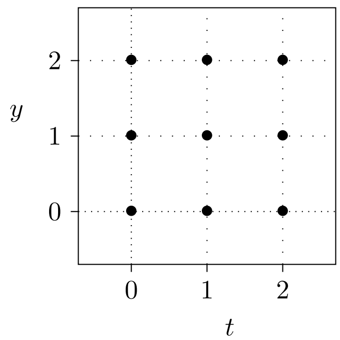
Math 2584, Fall 2023

Exam 1 will occur on Friday, February 9, 2024, at 2:00 p.m., in PHYS 133.
You are allowed a non-graphing calculator and a double-sided 3 inch by 5 inch card of notes.
The following theorem will be written on the cover page of the exam:

Picard-Lindelöf theorem. Consider $\frac{dy}{dt} = f(t, y)$. If the functions of two variables f and $\frac{\partial f}{\partial y}$ are continuous near the point (t_0, y_0), we call (t_0, y_0) a "good" point for f. If (t_0, y_0) is a "good" point for f, then exists is a unique solution to the initial value problem $\frac{dy}{dt} = f(y, t)$, $y(t_0) = y_0$.

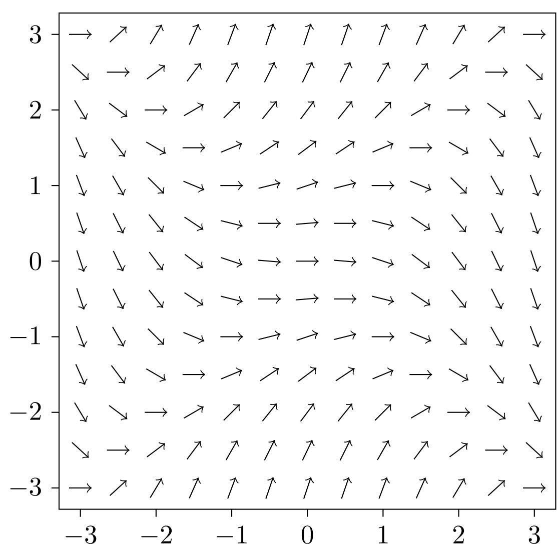
The solution will continue to exist and be unique until either it approaches a "bad" point, or until the solution becomes unbounded.

(AB 1) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.



(AB 2) Consider the differential equation $\frac{dy}{dx} = (y^2 - x^2)/3$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

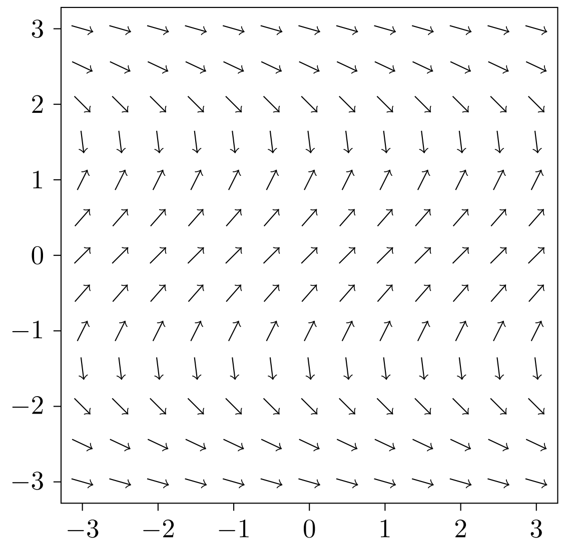
$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-2) = 1.$$



(AB 3) Consider the differential equation $\frac{dy}{dx} = \frac{2}{2-y^2}$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{2}{2-y^2}, \quad y(1) = 0.$$

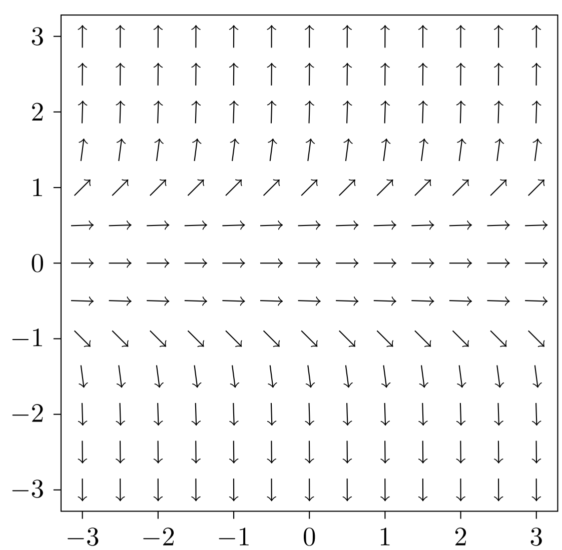
Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 4) Consider the differential equation $\frac{dy}{dx} = y^5$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = y^5, \quad y(1) = -1.$$

