Math 2584, Spring 2018

Exam 3 will occur during our regularly scheduled class time on Friday, April 17. You are allowed a non-graphing calculator and a double-sided, 3 inch by 5 inch card of notes.

Please check your final exam schedule. If you have 3 or more final exams scheduled for Monday, May 4, and you would like to reschedule the final exam for this class, please let me know by email by the day of the third exam.

(AB 1) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 2) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 3) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).
Hint: $\det \begin{pmatrix} 4-m & 1 & -1 \\ 1 & 4-m & -1 \\ -1 & -1 & 4-m \end{pmatrix} = -(m-3)^2(m-6).$

(AB 4) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, \qquad x(0) = 3, \quad y(0) = 2, \quad z(0) = 1.$$

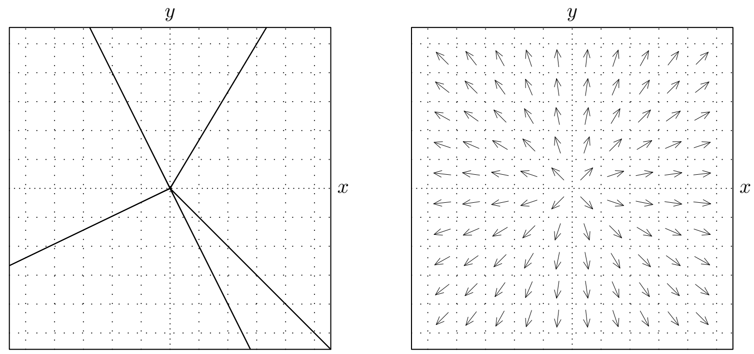
Express your final answer in terms of real functions (no complex numbers or complex exponentials).
Hint: $\det \begin{pmatrix} 3-m & 1 & 0 \\ 2 & 3-m & -2 \\ 0 & 1 & 3-m \end{pmatrix} = -(m-3)^3.$

(AB 5) Find the solution to the initial-value problem

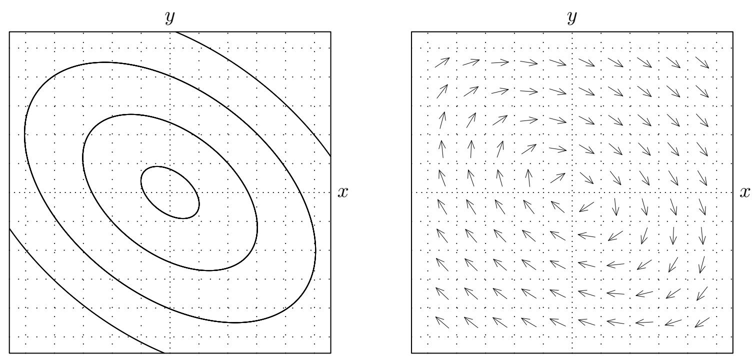
$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, x(0) = 4, \quad y(0) = 3, \quad z(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).
Hint: $\det \begin{pmatrix} 1-m & 1 & 1 \\ 1 & 3-m & -1 \\ 0 & 2 & 2-m \end{pmatrix} = -(m-2)^3.$

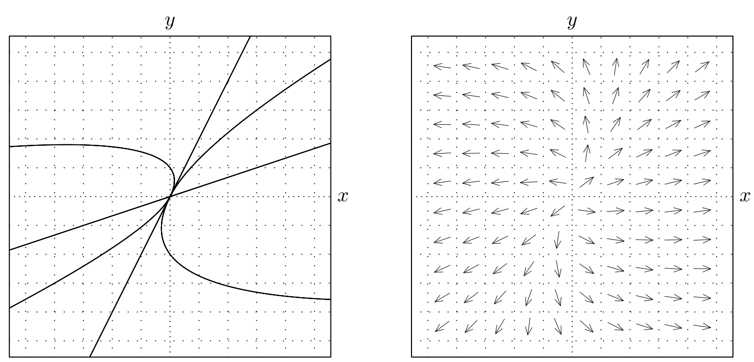
(AB 6) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



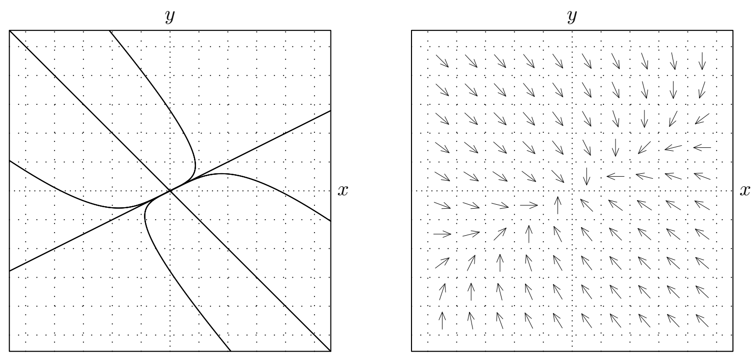
(AB 7) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



