Math 2584, Spring 2018

The final will occur on Monday, May 4. Final exam questions will be available on WebAssign at 12:45 p.m. on Monday, May 4. Please write your answers (by hand) on paper, a whiteboard, or a tablet, then scan, photograph or export your answers and upload them to Blackboard by 3:00 p.m. on Monday, May 4.

The exam is open book, open note, and open calculator; however, you are not allowed to discuss the exam in any way with any person other than Professor Barton while taking it.

You will need a table of Laplace transforms. You can write one by hand, use the one in your book, or print out the one provided on Blackboard.

Please complete the online course evaluation at
`https://courseval.uark.edu/etw/ets/et.asp?nxappid=WCQ&nxmid=start`
on or before Friday, May 1. If at least 80% of the class completes the course evaluation before the deadline, I will drop your 2 lowest lab scores; otherwise, I will drop your 1 lowest lab score.

(AB 1) Is $y = e^t$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 2) Is $y = e^{2t}$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 3) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 1$?

(AB 4) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 2$?

(AB 5) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 1$?

(AB 6) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 2$?

(AB 7) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + \arctan(t)\, y = e^t$, $y(3) = 7$.

(b) $\frac{1}{1+t^2}\frac{dy}{dt} - t^5 y = \cos(6t)$, $y(2) = -1$, $y'(2) = 3$.

(c) $\frac{d^2y}{dt^2} - 5\sin(t)\, y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.

(d) $e^t \frac{d^2y}{dt^2} + 3(t-4)\frac{dy}{dt} + 4t^6 y = 2$, $y(3) = 1$, $y'(3) = -1$.

(e) $\frac{d^2y}{dt^2} + \cos(t)\frac{dy}{dt} + 3\ln(1+t^2)\, y = 0$, $y(2) = 3$.

(f) $\frac{d^3y}{dt^3} - t^7\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\, y = t^3$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$, $y'''(1) = 3$.

(g) $(1+t^2)\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = t^3$, $y(3) = 9$, $y'(3) = 7$, $y''(3) = 5$.

(h) $\frac{d^3y}{dt^3} - e^t\frac{d^2y}{dt^2} + e^{2t}\frac{dy}{dt} + e^{3t}y = e^{4t}$, $y(-1) = 1$, $y'(-1) = 3$.

(i) $(2+\sin t)\frac{d^3y}{dt^3} + \cos t\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\, y = t^3$, $y(7) = 2$.

(AB 8) A lake initially has a population of 600 trout. The birth rate of trout is proportional to the number of trout living in the lake. Fishermen are allowed to harvest 30 trout/year from the lake. Write an initial value problem for the fish population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 9) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write an initial value problem for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 10) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen. Write an initial value problem for the amount of hydrogen in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 11) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C. Write an initial value problem for the temperature of the cup of coffee. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 12) Suppose that an isolated town has 300 households. In 1920, two families install telephones in their homes. Write an initial value problem for the number of telephones in the town if the rate at which families buy telephones is jointly proportional to the number of households with telephones and the number of households without telephones. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 13) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Write an initial value problem for the amount of salt in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 14) I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write an initial value problem for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 15) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write an initial value problem for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 16) According to the Stefan-Boltzmann law, a black body in a dark vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

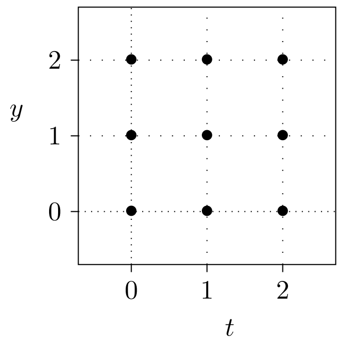
(AB 17) According to the Stefan-Boltzmann law, a black body in a vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). If there is some ambient light in the vacuum, then the black body absorbs energy at a rate proportional to the effective temperature of the light. The proportionality constant is the same in both cases; that is, if the object's temperature is equal to the effective temperature of the light, then its temperature will not change.

Suppose that a planet in space is currently at a temperature of 400 K. The effective temperature of its surroundings is 3 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 18) A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

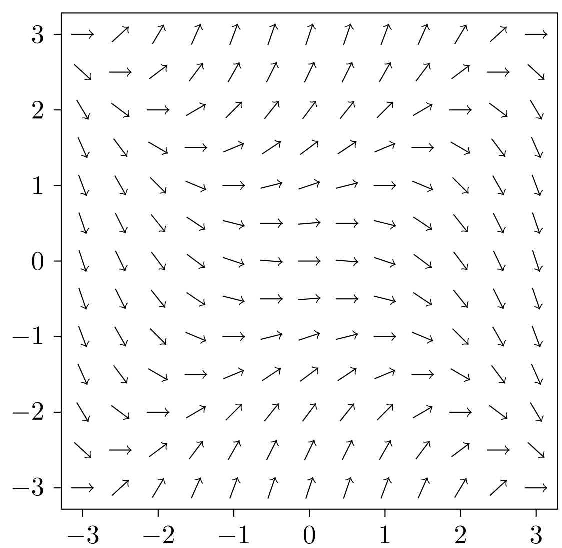
(AB 19) A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 20) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.



(AB 21) Consider the differential equation $\frac{dy}{dx} = (y^2 - x^2)/3$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

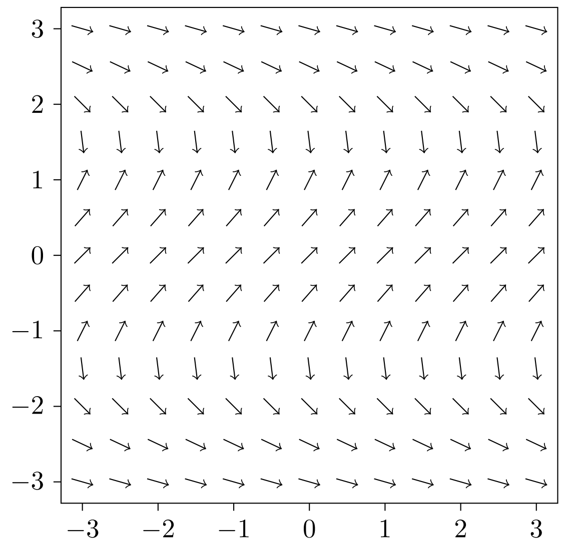
$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-2) = 1.$$



(AB 22) Consider the differential equation $\frac{dy}{dx} = \frac{2}{2-y^2}$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{2}{2-y^2}, \quad y(1) = 0.$$

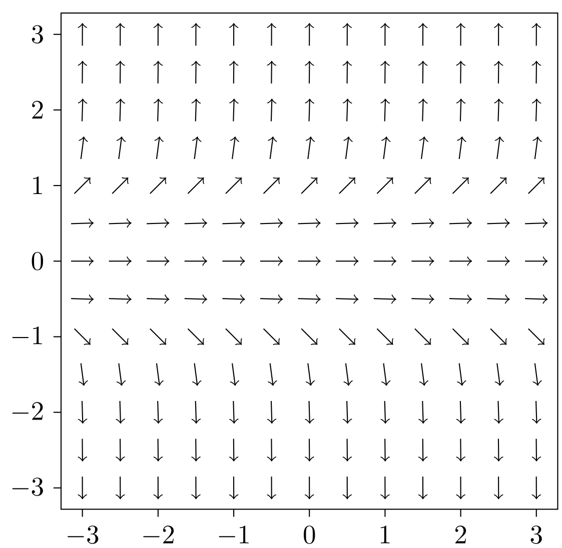
Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 23) Consider the differential equation $\frac{dy}{dx} = y^5$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = y^5, \quad y(1) = -1.$$

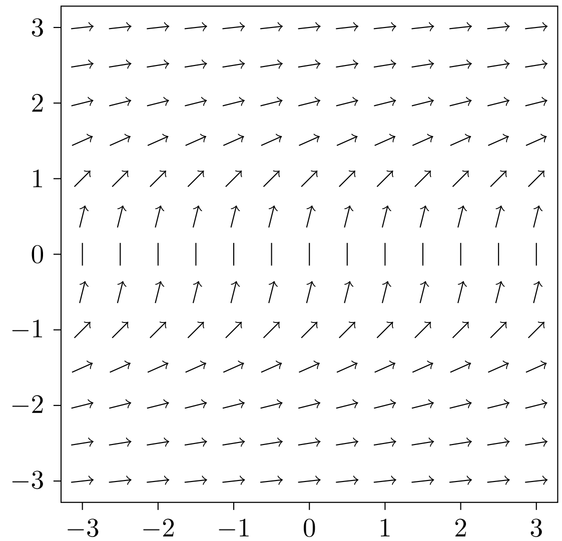
Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 24) Consider the differential equation $\frac{dy}{dx} = 1/y^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{1}{y^2}, \quad y(3) = 2.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 25) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y-2)(y+1)^2$. By hand, sketch some typical solutions.

(AB 26) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 27) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = y^2(y-2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 28) I owe my bank a debt of B dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

(a) Formulate a differential equation for the amount of money I owe.

(b) Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

(AB 29) A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

(a) Write the differential equation for the amount of salt in the tank.

(b) Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

(AB 30) A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where v is her velocity in meters per second.

(a) Write a differential equation for her velocity. Assume her velocity is always downwards.

(b) Find the (negative) critical points of this differential equation. Be sure to include units.

(c) What is the real-world meaning of these critical points?

(AB 31) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen.

(a) Write a differential equation for the amount of hydrogen left in the tank.
(b) Find the critical points of this differential equation. Be sure to include units.
(c) What is the real-world meaning of these critical points?

(AB 32) Suppose that the population y of catfish in a certain lake, absent human intervention, satisfies the logistic equation $\frac{dy}{dt} = ry(1 - y/K)$, where r and K are constants and t denotes time.

(a) Assuming that $r > 0$ and $K > 0$, find the critical points of this equation and classify them as to stability. What is the long term behavior of the population?
(b) Suppose that humans intervene by harvesting fish at a rate proportional to the fish population. Write a new differential equation for the fish population. Find the critical points of your new equation and classify them as to stability. What is the long term behavior of the population?
(c) Suppose that humans intervene by harvesting fish at a constant rate. Write a new differential equation for the fish population. Find the critical points of your new equation and classify them as to stability. What is the long term behavior of the population?

(AB 33) For each of the following differential equations, determine whether it is linear, separable, exact, homogeneous, a Bernoulli equation, or of the form $\frac{dy}{dt} = f(At+By+C)$. Then solve the differential equation.

(a) $\frac{dy}{dt} = \frac{t+\cos t}{\sin y - y}$
(b) $\ln y + y + t + \left(\frac{t}{y} + t\right)\frac{dy}{dt} = 0$
(c) $(t^2+1)\frac{dy}{dt} = ty - t^2 - 1$
(d) $t^2\frac{dy}{dt} = y^2 + t^2 - ty$
(e) $4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5)\frac{dy}{dt} = 0$
(f) $\frac{dy}{dt} = -y\tan 2t - y^3\cos 2t$
(g) $(t+y)\frac{dy}{dt} = 5y - 3t$
(h) $\frac{dy}{dt} = \frac{1}{3t+2y+7}$.
(i) $\frac{dy}{dt} = -y^3\cos(2t)$
(j) $4ty\frac{dy}{dt} = 3y^2 - 2t^2$
(k) $t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$
(l) $\frac{dy}{dt} = \csc^2(y-t)$.
(m) $\frac{dy}{dt} = 8y - y^8$
(n) $t\frac{dy}{dt} = -\cos t - 3y$
(o) $\frac{dy}{dt} = \cot(y/t) + y/t$
(p) $\frac{dy}{dt} = ty + t^2\sqrt{[3]y}$

(AB 34) The differential equation $e^x\cos y + (e^x\sin y + 1)\frac{dy}{dx} = 0$ is not exact.

(a) Find an integrating factor $\mu(y)$ depending only on y such that, upon multiplying by $\mu(y)$, the equation becomes exact.
(b) Solve the differential equation.

(AB 35) The differential equation $3x^2 + 2\sin 2y + x\cos 2y\frac{dy}{dx} = 0$ is not exact.

(a) Find an integrating factor $\mu(x)$ depending only on x such that, upon multiplying by $\mu(x)$, the equation becomes exact.
(b) Solve the differential equation.

(AB 36) For each of the following differential equations, determine whether it is linear, separable, exact, homogeneous, Bernoulli, or a function of a linear term. Then solve the given initial-value problem.

(a) $\cos(t+y^3)+2t+3y^2\cos(t+y^3)\frac{dy}{dt}=0$, $y(\pi/2)=0$

(b) $2ty\frac{dy}{dt}=4t^2-y^2$, $y(1)=3$.

(c) $t\frac{dy}{dt}=-1-y^2$, $y(1)=1$

(d) $\frac{dy}{dt}=(2y+2t-5)^2$, $y(0)=3$.

(e) $\frac{dy}{dt}=2y-\frac{6}{y^2}$, $y(0)=7$.

(f) $\frac{dy}{dt}=-3y-\sin t\, e^{-3t}$, $y(0)=2$

(AB 37) Solve the initial-value problem $\frac{dy}{dt}=\frac{t-5}{y}$, $y(0)=3$ and determine the range of t-values in which the solution is valid.

(AB 38) Solve the initial-value problem $\frac{dy}{dt}=y^2$, $y(0)=1/4$ and determine the range of t-values in which the solution is valid.

(AB 39) The function $y_1(t)=e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2}-(1+2t)\frac{dy}{dt}+(t+1)y=0$. Find the general solution to this differential equation on the interval $t>0$.

(AB 40) The function $y_1(t)=t$ is a solution to the differential equation $t^2\frac{d^2y}{dt^2}-(t^2+2t)\frac{dy}{dt}+(t+2)y=0$. Find the general solution to this differential equation on the interval $t>0$.

(AB 41) The function $y_1(x)=x^3$ is a solution to the differential equation $x^2\frac{d^2y}{dx^2}+(x^2\tan x-6x)\frac{dy}{dx}+(12-3\,x\tan x)y=0$. Find the general solution to this differential equation on the interval $0<x<\pi/2$.

(AB 42)

(a) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0,\quad y(0)=2,\quad y'(0)=7$$

is guaranteed a unique solution?

(b) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0,\quad y(2)=2,\quad y'(2)=7$$

is guaranteed a unique solution?

(c) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0,\quad y(5)=2,\quad y'(5)=7$$

is guaranteed a unique solution?

(d) You are given that $y_1=t$ is a solution to the differential equation $(t-3)(2t-3)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0$. Use the method of reduction of order to find the general solution.

(e) On what intervals are all of the the solutions you found in part (d) continuous?

(f) How many solutions are there to the initial value problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0,\quad y(3/2)=2,\quad y'(3/2)=7?$$

(AB 43) You are given that $y_1 = t - 3$ and $y_2 = \frac{1}{t-3}$ are both solutions to the differential equation $(t-3)^2 \frac{d^2y}{dt^2} + (t-3)\frac{dy}{dt} + y = 0$ for all t.

(a) How many solutions are there to the initial value problem

$$(t-3)^2 \frac{d^2y}{dt^2} + (t-3)\frac{dy}{dt} + y = 0, \quad y(3) = 1, \quad y'(3) = 0?$$

(b) How many solutions are there to the initial value problem

$$(t-3)^2 \frac{d^2y}{dt^2} + (t-3)\frac{dy}{dt} + y = 0, \quad y(3) = 0, \quad y'(3) = 1?$$

(AB 44) You are given that $y_1 = t^3 + 4t^2 + 4t$ and $y_2 = t^2 + 4t + 4$ are both solutions to the differential equation $(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$ for all t.

(a) What are the maximal intervals on which the general solution to $(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$ may be written $y = C_1(t^3 + 4t^2 + 4t) + C_2(t^2 + 4t + 4)$?

(b) Find a function $y_3(t)$ that is continuous and has continuous first and second derivatives for all t, that satisfies $(t+2)^2 \frac{d^2y_3}{dt^2} - (4t+8)\frac{dy_3}{dt} + 6y_3 = 0$ for all t, but such that there do not exist constants C_1 and C_2 such that $y_3(t) = C_1 y_1(t) + C_2 y_2(t) = C_1(t^3 + 4t^2 + 4t) + C_2(t^2 + 4t + 4)$ for all t.

(c) Find numbers a, b, and c such that the initial value problem

$$(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0, \quad y(a) = b, \quad y'(a) = c$$

has no solutions.

(d) Find numbers a, b, and c such that the initial value problem

$$(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0, \quad y(a) = b, \quad y'(a) = c$$

has at least two solutions.