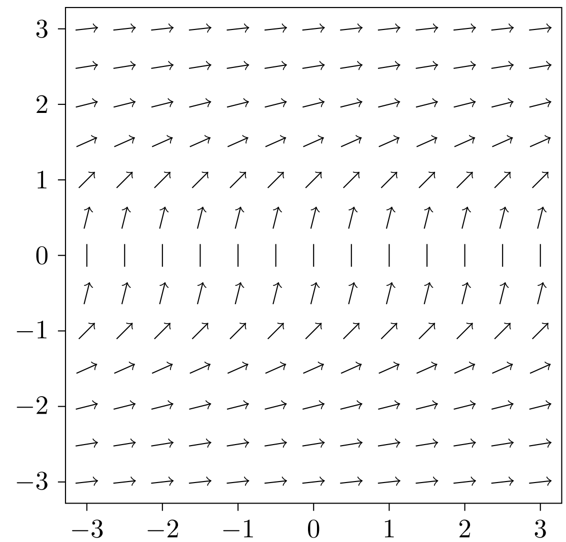
Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 5) Consider the differential equation $\frac{dy}{dx} = 1/y^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{1}{y^2}, \quad y(3) = 2.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 6) A lake initially has a population of 600 trout. The birth rate of trout is proportional to the number of trout living in the lake. Fishermen are allowed to harvest 30 trout/year from the lake. Write an initial value problem for the fish population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 7) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write an initial value problem for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 8) A tank contains nitrogen dioxide. Initially, there are 200 grams of NO_2. The gas decays (converts to molecular oxygen and nitrogen) at a rate proportional to the square of the amount of NO_2 remaining. Write an initial value problem for the amount of NO_2 in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 9) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen consumed. Write an initial value problem for the amount of hydrogen in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 10) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C. Write an initial value problem for the temperature of the cup of coffee. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 11) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Write an initial value problem for the amount of salt in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 12) I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write an initial value problem for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 13) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write an initial value problem for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 14) According to the Stefan-Boltzmann law, a black body in a dark vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 15) According to the Stefan-Boltzmann law, a black body in a vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). If there is some ambient light in the vacuum, then the black body absorbs energy at a rate proportional to the fourth power of the effective temperature of the light. The proportionality constant is the same in both cases.

Suppose that a planet in space is currently at a temperature of 400 K. The effective temperature of its surroundings is 3 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 16) A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 17) A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 18) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y-2)(y+1)^2$. By hand, sketch some typical solutions.

(AB 19) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 20) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = y^2(y-2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 21) Trout live in a lake. Each trout produces, on average, one baby trout every two years. Fishermen are allowed to harvest 100 trout/year from the lake. You may neglect other reasons why trout die.

(a) Formulate a differential equation for the number of trout in the lake.
(b) Find the critical points of this differential equation and classify them as to stability.
(c) What is the real-world meaning of the critical points?

(AB 22) I owe my bank a debt of B dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

(a) Formulate a differential equation for the amount of money I owe.
(b) Find the critical points of this differential equation and classify them as to stability.
(c) What is the real-world meaning of the critical points?

(AB 23) A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

(a) Write the differential equation for the amount of salt in the tank.
(b) Find the critical points of this differential equation and classify them as to stability.
(c) What is the real-world meaning of the critical points?

(AB 24) A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where v is her velocity in meters per second.

(a) Write a differential equation for her velocity. Assume her velocity is always downwards.
(b) Find the critical points of this differential equation and classify them as to stability. Be sure to include units.
(c) What is the real-world meaning of these critical points?

(AB 25) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen.

(a) Write a differential equation for the amount of hydrogen left in the tank.
(b) Find the critical points of this differential equation and classify them as to stability. Be sure to include units.
(c) What is the real-world meaning of these critical points?

(AB 26) Young oak trees grow in a large field with area 1 square kilometer. Initially, there are 30 trees in the field. Each tree shades 20 square meters. The trees produce acorns. Squirrels scatter the acorns across the field at random. Squirrels eat most of the acorns; on average, each tree produces two acorns per year that are not eaten by squirrels. Every acorn not eaten by squirrels and planted in sunlight sprouts.

(a) Write a differential equation for the number of trees in the field.
(b) Find the critical points of this differential equation and classify them as to stability.
(c) What is the real-world meaning of these critical points?

(AB 27) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate of 3 cm^3 per exposed cm2 per hour.

(a) Write a differential equation for the amount of water in the tank.
(b) Find the critical points of this differential equation and classify them as to stability.
(c) What is the real-world meaning of these critical points?

(AB 28) Is $y = e^t$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 29) Is $y = e^{2t}$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 30) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 1$?

(AB 31) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 2$?

(AB 32) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 1$?

(AB 33) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 2$?

(AB 34) For each of the following differential equations, determine whether it is linear or separable. Then solve the differential equation.