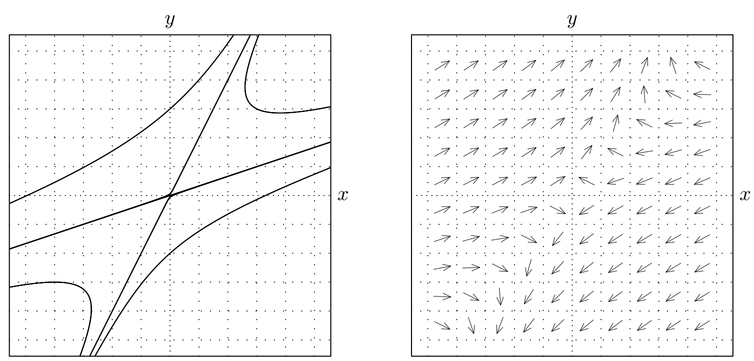
(AB 8) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



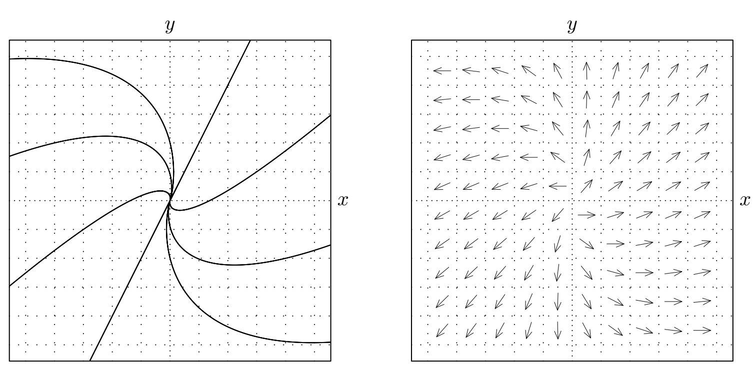
(AB 9) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



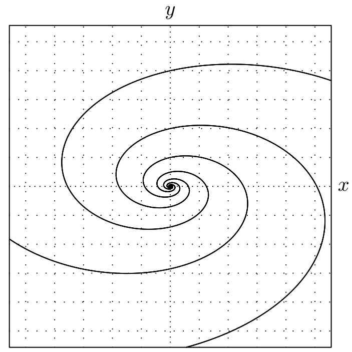
(AB 10) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?

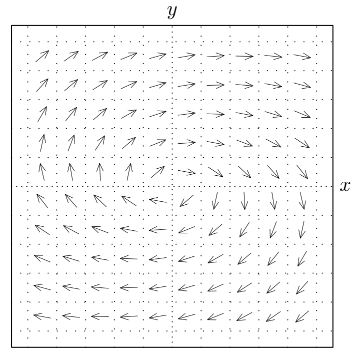


(AB 11) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



(AB 12) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?





(AB 13) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.
(b) Use the phase plane method to find an equation relating the number of susceptible and infected people.
(c) Use the phase plane method to find an equation relating the number of resistant and susceptible people.
(d) As $t \to \infty$, $I \to 0$ and the number of people who never contract the disease approaches a limit. Find an equation for S_∞, where S_∞ is the limiting number of people who never contract the disease.
(e) What is the maximum number of people that are infected at any one time?
(f) When is the peak infection time, that is, the time at which the number of infected people is at its maximum? You need not simplify your answer.

(AB 14) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

(a) Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(AB 15) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

(a) Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(b) Assume that there are initially no recovered or vaccinated people and 7 infected people. Use the phase plane method to find an equation relating S and I.

(c) What is the maximum number of people that are infected with the virus at any one time? (Round to the nearest whole number.)

(AB 16) Here is another way to model vaccination. Suppose that a disease is spreading through a town of 8000 people. Initially there are 10 infected people, 3000 vaccinated people, and no recovered people.

No other persons are vaccinated during the disease's spread. Thus, the population should be divided into *four* groups: people who are vaccinated, susceptible, infected, and recovered.

Each infected person encounters 10 people per day, who are distributed among susceptible, vaccinated, recovered and infected people according to their proportion of the total population. Each time an infected person encounters a susceptible person, there is a 4% chance that the susceptible person becomes infected. The vaccine is not perfect; each time an infected person encounters a vaccinated person, there is a 1% chance they become infected. (Once a vaccinated person becomes infected, they are identical to an infected person who was never vaccinated.) Each infected person has a 20% chance per day of recovering. A recovered person can never contract the disease again.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(b) Use the phase plane method to find an equation relating S and V.

(c) Use the phase plane method to find an equation relating I and V.

(d) What is the maximum number of people that are infected with the virus at any one time? (Round to the nearest whole number.)

(e) At what time does the peak number of infected people occur?

