Math 2584, Spring 2024

Exam 3 will occur on Friday, April 12, 2024, at 2:00 p.m., in PHYS 133.

You are allowed a non-graphing calculator and a double-sided 3 inch by 5 inch card of notes.

Please check your final exam schedule. If you have 3 or more final exams scheduled for the same day, and you would like to reschedule the final exam for this class, please let me know by email by Friday, April 19.

The following Laplace transforms will be written on the last page of the exam:

If α, β, a, and k are real numbers, $n \geq 0$ is an integer, $f(t)$ and $g(t)$ are functions with Laplace transforms, $F(s)$ is the Laplace transform of some function, and y is a differentiable function such that both y and $\frac{dy}{dt}$ have Laplace transforms, then

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\,dt \\
\mathcal{L}\{\alpha f(t) + \beta g(t)\} &= \alpha\,\mathcal{L}\{f(t)\} + \beta\,\mathcal{L}\{g(t)\} \\
\mathcal{L}\{1\} &= \frac{1}{s} && s > 0 \\
\mathcal{L}\{t\} &= \frac{1}{s^2} && s > 0 \\
\mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}} && s > 0 \\
\mathcal{L}\{e^{at}\} &= \frac{1}{s-a} && s > a \\
\mathcal{L}\{\cos(kt)\} &= \frac{s}{s^2+k^2} && s > 0 \\
\mathcal{L}\{\sin(kt)\} &= \frac{k}{s^2+k^2} && s > 0 \\
\mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\} &= f(t) \\
\mathcal{L}\{\mathcal{L}^{-1}\{F(s)\}\} &= F(s) \\
\mathcal{L}\{f(t)\} = F(s) &\text{ if and only if } \mathcal{L}^{-1}\{F(s)\} = f(t) \\
\mathcal{L}^{-1}\{F(s)\} &= e^{at}\,\mathcal{L}^{-1}\{F(s+a)\} \\
\text{If } \mathcal{L}\{f(t)\} = F(s) &\text{ then } \mathcal{L}\{e^{at} f(t)\} = F(s-a) \\
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= s\,\mathcal{L}\{y(t)\} - y(0) \\
\mathcal{L}\{t\,f(t)\} &= -\frac{d}{ds}\,\mathcal{L}\{f(t)\} \\
\mathcal{L}\left\{\int_0^t f(r)\,g(t-r)\,dr\right\} &= \mathcal{L}\left\{\int_0^t f(t-r)\,g(r)\,dr\right\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}
\end{aligned}
$$

(AB 1) Find the general solution to the equation $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3}\ln t$ on the interval $0 < t < \infty$.

(AB 2) Find the general solution to the equation $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$.

(AB 3) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - t\frac{dx}{dt} - 3x = 0$, $t > 0$, is $x(t) = C_1t^3 + C_2t^{-1}$. Find the general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

(AB 4) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - 2x = 0$, $t > 0$, is $x(t) = C_1t^2 + C_2t^{-1}$. Solve the initial-value problem $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(AB 5) You are given that the general solution to the differential equation $(1-t)\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = 0$ on the interval $t < 1$ is $x(t) = C_1t + C_2e^t$. Find the general solution to $(1-t)\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = (1-t)^2e^t$ on the interval $t < 1$.

(AB 6) Find the general solution to the following differential equations.

(a) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 3e^{4t}$.
(b) $16\frac{d^2y}{dt^2} - y = e^{t/4}\sin t$.
(c) $\frac{d^2y}{dt^2} + 49y = 3t\sin 7t$.
(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$.
(e) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = t\sin(3t)$.
(f) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$.
(g) $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 6t^2 + \cos(2t)$.
(h) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = t^2e^{3t}$.
(i) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}$.
(j) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t)$.
(k) $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t)$.
(l) $\frac{d^2y}{dt^2} + 9y = 5\sin(3t)$.
(m) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$.
(n) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$.
(o) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$.
(p) $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$.
(q) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$.

(AB 7) A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 $\text{N}\cdot\text{s/m}$ and is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(20t)$ N upwards. You may take the acceleration of gravity to be 9.8 meters/second^2.

Write the differential equation and initial conditions that describe the position of the object.