(AB 45) Find the general solution to the following differential equations.

(a) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$.

(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.

(c) $\frac{d^4y}{dt^4} + 7\frac{d^2y}{dt^2} - 144y = 0$.

(d) $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0$.

(AB 46) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 2y = 0$, $y(0) = 3$, $y'(0) = 2$.

(b) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$, $y(0) = 1$, $y'(0) = 4$.

(AB 47) An object weighing 5 lbs stretches a spring 4 inches. It is attached to a viscous damper with damping constant 16 lb $\cdot$ sec/ft. The object is pulled down an additional 2 inches and is released with initial velocity 3 ft/sec upwards.

Write the differential equation and initial conditions that describe the position of the object.

(AB 48) A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant β. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20\ \mathrm{N \cdot s/m}$, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30\ \mathrm{N \cdot s/m}$, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(AB 49) A 3-kg object is attached to a spring with constant k and to a viscous damper with damping constant 42 N·sec/m. The object is set in motion from its equilibrium position with initial velocity 5 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object.

If $k = 100$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $k = 200$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(AB 50) A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant β. The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

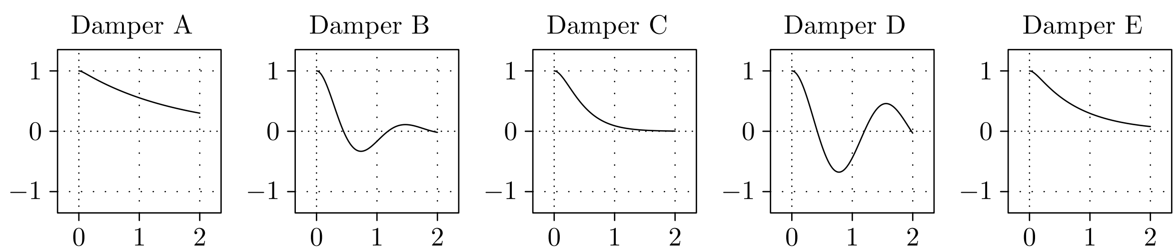
Write the differential equation and initial conditions that describe the position of the object. Then find the value of β for which the system is critically damped. Be sure to include units for β.

(AB 51) An object of mass m is attached to a spring with constant 80 N/m and to a viscous damper with damping constant 20 N·s/m. The object is pulled down to 5cm below its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of m for which the system is critically damped. Be sure to include units for m.

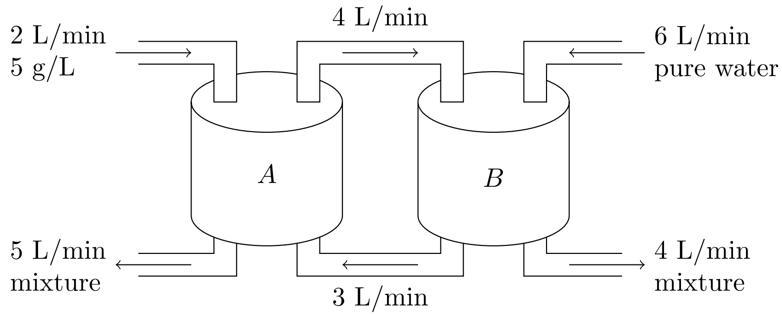
(AB 52) Five objects, each with mass 3 kg, are attached to five springs, each with constant 48 N/m. Five dampers with unknown constants are attached to the objects. In each case, the object is pulled down to a distance 1 cm below the equilibrium position and released from rest. You are given that the system is critically damped in exactly one of the five cases.

Here are the graphs of the objects' positions with respect to time:



(*a*) For which damper is the system critically damped?
(*b*) For which dampers is the system overdamped?
(*c*) For which dampers is the system underdamped?
(*d*) Which damper has the highest damping constant? Which damper has the lowest damping constant?

(AB 53) Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



Write the differential equations and initial conditions that describe the amount of salt in each tank.

(AB 54) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 55) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 56) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 57) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 58) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 59) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 60) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 61) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).
Hint: $\det \begin{pmatrix} 4-m & 1 & -1 \\ 1 & 4-m & -1 \\ -1 & -1 & 4-m \end{pmatrix} = -(m-3)^2(m-6).$

(AB 62) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, \qquad x(0) = 3, \quad y(0) = 2, \quad z(0) = 1.$$

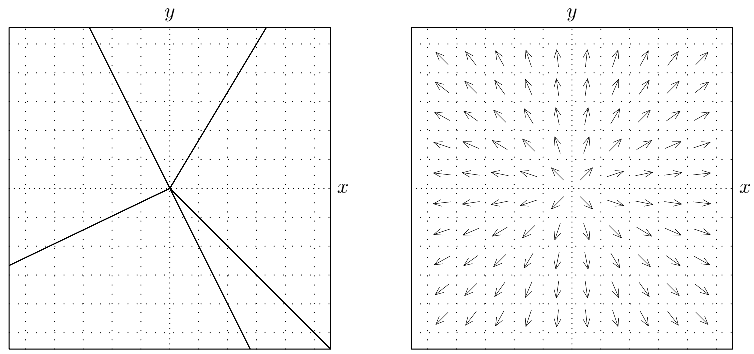
Express your final answer in terms of real functions (no complex numbers or complex exponentials).
Hint: $\det \begin{pmatrix} 3-m & 1 & 0 \\ 2 & 3-m & -2 \\ 0 & 1 & 3-m \end{pmatrix} = -(m-3)^3.$

(AB 63) Find the solution to the initial-value problem

$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, x(0) = 4, \quad y(0) = 3, \quad z(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).
Hint: $\det \begin{pmatrix} 1-m & 1 & 1 \\ 1 & 3-m & -1 \\ 0 & 2 & 2-m \end{pmatrix} = -(m-2)^3.$

(AB 64) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



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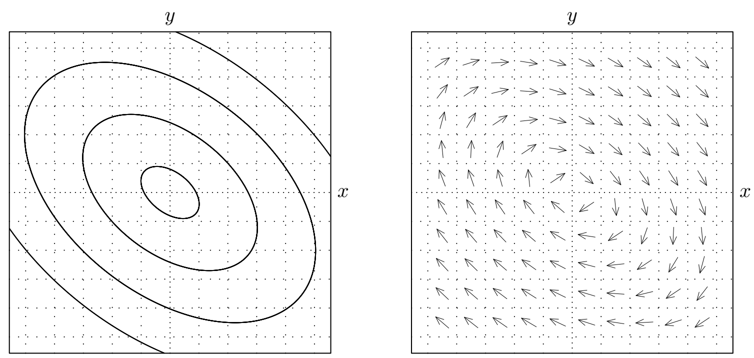

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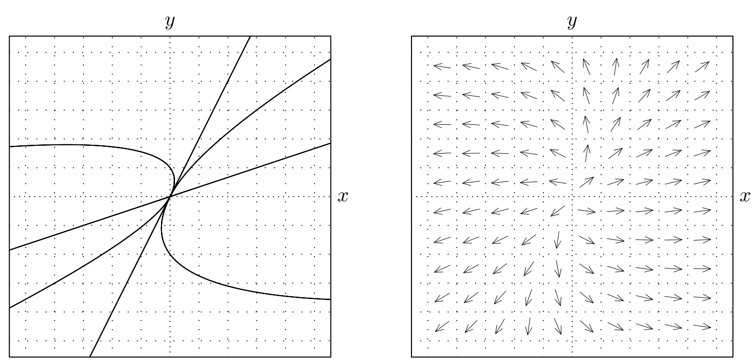
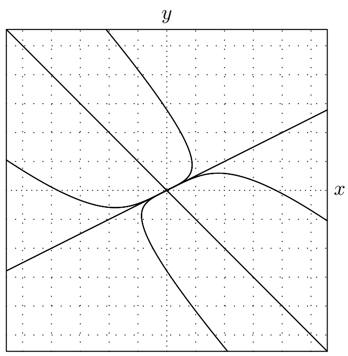
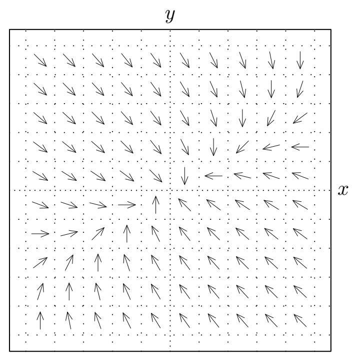
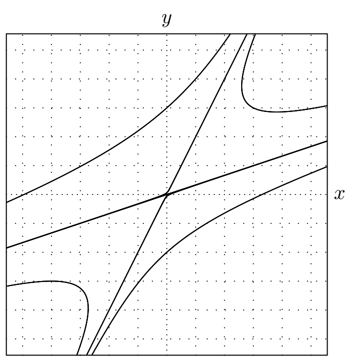


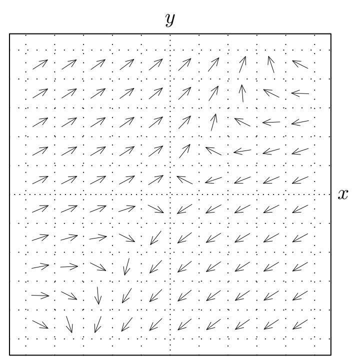
(AB 67) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



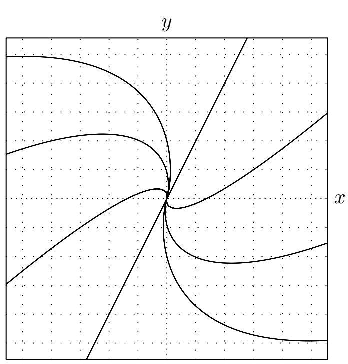


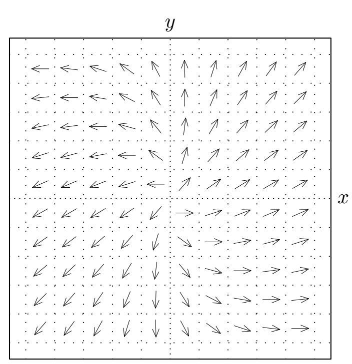
(AB 68) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



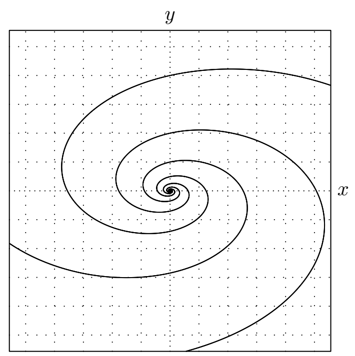


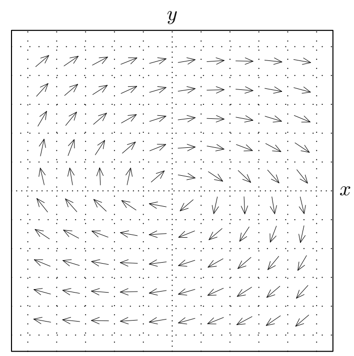
(AB 69) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?





(AB 70) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?





(AB 71) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.
(b) Use the phase plane method to find an equation relating the number of susceptible and infected people.
(c) Use the phase plane method to find an equation relating the number of resistant and susceptible people.
(d) As $t \to \infty$, $I \to 0$ and the number of people who never contract the disease approaches a limit. Find an equation for S_∞, where S_∞ is the limiting number of people who never contract the disease.
(e) What is the maximum number of people that are infected at any one time?
(f) When is the peak infection time, that is, the time at which the number of infected people is at its maximum? You need not simplify your answer.

(AB 72) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

(a) Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(AB 73) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

(a) Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(b) Assume that there are initially no recovered or vaccinated people and 7 infected people. Use the phase plane method to find an equation relating S and I.

(c) What is the maximum number of people that are infected with the virus at any one time? (Round to the nearest whole number.)

(AB 74) Here is another way to model vaccination. Suppose that a disease is spreading through a town of 8000 people. Initially there are 10 infected people, 3000 vaccinated people, and no recovered people.

No other persons are vaccinated during the disease's spread. Thus, the population should be divided into *four* groups: people who are vaccinated, susceptible, infected, and recovered.

Each infected person encounters 10 people per day, who are distributed among susceptible, vaccinated, recovered and infected people according to their proportion of the total population. Each time an infected person encounters a susceptible person, there is a 4% chance that the susceptible person becomes infected. The vaccine is not perfect; each time an infected person encounters a vaccinated person, there is a 1% chance they become infected. (Once a vaccinated person becomes infected, they are identical to an infected person who was never vaccinated.) Each infected person has a 20% chance per day of recovering. A recovered person can never contract the disease again.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(b) Use the phase plane method to find an equation relating S and V.

(c) Use the phase plane method to find an equation relating I and V.

(d) What is the maximum number of people that are infected with the virus at any one time? (Round to the nearest whole number.)

(e) At what time does the peak number of infected people occur?

(AB 75) Consider the system of differential equations $\frac{dx}{dt} = 3x - 4y$, $\frac{dy}{dt} = 4x - 3y$. Use the phase plane method to find an equation relating x and y.

(AB 76) Consider the system of differential equations $\frac{dx}{dt} = 3y - 2xy$, $\frac{dy}{dt} = 4xy - 3y$. Use the phase plane method to find an equation relating x and y.

(AB 77) Consider the system of differential equations $\frac{dx}{dt} = 3x - 4xy$, $\frac{dy}{dt} = 5xy - 2y$. Use the phase plane method to find an equation relating x and y.

(AB 78) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ (**not** the table in your notes, on Blackboard, or in your book), find the Laplace transforms of the following functions.

(a) $f(t) = e^{-11t}$

(b) $f(t) = t$

(c) $f(t) = \begin{cases} 3e^t, & 0 < t < 4, \\ 0, & 4 \leq t. \end{cases}$

(AB 79) Find the Laplace transforms of the following functions. You may use the table in your notes, on Blackboard, or in your book.

(a) $f(t) = t^4 + 5t^2 + 4$

(b) $f(t) = (t + 2)^3$

(c) $f(t) = 9e^{4t+7}$

(d) $f(t) = -e^{3(t-2)}$

(e) $f(t) = (e^t + 1)^2$

(f) $f(t) = 8\sin(3t) - 4\cos(3t)$

(g) $f(t) = t^2 e^{5t}$

(h) $f(t) = 7e^{3t}\cos 4t$

(i) $f(t) = 4e^{-t}\sin 5t$

(j) $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$

(k) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$

(l) $f(t) = \begin{cases} 0, & t < 1, \\ t - 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$

(m) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

(n) $f(t) = \begin{cases} 7t^2e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

(o) $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi \end{cases}$

(p) $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi \end{cases}$

(q) $f(t) = t\,e^t \sin t$

(r) $f(t) = t^2 \sin 5t$

(s) You are given that $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$. Find $\mathcal{L}\{t\,J_0(t)\}$. (The function J_0 is called a Bessel function and is important in the theory of partial differential equations in polar coordinates.)

(t) You are given that $\mathcal{L}\{\frac{1}{\sqrt{t}}e^{-1/t}\} = \frac{\sqrt{\pi}}{\sqrt{s}}e^{-2\sqrt{s}}$. Find $\mathcal{L}\{\sqrt{t}\,e^{-1/t}\}$.

(AB 80) For each of the following problems, find y.

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$

(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$

(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$

(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$

(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$

(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$

(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$

(h) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$

(i) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$

(j) $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$

(k) $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$

(l) $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$

(AB 81) Sketch the graph of $y = t^2 - t^2\,\mathcal{U}(t-1) + (2-t)\,\mathcal{U}(t-1)$.

(AB 82) Solve the following initial-value problems using the Laplace transform. Do you expect the graphs of y, $\frac{dy}{dt}$, or $\frac{d^2y}{dt^2}$ to show any corners or jump discontinuities? If so, at what values of t?