(a) $t + \cos t + (y - \sin y)\frac{dy}{dt} = 0$
(b) $1 + t^2 - ty + (t^2 + 1)\frac{dy}{dt} = 0$
(c) $y^3 \cos(2t) + \frac{dy}{dt} = 0$
(d) $t\frac{dz}{dt} = -\cos t - 3z$

(AB 35) For each of the following differential equations, find all the equilibrium solutions or state that no such solutions exist. You do not need to find the nonequilibrium solutions.

(a) $\frac{dy}{dt} = y^3 - yt$
(b) $\frac{dy}{dt} = t^2 e^y$
(c) $\frac{dy}{dt} = ty + t^3$
(d) $\frac{dy}{dt} = (y^2 + 3y + 2)\sin(t)$
(e) $\frac{dy}{dt} = \ln(y^t)$

(AB 36) Suppose that $\frac{dy}{dt} = \cos(t)\sin(y)$, $y(0) = 3\pi$. Find $y(2)$.

(AB 37) For each of the following differential equations, determine whether it is linear or separable. Then solve the given initial-value problem.

(a) $\frac{dy}{dt} = \sin t \cos y$, $y(\pi) = \pi/2$
(b) $1 + y^2 + t\frac{dy}{dt} = 0$, $y(1) = 1$
(c) $3y + e^{-3t}\sin t + \frac{dy}{dt} = 0$, $y(0) = 2$

(AB 38) Solve the initial-value problem $\frac{dy}{dt} = \frac{t-5}{y}$, $y(0) = 3$. Determine the domain of definition of the solution to the initial value problem. If the solution ceases to exist at a finite point, tell me whether the solution ceases to exist by approaching a "bad" point (in the sense of the Picard-Lindelöf theorem) or by becoming unbounded.

(AB 39) Solve the initial-value problem $\frac{dy}{dt} = y^2$, $y(0) = 1/4$. Determine the domain of definition of the solution to the initial value problem. If the solution ceases to exist at a finite point, tell me whether the solution ceases to exist by approaching a "bad" point (in the sense of the Picard-Lindelöf theorem) or by becoming unbounded.

(AB 40) Find all of the "bad" points (in the sense of the Picard-Lindelöf theorem) for the differential equation $\frac{dy}{dt} = \frac{1}{(t-3)\ln y}$.

(AB 41) Find all of the "bad" points (in the sense of the Picard-Lindelöf theorem) for the differential equation $\frac{dy}{dt} = \sqrt[3]{(y-4)(t-2)}$.

(AB 42)

(a) Find all equilibrium solutions to the differential equation $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$.
(b) Use the method of separation of variables to find a solution to the initial value problem $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$, $y(3) = 3$.
(c) Use the method of separation of variables to find a solution to the initial value problem $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$, $y(3) = 5$.
(d) You have found three solutions to the differential equation $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$. Do your solutions cross at any point?
(e) Find two piecewise-defined solutions to the initial value problem $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$, $y(3) = 3$ that are different from each other and also different from the solution you found in Problem (b).

(AB 43) The function $y_1(t) = e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2} - (1+2t)\frac{dy}{dt} + (t+1)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

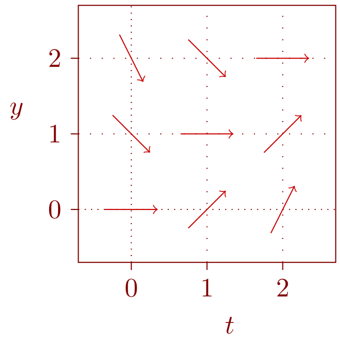
(AB 44) The function $y_1(t) = t$ is a solution to the differential equation $t^2\frac{d^2y}{dt^2} - (t^2+2t)\frac{dy}{dt} + (t+2)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

(AB 45) The function $y_1(x) = x^3$ is a solution to the differential equation $x^2\frac{d^2y}{dx^2} + (x^2\tan x - 6x)\frac{dy}{dx} + (12 - 3\,x\tan x)y = 0$. Find the general solution to this differential equation on the interval $0 < x < \pi/2$.

Answer key

(AB 1) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.

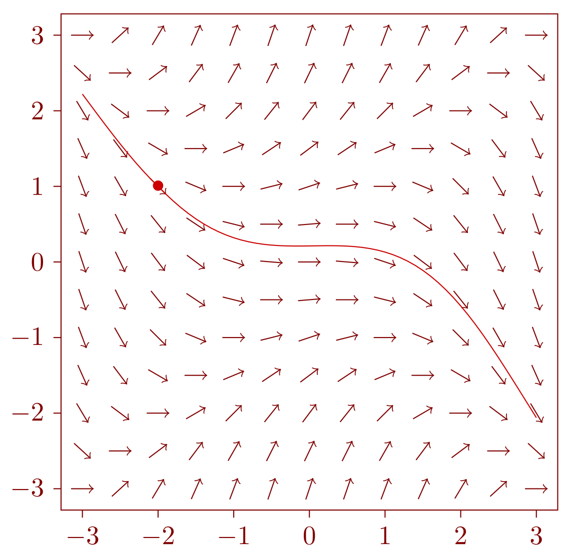
(Answer 1) Here is the direction field for the differential equation $\frac{dy}{dt} = t - y$.