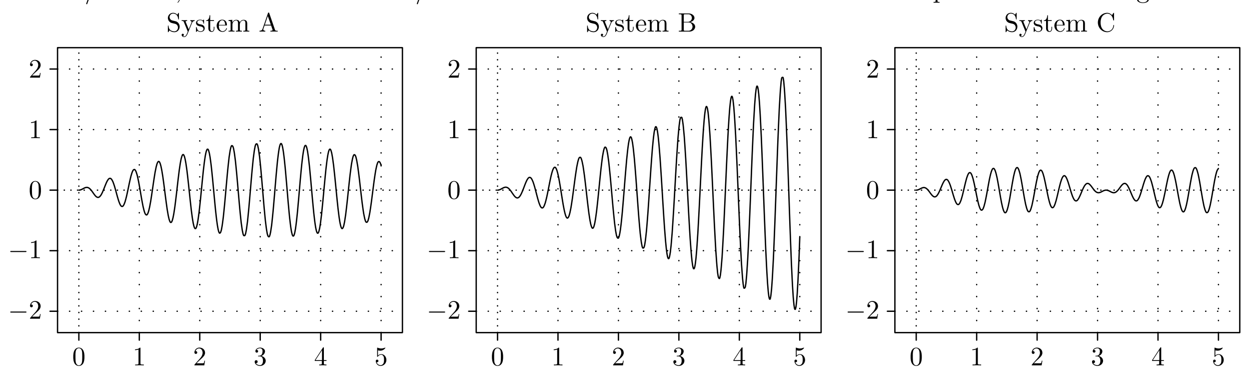
(AB 8) An object weighing 4 lbs stretches a spring 2 inches. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(\omega t)$ pounds, directed upwards. There is no damping. You may take the acceleration of gravity to be 32 feet/second^2.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of ω for which resonance occurs. Be sure to include units for ω.

(AB 9) A 4-kg object is suspended from a spring. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $7\sin(\omega t)$ newtons, directed upwards.

(a) Write the differential equation and initial conditions that describe the position of the object.
(b) It is observed that resonance occurs when $\omega = 20$ rad/sec. What is the constant of the spring? Be sure to include units.

(AB 10) A 1-kg object is suspended from a spring with constant 225 N/m. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $15\cos(\omega t)$ newtons, directed upwards. Illustrated are the object's position as a function of time for three different values of ω. You are given that the three values are $\omega = 15$ radians/second, $\omega = 16$ radians/second, and $\omega = 17$ radians/second. Determine the value of ω that will produce each image.



(AB 11) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ (**not** the table on the cover page of this exam), find the Laplace transforms of the following functions.

(a) $f(t) = e^{-11t}$
(b) $f(t) = t$
(c) $f(t) = \begin{cases} 3e^t, & 0 < t < 4, \\ 0, & 4 \leq t. \end{cases}$

(AB 12) Find the Laplace transforms of the following functions. You may use the table in your notes, on Blackboard, or in your book.

(a) $f(t) = t^4 + 5t^2 + 4$
(b) $f(t) = (t+2)^3$
(c) $f(t) = 9e^{4t+7}$
(d) $f(t) = -e^{3(t-2)}$
(e) $f(t) = (e^t + 1)^2$
(f) $f(t) = 8\sin(3t) - 4\cos(3t)$
(g) $f(t) = t^2 e^{5t}$
(h) $f(t) = 7e^{3t}\cos 4t$
(i) $f(t) = 4e^{-t}\sin 5t$
(j) $f(t) = t\,e^t \sin t$
(k) $f(t) = t^2 \sin 5t$
(l) $f(t) = \int_0^t e^{-4r}\sin(3r)\,dr$
(m) You are given that $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$. Find $\mathcal{L}\{t\,J_0(t)\}$. (The function J_0 is called a Bessel function and is important in the theory of partial differential equations in polar coordinates.)
(n) You are given that $\mathcal{L}\{\frac{1}{\sqrt{t}}e^{-1/t}\} = \frac{\sqrt{\pi}}{\sqrt{s}}e^{-2\sqrt{s}}$. Find $\mathcal{L}\{\sqrt{t}\,e^{-1/t}\}$.

(AB 13) For each of the following problems, find y.

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$
(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$
(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$
(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$
(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$
(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$
(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$
(h) $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$. *Hint*: Start by finding $\mathcal{L}\{t\sin 3t\}$ and $\mathcal{L}\{t\cos 3t\}$.
(i) $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$. You may express your answer as a definite integral.
(j) $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$. You may express your answer as a definite integral.
(k) $\mathcal{L}\{y\} = \frac{s}{s^4(s^2+36)}$. You may express your answer as a definite integral.