(a) $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$
(b) $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$
(c) $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$
(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$
(e) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$
(f) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$
(g) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$
(h) $\frac{d^2y}{dt^2} - 4y = e^t \sin(3t)$, $y(0) = 0$, $y'(0) = 0$
(i) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$
(j) $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \leq t < 4, \\ 0, & 4 \leq t \end{cases}$, $y(0) = 2$, $y'(0) = 0$
(k) $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \leq t < 2\pi, \\ 0, & 2\pi \leq t \end{cases}$, $y(0) = 0$, $y'(0) = 0$
(l) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \leq t < 10, \\ 0, & 10 \leq t \end{cases}$, $y(0) = 0$, $y'(0) = 0$
(m) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \leq t < 2, \\ 4, & 2 \leq t \end{cases}$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = 2$.
(n) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \leq t < 2, \\ 3, & 2 \leq t \end{cases}$, $y(0) = 2$, $y'(0) = 1$
(o) $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$
(p) $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$
(q) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$
(r) $y(t) + \int_0^t r\, y(t-r)\, dr = t$.
(s) $y(t) = te^t + \int_0^t (t-r)\, y(r)\, dr$.
(t) $\frac{dy}{dt} = 1 - \sin t - \int_0^t y(r)\, dr$, $y(0) = 0$.
(u) $\frac{dy}{dt} + 2y + 10\int_0^t e^{4r} y(t-r)\, dr = 0$, $y(0) = 7$.
(v) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$.
(w) $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$.
(x) $\frac{d^2y}{dt^2} + 4y = -2\delta(t-4\pi)$, $y(0) = 1/2$, $y'(0) = 0$
(y) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + \mathcal{U}(t-2)$, $y(0) = 1$, $y'(0) = 0$
(z) $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 5\delta(t-4)$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 2$.

(AB 83) At time $t = 0$, a group of 7 birds, all 1 year old, are blown onto an isolated island. A bird that is t years old on average produces te^{-t} chicks per year. Let $B(t)$ be the birth rate at time t years. Find an integrodifferential equation (and, if necessary, an initial condition) for the birth rate.

(AB 84) At time $t = 0$, a group of 11 birds, all 2 years old, are blown onto an isolated island. A bird that is t years old on average produces te^{-t} chicks per year. Let $B(t)$ be the birth rate at time t years. Find an integrodifferential equation (and, if necessary, an initial condition) for the birth rate.

(AB 85) Let I be the number of people with a rare infections disease. Suppose that 300 people simultaneously contract the disease at time $t = 0$. Each infected person infects three new people per day on average. A person who was infected r days ago has a probability of $4r^2e^{-2r}$ of recovering on day r. Write an integrodifferential equation (and, if necessary, an initial condition) for the number of infected people.

(AB 86) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y - 15, \qquad \frac{dy}{dt} = -5x + 6y - 8, \qquad x(0) = 2, \quad y(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 87) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 88) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y + e^{2t}\cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 89) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 90) Find the general solution to the equation $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3}\ln t$ on the interval $0 < t < \infty$.

(AB 91) Find the general solution to the equation $4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 8e^{-t/2}\arctan t$.

(AB 92) Find the general solution to the equation $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$.

(AB 93) Solve the initial-value problem $9\frac{d^2y}{dt^2} + y = \sec^2(t/3)$, $y(0) = 4$, $y'(0) = 2$, on the interval $-3\pi/2 < t < 3\pi/2$.

(AB 94) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - 2x = 0$, $t > 0$, is $x(t) = C_1 t^2 + C_2 t^{-1}$. Solve the initial-value problem $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(AB 95) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - t\frac{dx}{dt} - 3x = 0$, $t > 0$, is $x(t) = C_1 t^3 + C_2 t^{-1}$. Find the general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

(AB 96) Find the general solution to the following differential equations.

(a) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 3e^{4t}$.
(b) $16\frac{d^2y}{dt^2} - y = e^{t/4}\sin t$.
(c) $\frac{d^2y}{dt^2} + 49y = 3t\sin 7t$.
(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$.
(e) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = t\sin(3t)$.
(f) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$.
(g) $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 6t^2 + \cos(2t)$.
(h) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = t^2e^{3t}$.
(i) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}$.
(j) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t)$.
(k) $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t)$.
(l) $\frac{d^2y}{dt^2} + 9y = 5\sin(3t)$.
(m) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$.
(n) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$.
(o) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$.
(p) $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$.
(q) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$.

(AB 97) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $16\frac{d^2y}{dt^2} - y = 3e^t$, $y(0) = 1$, $y'(0) = 0$.
(b) $\frac{d^2y}{dt^2} + 49y = \sin 7t$, $y(\pi) = 3$, $y'(\pi) = 4$.
(c) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 25t^2$, $y(2) = 0$, $y'(2) = 3$.
(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$.

(AB 98) A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 $\mathrm{N\cdot s/m}$ and is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(20t)$ N upwards.

Write the differential equation and initial conditions that describe the position of the object.

(AB 99) An object weighing 4 lbs stretches a spring 2 inches. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(\omega t)$ pounds, directed upwards. There is no damping.

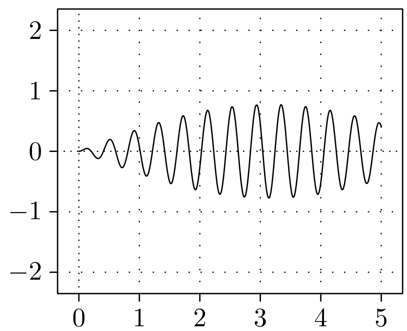
Write the differential equation and initial conditions that describe the position of the object. Then find the value of ω for which resonance occurs. Be sure to include units for ω.

(AB 100) A 4-kg object is suspended from a spring. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $7\sin(\omega t)$ newtons, directed upwards.

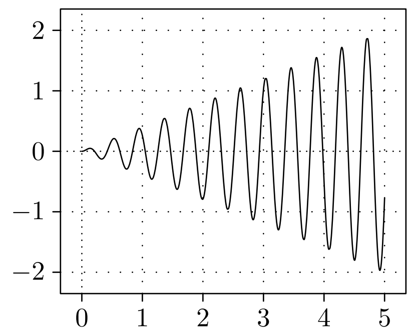
(a) Write the differential equation and initial conditions that describe the position of the object.
(b) It is observed that resonance occurs when $\omega = 20$ rad/sec. What is the constant of the spring? Be sure to include units.

(AB 101) A 1-kg object is suspended from a spring with constant 225 N/m. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $15\cos(\omega t)$ pounds, directed upwards. Illustrated are the object's position for three different values of ω. You are given that the three values are $\omega = 15$ radians/second, $\omega = 16$ radians/second, and $\omega = 17$ radians/second. Determine the value of ω that will produce each image.

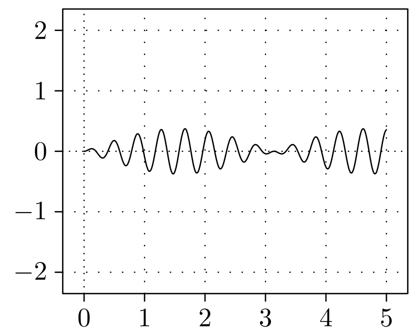
System A



System B



System C



Answer key

(AB 1) Is $y = e^t$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(Answer 1) No, $y = e^t$ is not a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(AB 2) Is $y = e^{2t}$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(Answer 2) Yes, $y = e^{2t}$ is a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(AB 3) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 1$?

(Answer 3) No.

(AB 4) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 2$?

(Answer 4) No.

(AB 5) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 1$?

(Answer 5) Yes.

(AB 6) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 2$?

(Answer 6) No.

(AB 7) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + \arctan(t)\,y = e^t$, $y(3) = 7$.
We expect a unique solution.

(b) $\frac{1}{1+t^2}\frac{dy}{dt} - t^5 y = \cos(6t)$, $y(2) = -1$, $y'(2) = 3$.
We do not expect any solutions.

(c) $\frac{d^2y}{dt^2} - 5\sin(t)\,y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.
We do not expect any solutions.

(d) $e^t\frac{d^2y}{dt^2} + 3(t-4)\frac{dy}{dt} + 4t^6 y = 2$, $y(3) = 1$, $y'(3) = -1$.
We expect a unique solution.

(e) $\frac{d^2y}{dt^2} + \cos(t)\frac{dy}{dt} + 3\ln(1+t^2)\,y = 0$, $y(2) = 3$.
We expect an infinite family of solutions.

(f) $\frac{d^3y}{dt^3} - t^7\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\,y = t^3$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$, $y'''(1) = 3$.
We do not expect any solutions.

(g) $(1+t^2)\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = t^3$, $y(3) = 9$, $y'(3) = 7$, $y''(3) = 5$.
We expect a unique solution.

(h) $\frac{d^3y}{dt^3} - e^t\frac{d^2y}{dt^2} + e^{2t}\frac{dy}{dt} + e^{3t}y = e^{4t}$, $y(-1) = 1$, $y'(-1) = 3$.
We expect an infinite family of solutions.

(i) $(2+\sin t)\frac{d^3y}{dt^3} + \cos t\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\,y = t^3$, $y(7) = 2$.
We expect an infinite family of solutions.

(AB 8) A lake initially has a population of 600 trout. The birth rate of trout is proportional to the number of trout living in the lake. Fishermen are allowed to harvest 30 trout/year from the lake. Write an initial value problem for the fish population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 8) Independent variable: t = time (in years).
Dependent variable: P = Number of trout in the lake
Initial condition: $P(0) = 600$.
Parameters: α = birth rate (in 1/years).
Differential equation: $\frac{dP}{dt} = \alpha P - 30$.

(AB 9) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write an initial value problem for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 9) Independent variable: t = time (in years).
Dependent variable: P = Number of birds on the island.
Parameters: α = birth rate parameter (in 1/years); β = death rate parameter (in 1/(bird·years)).
Initial condition: $P(0) = 5$.
Differential equation: $\frac{dP}{dt} = \alpha P - \beta P^2$.

(AB 10) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen. Write an initial value problem for the amount of hydrogen in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 10) Independent variable: t = time (in minutes).
Dependent variable: H = Amount of hydrogen in the tank (in grams).
Parameters: α = reaction rate parameter (in 1/second·grams).
Initial condition: $H(0) = 50$.
Differential equation: $\frac{dH}{dt} = -\alpha H(126.9H - 3345)$.

(AB 11) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C. Write an initial value problem for the temperature of the cup of coffee. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 11) Independent variable: t = time (in seconds).
Dependent variable: T = Temperature of the cup (in degrees Celsius)
Initial condition: $T(0) = 95$.
Differential equation: $\frac{dT}{dt} = -\alpha(T - 25)$, where α is a positive parameter (constant of proportionality) with units of 1/seconds.

(AB 12) Suppose that an isolated town has 300 households. In 1920, two families install telephones in their homes. Write an initial value problem for the number of telephones in the town if the rate at which families buy telephones is jointly proportional to the number of households with telephones and the number of households without telephones. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 12) Independent variable: t = time (in years).
Dependent variable: T = Number of telephones installed in the town.
Initial condition: $T(1920) = 2$.
Differential equation: $\frac{dT}{dt} = \alpha T(300 - T)$, where α is a positive parameter (constant of proportionality) with units of $1/\text{year} \cdot \text{telephone}$.

(AB 13) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Write an initial value problem for the amount of salt in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 13) Independent variable: t = time (in minutes).
Dependent variable: Q = amount of dissolved salt (in kilograms).
Initial condition: $Q(0) = 3$.
Differential equation: $\frac{dQ}{dt} = 0.01 - \frac{3Q}{600-t}$.

(AB 14) I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write an initial value problem for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 14) Independent variable: t = time (in years).
Dependent variable: B = balance of my loan (in dollars).
Initial condition: $B(0) = 300{,}000$.
Differential equation: $\frac{dB}{dt} = 0.04B - 12 \cdot 1600$

(AB 15) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write an initial value problem for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 15) Independent variable: t = time (in minutes).
Dependent variables:
h = depth of water in the hole (in centimeters)
V = volume of water in the hole (in cubic centimeters); notice that $V = \frac{1}{3}\pi(5h)^2 h$
Initial condition: $V(0) = 0$.
Differential equation: $\frac{dV}{dt} = 1 - \alpha\pi(5h)^2 = 1 - 25\alpha\pi(3V/25\pi)^{2/3}$, where α is a proportionality constant with units of cm/s.

(AB 16) According to the Stefan-Boltzmann law, a black body in a dark vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 16) Independent variable: t = time (in seconds).
Dependent variable: T = object's temperature (in kelvins)
Parameter: σ = proportionality constant (in 1/(seconds·kelvin^3))
Initial condition: $T(0) = 400$.
Differential equation: $\frac{dT}{dt} = -\sigma T^4$.

(AB 17) According to the Stefan-Boltzmann law, a black body in a vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). If there is some ambient light in the vacuum, then the black body absorbs energy at a rate proportional to the effective temperature of the light. The proportionality constant is the same in both cases; that is, if the object's temperature is equal to the effective temperature of the light, then its temperature will not change.

Suppose that a planet in space is currently at a temperature of 400 K. The effective temperature of its surroundings is 3 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 17) Independent variable: t = time (in seconds).

Dependent variable: T = object's temperature (in kelvins)

Parameter: σ = proportionality constant (in 1/(seconds·kelvin^3))

Initial condition: $T(0) = 400$.

Differential equation: $\frac{dT}{dt} = -\sigma T^4 + 81\sigma$.

(AB 18) A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 18) Independent variable: t = time (in seconds).

Dependent variable: v = velocity of the ball (in meters/second, where a positive velocity denotes upward motion).

Parameters: α = proportionality constant of the drag force (in newton·seconds/meter)

m = mass of the ball (in kilograms)

Initial condition: $v(0) = 10$.

Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/m)v$.

(AB 19) A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 19) Independent variable: t = time (in seconds).

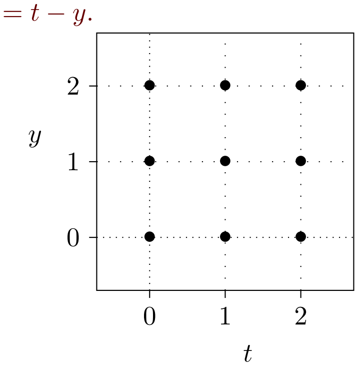
Dependent variable: v = velocity of the ball (in meters/second, where a positive velocity denotes upward motion).

Parameter: α = proportionality constant of the drag force (in newton·seconds/meter).

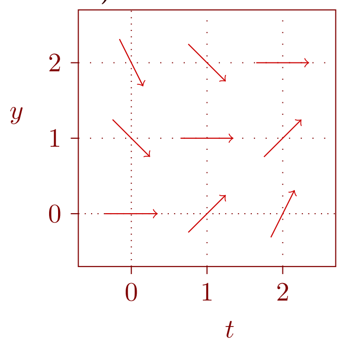
Initial condition: $v(0) = 20$.

Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/0.3)v|v|$, or $\frac{dv}{dt} = \begin{cases} -9.8 - (\alpha/0.3)v^2, & v \geq 0, \\ -9.8 + (\alpha/0.3)v^2, & v < 0. \end{cases}$

(AB 20) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.

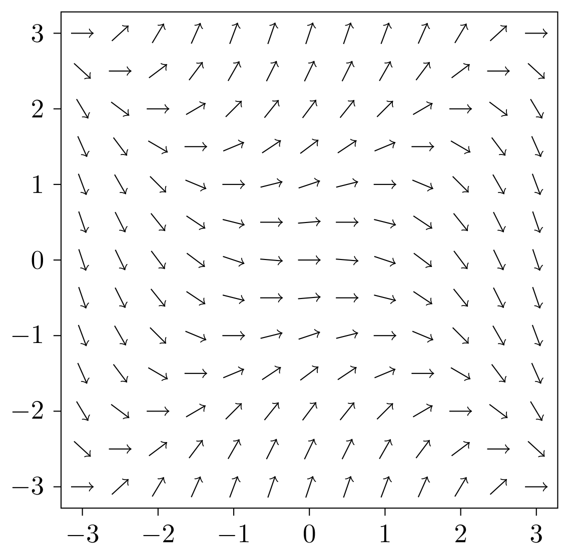


(Answer 20) Here is the direction field for the differential equation $\frac{dy}{dt} = t - y$.

