Math 2584, Spring 2018

Exam 2 will occur during our regularly scheduled class time on Friday, March 13. You are allowed a non-graphing calculator and a double-sided, 3 inch by 5 inch card of notes.

(AB 1) The function $y_1(t) = e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2} - (1+2t)\frac{dy}{dt} + (t+1)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

(AB 2) The function $y_1(t) = t$ is a solution to the differential equation $t^2\frac{d^2y}{dt^2} - (t^2+2t)\frac{dy}{dt} + (t+2)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

(AB 3) The function $y_1(x) = x^3$ is a solution to the differential equation $x^2\frac{d^2y}{dx^2} + (x^2\tan x - 6x)\frac{dy}{dx} + (12 - 3\,x\tan x)y = 0$. Find the general solution to this differential equation on the interval $0 < x < \pi/2$.

(AB 4)

(a) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0, \quad y(0) = 2, \quad y'(0) = 7$$

is guaranteed a unique solution?

(b) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0, \quad y(2) = 2, \quad y'(2) = 7$$

is guaranteed a unique solution?

(c) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0, \quad y(5) = 2, \quad y'(5) = 7$$

is guaranteed a unique solution?

(d) You are given that $y_1 = t$ is a solution to the differential equation $(t-3)(2t-3)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0$. Use the method of reduction of order to find the general solution.

(e) On what intervals are all of the the solutions you found in part (d) continuous?

(f) How many solutions are there to the initial value problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0, \quad y(3/2) = 2, \quad y'(3/2) = 7?$$

(AB 5) You are given that $y_1 = t - 3$ and $y_2 = \frac{1}{t-3}$ are both solutions to the differential equation $(t-3)^2\frac{d^2y}{dt^2} + (t-3)\frac{dy}{dt} + y = 0$ for all t.

(a) How many solutions are there to the initial value problem

$$(t-3)^2\frac{d^2y}{dt^2} + (t-3)\frac{dy}{dt} + y = 0, \quad y(3) = 1, \quad y'(3) = 0?$$

(b) How many solutions are there to the initial value problem

$$(t-3)^2\frac{d^2y}{dt^2} + (t-3)\frac{dy}{dt} + y = 0, \quad y(3) = 0, \quad y'(3) = 1?$$

(AB 6) You are given that $y_1 = t^3 + 4t^2 + 4t$ and $y_2 = t^2 + 4t + 4$ are both solutions to the differential equation $(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$ for all t.

(a) What are the maximal intervals on which the general solution to $(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$ may be written $y = C_1(t^3 + 4t^2 + 4t) + C_2(t^2 + 4t + 4)$?

(b) Find a function $y_3(t)$ that is continuous and has continuous first and second derivatives for all t, that satisfies $(t+2)^2 \frac{d^2y_3}{dt^2} - (4t+8)\frac{dy_3}{dt} + 6y_3 = 0$ for all t, but such that there do not exist constants C_1 and C_2 such that $y_3(t) = C_1 y_1(t) + C_2 y_2(t) = C_1(t^3 + 4t^2 + 4t) + C_2(t^2 + 4t + 4)$ for all t.

(c) Find numbers a, b, and c such that the initial value problem

$$(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0, \quad y(a) = b, \quad y'(a) = c$$

has no solutions.

(d) Find numbers a, b, and c such that the initial value problem

$$(t+2)^2 \frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0, \quad y(a) = b, \quad y'(a) = c$$

has at least two solutions.

(AB 7) Find the general solution to the following differential equations.

(a) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$.

(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.

(c) $\frac{d^4y}{dt^4} + 7\frac{d^2y}{dt^2} - 144y = 0$.

(d) $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0$.

(AB 8) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 2y = 0$, $y(0) = 3$, $y'(0) = 2$.

(b) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$, $y(0) = 1$, $y'(0) = 4$.

(AB 9) An object weighing 5 lbs stretches a spring 4 inches. It is attached to a viscous damper with damping constant 16 lb · sec/ft. The object is pulled down an additional 2 inches and is released with initial velocity 3 ft/sec upwards.

Write the differential equation and initial conditions that describe the position of the object.

(AB 10) A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant β. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20$ N · s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30$ N · s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(AB 11) A 3-kg object is attached to a spring with constant k and to a viscous damper with damping constant 42 N·sec/m. The object is set in motion from its equilibrium position with initial velocity 5 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object.