(AB 17) Consider the system of differential equations $\frac{dx}{dt} = 3x - 4y$, $\frac{dy}{dt} = 4x - 3y$. Use the phase plane method to find an equation relating x and y.

(AB 18) Consider the system of differential equations $\frac{dx}{dt} = 3y - 2xy$, $\frac{dy}{dt} = 4xy - 3y$. Use the phase plane method to find an equation relating x and y.

(AB 19) Consider the system of differential equations $\frac{dx}{dt} = 3x - 4xy$, $\frac{dy}{dt} = 5xy - 2y$. Use the phase plane method to find an equation relating x and y.

(AB 20) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ (**not** the table in your notes, on Blackboard, or in your book), find the Laplace transforms of the following functions.

(a) $f(t) = e^{-11t}$

(b) $f(t) = t$

(c) $f(t) = \begin{cases} 3e^t, & 0 < t < 4, \\ 0, & 4 \le t. \end{cases}$

(AB 21) Find the Laplace transforms of the following functions. You may use the table in your notes, on Blackboard, or in your book.

(a) $f(t) = t^4 + 5t^2 + 4$
(b) $f(t) = (t+2)^3$
(c) $f(t) = 9e^{4t+7}$
(d) $f(t) = -e^{3(t-2)}$
(e) $f(t) = (e^t + 1)^2$
(f) $f(t) = 8\sin(3t) - 4\cos(3t)$
(g) $f(t) = t^2 e^{5t}$
(h) $f(t) = 7e^{3t}\cos 4t$
(i) $f(t) = 4e^{-t}\sin 5t$
(j) $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$
(k) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$
(l) $f(t) = \begin{cases} 0, & t < 1, \\ t - 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$
(m) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$
(n) $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$
(o) $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi \end{cases}$
(p) $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi \end{cases}$

(AB 22) For each of the following problems, find y.

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$
(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$
(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$
(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$
(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$
(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$
(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$
(h) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$
(i) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$

(AB 23) Sketch the graph of $y = t^2 - t^2\,\mathcal{U}(t-1) + (2-t)\,\mathcal{U}(t-1)$.

(AB 24) Solve the following initial-value problems using the Laplace transform. Do you expect the graphs of y, $\frac{dy}{dt}$, or $\frac{d^2y}{dt^2}$ to show any corners or jump discontinuities? If so, at what values of t?

(a) $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$

(b) $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$

(c) $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

(e) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

(f) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$

(g) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$

(h) $\frac{d^2y}{dt^2} - 4y = e^t \sin(3t)$, $y(0) = 0$, $y'(0) = 0$

(i) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$

(j) $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \le t < 4, \\ 0, & 4 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 0$

(k) $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

(l) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

(m) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \le t < 2, \\ 4, & 2 \le t \end{cases}$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = 2$.

(n) $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t \end{cases}$, $y(0) = 1$, $y'(0) = 0$

(o) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$

(p) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$.

Answer key

(AB 1) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 1) If

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 4t - 5 \\ -2t + 3 \end{pmatrix}.$$

(AB 2) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 2) If

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 15t + 1 \\ (15/4)t + 4 \end{pmatrix}.$$

(AB 3) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).
Hint: $\det \begin{pmatrix} 4-m & 1 & -1 \\ 1 & 4-m & -1 \\ -1 & -1 & 4-m \end{pmatrix} = -(m-3)^2(m-6).$

(Answer 3) If

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{3t} \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} + e^{6t} \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}.$$

(AB 4) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, \qquad x(0) = 3, \quad y(0) = 2, \quad z(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 3-m & 1 & 0 \\ 2 & 3-m & -2 \\ 0 & 1 & 3-m \end{pmatrix} = -(m-3)^3.$

(Answer 4) If

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, \qquad x(0) = 3, \quad y(0) = 2, \quad z(0) = 1$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{3t} \begin{pmatrix} 2t^2 + 2t + 3 \\ 4t + 2 \\ 2t^2 + 2t + 1 \end{pmatrix}.$$

(AB 5) Find the solution to the initial-value problem

$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, x(0) = 4, \quad y(0) = 3, \quad z(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 1-m & 1 & 1 \\ 1 & 3-m & -1 \\ 0 & 2 & 2-m \end{pmatrix} = -(m-2)^3.$

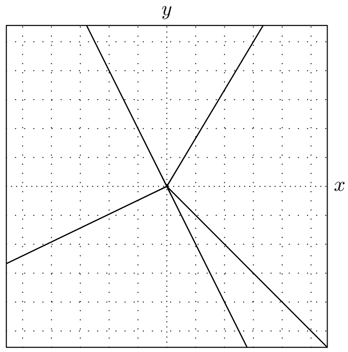
(Answer 5) If

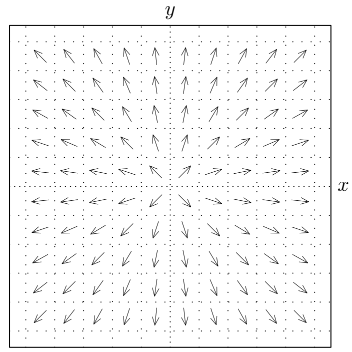
$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, \qquad x(0) = 4, \quad y(0) = 3, \quad z(0) = 5$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 4 + 4t + 2t^2 \\ 2t + 3 \\ 5 + 6t + 2t^2 \end{pmatrix}.$$