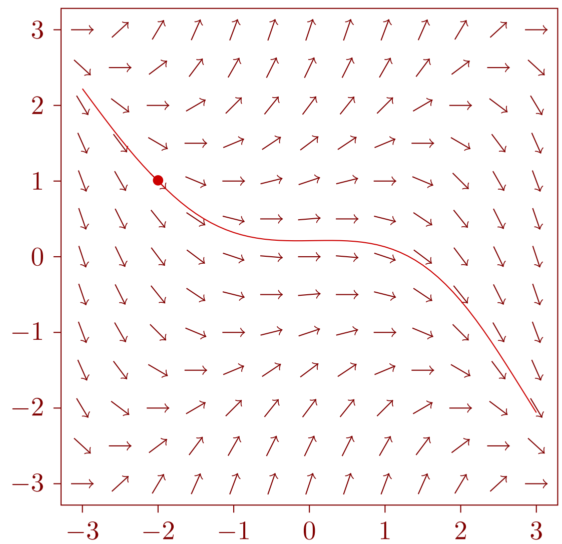


(AB 21) Consider the differential equation $\frac{dy}{dx} = (y^2 - x^2)/3$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-2) = 1.$$



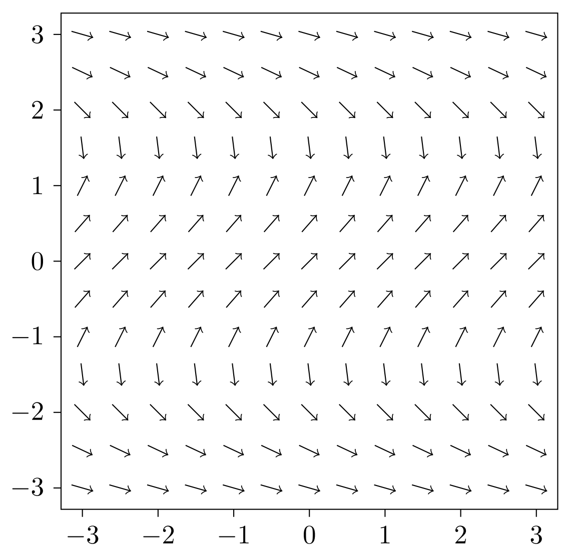
(Answer 21)



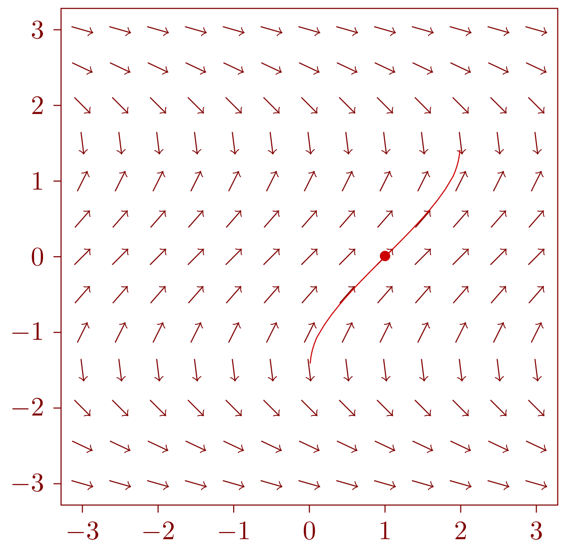
(AB 22) Consider the differential equation $\frac{dy}{dx} = \frac{2}{2-y^2}$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{2}{2-y^2}, \quad y(1) = 0.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(Answer 22)

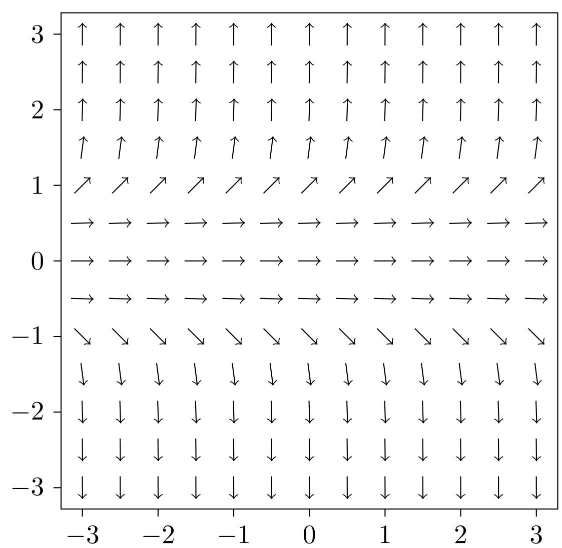


The domain of definition of the solution appears to be $0 < t < 2$.

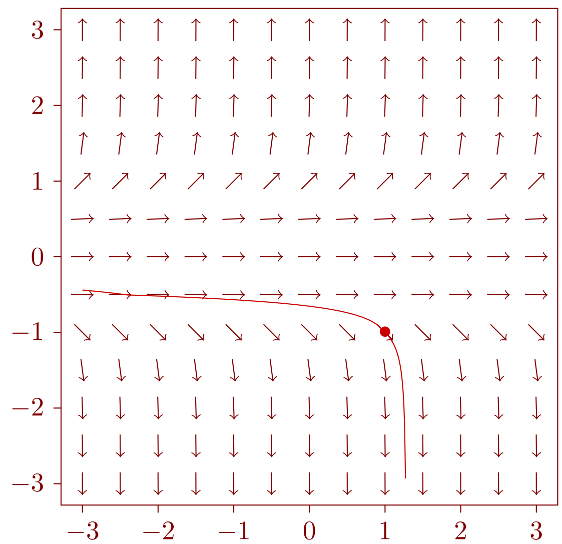
(AB 23) Consider the differential equation $\frac{dy}{dx} = y^5$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = y^5, \quad y(1) = -1.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(Answer 23)

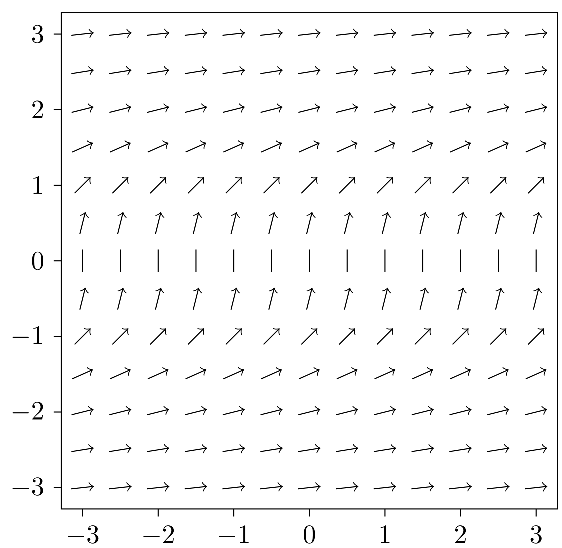


The domain of definition of the solution appears to be approximately $t < 1.3$.

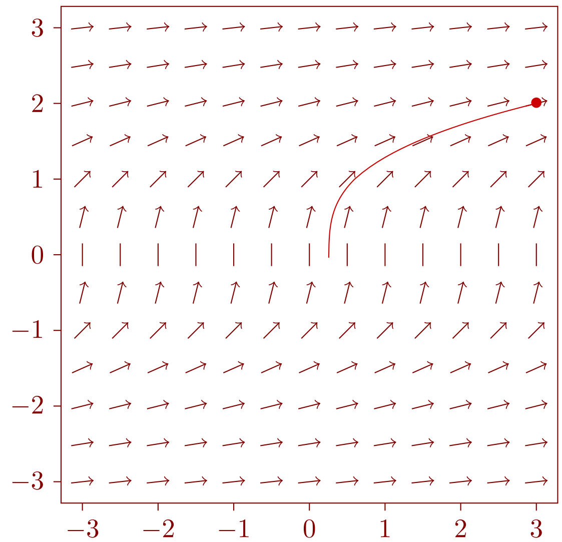
(AB 24) Consider the differential equation $\frac{dy}{dx} = 1/y^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{1}{y^2}, \quad y(3) = 2.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



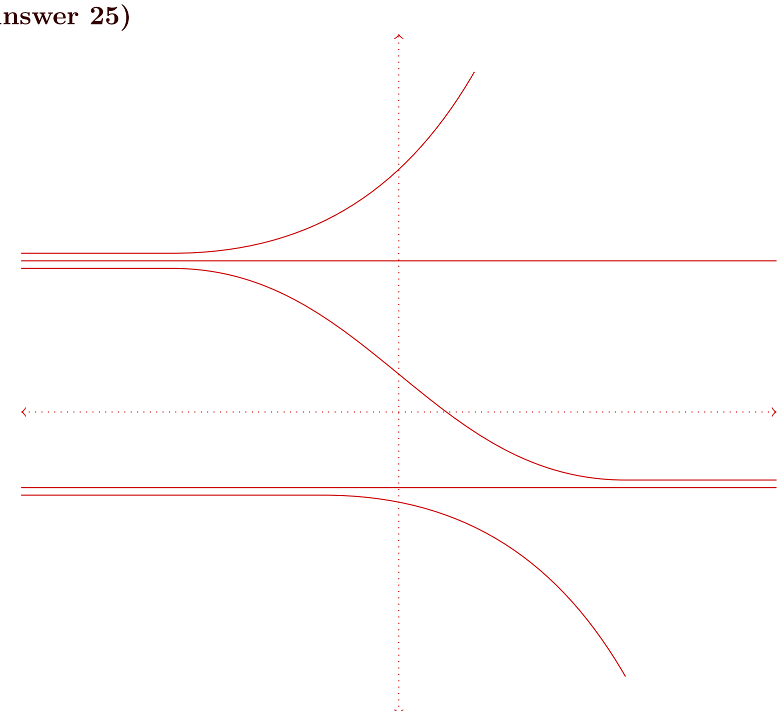
(Answer 24)



The domain of definition of the solution appears to be $\frac{1}{4} < t$.

(AB 25) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y-2)(y+1)^2$. By hand, sketch some typical solutions.

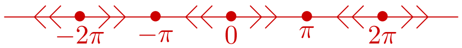
(Answer 25)



(AB 26) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(Answer 26)

Critical points: $y = k\pi$ for any integer k.

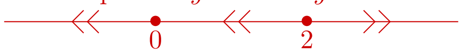


If k is even then $y = k\pi$ is unstable.
If k is odd then $y = k\pi$ is stable.

(AB 27) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = y^2(y-2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(Answer 27)

Critical points: $y = 0$ and $y = 2$.



$y = 0$ is semistable. $y = 2$ is unstable.

(AB 28) I owe my bank a debt of B dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

(a) Formulate a differential equation for the amount of money I owe.
$\frac{dB}{dt} = 0.05B - 19200$, where t denotes time in years.

(b) Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?
The critical point is $B = \$384{,}000$. It is unstable. If my initial debt is less than \$384,000, then I will eventually pay it off (have a debt of zero dollars), but if my initial debt is greater than \$384,000, my debt will grow exponentially. The critical point $B = 384{,}000$ corresponds to the balance that will allow me to make interest-only payments on my debt.

(AB 29) A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

(a) Write the differential equation for the amount of salt in the tank.
$\frac{dQ}{dt} = 10 - Q/300$, where Q denotes the amount of salt in grams and t denotes time in minutes.

(b) Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?
The critical point is $Q = 3000$. It is stable. No matter the initial amount of salt in the tank, as time passes the amount of salt will approach 3000 g or 3 kg.

(AB 30) A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where v is her velocity in meters per second.

(a) Write a differential equation for her velocity. Assume her velocity is always downwards.
If $v \le 0$ then $70\frac{dv}{dt} = -70 * 9.8 + 2v^2$. (If $v > 0$ then $70\frac{dv}{dt} = -70 * 9.8 - 2v^2$.)

(b) Find the (negative) critical points of this differential equation. Be sure to include units.
$v = -\sqrt{343}$ meters/second.

(c) What is the real-world meaning of these critical points?
As $t \to \infty$, her velocity will approach the stable critical point of $v = -\sqrt{343}$ meters/second.

(AB 31) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen.

(a) Write a differential equation for the amount of hydrogen left in the tank.
$\frac{dH}{dt} = -\alpha H(126.9H - 3345)$, where H is the amount of hydrogen remaining (in grams), t denotes time in minutes, and α is a positive parameter.

(b) Find the critical points of this differential equation. Be sure to include units.
$H = 0$ grams and $H = 26.36$ grams.

(c) What is the real-world meaning of these critical points?
As $t \to \infty$, the amount of hydrogen in the tank will tend towards 26.36 grams. The $H = 0$ critical point is not physically meaningful, as if $H = 0$ grams then the model predicts -3345 grams of iodine in the tank.

(AB 32) Suppose that the population y of catfish in a certain lake, absent human intervention, satisfies the logistic equation $\frac{dy}{dt} = ry(1 - y/K)$, where r and K are constants and t denotes time.

(a) Assuming that $r > 0$ and $K > 0$, find the critical points of this equation and classify them as to stability. What is the long term behavior of the population?
The critical points are $y = 0$ (unstable) and $y = K$ (stable). If any positive number of fish are present, then eventually the population will approach a level of $y = K$.

(b) Suppose that humans intervene by harvesting fish at a rate proportional to the fish population. Write a new differential equation for the fish population. Find the critical points of your new equation and classify them as to stability. What is the long term behavior of the population?
$\frac{dy}{dt} = ry(1 - y/K) - Ey$, where E is a proportionality constant. The critical points are $y = 0$ and $y = K - KE/r$.

If $E < r$, then $y = 0$ is unstable and $y = K - KE/r$ is stable, and the population will approach a level of $y = K - KE/r$.

If $E > r$, then $y = -KE/r + K$ is unstable (but has no physical significance) and 0 is stable. If $E = r$, then $y = 0$ is the only critical point and is semistable. If $E \leq r$, then the population will eventually tend towards extinction.

(c) Suppose that humans intervene by harvesting fish at a constant rate. Write a new differential equation for the fish population. Find the critical points of your new equation and classify them as to stability. What is the long term behavior of the population?
$\frac{dy}{dt} = ry(1 - y/K) - h$, where h is the harvesting rate. If $h < Kr/4$, then the critical points are $y = K/2 - \sqrt{(K/2)^2 - Kh/r}$ (unstable) and $y = K/2 + \sqrt{(K/2)^2 - Kh/r}$ (stable). Notice that both critical points are positive (i.e., correspond to the physically meaningful case of at least zero fish in the lake.) If $h = Kr/4$, then there is one critical point at $y = K/2$; it is semistable. Finally, if $h > Kr/4$, then there are no critical points and the fish population will decrease until it goes extinct.

(AB 33) For each of the following differential equations, determine whether it is linear, separable, exact, homogeneous, a Bernoulli equation, or of the form $\frac{dy}{dt} = f(At+By+C)$. Then solve the differential equation.

(a) $\frac{dy}{dt} = \frac{t+\cos t}{\sin y - y}$
$\frac{dy}{dt} = \frac{t+\cos t}{\sin y - y}$ is separable and has solution $\frac{1}{2}y^2 + \cos y = -\frac{1}{2}t^2 - \sin t + C$.

(b) $\ln y + y + t + \left(\frac{t}{y} + t\right)\frac{dy}{dt} = 0$
$\ln y + y + t + \left(\frac{t}{y} + t\right)\frac{dy}{dt} = 0$ is exact and has solution $t\ln y + ty + \frac{1}{2}t^2 = C$.

(c) $(t^2+1)\frac{dy}{dt} = ty - t^2 - 1$
$(t^2+1)\frac{dy}{dt} = ty - t^2 - 1$ is linear and has solution $y = -\sqrt{t^2+1}\ln(t+\sqrt{t^2+1}) + C\sqrt{t^2+1}$.

(d) $t^2\frac{dy}{dt} = y^2 + t^2 - ty$
$t^2\frac{dy}{dt} = y^2 + t^2 - ty$ is homogeneous and has solution $y = \frac{t}{C - \ln|t|} + t$.

(e) $4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5)\frac{dy}{dt} = 0$
$4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5)\frac{dy}{dt} = 0$ is exact and has solution $2t^2y^3 + \frac{3}{2}t^4 + 3y + \frac{1}{6}y^6 = C$.

(f) $\frac{dy}{dt} = -y\tan 2t - y^3\cos 2t$
$\frac{dy}{dt} = -y\tan 2t - y^3\cos 2t$ is Bernoulli. Let $v = y^{-2}$. Then $\frac{dv}{dt} = 2v\tan 2t + 2\cos 2t$, so $v = \frac{1}{2}\sin(2t) + t\sec(2t) + C\sec(2t)$ and $y = \frac{1}{\sqrt{\frac{1}{2}\sin(2t) + t\sec(2t) + C\sec(2t)}}$.

(g) $(t+y)\frac{dy}{dt} = 5y - 3t$
$(t+y)\frac{dy}{dt} = 5y - 3t$ is homogeneous and has solution $(y-3t)^2 = Ct^2(y-t)$.

(h) $\frac{dy}{dt} = \frac{1}{3t+2y+7}$.
If $\frac{dy}{dt} = \frac{1}{3t+2y+7}$, then $\frac{3t+2y+7}{3} - \frac{2}{9}\ln|3t+2y+7+2/3| = \ln|t| + C$.