(AB 14) If a and b are constants, find $\mathcal{L}\{a\sin(4t) + b\,t\cos(4t)\}$. Then find values of a and b such that $\mathcal{L}\{a\sin(4t) + b\,t\cos(4t)\} = \frac{1}{(s^2+16)^2}$.

(AB 15) Solve the following initial-value problems using the Laplace transform.

(a) $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$
(b) $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$
(c) $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$
(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$
(e) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$
(f) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$
(g) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$
(h) $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$
(i) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$
(j) $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$
(k) $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$
(l) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$
(m) $y(t) + \int_0^t r\,y(t-r)\,dr = t$.
(n) $y(t) = te^t + \int_0^t (t-r)\,y(r)\,dr$.
(o) $\frac{dy}{dt} = 1 - \sin t - \int_0^t y(r)\,dr$, $y(0) = 0$.
(p) $\frac{dy}{dt} + 2y + 10\int_0^t e^{4r}y(t-r)\,dr = 0$, $y(0) = 7$.

Answer key

(AB 1) Find the general solution to the equation $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3}\ln t$ on the interval $0 < t < \infty$.

(Answer 1) If $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3}\ln t$, then $y(t) = c_1\, e^{t/3} + c_2\, t\, e^{t/3} + \frac{1}{2}t^2e^{t/3}\ln t - \frac{3}{4}t^2e^{t/3}$ for all $t > 0$.

(AB 2) Find the general solution to the equation $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$.

(Answer 2) If $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$, then $y = c_1e^{-t} + c_2e^{-t/2} - 2e^{-t}\sin(e^{t/2})$.

(AB 3) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - t\frac{dx}{dt} - 3x = 0$, $t > 0$, is $x(t) = C_1t^3 + C_2t^{-1}$. Find the general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

(Answer 3) If $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$, then $y(t) = -\frac{3}{2}t^{-1}\ln t + C_1t^3 + C_2t^{-1}$ for all $t > 0$.

(AB 4) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - 2x = 0$, $t > 0$, is $x(t) = C_1t^2 + C_2t^{-1}$. Solve the initial-value problem $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(Answer 4) If $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$, then $y(t) = -4\sqrt{t} + 3t^2 + \frac{2}{t}$ for all $0 < t < \infty$.

(AB 5) You are given that the general solution to the differential equation $(1-t)\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = 0$ on the interval $t < 1$ is $x(t) = C_1t + C_2e^t$. Find the general solution to $(1-t)\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = (1-t)^2e^t$ on the interval $t < 1$.

(Answer 5) If $(1-t)\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = (1-t)^2e^t$, then $y(t) = te^t - \frac{1}{2}t^2e^t + C_1t + C_2e^t$.

(AB 6) Find the general solution to the following differential equations.

(a) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 3e^{4t}$.

The general solution to $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 0$ is $y_g = C_1e^{-t/2} + C_2e^{-t/3}$. To solve $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 2e^{4t}$ we make the guess $y_p = Ae^{4t}$. The solution is $y = C_1e^{-t/2} + C_2e^{-t/3} + \frac{1}{39}e^{4t}$.

(b) $16\frac{d^2y}{dt^2} - y = e^{t/4}\sin t$.

The general solution to $16\frac{d^2y}{dt^2} - y = 0$ is $y_g = C_1e^{t/4} + C_2e^{-t/4}$. To solve $16\frac{d^2y}{dt^2} - y = e^{t/4}\sin t$ we make the guess $y_p = Ae^{t/4}\sin t + Be^{t/4}\cos t$. The solution is $y = C_1e^{t/4} + C_2e^{-t/4} - (1/20)e^{t/4}\sin t - (1/40)e^{t/4}\cos t$.

(c) $\frac{d^2y}{dt^2} + 49y = 3t\sin 7t$.

The general solution to $\frac{d^2y}{dt^2} + 49y = 0$ is $y_g = C_1\sin 7t + C_2\cos 7t$. To solve $\frac{d^2y}{dt^2} + 49y = 3t\sin 7t$ we make the guess $y_p = At^2\sin 7t + Bt\sin 7t + Ct^2\cos 7t + Dt\cos 7t$. The solution is $y = C_1\sin 7t + C_2\cos 7t - (3/28)t^2\cos 7t + (3/4)t\sin 7t$.