(AB 2) Consider the differential equation $\frac{dy}{dx} = (y^2 - x^2)/3$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-2) = 1.$$

(Answer 2)

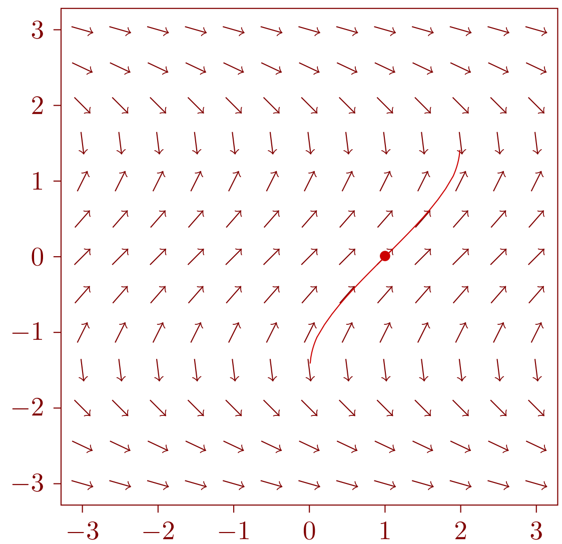


(AB 3) Consider the differential equation $\frac{dy}{dx} = \frac{2}{2-y^2}$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{2}{2 - y^2}, \quad y(1) = 0.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?

(Answer 3)



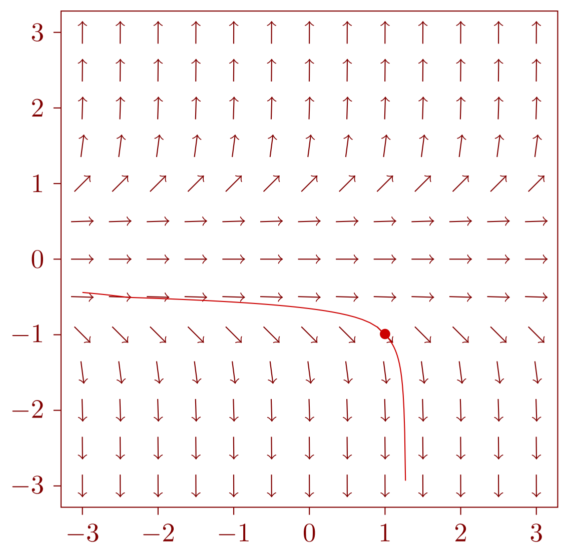
The domain of definition of the solution appears to be $0 < t < 2$.

(AB 4) Consider the differential equation $\frac{dy}{dx} = y^5$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = y^5, \quad y(1) = -1.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?

(Answer 4)



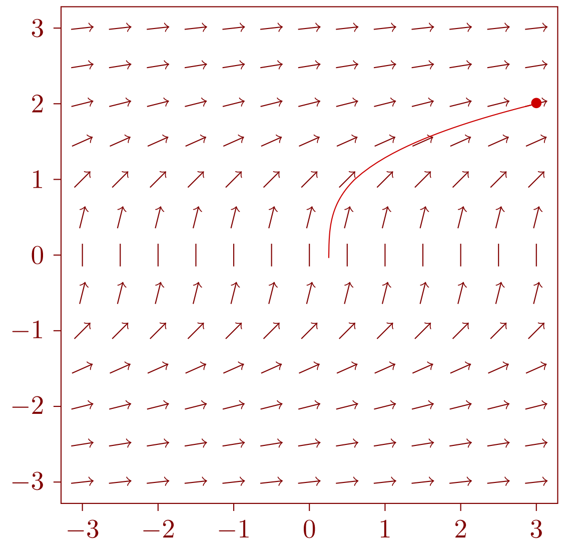
The domain of definition of the solution appears to be approximately $t < 1.3$.

(AB 5) Consider the differential equation $\frac{dy}{dx} = 1/y^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{1}{y^2}, \quad y(3) = 2.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?

(Answer 5)



The domain of definition of the solution appears to be $\frac{1}{4} < t$.

(AB 6) A lake initially has a population of 600 trout. The birth rate of trout is proportional to the number of trout living in the lake. Fishermen are allowed to harvest 30 trout/year from the lake. Write an initial value problem for the fish population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 6) Independent variable: t = time (in years).
Dependent variable: P = Number of trout in the lake
Initial condition: $P(0) = 600$.
Parameters: α = birth rate (in 1/years).
Differential equation: $\frac{dP}{dt} = \alpha P - 30$.