(i) $\frac{dy}{dt} = -y^3\cos(2t)$
$\frac{dy}{dt} = -y^3\cos(2t)$ is separable and has solution $y = \pm\frac{1}{\sqrt{\sin(2t)+C}}$.

(j) $4ty\frac{dy}{dt} = 3y^2 - 2t^2$
$4ty\frac{dy}{dt} = 3y^2 - 2t^2$ is homogeneous (and also Bernoulli) and has solution $2\ln(y^2/t^2 + 2) = -\ln|t| + C$.

(k) $t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$
$t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$ is a Bernoulli equation. Let $v = y^6$. Then $t\frac{dv}{dt} = 18v - 6t^2$, so $v = Ct^{18} + \frac{3}{8}t^2$ and $y = \sqrt[6]{Ct^{18} + \frac{3}{8}t^2}$.

(l) $\frac{dy}{dt} = \csc^2(y-t)$.
If $\frac{dy}{dt} = \csc^2(y-t)$, then $\tan(y-t) - y = C$.

(m) $\frac{dy}{dt} = 8y - y^8$
$\frac{dy}{dt} = 8y - y^8$ is Bernoulli (and also separable, but separating variables results in an impossible integral). Make the substitution $v = y^{-7}$. Then $\frac{dv}{dt} = -56v + 7$, so $v = Ce^{-56t} + \frac{1}{8}$ and $y = \frac{1}{\sqrt[7]{Ce^{-56t} + 1/8}}$.

(n) $t\frac{dy}{dt} = -\cos t - 3y$
$t\frac{dy}{dt} = -\cos t - 3y$ is linear and has solution $y = \frac{1}{t}\sin t + \frac{2}{t^2}\cos t - \frac{2}{t^3}\sin t + \frac{C}{t^3}$.

(o) $\frac{dy}{dt} = \cot(y/t) + y/t$
$\frac{dy}{dt} = \cot(y/t) + y/t$ is homogeneous and has solution $\sec(y/t) = Ct$.

(p) $\frac{dy}{dt} = ty + t^2\sqrt{[3]}y$
$\frac{dy}{dt} = ty + t^2\sqrt{[3]}y$ is Bernoulli. Make the substitution $v = y^{2/3}$. Then $v = -t - \frac{3}{2} + Ce^{t^2/3}$ and so $y = \sqrt{(-t - \frac{3}{2} + Ce^{t^2/3})^3}$.

(AB 34) The differential equation $e^x \cos y + (e^x \sin y + 1)\frac{dy}{dx} = 0$ is not exact.

(a) Find an integrating factor $\mu(y)$ depending only on y such that, upon multiplying by $\mu(y)$, the equation becomes exact.
$\mu(y) = \sec^2 y$.

(b) Solve the differential equation.
$e^x \sec y + \tan y = C$.

(AB 35) The differential equation $3x^2 + 2\sin 2y + x\cos 2y\frac{dy}{dx} = 0$ is not exact.

(a) Find an integrating factor $\mu(x)$ depending only on x such that, upon multiplying by $\mu(x)$, the equation becomes exact.
$\mu(x) = x^3$.

(b) Solve the differential equation.
$\frac{1}{2}x^4 \sin 2y + \frac{1}{2}x^6 = C$.

(AB 36) For each of the following differential equations, determine whether it is linear, separable, exact, homogeneous, Bernoulli, or a function of a linear term. Then solve the given initial-value problem.

(a) $\cos(t + y^3) + 2t + 3y^2\cos(t + y^3)\frac{dy}{dt} = 0$, $y(\pi/2) = 0$
If $\cos(t + y^3) + 2t + 3y^2\cos(t + y^3)\frac{dy}{dt} = 0$, $y(\pi/2) = 0$, then $\sin(t + y^3) + t^2 = 1 + \pi^2/4$.

(b) $2ty\frac{dy}{dt} = 4t^2 - y^2$, $y(1) = 3$.
If $2ty\frac{dy}{dt} = 4t^2 - y^2$, $y(1) = 3$, then $y = \sqrt{\frac{4}{3}t^2 + \frac{23}{3t}}$.

(c) $t\frac{dy}{dt} = -1 - y^2$, $y(1) = 1$
If $t\frac{dy}{dt} = -1 - y^2$, $y(1) = 1$, then $y = \tan(\pi/4 - \ln t)$.

(d) $\frac{dy}{dt} = (2y + 2t - 5)^2$, $y(0) = 3$.
If $\frac{dy}{dt} = (2y + 2t - 5)^2$, $y(0) = 3$, then $y = \frac{1}{2}\tan(2t + \pi/4) + \frac{5}{2} - t$.

(e) $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$.
If $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$, then $y = \sqrt[3]{340e^{6t} + 3}$.

(f) $\frac{dy}{dt} = -3y - \sin t\, e^{-3t}$, $y(0) = 2$
If $\frac{dy}{dt} = -3y - \sin t\, e^{-3t}$, $y(0) = 2$, then $y = e^{-3t}\cos t + e^{-3t}$.

(AB 37) Solve the initial-value problem $\frac{dy}{dt} = \frac{t-5}{y}$, $y(0) = 3$ and determine the range of t-values in which the solution is valid.

(Answer 37) $y = \sqrt{(t-5)^2 - 16} = \sqrt{(t-1)(t-9)} = \sqrt{t^2 - 10t + 9}$. The solution is valid for all $t < 1$.

(AB 38) Solve the initial-value problem $\frac{dy}{dt} = y^2$, $y(0) = 1/4$ and determine the range of t-values in which the solution is valid.

(Answer 38) $y = \frac{1}{4-t}$. The solution is valid for all $t < 4$.

(AB 39) The function $y_1(t) = e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2} - (1 + 2t)\frac{dy}{dt} + (t + 1)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

(Answer 39) $y(t) = C_1\, e^t + C_2\, t^2 e^t$.

(AB 40) The function $y_1(t) = t$ is a solution to the differential equation $t^2\frac{d^2y}{dt^2} - (t^2 + 2t)\frac{dy}{dt} + (t + 2)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

(Answer 40) $y(t) = C_1\, t + C_2\, te^t$.

(AB 41) The function $y_1(x) = x^3$ is a solution to the differential equation $x^2 \frac{d^2y}{dx^2} + (x^2 \tan x - 6x)\frac{dy}{dx} + (12 - 3\,x \tan x)y = 0$. Find the general solution to this differential equation on the interval $0 < x < \pi/2$.

(Answer 41) $y(x) = C_1\, x^3 + C_2\, x^3 \sin x$.

(AB 42)

(a) What is the largest interval on which the problem

$$(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0, \quad y(0) = 2, \quad y'(0) = 7$$

is guaranteed a unique solution?
$(-\infty, 3/2)$.

(b) What is the largest interval on which the problem

$$(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0, \quad y(2) = 2, \quad y'(2) = 7$$

is guaranteed a unique solution?
$(3/2, 3)$.

(c) What is the largest interval on which the problem

$$(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0, \quad y(5) = 2, \quad y'(5) = 7$$

is guaranteed a unique solution?
$(3, \infty)$.

(d) You are given that $y_1 = t$ is a solution to the differential equation $(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$. Use the method of reduction of order to find the general solution.
$y(t) = C_1\, t + \frac{C_2}{t-3}$.

(e) On what intervals are all of the the solutions you found in part (d) continuous?
On the intervals $(-\infty, 3)$ and $(3, \infty)$.

(f) How many solutions are there to the initial value problem

$$(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0, \quad y(3/2) = 2, \quad y'(3/2) = 7?$$

This initial value problem has no solutions.

(AB 43) You are given that $y_1 = t^3 + 4t^2 + 4t$ and $y_2 = t^2 + 4t + 4$ are both solutions to the differential equation
$(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$ for all t.

(a) How many solutions are there to the initial value problem

$$(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0, \quad y(3) = 1, \quad y'(3) = 0?$$

There are no solutions to this problem.

(b) How many solutions are there to the initial value problem

$$(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0, \quad y(3) = 0, \quad y'(3) = 1?$$

There is one solution (a unique solution). Any solution must satisfy $y = C_1(t-3) + \frac{C_2}{t-3}$ for all $t > 3$ and $y = C_3(t-3) + \frac{C_4}{t-3}$ for all $t < 3$. If $y'(3)$ exists, then y must be continuous at $t = 3$, and so $C_2 = C_4$. Thus, $y = \begin{cases} C_3(t-3), & t \leq 3, \\ C_1(t-3), & t \geq 3. \end{cases}$ In order for $y'(3)$ to exist, we must have $C_1 = C_3$ and so $y(t) = C_1(t-3)$ for all t; if $y'(3) = 1$ then $y = t - 3$.

(AB 44) You are given that $y_1 = t^3 + 4t^2 + 4t$ and $y_2 = t^2 + 4t + 4$ are both solutions to the differential equation $(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$ for all t.

(a) What are the maximal intervals on which the general solution to $(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$ may be written $y = C_1(t^3 + 4t^2 + 4t) + C_2(t^2 + 4t + 4)$?
$(-\infty, -2)$ and $(-2, \infty)$.

(b) Find a function $y_3(t)$ that is continuous and has continuous first and second derivatives for all t, that satisfies $(t+2)^2 \frac{d^2y_3}{dt^2} - (4t+8)\frac{dy_3}{dt} + 6y_3 = 0$ for all t, but such that there do not exist constants C_1 and C_2 such that $y_3(t) = C_1y_1(t) + C_2y_2(t) = C_1(t^3 + 4t^2 + 4t) + C_2(t^2 + 4t + 4)$ for all t.
There are many possible answers. They include:
$y(t) = \begin{cases} t^3 + 4t^2 + 4t, & t \leq 2, \\ -2t^2 - 8t - 8, & t \geq 2, \end{cases}$
$y(t) = \begin{cases} -2t^2 - 8t - 8, & t \leq 2, \\ t^3 + 4t^2 + 4t, & t \geq 2, \end{cases}$
$y(t) = \begin{cases} t^3 + 6t^2 + 12t + 8, & t \leq 2, \\ 0, & t \geq 2. \end{cases}$

(c) Find numbers a, b, and c such that the initial value problem

$$(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0, \quad y(a) = b, \quad y'(a) = c$$

has no solutions.
We must choose $a = -2$, as $(t+2)^2$, $-(4t+8)$, and $+6$ are continuous for all t and $(t+2)^2 = 0$ only for $t = -2$. There is no solution if $c \neq 0$ or if $b \neq 0$.

(d) Find numbers a, b, and c such that the initial value problem

$$(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0, \quad y(a) = b, \quad y'(a) = c$$

has at least two solutions.
As before, $a = -2$. If we choose $b = c = 0$, then $y(t) = 0$, $y(t) = t^3 + 4t^2 + 4t$ and $y(t) = t^2 + 4t + 4$ are all solutions.

(AB 45) Find the general solution to the following differential equations.

(a) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$.

If $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$, then $y = C_1 e^{-6t}\cos(7t) + C_2 e^{-6t}\sin(7t)$.

(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.

If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$, then $y = C_1 e^{(-2+\sqrt{2})t} + C_2 e^{(-2-\sqrt{2})t}$.

(c) $\frac{d^4y}{dt^4} + 7\frac{d^2y}{dt^2} - 144y = 0$.

If $\frac{d^4y}{dt^4} + 7\frac{d^2y}{dt^2} - 144y = 0$, then $y = C_1 e^{3t} + C_2 e^{-3t} + C_3\cos 4t + C_4 \sin 4t$.

(d) $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0$.

If $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0$, then $y = C_1 e^{2t} + C_2 te^{2t} + C_3 e^{-2t} + C_4 te^{-2t}$.

(AB 46) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 2y = 0$, $y(0) = 3$, $y'(0) = 2$.

If $9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 2y = 0$, $y(0) = 3$, $y'(0) = 2$, then $y = 3e^{-t/3}\cos(t/3) + 9e^{-t/3}\sin(t/3)$.

(b) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$, $y(0) = 1$, $y'(0) = 4$.

If $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$, $y(0) = 1$, $y'(0) = 4$, then $y = e^{-5t} + 9te^{-5t}$.

(AB 47) An object weighing 5 lbs stretches a spring 4 inches. It is attached to a viscous damper with damping constant 16 lb · sec/ft. The object is pulled down an additional 2 inches and is released with initial velocity 3 ft/sec upwards.

Write the differential equation and initial conditions that describe the position of the object.

(Answer 47) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then

$$\frac{5}{32}\frac{d^2x}{dt^2} + 16\frac{dx}{dt} + 15x = 0, \qquad x(0) = -\frac{1}{6}, \quad x'(0) = 3.$$

(AB 48) A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant β. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20$ N · s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30$ N · s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(Answer 48) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$2\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.1, \quad x'(0) = 0.$$

If $\beta = 20$ N · s/m, then the system is underdamped, and we do expect to see decaying oscillations.

If $\beta = 30$ N · s/m, then the system overdamped, and we do not expect to see decaying oscillations.

(AB 49) A 3-kg object is attached to a spring with constant k and to a viscous damper with damping constant 42 N·sec/m. The object is set in motion from its equilibrium position with initial velocity 5 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object.

If $k = 100$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $k = 200$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(Answer 49) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$3\frac{d^2x}{dt^2} + 42\frac{dx}{dt} + kx = 0, \qquad x(0) = 0, \quad x'(0) = -5.$$

If $k = 100$ N/m, then the system is overdamped, and we do not expect to see decaying oscillations.

If $k = 200$ N/m, then the system underdamped, and we do expect to see decaying oscillations.

(AB 50) A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant β. The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of β for which the system is critically damped. Be sure to include units for β.

(Answer 50) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$4\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 70x = 0, \qquad x(0) = 0.05, \quad x'(0) = -3.$$

Critical damping occurs when $\beta = 4\sqrt{70}\ \mathrm{N\cdot s/m}$.

(AB 51) An object of mass m is attached to a spring with constant 80 N/m and to a viscous damper with damping constant 20 N·s/m. The object is pulled down to 5cm below its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of m for which the system is critically damped. Be sure to include units for m.

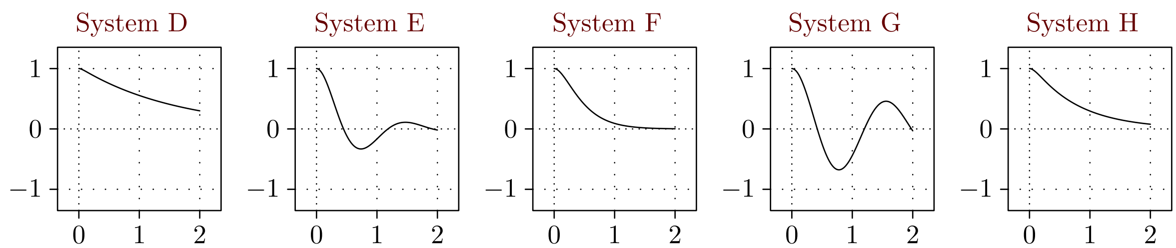
(Answer 51) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$m\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.05, \quad x'(0) = -3.$$

Critical damping occurs when $m = \frac{5}{4}$ kg.

(AB 52) Five objects, each with mass 3 kg, are attached to five springs, each with constant 48 N/m. Five dampers with unknown constants are attached to the objects. In each case, the object is pulled down to a distance 1 cm below the equilibrium position and released from rest. You are given that the system is critically damped in exactly one of the five cases.

Here are the graphs of the objects' positions with respect to time:



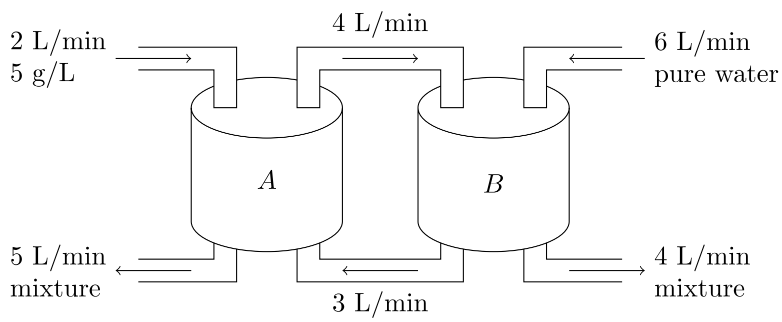
(a) For which damper is the system critically damped?
The system is critically damped for Damper F.

(b) For which dampers is the system overdamped?
The system overdamped for Dampers H and D.

(c) For which dampers is the system underdamped?
The system underdamped for Dampers E and G.