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$.

The general solution to $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 0$ is $y_g = C_1 + C_2e^{4t}$. To solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$ we make the guess $y_p = Ae^{3t}$. The solution is $y = C_1 + C_2e^{4t} - (4/3)e^{3t}$.

(e) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = t\sin(3t)$.

The general solution to $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$ is $y_g = C_1e^{-6t}\cos(7t) + C_2e^{-6t}\sin(7t)$. To solve $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = t\sin(3t)$ we make the guess $y_p = At\sin(3t) + Bt\cos(3t) + C\sin(3t) + D\cos(3t)$. The solution is $y = C_1e^{-6t}\cos(7t) + C_2e^{-6t}\sin(7t) + \frac{19}{1768}t\sin(3t) - \frac{9}{1768}t\cos(3t) - \frac{1353}{781456}\sin(3t) + \frac{606}{781456}\cos(3t)$.

(f) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$.

The general solution to $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 0$ is $y_g = C_1e^{2t} + C_2e^{-5t}$. To solve $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$ we make the guess $y_p = Ate^{-5t}$. The solution is $y = C_1e^{2t} + C_2e^{-5t} - (1/7)te^{-5t}$.

(g) $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 6t^2 + \cos(2t)$.

The general solution to $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 0$ is $y_g = C_1e^{(3/4)t} + C_2te^{(3/4)t}$. To solve $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 6t^2 + \cos(2t)$ we make the guess $y_p = At^2 + Bt + C + D\cos(2t) + E\sin(2t)$. The solution is $y = C_1e^{(3/4)t} + C_2te^{(3/4)t} + (2/3)t^2 + (32/9)t + 64/9 - (48/5329)\cos(2t) - (55/5329)\sin(2t)$.

(h) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = t^2e^{3t}$.

The general solution to $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = 0$ is $y_g = C_1e^{3t}\cos(4t) + C_2e^{3t}\sin(4t)$. To solve $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = t^2e^{3t}$ we make the guess $y_p = At^2e^{3t} + Bte^{3t} + Ce^{3t}$. The solution is $y = C_1e^{3t}\cos(4t) + C_2e^{3t}\sin(4t) + (1/16)t^2e^{3t} - (1/128)e^{3t}$.

(i) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}$.

The general solution to $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$ is $y_g = C_1e^{-5t} + C_2te^{-5t}$. To solve $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}$ we make the guess $y_p = At^2e^{-5t}$. The solution is $y = C_1e^{-5t} + C_2te^{-5t} + (3/2)t^2e^{-5t}$.

(j) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t)$.

The general solution to $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$ is $y_g = C_1e^{-2t} + C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t)$, we make the guess $y_p = A\cos(2t) + B\sin(2t)$. The solution is $y = C_1e^{-2t} + C_2e^{-3t} + \frac{15}{52}\sin(2t) + \frac{3}{52}\cos(2t)$.

(k) $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t)$.

The general solution to $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$ is $y_g = C_1e^{-3t} + C_2te^{-3t}$. To solve $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t)$, we make the guess $y_p = A\cos(4t) + B\sin(4t)$. The solution is $y = C_1e^{-3t} + C_2te^{-3t} - \frac{7}{125}\sin(4t) - \frac{24}{125}\cos(4t)$.

(l) $\frac{d^2y}{dt^2} + 9y = 5\sin(3t)$.

The general solution to $\frac{d^2y}{dt^2} + 9y = 0$ is $y_g = C_1\cos(3t) + C_2\sin(3t)$. To solve $\frac{d^2y}{dt^2} + 9y = 5\sin(3t)$, we make the guess $y_p = C_1t\cos(3t) + C_2t\sin(3t)$. The solution is $y = C_1\cos(3t) + C_2\sin(3t) - \frac{5}{6}t\cos(3t)$.

(m) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$.

The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ is $y_g = C_1 e^{-t} + C_2 t e^{-t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$, we make the guess $y_p = At^2 + Bt + C$. The solution is $y = C_1 e^{-t} + C_2 t e^{-t} + 2t^2 - 8t + 12$.

(n) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$.

The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 0$ is $y_g = C_1 + C_2 e^{-2t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$, we make the guess $y_p = At^2 + Bt$. The solution is $y = C_1 + C_2 e^{-2t} + \frac{3}{4}t^2 - \frac{3}{4}t$.