If $k = 100$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $k = 200$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(AB 12) A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant β. The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

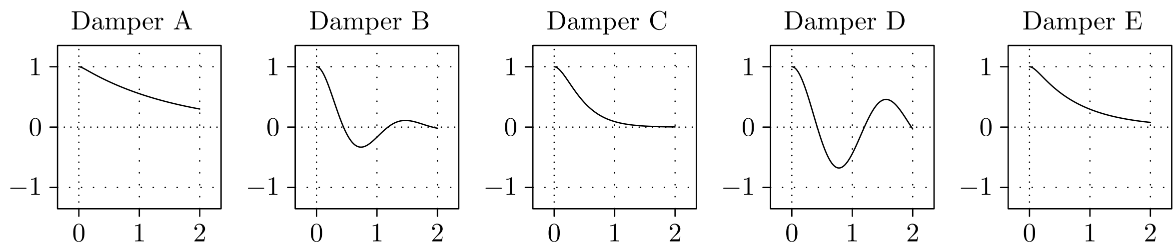
Write the differential equation and initial conditions that describe the position of the object. Then find the value of β for which the system is critically damped. Be sure to include units for β.

(AB 13) An object of mass m is attached to a spring with constant 80 N/m and to a viscous damper with damping constant 20 N·s/m. The object is pulled down to 5cm below its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of m for which the system is critically damped. Be sure to include units for m.

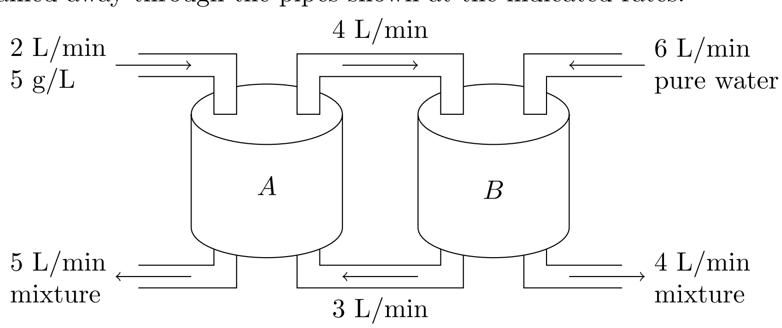
(AB 14) Five objects, each with mass 3 kg, are attached to five springs, each with constant 48 N/m. Five dampers with unknown constants are attached to the objects. In each case, the object is pulled down to a distance 1 cm below the equilibrium position and released from rest. You are given that the system is critically damped in exactly one of the five cases.

Here are the graphs of the objects' positions with respect to time:



(*a*) For which damper is the system critically damped?
(*b*) For which dampers is the system overdamped?
(*c*) For which dampers is the system underdamped?
(*d*) Which damper has the highest damping constant? Which damper has the lowest damping constant?

(AB 15) Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



Write the differential equations and initial conditions that describe the amount of salt in each tank.

(AB 16) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 17) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 18) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 19) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 20) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 21) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 22) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Answer key

(Answer 1) $y(t) = C_1\, e^t + C_2\, t^2 e^t$.

(Answer 2) $y(t) = C_1\, t + C_2\, te^t$.

(Answer 3) $y(x) = C_1\, x^3 + C_2\, x^3 \sin x$.

(Answer 4)

(a) $(-\infty, 3/2)$.

(b) $(3/2, 3)$.

(c) $(3, \infty)$.

(d) $y(t) = C_1\, t + \frac{C_2}{t-3}$.

(e) On the intervals $(-\infty, 3)$ and $(3, \infty)$.

(f) This initial value problem has no solutions.

(Answer 5)

(a) There are no solutions to this problem.

(b) There is one solution (a unique solution). Any solution must satisfy $y = C_1(t-3) + \frac{C_2}{t-3}$ for all $t > 3$ and $y = C_3(t-3) + \frac{C_4}{t-3}$ for all $t < 3$. If $y'(3)$ exists, then y must be continuous at $t = 3$, and so $C_2 = C_4$. Thus, $y = \begin{cases} C_3(t-3), & t \leq 3, \\ C_1(t-3), & t \geq 3. \end{cases}$ In order for $y'(3)$ to exist, we must have $C_1 = C_3$ and so $y(t) = C_1(t-3)$ for all t; if $y'(3) = 1$ then $y = t - 3$.

(Answer 6)

(a) $(-\infty, -2)$ and $(-2, \infty)$.