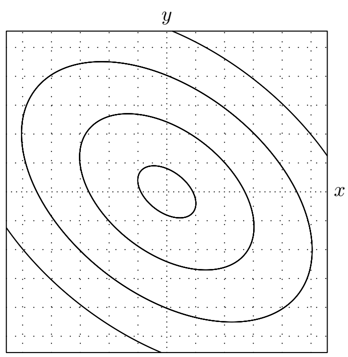
(AB 6) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?

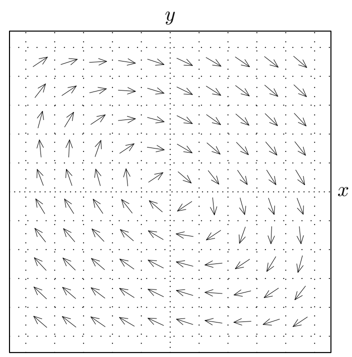




(Answer 6) This is a star. Every vector is an eigenvector. There is only one eigenvalue and it is positive. We must have that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ for some positive number r.

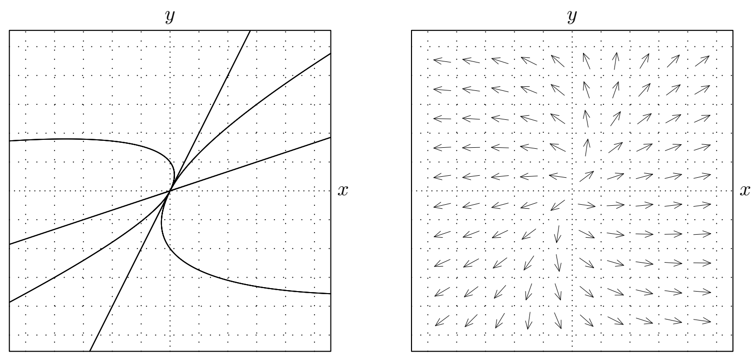
(AB 7) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?





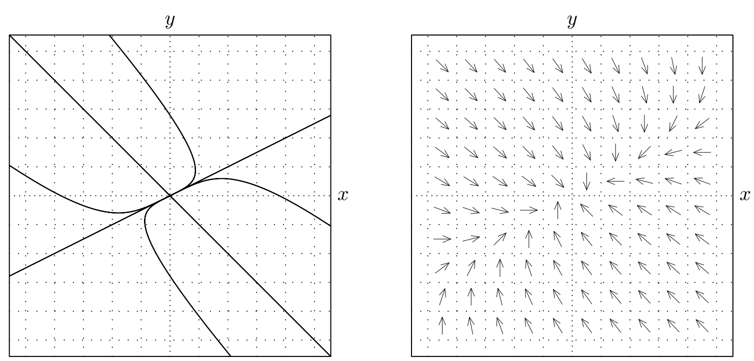
(Answer 7) This is a center. The eigenvalues are purely imaginary. The solutions consist of sines and cosines (no exponentials or powers of t).

(AB 8) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



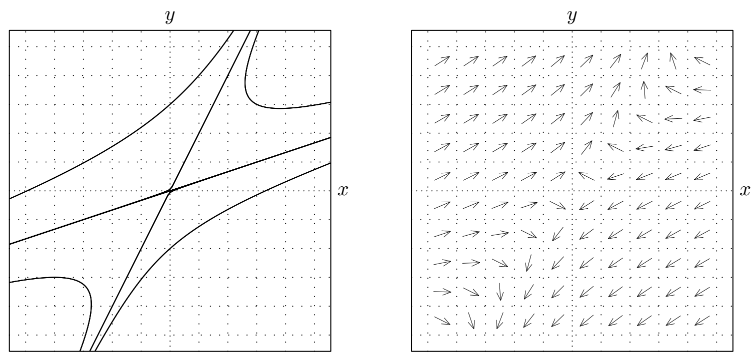
(Answer 8) This is a node. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two distinct real positive eigenvalues. The solutions take the form $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{st} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, where r and s are the eigenvalues and $0 < r < s$.