(o) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$.

The general solution to $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 0$ is $y_g = C_1 e^{3t} + C_2 e^{4t}$. To solve $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$, we make the guess $y_p = At + B + C\cos(2t) + D\sin(2t)$. The solution is $y = C_1 e^{3t} + C_2 e^{4t} + \frac{5}{12}t + \frac{35}{144} + \frac{16}{177}\cos 2t - \frac{28}{177}\sin 2t$.

(p) $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$.

The general solution to $\frac{d^2y}{dt^2} - 9y = 0$ is $y_g = C_1 e^{3t} + C_2 e^{-3t}$. To solve $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$, we make the guess $y_p = Ae^t + Be^{-t} + Ct + D$. The solution is $y = C_1 e^{3t} + C_2 e^{-3t} - \frac{1}{4}e^t - \frac{1}{8}e^{-t} - \frac{5}{9}t - \frac{2}{9}$.

(q) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$.

The general solution to $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$ is $y_g = C_1 e^{2t} + C_2 t e^{2t}$. To solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$, we make the guess $y_p = At^2 e^{2t} + B\cos t + C\sin t$. The solution is $y = C_1 e^{2t} + C_2 t e^{2t} + \frac{3}{2}t^2 e^{2t} + \frac{3}{5}\cos t - \frac{4}{5}\sin t$.

(AB 7) A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 N$\cdot$s/m and is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(20t)$ N upwards. You may take the acceleration of gravity to be 9.8 meters/second^2.

Write the differential equation and initial conditions that describe the position of the object.

(Answer 7) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters).

Then

$$3\frac{d^2x}{dt^2} + 27\frac{dx}{dt} + 588x = 3\cos(20t), \qquad u(0) = 0, \quad u'(0) = 0.$$

(AB 8) An object weighing 4 lbs stretches a spring 2 inches. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(\omega t)$ pounds, directed upwards. There is no damping. You may take the acceleration of gravity to be 32 feet/second^2.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of ω for which resonance occurs. Be sure to include units for ω.

(Answer 8) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then

$$\frac{4}{32}\frac{d^2x}{dt^2} + 24x = 3\cos(\omega t), \qquad x(0) = 0, \quad x'(0) = 0.$$

Resonance occurs when $\omega = 8\sqrt{3}$ s^{-1}.

(AB 9) A 4-kg object is suspended from a spring. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $7\sin(\omega t)$ newtons, directed upwards.

(a) Write the differential equation and initial conditions that describe the position of the object.
Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Let k denote the constant of the spring (in N·s/m). Then

$$4\frac{d^2x}{dt^2} + kx = 7\sin(\omega t), \qquad x(0) = 0, \quad x'(0) = 0.$$

(b) It is observed that resonance occurs when $\omega = 20$ rad/sec. What is the constant of the spring? Be sure to include units.
The spring constant is $k = 1600$ N·s/m.

(AB 10) A 1-kg object is suspended from a spring with constant 225 N/m. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $15\cos(\omega t)$ newtons, directed upwards. Illustrated are the object's position as a function of time for three different values of ω. You are given that the three values are $\omega = 15$ radians/second, $\omega = 16$ radians/second, and $\omega = 17$ radians/second. Determine the value of ω that will produce each image.

(Answer 10) $\omega = 15$ radians/second in Picture B. $\omega = 16$ radians/second in Picture A. $\omega = 17$ radians/second in Picture C.

(AB 11) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ (**not** the table on the cover page of this exam), find the Laplace transforms of the following functions.

(a) $f(t) = e^{-11t}$
(b) $f(t) = t$
(c) $f(t) = \begin{cases} 3e^t, & 0 < t < 4, \\ 0, & 4 \leq t. \end{cases}$

(Answer 11)

(a) $\mathcal{L}\{t\} = \frac{1}{s^2}$.
(b) $\mathcal{L}\{e^{-11t}\} = \frac{1}{s+11}$.
(c) $\mathcal{L}\{f(t)\} = \frac{3-3e^{4-4s}}{s-1}$.

(AB 12) Find the Laplace transforms of the following functions. You may use the table in your notes, on Blackboard, or in your book.

(a) $f(t) = t^4 + 5t^2 + 4$
$\mathcal{L}\{t^4 + 5t^2 + 4\} = \frac{96}{s^5} + \frac{10}{s^3} + \frac{4}{s}$.