(b) There are many possible answers. They include:

$$y(t) = \begin{cases} t^3 + 4t^2 + 4t, & t \leq 2, \\ -2t^2 - 8t - 8, & t \geq 2, \end{cases}$$

$$y(t) = \begin{cases} -2t^2 - 8t - 8, & t \leq 2, \\ t^3 + 4t^2 + 4t, & t \geq 2, \end{cases}$$

$$y(t) = \begin{cases} t^3 + 6t^2 + 12t + 8, & t \leq 2, \\ 0, & t \geq 2. \end{cases}$$

(c) We must choose $a = -2$, as $(t+2)^2$, $-(4t+8)$, and $+6$ are continuous for all t and $(t+2)^2 = 0$ only for $t = -2$. There is no solution if $c \neq 0$ or if $b \neq 0$.

(d) As before, $a = -2$. If we choose $b = c = 0$, then $y(t) = 0$, $y(t) = t^3 + 4t^2 + 4t$ and $y(t) = t^2 + 4t + 4$ are all solutions.

(Answer 7)

(a) If $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$, then $y = C_1 e^{-6t}\cos(7t) + C_2 e^{-6t}\sin(7t)$.

(b) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$, then $y = C_1 e^{(-2+\sqrt{2})t} + C_2 e^{(-2-\sqrt{2})t}$.

(c) If $\frac{d^4y}{dt^4} + 7\frac{d^2y}{dt^2} - 144y = 0$, then $y = C_1 e^{3t} + C_2 e^{-3t} + C_3 \cos 4t + C_4 \sin 4t$.

(d) If $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0$, then $y = C_1 e^{2t} + C_2 te^{2t} + C_3 e^{-2t} + C_4 te^{-2t}$.

(Answer 8)

(a) If $9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 2y = 0$, $y(0) = 3$, $y'(0) = 2$, then $y = 3e^{-t/3}\cos(t/3) + 9e^{-t/3}\sin(t/3)$.

(b) If $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$, $y(0) = 1$, $y'(0) = 4$, then $y = e^{-5t} + 9te^{-5t}$.

(Answer 9) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then

$$\frac{5}{32}\frac{d^2x}{dt^2} + 16\frac{dx}{dt} + 15x = 0, \qquad x(0) = -\frac{1}{6}, \quad x'(0) = 3.$$

(Answer 10) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$2\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.1, \quad x'(0) = 0.$$

If $\beta = 20$ N·s/m, then the system is underdamped, and we do expect to see decaying oscillations.
If $\beta = 30$ N·s/m, then the system overdamped, and we do not expect to see decaying oscillations.

(Answer 11) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$3\frac{d^2x}{dt^2} + 42\frac{dx}{dt} + kx = 0, \qquad x(0) = 0, \quad x'(0) = -5.$$

If $k = 100$ N/m, then the system is overdamped, and we do not expect to see decaying oscillations.
If $k = 200$ N/m, then the system underdamped, and we do expect to see decaying oscillations.

(Answer 12) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$4\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 70x = 0, \qquad x(0) = 0.05, \quad x'(0) = -3.$$

Critical damping occurs when $\beta = 4\sqrt{70}$ N·s/m.

(Answer 13) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$m\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.05, \quad x'(0) = -3.$$

Critical damping occurs when $m = \frac{5}{4}$ kg.

(Answer 14)
(a) The system is critically damped for Damper C.
(b) The system overdamped for Dampers E and A.
(c) The system underdamped for Dampers B and D.
(d) Damper A has the highest damping constant. Damper D has the lowest damping constant.

(Answer 15) Let t denote time (in minutes).
Let x denote the amount of salt (in grams) in tank A.
Let y denote the amount of salt (in grams) in tank B.
Then $x(0) = 3000$ and $y(0) = 2000$.
If $t < 50$, then

$$\frac{dx}{dt} = -\frac{9x}{200-4t} + \frac{3y}{300+3t}, \qquad \frac{dy}{dt} = \frac{4x}{200-4t} - \frac{7y}{300+3t}.$$

(Answer 16) If

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 4t - 5 \\ -2t + 3 \end{pmatrix}.$$

(Answer 17) If

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} -3\cos t + 16\sin t \\ 2\cos t + 7\sin t \end{pmatrix}.$$

(Answer 18) If

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-20t} \begin{pmatrix} 3/20 \\ 9/10 \end{pmatrix} + e^{10t} \begin{pmatrix} -3/20 \\ 1/10 \end{pmatrix}.$$

(Answer 19) If

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 5\cos 12t + \sin 12t \\ 2\cos 12t + 2\sin 12t \end{pmatrix}.$$

(Answer 20) If

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos 3t - 11\sin 3t \\ -3\cos 3t - (23/3)\sin 3t \end{pmatrix}.$$

(Answer 21) If

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 15t + 1 \\ (15/4)t + 4 \end{pmatrix}.$$

(Answer 22) If

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} + e^{-6t} \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix}.$$