(AB 7) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write an initial value problem for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 7) Independent variable: t = time (in years).
Dependent variable: P = Number of birds on the island.
Parameters: α = birth rate parameter (in 1/years); β = death rate parameter (in 1/(bird·years)).
Initial condition: $P(0) = 5$.
Differential equation: $\frac{dP}{dt} = \alpha P - \beta P^2$.

(AB 8) A tank contains nitrogen dioxide. Initially, there are 200 grams of NO_2. The gas decays (converts to molecular oxygen and nitrogen) at a rate proportional to the square of the amount of NO_2 remaining. Write an initial value problem for the amount of NO_2 in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 8) Independent variable: $t =$ time (in minutes).
Dependent variable: $M =$ Amount of NO_2 in the tank (in grams).
Parameters: $\alpha =$ reaction rate parameter (in 1/second·grams).
Initial condition: $M(0) = 200$.
Differential equation: $\frac{dM}{dt} = -\alpha M^2$.

(AB 9) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen consumed. Write an initial value problem for the amount of hydrogen in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 9) Independent variable: $t =$ time (in minutes).
Dependent variable: $H =$ Amount of hydrogen in the tank (in grams).
Parameters: $\alpha =$ reaction rate parameter (in 1/minute·grams).
Initial condition: $H(0) = 50$.
Differential equation: $\frac{dH}{dt} = -\alpha H(126.9H - 3345)$.

(AB 10) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C. Write an initial value problem for the temperature of the cup of coffee. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 10) Independent variable: $t =$ time (in seconds).
Dependent variable: $T =$ Temperature of the cup (in degrees Celsius)
Initial condition: $T(0) = 95$.
Differential equation: $\frac{dT}{dt} = -\alpha(T - 25)$, where α is a positive parameter (constant of proportionality) with units of 1/seconds.

(AB 11) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Write an initial value problem for the amount of salt in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 11) Independent variable: $t =$ time (in minutes).
Dependent variable: $Q =$ amount of dissolved salt (in kilograms).
Initial condition: $Q(0) = 3$.
Differential equation: $\frac{dQ}{dt} = 0.01 - \frac{3Q}{600-t}$.

(AB 12) I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write an initial value problem for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 12) Independent variable: t = time (in years).

Dependent variable: B = balance of my loan (in dollars).

Initial condition: $B(0) = 300{,}000$.

Differential equation: $\frac{dB}{dt} = 0.04B - 12 \cdot 1600$

(AB 13) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write an initial value problem for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 13) Independent variable: t = time (in minutes).

Dependent variables:

h = depth of water in the hole (in centimeters)

V = volume of water in the hole (in cubic centimeters); notice that $V = \frac{1}{3}\pi(5h)^2 h$

Initial condition: $V(0) = 0$.

Differential equation: $\frac{dV}{dt} = 1 - \alpha\pi(5h)^2 = 1 - 25\alpha\pi(3V/25\pi)^{2/3}$, where α is a proportionality constant with units of cm/s.

(AB 14) According to the Stefan-Boltzmann law, a black body in a dark vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 14) Independent variable: t = time (in seconds).

Dependent variable: T = object's temperature (in kelvins)

Parameter: σ = proportionality constant (in 1/(seconds·kelvin^3))

Initial condition: $T(0) = 400$.

Differential equation: $\frac{dT}{dt} = -\sigma T^4$.

(AB 15) According to the Stefan-Boltzmann law, a black body in a vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). If there is some ambient light in the vacuum, then the black body absorbs energy at a rate proportional to the fourth power of the effective temperature of the light. The proportionality constant is the same in both cases.

Suppose that a planet in space is currently at a temperature of 400 K. The effective temperature of its surroundings is 3 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 15) Independent variable: t = time (in seconds).

Dependent variable: T = object's temperature (in kelvins)

Parameter: σ = proportionality constant (in 1/(seconds·kelvin^3))

Initial condition: $T(0) = 400$.

Differential equation: $\frac{dT}{dt} = -\sigma T^4 + 81\sigma$.

(AB 16) A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 16) Independent variable: $t =$ time (in seconds).

Dependent variable: $v =$ velocity of the ball (in meters/second, where a positive velocity denotes upward motion).

Parameters: $\alpha =$ proportionality constant of the drag force (in newton·seconds/meter)
$m =$ mass of the ball (in kilograms)

Initial condition: $v(0) = 10$.

Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/m)v$.

(AB 17) A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 17) Independent variable: $t =$ time (in seconds).

Dependent variable: $v =$ velocity of the ball (in meters/second, where a positive velocity denotes upward motion).