(d) Which damper has the highest damping constant? Which damper has the lowest damping constant?
Damper D has the highest damping constant. Damper G has the lowest damping constant.

(AB 53) Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



Write the differential equations and initial conditions that describe the amount of salt in each tank.

(Answer 53) Let t denote time (in minutes).
Let x denote the amount of salt (in grams) in tank A.
Let y denote the amount of salt (in grams) in tank B.
Then $x(0) = 3000$ and $y(0) = 2000$.
If $t < 50$, then

$$\frac{dx}{dt} = 10 - \frac{9x}{200-4t} + \frac{3y}{300+3t}, \qquad \frac{dy}{dt} = \frac{4x}{200-4t} - \frac{7y}{300+3t}.$$

(AB 54) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 54) If

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 4t - 5 \\ -2t + 3 \end{pmatrix}.$$

(AB 55) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 55) If

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} -3\cos t + 16\sin t \\ 2\cos t + 7\sin t \end{pmatrix}.$$

(AB 56) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 56) If

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-20t} \begin{pmatrix} 3/20 \\ 9/10 \end{pmatrix} + e^{10t} \begin{pmatrix} -3/20 \\ 1/10 \end{pmatrix}.$$

(AB 57) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 57) If

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 5\cos 12t + \sin 12t \\ 2\cos 12t + 2\sin 12t \end{pmatrix}.$$

(AB 58) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 58) If

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos 3t - 11\sin 3t \\ -3\cos 3t - (23/3)\sin 3t \end{pmatrix}.$$

(AB 59) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 59) If

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t}\begin{pmatrix} 15t + 1 \\ (15/4)t + 4 \end{pmatrix}.$$

(AB 60) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 60) If

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t}\begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} + e^{-6t}\begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix}.$$

(AB 61) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det\begin{pmatrix} 4-m & 1 & -1 \\ 1 & 4-m & -1 \\ -1 & -1 & 4-m \end{pmatrix} = -(m-3)^2(m-6).$

(Answer 61) If

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{3t} \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} + e^{6t} \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}.$$

(AB 62) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, \qquad x(0) = 3, \quad y(0) = 2, \quad z(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).
Hint: $\det \begin{pmatrix} 3-m & 1 & 0 \\ 2 & 3-m & -2 \\ 0 & 1 & 3-m \end{pmatrix} = -(m-3)^3$.

(Answer 62) If

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, \qquad x(0) = 3, \quad y(0) = 2, \quad z(0) = 1$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{3t} \begin{pmatrix} 2t^2 + 2t + 3 \\ 4t + 2 \\ 2t^2 + 2t + 1 \end{pmatrix}.$$

(AB 63) Find the solution to the initial-value problem

$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, x(0) = 4, \quad y(0) = 3, \quad z(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).
Hint: $\det \begin{pmatrix} 1-m & 1 & 1 \\ 1 & 3-m & -1 \\ 0 & 2 & 2-m \end{pmatrix} = -(m-2)^3$.

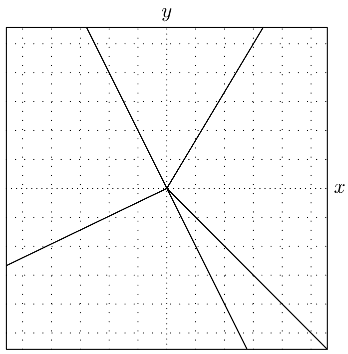
(Answer 63) If

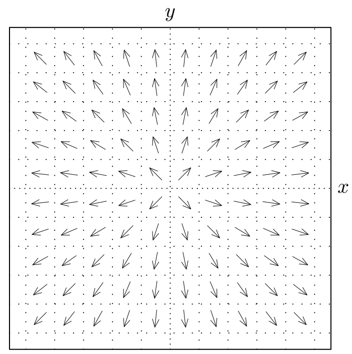
$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, \qquad x(0) = 4, \quad y(0) = 3, \quad z(0) = 5$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 4 + 4t + 2t^2 \\ 2t + 3 \\ 5 + 6t + 2t^2 \end{pmatrix}.$$

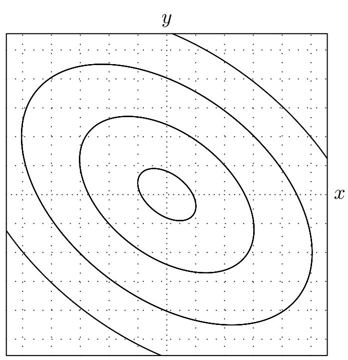
(AB 64) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?

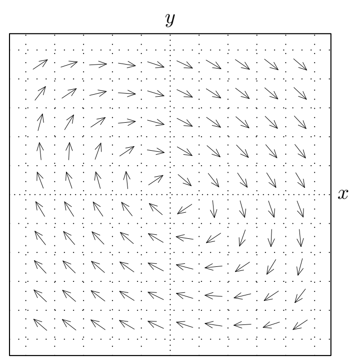




(Answer 64) This is a star. Every vector is an eigenvector. There is only one eigenvalue and it is positive. We must have that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ for some positive number r.

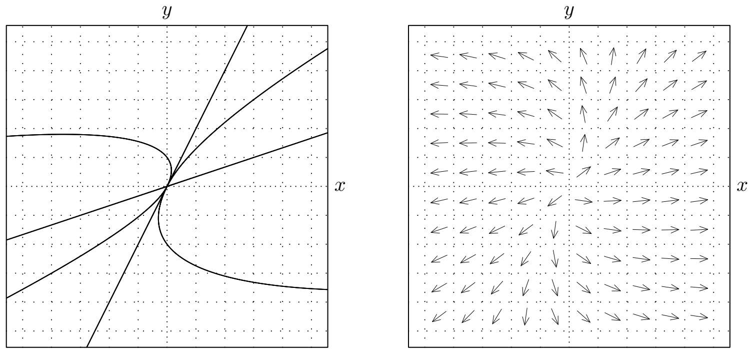
(AB 65) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?





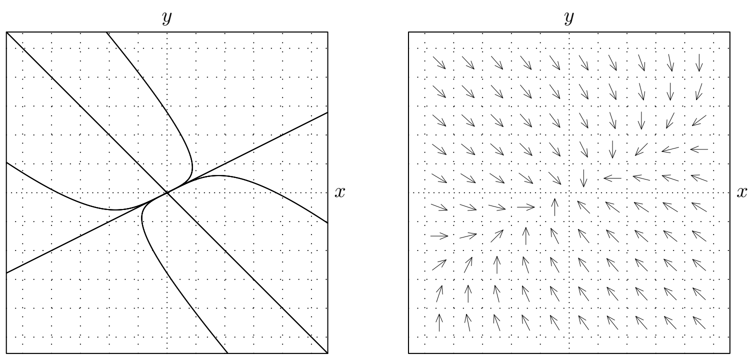
(Answer 65) This is a center. The eigenvalues are purely imaginary. The solutions consist of sines and cosines (no exponentials or powers of t).

(AB 66) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



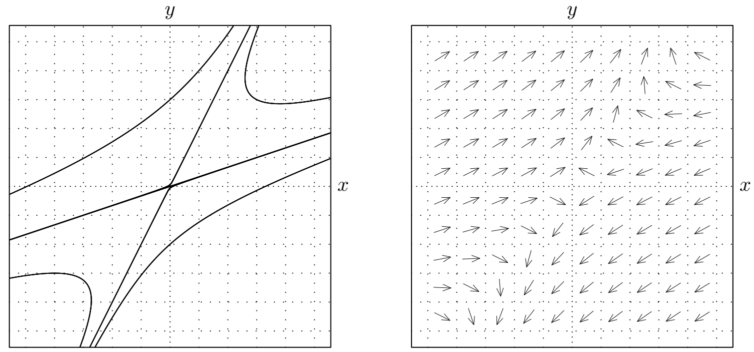
(Answer 66) This is a node. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two distinct real positive eigenvalues. The solutions take the form $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{st} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, where r and s are the eigenvalues and $0 < r < s$.

(AB 67) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



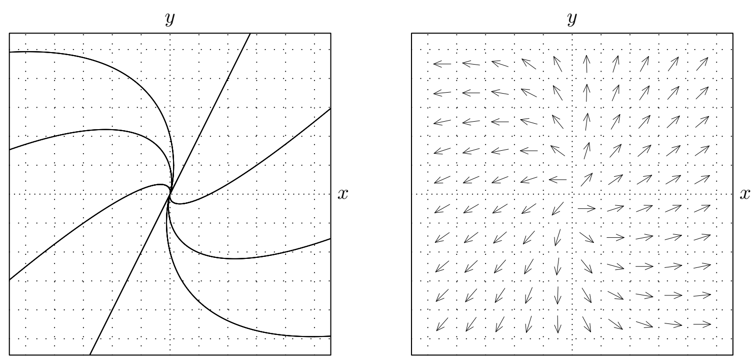
(Answer 67) This is a node. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two distinct real negative eigenvalues. The solutions take the form $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{st} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, where r and s are the eigenvalues and $s < r < 0$.

(AB 68) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



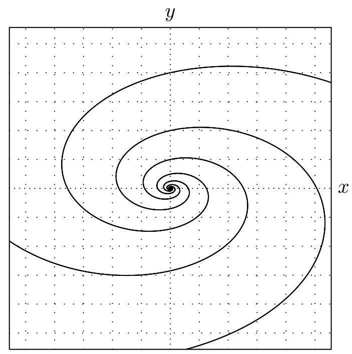
(Answer 68) This is a saddle point. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two real eigenvalues, one positive and one negative. The solutions take the form $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{st} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, where r and s are the eigenvalues and $s < 0 < r$.

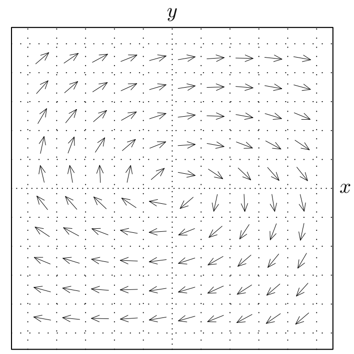
(AB 69) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



(Answer 69) This is a degenerate node. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has only one eigenvalue and it is positive. If C is a constant then $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, where $r > 0$ is the eigenvalue. The general solution is $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{rt} \begin{pmatrix} t + A \\ 2t + B \end{pmatrix}$, where A and B are constants.

(AB 70) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?





(Answer 70) This is a spiral point. The eigenvalues are complex and take the form $\mu \pm \lambda i$, where $\mu > 0$ is real and λ is real. The solutions are exponentials multiplied by sines and cosines.

(AB 71) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

(*a*) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.
Let t denote time (in days), let S denote the number of susceptible people (who have never had the disease), let I denote the number of infected people (who currently have the disease), and let R denote the number of recovered people (who now are resistant, that is, cannot get the disease again). Then $\boxed{\frac{dS}{dt} = -\frac{1}{22500}SI,\ \frac{dI}{dt} = \frac{1}{22500}SI - 0.17I,\ \frac{dR}{dt} = 0.17I,\ S(0) = 8997,\ I(0) = 3,\ R(0) = 0.}$

(*b*) Use the phase plane method to find an equation relating the number of susceptible and infected people.
$\frac{dI}{dS} = \frac{3825}{S} - 1$, so $\boxed{I = 9000 - S - 3825 \ln \frac{8997}{S}.}$

(*c*) Use the phase plane method to find an equation relating the number of resistant and susceptible people.
$\frac{dR}{dS} = -\frac{3825}{S}$, so $\boxed{R = 3825 \ln \frac{8997}{S}.}$

(*d*) As $t \to \infty$, $I \to 0$ and the number of people who never contract the disease approaches a limit. Find an equation for S_∞, where S_∞ is the limiting number of people who never contract the disease.
$\boxed{9000 - S_\infty - 3825 \ln \frac{8997}{S_\infty} = 0.}$ If you have access to `wolframalpha.com` or another numerical solver during the exam, you can solve this equation by typing `solve(9000-S-3825*ln(8997/S)=0,S)` and discover that $\boxed{S_\infty = 1158}$ (when rounded to the nearest whole number).

(*e*) What is the maximum number of people that are infected at any one time?
The maximum occurs when $S = 3825$, at which time $\boxed{I = 9000 - 3825 - 3825 \ln \frac{62979}{33750}.}$

(*f*) When is the peak infection time, that is, the time at which the number of infected people is at its maximum? You need not simplify your answer.
$$\boxed{\int_{3825}^{8997} \frac{22500}{S(9000 - S - 3825 \ln \frac{8997}{S})}\, dS.}$$

(AB 72) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

(a) Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)
$\frac{dS}{dt} = -\frac{0.2}{5000}SI - 15$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I + 15$.

(AB 73) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

(a) Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)
$\boxed{\frac{dS}{dt} = -\frac{0.2}{5000}SI - 15\frac{S}{5000},\ \frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I,\ \frac{dR}{dt} = 0.1I + 15\frac{S}{5000}.}$

(b) Assume that there are initially no recovered or vaccinated people and 7 infected people. Use the phase plane method to find an equation relating S and I.
$\frac{dI}{dS} = \frac{0.2S-500}{S}\frac{I}{-0.2I-15}$, so $\boxed{-0.2(I-7) - 15\ln(I/7) = 0.2(S-4993) - 500\ln(S/4993).}$

(c) What is the maximum number of people that are infected with the virus at any one time? (Round to the nearest whole number.)
The maximum I value occurs when $S = 2500$. The maximum I value then satisfies
$\boxed{-0.2(I-7) - 15\ln(I/7) = 0.2(2500-4993) - 500\ln(2500/4993).}$ Were you given this problem on the exam, you would be allowed to either stop there or to use `wolframalpha.com`, which would let you solve the equation and see that the maximum I value is $\boxed{I = 457}$.

(AB 74) Here is another way to model vaccination. Suppose that a disease is spreading through a town of 8000 people. Initially there are 10 infected people, 3000 vaccinated people, and no recovered people.

No other persons are vaccinated during the disease's spread. Thus, the population should be divided into *four* groups: people who are vaccinated, susceptible, infected, and recovered.

Each infected person encounters 10 people per day, who are distributed among susceptible, vaccinated, recovered and infected people according to their proportion of the total population. Each time an infected person encounters a susceptible person, there is a 4% chance that the susceptible person becomes infected. The vaccine is not perfect; each time an infected person encounters a vaccinated person, there is a 1% chance they become infected. (Once a vaccinated person becomes infected, they are identical to an infected person who was never vaccinated.) Each infected person has a 20% chance per day of recovering. A recovered person can never contract the disease again.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

Let t denote time (in days). Let V denote the number of never-infected vaccinated people, S denote the number of never-infected susceptible people, I denote the number of infected people, and R denote the number of recovered, disease-resistant people. Then

$$\frac{dS}{dt} = -\frac{0.4}{8000}IS, \quad \frac{dV}{dt} = -\frac{0.1}{8000}IV, \quad \frac{dI}{dt} = \frac{0.4}{8000}IS + \frac{0.1}{8000}IV - 0.2I, \quad \frac{dR}{dt} = 0.2I$$

and

$$S(0) = 4990, \quad V(0) = 3000, \quad I(0) = 10, \quad R(0) = 0.$$

(b) Use the phase plane method to find an equation relating S and V.

$\frac{dS}{dV} = 4\frac{S}{V}$, so $\boxed{S = \frac{4990}{3000^4}V^4.}$

(c) Use the phase plane method to find an equation relating I and V.

$\frac{dI}{dV} = -\frac{4V^4 4990/3000^4 + V - 16000}{V}$, so $\boxed{I = -\frac{4990}{3000^4}V^4 + 2000 + V - 16000\ln(V/3000)}$.

(d) What is the maximum number of people that are infected with the virus at any one time? (Round to the nearest whole number.)

The maximum I value occurs when $4\frac{4990}{3000^4}V^4 + V - 16000 = 0$. According to `wolframalpha.com` this occurs when $V = 2710$. Thus, the maximum number of infected people is is $\boxed{3014.}$

(e) At what time does the peak number of infected people occur?

If t denotes the time at which 2710 people are vaccinated, then

$$\boxed{t = \int_{2710}^{3000} \frac{80000\,dV}{V\left(-\frac{4990}{3000^4}V^4 + 2000 + V - 16000\ln(V/3000)\right)}.}$$

(AB 75) Consider the system of differential equations $\frac{dx}{dt} = 3x - 4y$, $\frac{dy}{dt} = 4x - 3y$. Use the phase plane method to find an equation relating x and y.