(AB 9) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



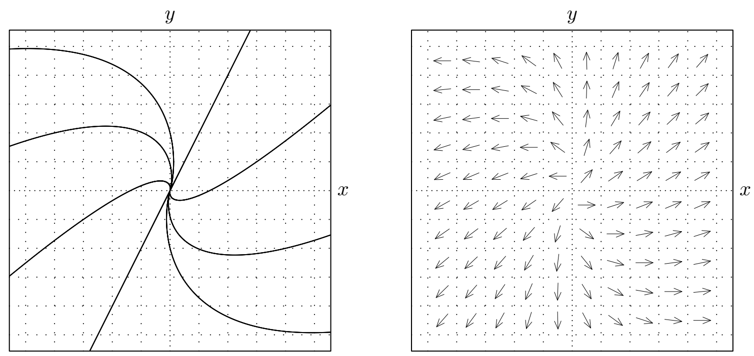
(Answer 9) This is a node. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two distinct real negative eigenvalues. The solutions take the form $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{st} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, where r and s are the eigenvalues and $s < r < 0$.

(AB 10) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



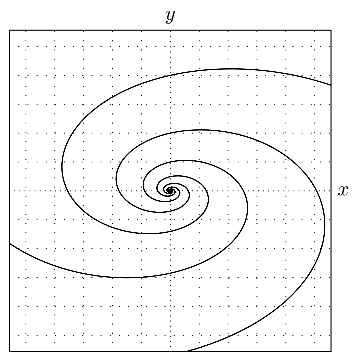
(Answer 10) This is a saddle point. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two real eigenvalues, one positive and one negative. The solutions take the form $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{st} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, where r and s are the eigenvalues and $s < 0 < r$.

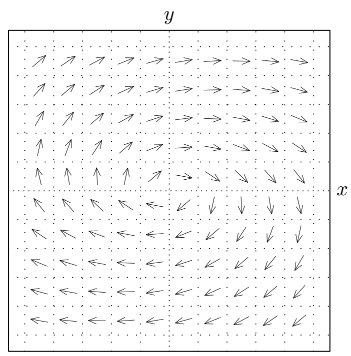
(AB 11) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?



(Answer 11) This is a degenerate node. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has only one eigenvalue and it is positive. If C is a constant then $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, where $r > 0$ is the eigenvalue. The general solution is $\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{rt} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{rt} \begin{pmatrix} t + A \\ 2t + B \end{pmatrix}$, where A and B are constants.

(AB 12) Here is a direction field and phase plane corresponding to the system $\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. What can you say about the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, or the solutions to the system?





(Answer 12) This is a spiral point. The eigenvalues are complex and take the form $\mu \pm \lambda i$, where $\mu > 0$ is real and λ is real. The solutions are exponentials multiplied by sines and cosines.

(AB 13) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.
Let t denote time (in days), let S denote the number of susceptible people (who have never had the disease), let I denote the number of infected people (who currently have the disease), and let R denote the number of recovered people (who now are resistant, that is, cannot get the disease again). Then $\boxed{\frac{dS}{dt} = -\frac{1}{22500}SI,\ \frac{dI}{dt} = \frac{1}{22500}SI - 0.17I,\ \frac{dR}{dt} = 0.17I,\ S(0) = 8997,\ I(0) = 3,\ R(0) = 0.}$

(b) Use the phase plane method to find an equation relating the number of susceptible and infected people.
$\frac{dI}{dS} = \frac{3825}{S} - 1$, so $\boxed{I = 9000 - S - 3825 \ln \frac{8997}{S}.}$

(c) Use the phase plane method to find an equation relating the number of resistant and susceptible people.
$\frac{dR}{dS} = -\frac{3825}{S}$, so $\boxed{R = 3825 \ln \frac{8997}{S}.}$

(d) As $t \to \infty$, $I \to 0$ and the number of people who never contract the disease approaches a limit. Find an equation for S_∞, where S_∞ is the limiting number of people who never contract the disease.
$\boxed{9000 - S_\infty - 3825 \ln \frac{8997}{S_\infty} = 0.}$ If you have access to `wolframalpha.com` or another numerical solver during the exam, you can solve this equation by typing `solve(9000-S-3825*ln(8997/S)=0,S)` and discover that $\boxed{S_\infty = 1158}$ (when rounded to the nearest whole number).

(e) What is the maximum number of people that are infected at any one time?
The maximum occurs when $S = 3825$, at which time $\boxed{I = 9000 - 3825 - 3825 \ln \frac{62979}{33750}.}$

(f) When is the peak infection time, that is, the time at which the number of infected people is at its maximum? You need not simplify your answer.

$$\boxed{\int_{3825}^{8997} \frac{22500}{S(9000 - S - 3825 \ln \frac{8997}{S})}\, dS.}$$

(AB 14) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

(a) Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)
$\frac{dS}{dt} = -\frac{0.2}{5000}SI - 15$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I + 15$.