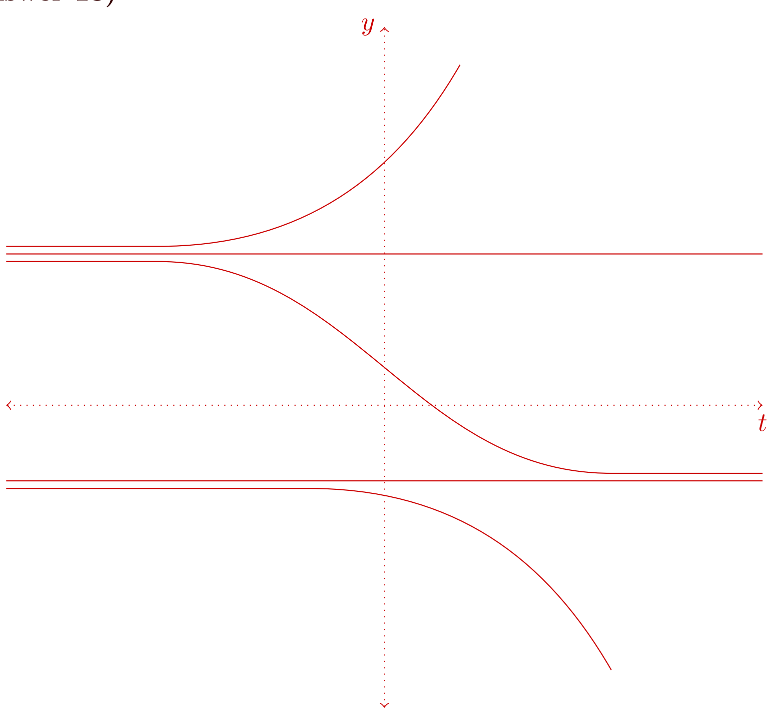
Parameter: $\alpha =$ proportionality constant of the drag force (in newton·seconds/meter).

Initial condition: $v(0) = 20$.

Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/0.3)v|v|$, or $\frac{dv}{dt} = \begin{cases} -9.8 - (\alpha/0.3)v^2, & v \geq 0, \\ -9.8 + (\alpha/0.3)v^2, & v < 0. \end{cases}$

(AB 18) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y-2)(y+1)^2$. By hand, sketch some typical solutions.

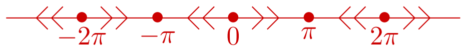
(Answer 18)



(AB 19) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(Answer 19)

Critical points: $y = k\pi$ for any integer k.

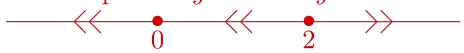


If k is even then $y = k\pi$ is unstable.
If k is odd then $y = k\pi$ is stable.

(AB 20) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = y^2(y-2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(Answer 20)

Critical points: $y = 0$ and $y = 2$.



$y = 0$ is semistable. $y = 2$ is unstable.

(AB 21) Trout live in a lake. Each trout produces, on average, one baby trout every two years. Fishermen are allowed to harvest 100 trout/year from the lake. You may neglect other reasons why trout die.

(a) Formulate a differential equation for the number of trout in the lake.
$\frac{dP}{dt} = \frac{1}{2}P - 100$, where t denotes time in years and P denotes the number of trout in the lake.

(b) Find the critical points of this differential equation and classify them as to stability.
The critical point is $P = 200$. It is unstable.

(c) What is the real-world meaning of the critical points?
If there are initially fewer than 200 trout in the lake, then eventually the trout will go extinct. If there are initially more than 200 trout in the lake, the trout population will grow without limit (or at least, until the present model stops being applicable).

(AB 22) I owe my bank a debt of B dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

(a) Formulate a differential equation for the amount of money I owe.
$\frac{dB}{dt} = 0.05B - 19200$, where t denotes time in years.

(b) Find the critical points of this differential equation and classify them as to stability.
The critical point is $B = \$384{,}000$. It is unstable.

(c) What is the real-world meaning of the critical points?
If my initial debt is less than \$384,000, then I will eventually pay it off (have a debt of zero dollars), but if my initial debt is greater than \$384,000, my debt will grow exponentially. The critical point $B = 384{,}000$ corresponds to the balance that will allow me to make interest-only payments on my debt.

(AB 23) A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

(a) Write the differential equation for the amount of salt in the tank.
$\frac{dQ}{dt} = 10 - Q/300$, where Q denotes the amount of salt in grams and t denotes time in minutes.

(b) Find the critical points of this differential equation and classify them as to stability.
The critical point is $Q = 3000$. It is stable.

(c) What is the real-world meaning of the critical points?
No matter the initial amount of salt in the tank, as time passes the amount of salt will approach 3000 g or 3 kg.

(AB 24) A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where v is her velocity in meters per second.

(a) Write a differential equation for her velocity. Assume her velocity is always downwards.
If $v \leq 0$ then $70\frac{dv}{dt} = -70 * 9.8 + 2v^2$. (If $v > 0$ then $70\frac{dv}{dt} = -70 * 9.8 - 2v^2$.)

(b) Find the critical points of this differential equation and classify them as to stability. Be sure to include units.
$v = -\sqrt{343}$ meters/second is a stable critical point.