(Answer 75) The trajectories of solutions to $\frac{dx}{dt} = 3x - 4y$, $\frac{dy}{dt} = 4x - 3y$ satisfy $4y^2 - 6xy + 4x^2 = C$ for constants C.

(AB 76) Consider the system of differential equations $\frac{dx}{dt} = 3y - 2xy$, $\frac{dy}{dt} = 4xy - 3y$. Use the phase plane method to find an equation relating x and y.

(Answer 76) If $\frac{dx}{dt} = 3y - 2xy$, $\frac{dy}{dt} = 4xy - 3y$, then $y = -2x - \frac{3}{2}\ln|x - 3/2| + C$.

(AB 77) Consider the system of differential equations $\frac{dx}{dt} = 3x - 4xy$, $\frac{dy}{dt} = 5xy - 2y$. Use the phase plane method to find an equation relating x and y.

(Answer 77) If $\frac{dx}{dt} = 3x - 4xy$, $\frac{dy}{dt} = 5xy - 2y$, then $3\ln|y| + 2\ln|x| - 4y - 5x = C$.

(AB 78) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ (**not** the table in your notes, on Blackboard, or in your book), find the Laplace transforms of the following functions.

(a) $f(t) = e^{-11t}$
(b) $f(t) = t$
(c) $f(t) = \begin{cases} 3e^t, & 0 < t < 4, \\ 0, & 4 \leq t. \end{cases}$

(Answer 78)

(a) $\mathcal{L}\{t\} = \frac{1}{s^2}$.
(b) $\mathcal{L}\{e^{-11t}\} = \frac{1}{s+11}$.
(c) $\mathcal{L}\{f(t)\} = \frac{3-3e^{4-4s}}{s-1}$.

(AB 79) Find the Laplace transforms of the following functions. You may use the table in your notes, on Blackboard, or in your book.

(a) $f(t) = t^4 + 5t^2 + 4$
$\mathcal{L}\{t^4 + 5t^2 + 4\} = \frac{96}{s^5} + \frac{10}{s^3} + \frac{4}{s}$.

(b) $f(t) = (t+2)^3$
$\mathcal{L}\{(t+2)^3\} = \mathcal{L}\{t^3 + 6t^2 + 12t + 8\} = \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{8}{s}$.

(c) $f(t) = 9e^{4t+7}$
$\mathcal{L}\{9e^{4t+7}\} = \mathcal{L}\{9e^7 e^{4t}\} = \frac{9e^7}{s-4}$.

(d) $f(t) = -e^{3(t-2)}$
$\mathcal{L}\{-e^{3(t-2)}\} = \mathcal{L}\{-e^{-6}e^{3t}\} = -\frac{e^{-6}}{s-3}$.

(e) $f(t) = (e^t + 1)^2$
$\mathcal{L}\{(e^t+1)^2\} = \mathcal{L}\{e^{2t} + 2e^t + 1\} = \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s}$.

(f) $f(t) = 8\sin(3t) - 4\cos(3t)$
$\mathcal{L}\{8\sin(3t) - 4\cos(3t)\} = \frac{24}{s^2+9} - \frac{4s}{s^2+9}$.

(g) $f(t) = t^2 e^{5t}$
$\mathcal{L}\{t^2 e^{5t}\} = \frac{2}{(s-5)^3}$.

(h) $f(t) = 7e^{3t}\cos 4t$
$\mathcal{L}\{7e^{3t}\cos 4t\} = \frac{7s-21}{(s-3)^2+16}$.

(i) $f(t) = 4e^{-t}\sin 5t$
$\mathcal{L}\{4e^{-t}\sin 5t\} = \frac{20}{(s+1)^2+25}$.

(j) $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$

If $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-3s}e^3}{s-1}$.

(k) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$

If $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$ then $\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{1}{s} + \frac{2}{s^3}\right)$.

(l) $f(t) = \begin{cases} 0, & t < 1, \\ t-2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$

If $f(t) = \begin{cases} 0, & t < 1, \\ t-2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2}$.

(m) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

If $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{5}{s-2} - \frac{5e^6 e^{-3s}}{s-2}$.

(n) $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

If $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{14}{(s+1)^3} - \frac{14e^{-3s-3}}{(s+1)^3} - \frac{42e^{-3s-3}}{(s+1)^2} - \frac{63e^{-3s-3}}{s+1}$.

(o) $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi \end{cases}$

If $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{s}{s^2+9} + e^{-\pi s}\frac{s}{s^2+9} - e^{-\pi s}\frac{3}{s^2+9}$.

(p) $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi \end{cases}$

If $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s/2}\frac{1}{s^2+1} + e^{-\pi s}\frac{s}{s^2+1}$.

(q) $f(t) = t\,e^t \sin t$
$\mathcal{L}\{t\,e^t \sin t\} = \frac{2s-2}{(s^2-2s+2)^2}$.

(r) $f(t) = t^2 \sin 5t$
$\mathcal{L}\{t^2 \sin 5t\} = \frac{40s^2-10(s^2+25)}{(s^2+25)^3}$.

(s) You are given that $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$. Find $\mathcal{L}\{t\,J_0(t)\}$. (The function J_0 is called a Bessel function and is important in the theory of partial differential equations in polar coordinates.)
$\mathcal{L}\{t\,J_0(t)\} = \frac{s}{(s^2+1)^{3/2}}$.

(t) You are given that $\mathcal{L}\{\frac{1}{\sqrt{t}}e^{-1/t}\} = \frac{\sqrt{\pi}}{\sqrt{s}}e^{-2\sqrt{s}}$. Find $\mathcal{L}\{\sqrt{t}\,e^{-1/t}\}$.
$\mathcal{L}\{\sqrt{t}\,e^{-1/t}\} = \frac{\sqrt{\pi}(1+2\sqrt{s})}{2s\sqrt{s}}e^{-2\sqrt{s}}$

(AB 80) For each of the following problems, find y.

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$
If $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$, then $y = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$.

(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$
If $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$, then $y = \frac{5}{2}t^2 - \frac{7}{6}t^3$.

(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$
If $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$, then $y = \frac{1}{2}t^2e^{-t} + \frac{1}{6}t^3e^{-t}$.

(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$
If $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$, then $y = (1/4)e^{2t} + (7/4)e^{-2t}$.

(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$
If $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$, then $y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{1}{9}t\,e^{3t}$.

(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$
If $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$, then $y = \frac{1}{2} - \frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t)$.

(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$
If $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$, then $y = \frac{1}{8}\cos t - \frac{1}{8}\cos 3t$.

(h) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$
If $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$, then $y = 2\,\mathcal{U}(t-2)\,e^{t-2}\,\cos(t-2) + \mathcal{U}(t-2)\,e^{t-2}\,\sin(t-2)$.

(i) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$
If $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$, then $y = \frac{1}{2}\,\mathcal{U}(t-1)\,e^{3(t-1)} + \frac{1}{2}\,\mathcal{U}(t-1)\,e^{t-1}$.

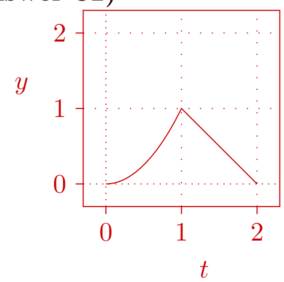
(j) $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$
If $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$, then $y = \int_0^t \frac{1}{3}\sin 3r\cos(3t-3r)\,dr = \frac{1}{6}t\sin(3t)$.

(k) $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$
If $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$, then $y = e^{-2t}\int_0^t \sin(2r)\sin(2t-2r)\,dr = \frac{1}{4}e^{-2t}\sin(2t) - \frac{1}{2}e^{-2t}t\cos(2t)$.

(l) $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$
If $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$, then $y = \int_0^t \frac{e^{3r}+e^{-3r}}{2}\sqrt{t-r}\,dr$.

(AB 81) Sketch the graph of $y = t^2 - t^2\,\mathcal{U}(t-1) + (2-t)\,\mathcal{U}(t-1)$.

(Answer 81)



(AB 82) Solve the following initial-value problems using the Laplace transform. Do you expect the graphs of y, $\frac{dy}{dt}$, or $\frac{d^2y}{dt^2}$ to show any corners or jump discontinuities? If so, at what values of t?

(a) $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$

If $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$, then $y(t) = -\frac{1}{30}\sin 3t - \frac{1}{90}\cos 3t + \frac{31}{30}e^{9t}$.

(b) $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$

If $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$, then $y = 3te^{2t} + 2e^{2t}$.

(c) $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$

If $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$, then $y = \frac{1}{5}t^3 - \frac{3}{25}t^2 + \frac{6}{125}t - \frac{6}{625} + \frac{1881}{625}e^{-5t}$.

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$, then $y(t) = e^{2t} - te^{2t}$.

(e) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$, then $y(t) = \frac{1}{5}(e^{-t} - e^t\cos t + 7e^t\sin t)$.

(f) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$

If $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$, then $y(t) = 4e^{3t} - 2e^{4t} - te^{3t}$.

(g) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$

If $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$, then $y(t) = -\frac{15}{8}e^{4t} + \frac{39}{8}e^{2t} - \frac{1}{4}te^{2t} - \frac{1}{2}t^2e^{2t} - t^3e^{2t}$.

(h) $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{40}e^{2t} - \frac{1}{72}e^{-2t} - \frac{1}{90}e^t\cos(3t) - \frac{1}{45}e^t\sin(3t)$.

(i) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$

If $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$, then $y(t) = \frac{1}{5}\cos 2t + \cos 3t + \frac{5}{3}\sin 3t$.

(j) $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \le t < 4, \\ 0, & 4 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 0$

If $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \le t < 4, \\ 0, & 4 \le t \end{cases}$, $y(0) = 2$, then

$$y(t) = \frac{2}{3} + \frac{4}{3}e^{-3t} - \frac{2}{3}\mathcal{U}(t-4) + \frac{2}{3}\mathcal{U}(t-4)e^{12-3t}.$$

The graph of y has a corner at $t = 4$. The graph of $\frac{dy}{dt}$ has a jump at $t = 4$.

(k) $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = (1/6)(1 - \mathcal{U}(t-2\pi))(2\sin t - \sin 2t).$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2\pi$.

(l) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - \mathcal{U}(t-10)\left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right].$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 10$. The graph of $\frac{d^2y}{dt^2}$ has a jump discontinuity at $t = 10$.

(m) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \le t < 2, \\ 4, & 2 \le t \end{cases}$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = 2$.

If $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \le t < 2, \\ 4, & 2 \le t \end{cases}$, $y(0) = 3$, $y'(0) = 2$, $y''(0) = 1$, then

$$y(t) = 3e^{-t} + 5te^{-t} + 4t^2e^{-t} + \mathcal{U}(t-2)(4 - 4e^{2-t} - 4te^{2-t} - 2(t-2)^2e^{2-t}).$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2$. The graph of $\frac{d^3y}{dt^3}$ has a jump discontinuity at $t = 2$.

(n) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$, then

$$y(t) = 2e^{-2t} + 5te^{-2t} + \frac{3}{4}u_2(t) - \frac{3}{4}e^{-2t+4}u_2(t) - \frac{3}{2}(t-2)e^{-2t+4}u_2(t).$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 2$. The graph of $\frac{d^2y}{dt^2}$ has a jump at $t = 2$.

(o) $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t r\sin 3r \sin(3t - 3r)\,dr$.

(p) $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t \sin 3r \sin(3t - 3r)\,dr = \frac{1}{6}\sin 3t - \frac{1}{2}t\cos 3t$.

(q) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$, then $y(t) = 7e^{-2t} - 5e^{-3t} + \int_0^t (e^{-2r} - e^{-3r})\sqrt{t - r + 1}\,dr$.

(r) $y(t) + \int_0^t r\,y(t-r)\,dr = t$.

If $y(t) + \int_0^t y(r)(t-r)\,dr = t$, then $y(t) = \sin t$.

(s) $y(t) = te^t + \int_0^t (t-r)\,y(r)\,dr$.

If $y(t) = te^t + \int_0^t (t-r)\,y(r)\,dr$, then $y(t) = -\frac{1}{8}e^{-t} + \frac{1}{8}e^t + \frac{3}{4}te^t + \frac{1}{4}t^2e^t$.

(t) $\frac{dy}{dt} = 1 - \sin t - \int_0^t y(r)\,dr$, $y(0) = 0$.

If $\frac{dy}{dt} = 1 - \sin t - \int_0^t y(r)\,dr$, $y(0) = 0$, then $y(t) = \sin t - \frac{1}{2}t\sin t$.

(u) $\frac{dy}{dt} + 2y + 10\int_0^t e^{4r}y(t-r)\,dr = 0$, $y(0) = 7$.

If $\frac{dy}{dt} + 2y + 10\int_0^t e^{4r}y(t-r)\,dr = 0$, $y(0) = 7$, then $y = 7e^t\cos t - 21e^t\sin t$.

(v) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$.

If $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$, then

$$y = 6e^{-t/3} - 6e^{-t/2} + 4\mathcal{U}(t-2) - 12\mathcal{U}(t-2)e^{-(t-2)/3} + 8\mathcal{U}(t-2)e^{-(t-2)/2}.$$

The graph of $y'(t)$ has a corner at $t = 2$, and the graph of $y''(t)$ has a jump at $t = 2$.

(w) $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$.

If $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$, then $y(t) = 3e^{-9t} + 7\mathcal{U}(t-2)e^{-9t+18}$. The graph of $y(t)$ has a jump at $t = 2$.

(x) $\frac{d^2y}{dt^2} + 4y = -2\delta(t - 4\pi)$, $y(0) = 1/2$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 4y = \delta(t - 4\pi)$, $y(0) = 1/2$, $y'(0) = 0$, then

$$y = \frac{1}{2}\cos(2t) - \mathcal{U}(t - 4\pi)\sin(2t).$$

The graph of $y(t)$ has a corner at $t = 4\pi$, and graph of $y'(t)$ has a jump at $t = 4\pi$.

(y) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + \mathcal{U}(t-2)$, $y(0) = 1$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + \mathcal{U}(t-2)$, $y(0) = 1$, $y'(0) = 0$, then

$$y = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} + \frac{1}{2}\mathcal{U}(t-1)e^{-t+1} - \frac{1}{2}\mathcal{U}(t-1)e^{-3t+3} + \frac{1}{3}\mathcal{U}(t-2) - \frac{1}{2}e^{-t+2}\mathcal{U}(t-2) + \frac{1}{6}\mathcal{U}(t-2)e^{-3t+6}.$$

The graph of $y(t)$ has a corner at $t=1$. The graph of $y'(t)$ has a corner at $t=2$, and a jump at $t=1$. $y''(t)$ has an impulse at $t=1$, and a jump at $t=2$.

(z) $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 5\delta(t-4)$, $y(0)=1$, $y'(0)=0$, $y''(0)=2$.
If $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - \frac{dy}{dt} + 2y = 5\delta(t-4)$, $y(0)=3$, $y'(0)=0$, $y''(0)=0$, then

$$y = \frac{3}{5}e^{2t} + \frac{4}{5}\cos t - \frac{2}{5}\sin t + \mathcal{U}(t-4)(e^{2t} - \cos t - 2\sin t)$$

The graph of $\frac{dy}{dt}$ has a corner at $t=4$, and the graph of $\frac{d^2y}{dt^2}$ has a jump at $t=4$.

(AB 83) At time $t=0$, a group of 7 birds, all 1 year old, are blown onto an isolated island. A bird that is t years old on average produces te^{-t} chicks per year. Let $B(t)$ be the birth rate at time t years. Find an integrodifferential equation (and, if necessary, an initial condition) for the birth rate.

(Answer 83) $B(t) = 7te^{-t} + \int_0^t B(t-r)\,re^{-r}\,dr$.

(AB 84) At time $t=0$, a group of 11 birds, all 2 years old, are blown onto an isolated island. A bird that is t years old on average produces te^{-t} chicks per year. Let $B(t)$ be the birth rate at time t years. Find an integrodifferential equation (and, if necessary, an initial condition) for the birth rate.

(Answer 84) $B(t) = 7te^{-t} + \int_0^t B(t-r)\,re^{-r}\,dr$.

(AB 85) Let I be the number of people with a rare infections disease. Suppose that 300 people simultaneously contract the disease at time $t=0$. Each infected person infects three new people per day on average. A person who was infected r days ago has a probability of $4r^2e^{-2r}$ of recovering on day r. Write an integrodifferential equation (and, if necessary, an initial condition) for the number of infected people.

