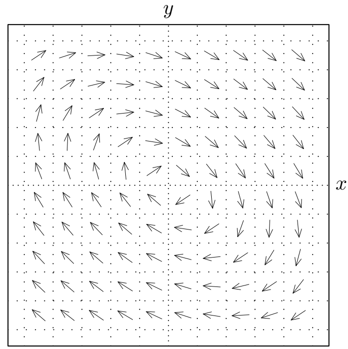




(Answer 110) This is a star. The system is unstable. Every vector is an eigenvector. There is only one eigenvalue and it is positive. We must have that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ for some positive number r.

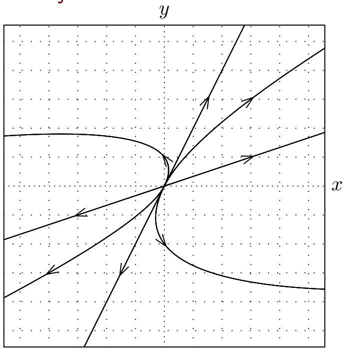
(AB 111) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?

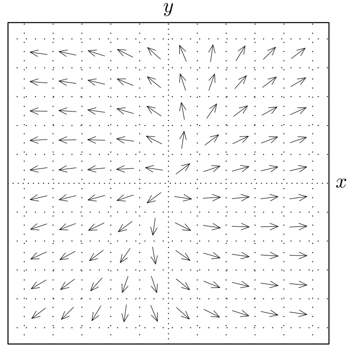




(Answer 111) This is a center. The system is stable but not asymptotically stable. The eigenvalues are purely imaginary. The solutions consist of sines and cosines (no exponentials or powers of t).

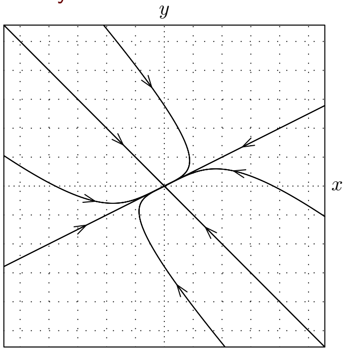
(AB 112) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?

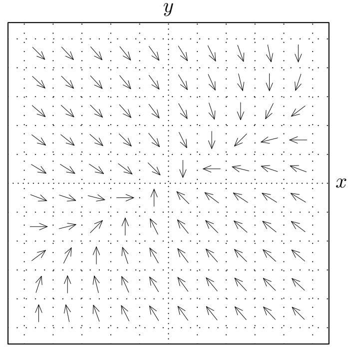




(Answer 112) This is a node. The system is unstable. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two distinct real positive eigenvalues. The solutions take the form $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{st} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, where r and s are the eigenvalues and $0 < r < s$.

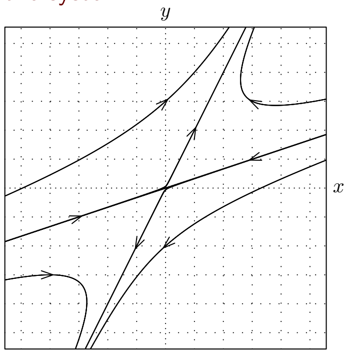
(AB 113) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?

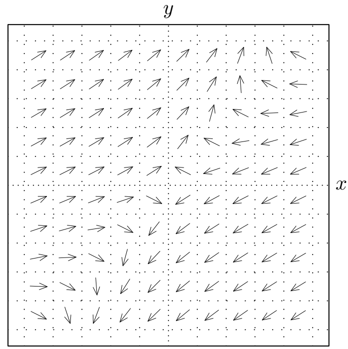




(Answer 113) This is a node. The system is asymptotically stable. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two distinct real negative eigenvalues. The solutions take the form $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{st} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, where r and s are the eigenvalues and $s < r < 0$.

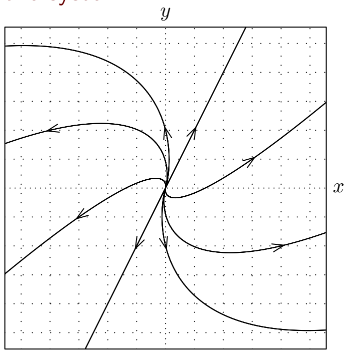
(AB 114) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?

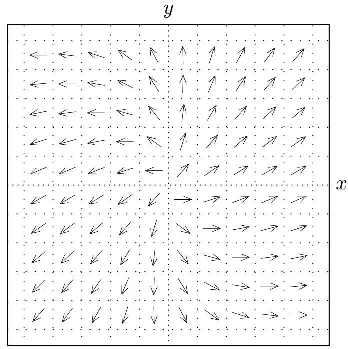




(Answer 114) This is a saddle point. The system is unstable. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two real eigenvalues, one positive and one negative. The solutions take the form $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{st} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, where r and s are the eigenvalues and $s < 0 < r$.

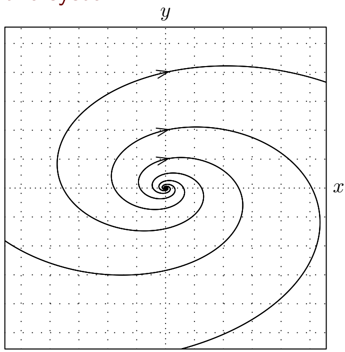
(AB 115) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?