(c) What is the real-world meaning of these critical points?
As $t \to \infty$, her velocity will approach the stable critical point of $v = -\sqrt{343}$ meters/second.

(AB 25) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen.

(a) Write a differential equation for the amount of hydrogen left in the tank.
$\frac{dH}{dt} = -\alpha H(126.9H - 3345)$, where H is the amount of hydrogen remaining (in grams), t denotes time in minutes, and α is a positive parameter.

(b) Find the critical points of this differential equation and classify them as to stability. Be sure to include units.
$H = 0$ grams (unstable) and $H = 26.36$ grams (stable).

(c) What is the real-world meaning of these critical points?
As $t \to \infty$, the amount of hydrogen in the tank will tend towards 26.36 grams. The $H = 0$ critical point is not physically meaningful, as if $H = 0$ grams then the model predicts -3345 grams of iodine in the tank.

(AB 26) Young oak trees grow in a large field with area 1 square kilometer. Initially, there are 30 trees in the field. Each tree shades 20 square meters. The trees produce acorns. Squirrels scatter the acorns across the field at random. Squirrels eat most of the acorns; on average, each tree produces two acorns per year that are not eaten by squirrels. Every acorn not eaten by squirrels and planted in sunlight sprouts.

(a) Write a differential equation for the number of trees in the field.
$\frac{dP}{dt} = 2P(1 - 20P/1{,}000{,}000)$, where P is the number of trees in the field and t denotes time in years.

(b) Find the critical points of this differential equation and classify them as to stability.
$P = 0$ (unstable) and $P = 50{,}000$ trees (stable).

(c) What is the real-world meaning of these critical points?
If there are initially no trees in the field, then the number of trees will remain at the $P = 0$ equilibrium. However, if there are initially any trees in the field, then as $t \to \infty$, the number of trees will approach 50,000.

(AB 27) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate of 3 cm^3 per exposed cm2 per hour.

(a) Write a differential equation for the amount of water in the tank.
$\frac{dV}{dt} = 1 - 75\pi(3V/25\pi)^{2/3}$, where t denotes time in hours and V is the volume of water in the hole in cm^3.

(b) Find the critical points of this differential equation and classify them as to stability.
$V = (25/3)(1/75\pi)^{3/2}$. The critical point is stable.

(c) What is the real-world meaning of these critical points?
As $t \to \infty$, the amount of water in the hole will approach $(25/3)(1/75\pi)^{3/2}$ cubic centimeters.

(AB 28) Is $y = e^t$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(Answer 28) No, $y = e^t$ is not a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(AB 29) Is $y = e^{2t}$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(Answer 29) Yes, $y = e^{2t}$ is a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(AB 30) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 1$?

(Answer 30) No.

(AB 31) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 2$?

(Answer 31) No.

(AB 32) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 1$?

(Answer 32) Yes.

(AB 33) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 2$?

(Answer 33) No.

(AB 34) For each of the following differential equations, determine whether it is linear or separable. Then solve the differential equation.

(a) $t + \cos t + (y - \sin y)\frac{dy}{dt} = 0$
$t+\cos t+(y-\sin y)\frac{dy}{dt} = 0$ is separable (and also exact) and has solution $\frac{1}{2}y^2+\cos y = -\frac{1}{2}t^2-\sin t+C$.

(b) $1 + t^2 - ty + (t^2+1)\frac{dy}{dt} = 0$
$1 + t^2 - ty + (t^2+1)\frac{dy}{dt} = 0$ is linear and has solution $y = -\sqrt{t^2+1}\ln(t+\sqrt{t^2+1}) + C\sqrt{t^2+1}$.

(c) $y^3\cos(2t) + \frac{dy}{dt} = 0$
$y^3\cos(2t) + \frac{dy}{dt} = 0$ is separable and has solution $y = \pm\frac{1}{\sqrt{\sin(2t)+C}}$ or $y = 0$.

(d) $t\frac{dz}{dt} = -\cos t - 3z$
$t\frac{dz}{dt} = -\cos t - 3z$ is linear and has solution $z = \frac{1}{t}\sin t + \frac{2}{t^2}\cos t - \frac{2}{t^3}\sin t + \frac{C}{t^3}$.

(AB 35) For each of the following differential equations, find all the equilibrium solutions or state that no such solutions exist. You do not need to find the nonequilibrium solutions.

(a) $\frac{dy}{dt} = y^3 - yt$
$y = 0$.

(b) $\frac{dy}{dt} = t^2e^y$
There are no equilibrium solutions.

(c) $\frac{dy}{dt} = ty + t^3$
There are no equilibrium solutions.

(d) $\frac{dy}{dt} = (y^2+3y+2)\sin(t)$
$y = -1$ and $y = -2$.

(e) $\frac{dy}{dt} = \ln(y^t)$
$y = 1$.

(AB 36) Suppose that $\frac{dy}{dt} = \cos(t)\sin(y)$, $y(0) = 3\pi$. Find $y(2)$.