(b) $f(t) = (t+2)^3$
$\mathcal{L}\{(t+2)^3\} = \mathcal{L}\{t^3 + 6t^2 + 12t + 8\} = \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{8}{s}$.

(c) $f(t) = 9e^{4t+7}$
$\mathcal{L}\{9e^{4t+7}\} = \mathcal{L}\{9e^7 e^{4t}\} = \frac{9e^7}{s-4}$.

(d) $f(t) = -e^{3(t-2)}$
$\mathcal{L}\{-e^{3(t-2)}\} = \mathcal{L}\{-e^{-6}e^{3t}\} = -\frac{e^{-6}}{s-3}$.

(e) $f(t) = (e^t + 1)^2$
$\mathcal{L}\{(e^t+1)^2\} = \mathcal{L}\{e^{2t} + 2e^t + 1\} = \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s}$.

(f) $f(t) = 8\sin(3t) - 4\cos(3t)$
$\mathcal{L}\{8\sin(3t) - 4\cos(3t)\} = \frac{24}{s^2+9} - \frac{4s}{s^2+9}$.

(g) $f(t) = t^2 e^{5t}$
$\mathcal{L}\{t^2 e^{5t}\} = \frac{2}{(s-5)^3}$.

(h) $f(t) = 7e^{3t}\cos 4t$
$\mathcal{L}\{7e^{3t}\cos 4t\} = \frac{7s-21}{(s-3)^2+16}$.

(i) $f(t) = 4e^{-t}\sin 5t$
$\mathcal{L}\{4e^{-t}\sin 5t\} = \frac{20}{(s+1)^2+25}$.

(j) $f(t) = t\,e^t \sin t$
$\mathcal{L}\{t\,e^t\sin t\} = \frac{2s-2}{(s^2-2s+2)^2}$.

(k) $f(t) = t^2 \sin 5t$
$\mathcal{L}\{t^2\sin 5t\} = \frac{30s^2-250}{(s^2+25)^3}$.

(l) $f(t) = \int_0^t e^{-4r}\sin(3r)\,dr$
$\mathcal{L}\left\{\int_0^t e^{-4r}\sin(3r)\,dr\right\} = \frac{3}{s((s+4)^2+9)}$.

(m) You are given that $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$. Find $\mathcal{L}\{t\,J_0(t)\}$. (The function J_0 is called a Bessel function and is important in the theory of partial differential equations in polar coordinates.)
$\mathcal{L}\{t\,J_0(t)\} = \frac{s}{(s^2+1)^{3/2}}$.

(n) You are given that $\mathcal{L}\{\frac{1}{\sqrt{t}}e^{-1/t}\} = \frac{\sqrt{\pi}}{\sqrt{s}}e^{-2\sqrt{s}}$. Find $\mathcal{L}\{\sqrt{t}\,e^{-1/t}\}$.
$\mathcal{L}\{\sqrt{t}\,e^{-1/t}\} = \frac{\sqrt{\pi}(1+2\sqrt{s})}{2s\sqrt{s}}e^{-2\sqrt{s}}$

(AB 13) For each of the following problems, find y.

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$

If $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$, then $y = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$.

(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$

If $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$, then $y = \frac{5}{2}t^2 - \frac{7}{6}t^3$.

(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$

If $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$, then $y = \frac{1}{2}t^2e^{-t} + \frac{1}{6}t^3e^{-t}$.

(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$

If $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$, then $y = (1/4)e^{2t} + (7/4)e^{-2t}$.

(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$

If $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$, then $y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{1}{9}t\,e^{3t}$.

(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$

If $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$, then $y = \frac{1}{2} - \frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t)$.

(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$

If $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$, then $y = \frac{1}{8}\cos t - \frac{1}{8}\cos 3t$.

(h) $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$. *Hint*: Start by finding $\mathcal{L}\{t\sin 3t\}$ and $\mathcal{L}\{t\cos 3t\}$.

If $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$, then $y = \frac{1}{6}t\sin(3t)$.

(i) $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$. You may express your answer as a definite integral.

If $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$, then $y = e^{-2t}\int_0^t \sin(2r)\sin(2t-2r)\,dr = \frac{1}{4}e^{-2t}\sin(2t) - \frac{1}{2}e^{-2t}t\cos(2t)$.

(j) $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$. You may express your answer as a definite integral.