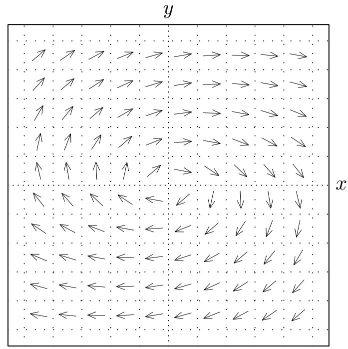




(Answer 115) This is a degenerate node. The system is unstable. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has only one eigenvalue and it is positive. If C is a constant then $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, where $r > 0$ is the eigenvalue. The general solution is $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{rt} \begin{pmatrix} t + A \\ 2t + B \end{pmatrix}$, where A and B are constants.

(AB 116) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



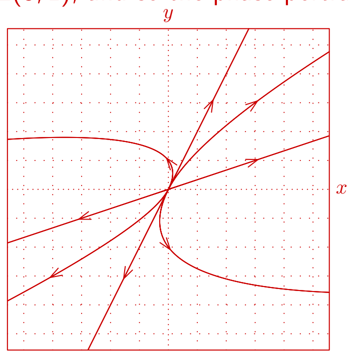


(Answer 116) This is a spiral point. The system is unstable. The eigenvalues are complex and take the form $\mu \pm \lambda i$, where $\mu > 0$ is real and λ is real. The solutions are exponentials multiplied by sines and cosines.

(AB 117) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

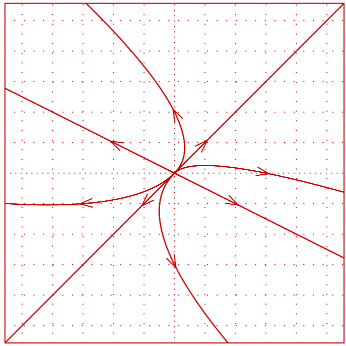
(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

The straight line solutions should pass through the points $\pm(1, 2)$ and $\pm(3, 1)$, and so the phase portrait must be



(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{6t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

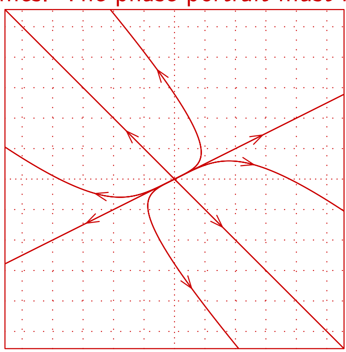
The straight line solutions should pass through the points $\pm(1, 1)$ and $\pm(2, -1)$, and so the phase portrait must be



(AB 118) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

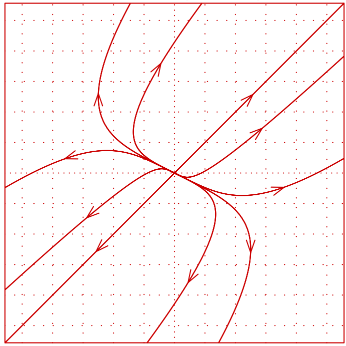
(*a*) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ has a smaller eigenvalue than the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and so the $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ will dominate solutions for negative times and the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ will dominate solutions for positive times. The phase portrait must be



(*b*) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

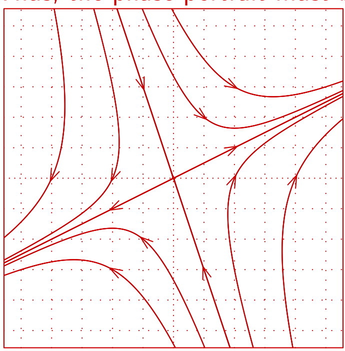
The eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ has a larger eigenvalue than the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and so the $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ will dominate solutions for large positive times and the eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ will dominate solutions for negative times. The phase portrait must be



(AB 119) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

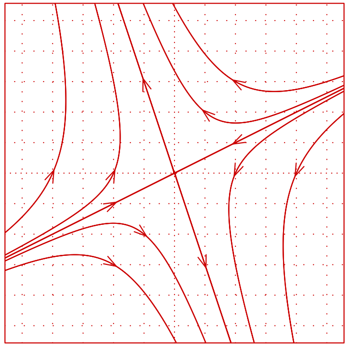
(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 9 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-14t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

The eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ corresponds to the positive eigenvalue 7, and so solutions should diverge away from the origin along the line $x = 2y$. Thus, the phase portrait must be



(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -4 & -6 \\ -9 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{14t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

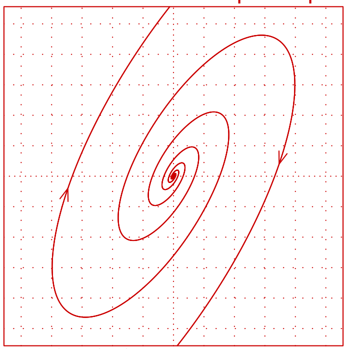
The eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ corresponds to the negative eigenvalue -7, and so solutions should approach the origin along the line $x = 2y$. Thus, the phase portrait must be



(AB 120) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(*a*) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -8 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, eigenvalues $r = 1 \pm 4i$

This system has eigenvalues with positive real part, so it must be unstable, and therefore must have phase portrait



(*b*) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 8 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, eigenvalues $r = -1 \pm 4i$

This system has eigenvalues with negative real part, so it must be asymptotically stable, and therefore must be