(Answer 36) $y(t) = 3\pi$ is an equilibrium solution to the differential equation and satisfies $y(0) = 3\pi$, so we must have that $y(t) = 3\pi$ for all t. In particular, $y(2) = 3\pi$.

(AB 37) For each of the following differential equations, determine whether it is linear or separable. Then solve the given initial-value problem.

(a) $\frac{dy}{dt} = \sin t \cos y$, $y(\pi) = \pi/2$

If $\frac{dy}{dt} = \sin t \cos y$, $y(\pi) = \pi/2$, then $y = \pi/2$ for all t.

(b) $1 + y^2 + t\frac{dy}{dt} = 0$, $y(1) = 1$

If $1 + y^2 + t\frac{dy}{dt} = 0$, $y(1) = 1$, then $y = \tan(\pi/4 - \ln t)$.

(c) $3y + e^{-3t} \sin t + \frac{dy}{dt} = 0$, $y(0) = 2$

If $3y + e^{-3t} \sin t + \frac{dy}{dt} = 0$, $y(0) = 2$, then $y = e^{-3t} \cos t + e^{-3t}$.

(AB 38) Solve the initial-value problem $\frac{dy}{dt} = \frac{t-5}{y}$, $y(0) = 3$. Determine the domain of definition of the solution to the initial value problem. If the solution ceases to exist at a finite point, tell me whether the solution ceases to exist by approaching a "bad" point (in the sense of the Picard-Lindelöf theorem) or by becoming unbounded.

(Answer 38) $y = \sqrt{(t-5)^2 - 16} = \sqrt{(t-1)(t-9)} = \sqrt{t^2 - 10t + 9}$. The solution is valid for all $t < 1$. The solution ceases to exist by approaching a "bad" point.

(AB 39) Solve the initial-value problem $\frac{dy}{dt} = y^2$, $y(0) = 1/4$. Determine the domain of definition of the solution to the initial value problem. If the solution ceases to exist at a finite point, tell me whether the solution ceases to exist by approaching a "bad" point (in the sense of the Picard-Lindelöf theorem) or by becoming unbounded.

(Answer 39) $y = \frac{1}{4-t}$. The solution is valid for all $t < 4$. The solution ceases to exist by becoming unbounded.

(AB 40) Find all of the "bad" points (in the sense of the Picard-Lindelöf theorem) for the differential equation $\frac{dy}{dt} = \frac{1}{(t-3)\ln y}$.

(Answer 40) The "bad" points occur when either $t = 3$ or $y = 1$.

(AB 41) Find all of the "bad" points (in the sense of the Picard-Lindelöf theorem) for the differential equation $\frac{dy}{dt} = \sqrt[3]{(y-4)(t-2)}$.

(Answer 41) The "bad" points occur when $y = 4$.

(AB 42)

(a) Find all equilibrium solutions to the differential equation $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$.
The equilibrium solution is $y = 4$.

(b) Use the method of separation of variables to find a solution to the initial value problem $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$, $y(3) = 3$.
One such solution is $y = 4 - (t-2)^2 = 4t - t^2$.

(c) Use the method of separation of variables to find a solution to the initial value problem $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$, $y(3) = 5$.
One such solution is $y = 4 + (t-2)^2$.

(d) You have found three solutions to the differential equation $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$. Do your solutions cross at any point?
Yes, all three solutions cross at the point $(2, 4)$.

(e) Find two piecewise-defined solutions to the initial value problem $\frac{dy}{dt} = 2\sqrt[3]{(y-4)(t-2)}$, $y(3) = 3$ that are different from each other and also different from the solution you found in Problem (b).
One such solution is

$$y(t) = \begin{cases} 4, & t < 2, \\ 4 - (t-2)^2, & 2 \leq t. \end{cases}$$

Another such solution is

$$y(t) = \begin{cases} 4 + (t-2)^2, & t < 2, \\ 4 - (t-2)^2, & 2 \leq t. \end{cases}$$

(AB 43) The function $y_1(t) = e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2} - (1+2t)\frac{dy}{dt} + (t+1)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

(Answer 43) $y(t) = C_1\, e^t + C_2\, t^2 e^t$.

(AB 44) The function $y_1(t) = t$ is a solution to the differential equation $t^2\frac{d^2y}{dt^2} - (t^2+2t)\frac{dy}{dt} + (t+2)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

(Answer 44) $y(t) = C_1\, t + C_2\, te^t$.

(AB 45) The function $y_1(x) = x^3$ is a solution to the differential equation $x^2\frac{d^2y}{dx^2} + (x^2\tan x - 6x)\frac{dy}{dx} + (12 - 3\,x\tan x)y = 0$. Find the general solution to this differential equation on the interval $0 < x < \pi/2$.

(Answer 45) $y(x) = C_1\, x^3 + C_2\, x^3 \sin x$.