(Answer 85)

$$\frac{dI}{dt} = 3I - \int_0^t 12r^2e^{-2r}\,I(t-r)\,dr - 1200t^2e^{-2t}, \qquad I(0)=300.$$

(AB 86) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y - 15, \qquad \frac{dy}{dt} = -5x + 6y - 8, \qquad x(0)=2, \quad y(0)=5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 86) If

$$\frac{dx}{dt} = -6x + 9y - 15, \qquad \frac{dy}{dt} = -5x + 6y - 8, \qquad x(0)=2, \quad y(0)=5$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 6\sin 3t \\ 4\sin 3t + 2\cos 3t \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

(AB 87) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y + t^8e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0)=0, \quad y(0)=0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 87) If

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 0, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} (38/45)t^{10} + (1/9)t^9 \\ -(29/45)t^{10} \end{pmatrix}.$$

(AB 88) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y + e^{2t} \cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 88) If

$$\frac{dx}{dt} = 5y + e^{2t} \cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 0, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} -t \sin t + (1/2)t \cos t + (1/2) \sin t \\ -(1/2)t \sin t \end{pmatrix}.$$

(AB 89) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 89) If

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 4 - 4\cos(e^t) \\ 4\cos(e^t) - 4 \end{pmatrix} + e^{-2t} \begin{pmatrix} 3\sin(e^t) - 3e^t \cos(e^t) - 3\sin 1 + 3\cos 1 \\ 4e^t \cos(e^t) - 4\sin(e^t) - 4\cos 1 + 4\sin 1 \end{pmatrix}.$$

(AB 90) Find the general solution to the equation $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3} \ln t$ on the interval $0 < t < \infty$.

(Answer 90) If $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3} \ln t$, then $y(t) = c_1 e^{t/3} + c_2\, t\, e^{t/3} + \frac{1}{2}t^2 e^{t/3} \ln t - \frac{3}{4}t^2 e^{t/3}$ for all $t > 0$.

(AB 91) Find the general solution to the equation $4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 8e^{-t/2} \arctan t$.

(Answer 91) If $4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 8e^{-t/2} \arctan t$, then $y(t) = c_1 e^{-t/2} + c_2\, t\, e^{-t/2} + e^{-t/2}\big(t^2 \arctan t + t - \arctan t - t \ln(1 + t^2)\big)$.

(AB 92) Find the general solution to the equation $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$.

(Answer 92) If $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$, then $y = c_1 e^{-t} + c_2 e^{-2t} - 2e^{-t} \sin(e^{t/2})$.

(AB 93) Solve the initial-value problem $9\frac{d^2y}{dt^2} + y = \sec^2(t/3)$, $y(0) = 4$, $y'(0) = 2$, on the interval $-3\pi/2 < t < 3\pi/2$.

(Answer 93) If $9\frac{d^2y}{dt^2} + y = \sec^2(t/3)$, $y(0) = 4$, $y'(0) = 2$, then

$$y(t) = 5\cos(t/3) + 6\sin(t/3) + (\sin(t/3))\ln(\tan(t/3) + \sec(t/3)) - 1$$

for all $-3\pi/2 < t < 3\pi/2$.

(AB 94) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - 2x = 0$, $t > 0$, is $x(t) = C_1t^2 + C_2t^{-1}$. Solve the initial-value problem $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(Answer 94) If $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$, then $y(t) = -4\sqrt{t} + 3t^2 + \frac{2}{t}$ for all $0 < t < \infty$.

(AB 95) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - t\frac{dx}{dt} - 3x = 0$, $t > 0$, is $x(t) = C_1t^3 + C_2t^{-1}$. Find the general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

(Answer 95) If $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$, then $y(t) = -\frac{3}{2}t^{-1}\ln t + C_1t^3 + C_2t^{-1}$ for all $t > 0$.

(AB 96) Find the general solution to the following differential equations.

(a) $6\frac{d^2y}{dt^2}+5\frac{dy}{dt}+y=3e^{4t}$.

The general solution to $6\frac{d^2y}{dt^2}+5\frac{dy}{dt}+y=0$ is $y_g=C_1e^{-t/2}+C_2e^{-t/3}$. To solve $6\frac{d^2y}{dt^2}+5\frac{dy}{dt}+y=2e^{4t}$ we make the guess $y_p=Ae^{4t}$. The solution is $y=C_1e^{-t/2}+C_2e^{-t/3}+\frac{1}{39}e^{4t}$.

(b) $16\frac{d^2y}{dt^2}-y=e^{t/4}\sin t$.

The general solution to $16\frac{d^2y}{dt^2}-y=0$ is $y_g=C_1e^{t/4}+C_2e^{-t/4}$. To solve $16\frac{d^2y}{dt^2}-y=e^{t/4}\sin t$ we make the guess $y_p=Ae^{t/4}\sin t+Be^{t/4}\cos t$. The solution is $y=C_1e^{t/4}+C_2e^{-t/4}-(1/20)e^{t/4}\sin t-(1/40)e^{t/4}\cos t$.

(c) $\frac{d^2y}{dt^2}+49y=3t\sin 7t$.

The general solution to $\frac{d^2y}{dt^2}+49y=0$ is $y_g=C_1\sin 7t+C_2\cos 7t$. To solve $\frac{d^2y}{dt^2}+49y=3t\sin 7t$ we make the guess $y_p=At^2\sin 7t+Bt\sin 7t+Ct^2\cos 7t+Dt\cos 7t$. The solution is $y=C_1\sin 7t+C_2\cos 7t-(3/28)t^2\cos 7t+(3/4)t\sin 7t$.

(d) $\frac{d^2y}{dt^2}-4\frac{dy}{dt}=4e^{3t}$.

The general solution to $\frac{d^2y}{dt^2}-4\frac{dy}{dt}=0$ is $y_g=C_1+C_2e^{4t}$. To solve $\frac{d^2y}{dt^2}-4\frac{dy}{dt}=4e^{3t}$ we make the guess $y_p=Ae^{3t}$. The solution is $y=C_1+C_2e^{4t}-(4/3)e^{3t}$.

(e) $\frac{d^2y}{dt^2}+12\frac{dy}{dt}+85y=t\sin(3t)$.

The general solution to $\frac{d^2y}{dt^2}+12\frac{dy}{dt}+85y=0$ is $y_g=C_1e^{-6t}\cos(7t)+C_2e^{-6t}\sin(7t)$. To solve $\frac{d^2y}{dt^2}+12\frac{dy}{dt}+85y=t\sin(3t)$ we make the guess $y_p=At\sin(3t)+Bt\cos(3t)+C\sin(3t)+D\cos(3t)$. The solution is

$$y=C_1e^{-6t}\cos(7t)+C_2e^{-6t}\sin(7t)+\frac{19}{1768}t\sin(3t)-\frac{9}{1768}t\cos(3t)-\frac{1353}{781456}\sin(3t)+\frac{606}{781456}\cos(3t).$$

(f) $\frac{d^2y}{dt^2}+3\frac{dy}{dt}-10y=7e^{-5t}$.

The general solution to $\frac{d^2y}{dt^2}+3\frac{dy}{dt}-10y=0$ is $y_g=C_1e^{2t}+C_2e^{-5t}$. To solve $\frac{d^2y}{dt^2}+3\frac{dy}{dt}-10y=7e^{-5t}$ we make the guess $y_p=Ate^{-5t}$. The solution is $y=C_1e^{2t}+C_2e^{-5t}-(1/7)te^{-5t}$.

(g) $16\frac{d^2y}{dt^2}-24\frac{dy}{dt}+9y=6t^2+\cos(2t)$.

The general solution to $16\frac{d^2y}{dt^2}-24\frac{dy}{dt}+9y=0$ is $y_g=C_1e^{(3/4)t}+C_2te^{(3/4)t}$. To solve $16\frac{d^2y}{dt^2}-24\frac{dy}{dt}+9y=6t^2+\cos(2t)$ we make the guess $y_p=At^2+Bt+C+D\cos(2t)+E\sin(2t)$. The solution is $y=C_1e^{(3/4)t}+C_2te^{(3/4)t}+(2/3)t^2+(32/9)t+64/9-(48/5329)\cos(2t)-(55/5329)\sin(2t)$.

(h) $\frac{d^2y}{dt^2}-6\frac{dy}{dt}+25y=t^2e^{3t}$.

The general solution to $\frac{d^2y}{dt^2}-6\frac{dy}{dt}+25y=0$ is $y_g=C_1e^{3t}\cos(4t)+C_2e^{3t}\sin(4t)$. To solve $\frac{d^2y}{dt^2}-6\frac{dy}{dt}+25y=t^2e^{3t}$ we make the guess $y_p=At^2e^{3t}+Bte^{3t}+Ce^{3t}$. The solution is $y=C_1e^{3t}\cos(4t)+C_2e^{3t}\sin(4t)+(1/16)t^2e^{3t}-(1/128)e^{3t}$.

(i) $\frac{d^2y}{dt^2}+10\frac{dy}{dt}+25y=3e^{-5t}$.

The general solution to $\frac{d^2y}{dt^2}+10\frac{dy}{dt}+25y=0$ is $y_g=C_1e^{-5t}+C_2te^{-5t}$. To solve $\frac{d^2y}{dt^2}+10\frac{dy}{dt}+25y=3e^{-5t}$ we make the guess $y_p=At^2e^{-5t}$. The solution is $y=C_1e^{-5t}+C_2te^{-5t}+(3/2)t^2e^{-5t}$.

(j) $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=3\cos(2t)$.

The general solution to $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=0$ is $y_g=C_1e^{-2t}+C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=3\cos(2t)$, we make the guess $y_p=A\cos(2t)+B\sin(2t)$. The solution is $y=C_1e^{-2t}+C_2e^{-3t}+\frac{15}{52}\sin(2t)+\frac{3}{52}\cos(2t)$.

(k) $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=5\sin(4t)$.

The general solution to $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=0$ is $y_g=C_1e^{-3t}+C_2te^{-3t}$. To solve $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=5\sin(4t)$, we make the guess $y_p=A\cos(4t)+B\sin(4t)$.

(l) $\frac{d^2y}{dt^2}+9y=5\sin(3t)$.

The general solution to $\frac{d^2y}{dt^2}+9y=0$ is $y_g=C_1\cos(3t)+C_2\sin(3t)$. To solve $\frac{d^2y}{dt^2}+9y=5\sin(3t)$, we make the guess $y_p=C_1t\cos(3t)+C_2t\sin(3t)$. The solution is $y=C_1\cos(3t)+C_2\sin(3t)-\frac{5}{6}t\cos(3t)$.

(m) $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=2t^2$.

The general solution to $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=0$ is $y_g=C_1e^{-t}+C_2te^{-t}$. To solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=2t^2$, we make the guess $y_p=At^2+Bt+C$. The solution is $y=C_1e^{-t}+C_2te^{-t}+2t^2-8t+12$.

(n) $\frac{d^2y}{dt^2}+2\frac{dy}{dt}=3t$.

The general solution to $\frac{d^2y}{dt^2}+2\frac{dy}{dt}=0$ is $y_g=C_1+C_2e^{-2t}$. To solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}=3t$, we make the guess $y_p=At^2+Bt$. The solution is $y=C_1+C_2e^{-2t}+\frac{3}{4}t^2-\frac{3}{4}t$.

(o) $\frac{d^2y}{dt^2}-7\frac{dy}{dt}+12y=5t+\cos(2t)$.

The general solution to $\frac{d^2y}{dt^2}-7\frac{dy}{dt}+12y=0$ is $y_g=C_1e^{3t}+C_2e^{4t}$. To solve $\frac{d^2y}{dt^2}-7\frac{dy}{dt}+12y=5t+\cos(2t)$, we make the guess $y_p=At+B+C\cos(2t)+D\sin(2t)$. The solution is $y=C_1e^{3t}+C_2e^{4t}+\frac{5}{12}t+\frac{35}{144}+\frac{16}{177}\cos 2t-\frac{28}{177}\sin 2t$.

(p) $\frac{d^2y}{dt^2}-9y=2e^t+e^{-t}+5t+2$.

The general solution to $\frac{d^2y}{dt^2}-9y=0$ is $y_g=C_1e^{3t}+C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2}-9y=2e^t+e^{-t}+5t+2$, we make the guess $y_p=Ae^t+Be^{-t}+Ct+D$.

(q) $\frac{d^2y}{dt^2}-4\frac{dy}{dt}+4y=3e^{2t}+5\cos t$.

The general solution to $\frac{d^2y}{dt^2}-4\frac{dy}{dt}+4y=0$ is $y_g=C_1e^{2t}+C_2te^{2t}$. To solve $\frac{d^2y}{dt^2}-4\frac{dy}{dt}+4y=3e^{2t}+5\cos t$, we make the guess $y_p=At^2e^{2t}+B\cos t+C\sin t$.

(AB 97) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $16\frac{d^2y}{dt^2}-y=3e^t$, $y(0)=1$, $y'(0)=0$.

If $16\frac{d^2y}{dt^2}-y=3e^t$, $y(0)=1$, $y'(0)=0$, then $y(t)=\frac{1}{5}e^t+\frac{4}{5}e^{-t/4}$.

(b) $\frac{d^2y}{dt^2}+49y=\sin 7t$, $y(\pi)=3$, $y'(\pi)=4$.

If $\frac{d^2y}{dt^2}+49y=\sin 7t$, $y(\pi)=3$, $y'(\pi)=4$, then $y(t)=-\frac{1}{14}t\cos(7t)+\frac{\pi-42}{14}\cos(7t)-\frac{55}{98}\sin(7t)$.

(c) $\frac{d^2y}{dt^2}+3\frac{dy}{dt}-10y=25t^2$, $y(2)=0$, $y'(2)=3$.

If $\frac{d^2y}{dt^2}+3\frac{dy}{dt}-10y=25t^2$, $y(2)=0$, $y'(2)=0$, then $y(t)=-\frac{5}{2}t^2-\frac{3}{2}t+\frac{19}{20}+\frac{11}{60}e^{2t-4}-\frac{356}{30}e^{-5t+10}$.

(d) $\frac{d^2y}{dt^2}-4\frac{dy}{dt}=4e^{3t}$, $y(0)=1$, $y'(0)=3$.

If $\frac{d^2y}{dt^2}-4\frac{dy}{dt}=4e^{3t}$, $y(0)=1$, $y'(0)=3$, then $y(t)=-\frac{4}{3}e^{3t}+\frac{7}{4}e^{4t}+\frac{7}{12}$.

(AB 98) A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 $\mathrm{N\cdot s/m}$ and is initially at rest at equilibrium. At time $t=0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(20t)$ N upwards.

Write the differential equation and initial conditions that describe the position of the object.

(Answer 98) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters).

Then

$$3\frac{d^2x}{dt^2}+27\frac{dx}{dt}+588x=3\cos(20t),\qquad u(0)=0,\quad u'(0)=0.$$

(AB 99) An object weighing 4 lbs stretches a spring 2 inches. It is initially at rest at equilibrium. At time $t=0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(\omega t)$ pounds, directed upwards. There is no damping.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of ω for which resonance occurs. Be sure to include units for ω.

(Answer 99) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then

$$\frac{4}{32}\frac{d^2x}{dt^2} + 24x = 3\cos(\omega t), \qquad x(0) = 0, \quad x'(0) = 0.$$

Resonance occurs when $\omega = 8\sqrt{3}$ s^{-1}.

(AB 100) A 4-kg object is suspended from a spring. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $7\sin(\omega t)$ newtons, directed upwards.

(a) Write the differential equation and initial conditions that describe the position of the object.
Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Let k denote the constant of the spring (in N·s/m). Then

$$4\frac{d^2x}{dt^2} + kx = 7\sin(\omega t), \qquad x(0) = 0, \quad x'(0) = 0.$$

(b) It is observed that resonance occurs when $\omega = 20$ rad/sec. What is the constant of the spring? Be sure to include units.
The spring constant is $k = 1600$ N·s/m.

(AB 101) A 1-kg object is suspended from a spring with constant 225 N/m. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $15\cos(\omega t)$ pounds, directed upwards. Illustrated are the object's position for three different values of ω. You are given that the three values are $\omega = 15$ radians/second, $\omega = 16$ radians/second, and $\omega = 17$ radians/second. Determine the value of ω that will produce each image.

(Answer 101) $\omega = 15$ radians/second in Picture B. $\omega = 16$ radians/second in Picture A. $\omega = 17$ radians/second in Picture C.