(AB 15) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

(a) Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)
$\boxed{\frac{dS}{dt} = -\frac{0.2}{5000}SI - 15\frac{S}{5000},\ \frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I,\ \frac{dR}{dt} = 0.1I + 15\frac{S}{5000}.}$

(b) Assume that there are initially no recovered or vaccinated people and 7 infected people. Use the phase plane method to find an equation relating S and I.
$\frac{dI}{dS} = \frac{0.2S-500}{S}\frac{I}{-0.2I-15}$, so $\boxed{-0.2(I-7) - 15\ln(I/7) = 0.2(S-4993) - 500\ln(S/4993).}$

(c) What is the maximum number of people that are infected with the virus at any one time? (Round to the nearest whole number.)
The maximum I value occurs when $S = 2500$. The maximum I value then satisfies
$\boxed{-0.2(I-7) - 15\ln(I/7) = 0.2(2500-4993) - 500\ln(2500/4993).}$ Were you given this problem on the exam, you would be allowed to either stop there or to use `wolframalpha.com`, which would let you solve the equation and see that the maximum I value is $\boxed{I = 457}$.

(AB 16) Here is another way to model vaccination. Suppose that a disease is spreading through a town of 8000 people. Initially there are 10 infected people, 3000 vaccinated people, and no recovered people.

No other persons are vaccinated during the disease's spread. Thus, the population should be divided into *four* groups: people who are vaccinated, susceptible, infected, and recovered.

Each infected person encounters 10 people per day, who are distributed among susceptible, vaccinated, recovered and infected people according to their proportion of the total population. Each time an infected person encounters a susceptible person, there is a 4% chance that the susceptible person becomes infected. The vaccine is not perfect; each time an infected person encounters a vaccinated person, there is a 1% chance they become infected. (Once a vaccinated person becomes infected, they are identical to an infected person who was never vaccinated.) Each infected person has a 20% chance per day of recovering. A recovered person can never contract the disease again.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.
Let t denote time (in days). Let V denote the number of never-infected vaccinated people, S denote the number of never-infected susceptible people, I denote the number of infected people, and R denote the number of recovered, disease-resistant people. Then

$$\frac{dS}{dt} = -\frac{0.4}{8000}IS, \quad \frac{dV}{dt} = -\frac{0.1}{8000}IV, \quad \frac{dI}{dt} = \frac{0.4}{8000}IS + \frac{0.1}{8000}IV - 0.2I, \quad \frac{dR}{dt} = 0.2I$$

and

$$S(0) = 4990, \quad V(0) = 3000, \quad I(0) = 10, \quad R(0) = 0.$$

(b) Use the phase plane method to find an equation relating S and V.
$\frac{dS}{dV} = 4\frac{S}{V}$, so $\boxed{S = \frac{4990}{3000^4}V^4.}$

(c) Use the phase plane method to find an equation relating I and V.
$\frac{dI}{dV} = -\frac{4V^4 4990/3000^4 + V - 16000}{V}$, so $\boxed{I = -\frac{4990}{3000^4}V^4 + 2000 + V - 16000\ln(V/3000)}$.

(d) What is the maximum number of people that are infected with the virus at any one time? (Round to the nearest whole number.)
The maximum I value occurs when $4\frac{4990}{3000^4}V^4 + V - 16000 = 0$. According to `wolframalpha.com` this occurs when $V = 2710$. Thus, the maximum number of infected people is is $\boxed{3014.}$

(e) At what time does the peak number of infected people occur?
If t denotes the time at which 2710 people are vaccinated, then
$$\boxed{t = \int_{2710}^{3000} \frac{80000\,dV}{V\left(-\frac{4990}{3000^4}V^4 + 2000 + V - 16000\ln(V/3000)\right)}}.$$

(AB 17) Consider the system of differential equations $\frac{dx}{dt} = 3x - 4y$, $\frac{dy}{dt} = 4x - 3y$. Use the phase plane method to find an equation relating x and y.

(Answer 17) The trajectories of solutions to $\frac{dx}{dt} = 3x - 4y$, $\frac{dy}{dt} = 4x - 3y$ satisfy $4y^2 - 6xy + 4x^2 = C$ for constants C.

(AB 18) Consider the system of differential equations $\frac{dx}{dt} = 3y - 2xy$, $\frac{dy}{dt} = 4xy - 3y$. Use the phase plane method to find an equation relating x and y.

(Answer 18) If $\frac{dx}{dt} = 3y - 2xy$, $\frac{dy}{dt} = 4xy - 3y$, then $y = -2x - \frac{3}{2}\ln|x - 3/2| + C$.

(AB 19) Consider the system of differential equations $\frac{dx}{dt} = 3x - 4xy$, $\frac{dy}{dt} = 5xy - 2y$. Use the phase plane method to find an equation relating x and y.

(Answer 19) If $\frac{dx}{dt} = 3x - 4xy$, $\frac{dy}{dt} = 5xy - 2y$, then $3\ln|y| + 2\ln|x| - 4y - 5x = C$.

(AB 20) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\, dt$ (**not** the table in your notes, on Blackboard, or in your book), find the Laplace transforms of the following functions.