If $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$, then $y = \int_0^t \frac{e^{3r}+e^{-3r}}{2}\sqrt{t-r}\,dr$.

(k) $\mathcal{L}\{y\} = \frac{s}{s^4(s^2+36)}$. You may express your answer as a definite integral.

$y = \int_0^t \cos(6r)\frac{1}{6}(t-r)^3\,dr$ or $y = \int_0^t \cos(6(t-r))\frac{1}{6}r^3\,dr$.

(AB 14) If a and b are constants, find $\mathcal{L}\{a\sin(4t) + b\,t\cos(4t)\}$. Then find values of a and b such that $\mathcal{L}\{a\sin(4t) + b\,t\cos(4t)\} = \frac{1}{(s^2+16)^2}$.

(Answer 14)

$$\mathcal{L}\{a\sin(4t) + b\,t\cos(4t)\} = \frac{4a}{s^2+16} + \frac{bs^2-16b}{(s^2+16)^2} = \frac{4as^2+bs^2+64a-16b}{(s^2+16)^2}.$$

Thus

$$\mathcal{L}\left\{\frac{1}{128}\sin(4t) - \frac{1}{32}\,t\cos(4t)\right\} = \frac{1}{(s^2+16)^2}.$$

(AB 15) Solve the following initial-value problems using the Laplace transform.

(a) $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$

If $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$, then $y(t) = -\frac{1}{30}\sin 3t - \frac{1}{90}\cos 3t + \frac{31}{30}e^{9t}$.

(b) $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$

If $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$, then $y = 3te^{2t} + 2e^{2t}$.

(c) $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$

If $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$, then $y = \frac{1}{5}t^3 - \frac{3}{25}t^2 + \frac{6}{125}t - \frac{6}{625} + \frac{1881}{625}e^{-5t}$.

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$, then $y(t) = e^{2t} - te^{2t}$.

(e) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$, then $y(t) = \frac{1}{5}(e^{-t} - e^t\cos t + 7e^t \sin t)$.

(f) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$

If $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$, then $y(t) = 4e^{3t} - 2e^{4t} - te^{3t}$.

(g) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$

If $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$, then $y(t) = -\frac{15}{8}e^{4t} + \frac{39}{8}e^{2t} - \frac{1}{4}te^{2t} - \frac{1}{2}t^2e^{2t} - t^3e^{2t}$.

(h) $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{40}e^{2t} - \frac{1}{72}e^{-2t} - \frac{1}{90}e^t\cos(3t) - \frac{1}{45}e^t\sin(3t)$.

(i) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$

If $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$, then $y(t) = \frac{1}{5}\cos 2t + \cos 3t + \frac{5}{3}\sin 3t$.

(j) $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t r\sin 3r \sin(3t - 3r)\,dr$.

(k) $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t \sin 3r\sin(3t - 3r)\,dr = \frac{1}{6}\sin 3t - \frac{1}{2}t\cos 3t$.

(l) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$, then $y(t) = 7e^{-2t} - 5e^{-3t} + \int_0^t (e^{-2r} - e^{-3r})\sqrt{t - r + 1}\,dr$.

(m) $y(t) + \int_0^t r\,y(t-r)\,dr = t$.

If $y(t) + \int_0^t y(r)(t-r)\,dr = t$, then $y(t) = \sin t$.

(n) $y(t) = te^t + \int_0^t (t-r)\,y(r)\,dr$.

If $y(t) = te^t + \int_0^t (t-r)\,y(r)\,dr$, then $y(t) = -\frac{1}{8}e^{-t} + \frac{1}{8}e^t + \frac{3}{4}te^t + \frac{1}{4}t^2e^t$.

(o) $\frac{dy}{dt} = 1 - \sin t - \int_0^t y(r)\,dr$, $y(0) = 0$.

If $\frac{dy}{dt} = 1 - \sin t - \int_0^t y(r)\,dr$, $y(0) = 0$, then $y(t) = \sin t - \frac{1}{2}t\sin t$.

(p) $\frac{dy}{dt} + 2y + 10\int_0^t e^{4r}y(t-r)\,dr = 0$, $y(0) = 7$.

If $\frac{dy}{dt} + 2y + 10\int_0^t e^{4r}y(t-r)\,dr = 0$, $y(0) = 7$, then $y = 7e^t\cos t - 21e^t\sin t$.