(a) $f(t) = e^{-11t}$

(b) $f(t) = t$

(c) $f(t) = \begin{cases} 3e^t, & 0 < t < 4, \\ 0, & 4 \leq t. \end{cases}$

(Answer 20)

(a) $\mathcal{L}\{t\} = \frac{1}{s^2}$.

(b) $\mathcal{L}\{e^{-11t}\} = \frac{1}{s+11}$.

(c) $\mathcal{L}\{f(t)\} = \frac{3-3e^{4-4s}}{s-1}$.

(AB 21) Find the Laplace transforms of the following functions. You may use the table in your notes, on Blackboard, or in your book.

(a) $f(t) = t^4 + 5t^2 + 4$

$\mathcal{L}\{t^4 + 5t^2 + 4\} = \frac{96}{s^5} + \frac{10}{s^3} + \frac{4}{s}$.

(b) $f(t) = (t+2)^3$

$\mathcal{L}\{(t+2)^3\} = \mathcal{L}\{t^3 + 6t^2 + 12t + 8\} = \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{8}{s}$.

(c) $f(t) = 9e^{4t+7}$

$\mathcal{L}\{9e^{4t+7}\} = \mathcal{L}\{9e^7 e^{4t}\} = \frac{9e^7}{s-4}$.

(d) $f(t) = -e^{3(t-2)}$

$\mathcal{L}\{-e^{3(t-2)}\} = \mathcal{L}\{-e^{-6}e^{3t}\} = -\frac{e^{-6}}{s-3}$.

(e) $f(t) = (e^t + 1)^2$

$\mathcal{L}\{(e^t+1)^2\} = \mathcal{L}\{e^{2t} + 2e^t + 1\} = \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s}$.

(f) $f(t) = 8\sin(3t) - 4\cos(3t)$

$\mathcal{L}\{8\sin(3t) - 4\cos(3t)\} = \frac{24}{s^2+9} - \frac{4s}{s^2+9}$.

(g) $f(t) = t^2 e^{5t}$

$\mathcal{L}\{t^2 e^{5t}\} = \frac{2}{(s-5)^3}$.

(h) $f(t) = 7e^{3t}\cos 4t$

$\mathcal{L}\{7e^{3t}\cos 4t\} = \frac{7s-21}{(s-3)^2+16}$.

(i) $f(t) = 4e^{-t}\sin 5t$

$\mathcal{L}\{4e^{-t}\sin 5t\} = \frac{20}{(s+1)^2+25}$.

(j) $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$

If $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-3s}e^3}{s-1}$.

(k) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$

If $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$ then $\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{1}{s} + \frac{2}{s^3}\right)$.

(l) $f(t) = \begin{cases} 0, & t < 1, \\ t-2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$

If $f(t) = \begin{cases} 0, & t < 1, \\ t-2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2}$.

(m) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

If $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{5}{s-2} - \frac{5e^6 e^{-3s}}{s-2}$.

(n) $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

If $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{14}{(s+1)^3} - \frac{14e^{-3s-3}}{(s+1)^3} - \frac{42e^{-3s-3}}{(s+1)^2} - \frac{63e^{-3s-3}}{s+1}$.

(o) $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi \end{cases}$

If $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{s}{s^2+9} + e^{-\pi s}\frac{s}{s^2+9} - e^{-\pi s}\frac{3}{s^2+9}$.

(p) $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi \end{cases}$

If $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s/2}\frac{1}{s^2+1} + e^{-\pi s}\frac{s}{s^2+1}$.

(AB 22) For each of the following problems, find y.

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$

If $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$, then $y = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$.

(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$

If $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$, then $y = \frac{5}{2}t^2 - \frac{7}{6}t^3$.

(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$

If $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$, then $y = \frac{1}{2}t^2e^{-t} + \frac{1}{6}t^3e^{-t}$.

(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$

If $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$, then $y = (1/4)e^{2t} + (7/4)e^{-2t}$.

(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$

If $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$, then $y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{1}{9}t\,e^{3t}$.

(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$

If $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$, then $y = \frac{1}{2} - \frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t)$.

(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$

If $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$, then $y = \frac{1}{8}\cos t - \frac{1}{8}\cos 3t$.

(h) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$

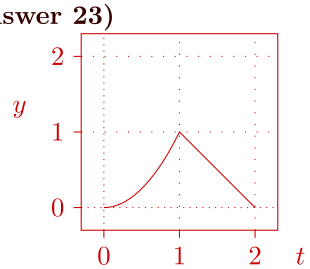
If $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$, then $y = 2\,\mathcal{U}(t-2)\,e^{t-2}\,\cos(t-2) + \mathcal{U}(t-2)\,e^{t-2}\,\sin(t-2)$.

(i) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$

If $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$, then $y = \frac{1}{2}\,\mathcal{U}(t-1)\,e^{3(t-1)} + \frac{1}{2}\,\mathcal{U}(t-1)\,e^{t-1}$.

(AB 23) Sketch the graph of $y = t^2 - t^2\,\mathcal{U}(t-1) + (2-t)\,\mathcal{U}(t-1)$.

(Answer 23)



(AB 24) Solve the following initial-value problems using the Laplace transform. Do you expect the graphs of y, $\frac{dy}{dt}$, or $\frac{d^2y}{dt^2}$ to show any corners or jump discontinuities? If so, at what values of t?

(a) $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$

If $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$, then $y(t) = -\frac{1}{30}\sin 3t - \frac{1}{90}\cos 3t + \frac{31}{30}e^{9t}$.

(b) $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$

If $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$, then $y = 3te^{2t} + 2e^{2t}$.

(c) $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$

If $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$, then $y = \frac{1}{5}t^3 - \frac{3}{25}t^2 + \frac{6}{125}t - \frac{6}{625} + \frac{1881}{625}e^{-5t}$.

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$, then $y(t) = e^{2t} - te^{2t}$.

(e) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$, then $y(t) = \frac{1}{5}(e^{-t} - e^t\cos t + 7e^t \sin t)$.

(f) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$

If $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$, then $y(t) = 4e^{3t} - 2e^{4t} - te^{3t}$.

(g) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$

If $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$, then $y(t) = -\frac{15}{8}e^{4t} + \frac{39}{8}e^{2t} - \frac{1}{4}te^{2t} - \frac{1}{2}t^2e^{2t} - t^3e^{2t}$.

(h) $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{40}e^{2t} - \frac{1}{72}e^{-2t} - \frac{1}{90}e^t\cos(3t) - \frac{1}{45}e^t\sin(3t)$.

(i) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$

If $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$, then $y(t) = \frac{1}{5}\cos 2t + \cos 3t + \frac{5}{3}\sin 3t$.

(j) $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \le t < 4, \\ 0, & 4 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 0$

If $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \le t < 4, \\ 0, & 4 \le t \end{cases}$, $y(0) = 2$, then

$$y(t) = \frac{2}{3} + \frac{4}{3}e^{-3t} - \frac{2}{3}\mathcal{U}(t-4) + \frac{2}{3}\mathcal{U}(t-4)e^{12-3t}.$$

The graph of y has a corner at $t = 4$. The graph of $\frac{dy}{dt}$ has a jump at $t = 4$.

(k) $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = (1/6)(1 - \mathcal{U}(t - 2\pi))(2\sin t - \sin 2t).$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2\pi$.

(l) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - \mathcal{U}(t-10)\left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right].$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 10$. The graph of $\frac{d^2y}{dt^2}$ has a jump discontinuity at $t = 10$.

(m) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \le t < 2, \\ 4, & 2 \le t \end{cases}$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = 2$.

If $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \le t < 2, \\ 4, & 2 \le t \end{cases}$, $y(0) = 3$, $y'(0) = 2$, $y''(0) = 1$, then

$$y(t) = 3e^{-t} + 5te^{-t} + 4t^2e^{-t} + \mathcal{U}(t-2)(4 - 4e^{2-t} - 4te^{2-t} - 2(t-2)^2e^{2-t}).$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2$. The graph of $\frac{d^3y}{dt^3}$ has a jump discontinuity at $t = 2$.

(n) $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t \end{cases}$, $y(0) = 1$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t \end{cases}$, $y(0) = 1$, $y'(0) = 0$, then

$$\begin{aligned} y(t) = {} & e^{-t/2}\cos t + \frac{1}{2}e^{-t/2}\sin t \\ & + \mathcal{U}(t-\pi)\left(-\frac{16}{17}\cos(t-\pi) + \frac{4}{17}\sin(t-\pi)\right) \\ & + \mathcal{U}(t-\pi)\left(\frac{4}{17}e^{-(t-\pi)/2}\sin(t-\pi) + \frac{16}{17}e^{-(t-\pi)/2}\cos(t-\pi)\right). \end{aligned}$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = \pi$.

(o) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$, then

$$y(t) = 2e^{-2t} + 5te^{-2t} + \frac{3}{4}u_2(t) - \frac{3}{4}e^{-2t+4}u_2(t) - \frac{3}{2}(t-2)e^{-2t+4}u_2(t).$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 2$. The graph of $\frac{d^2y}{dt^2}$ has a jump at $t = 2$.

(p) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$.

If $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$, then

$$y = 6e^{-t/3} - 6e^{-t/2} + 4\mathcal{U}(t-2) - 12\mathcal{U}(t-2)e^{-(t-2)/3} + 8\mathcal{U}(t-2)e^{-(t-2)/2}.$$

The graph of $y'(t)$ has a corner at $t = 2$, and the graph of $y''(t)$ has a jump at $t = 2$.