Graded and supplementary homework
Math 3083, Sections 2 and 3, Spring 2019

(AB 1) Solve the system

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 = 20, \\ -5x_1 + 1x_2 + 2x_3 = -7, \\ 3x_1 + 4x_2 - 5x_3 = -24. \end{cases}$$

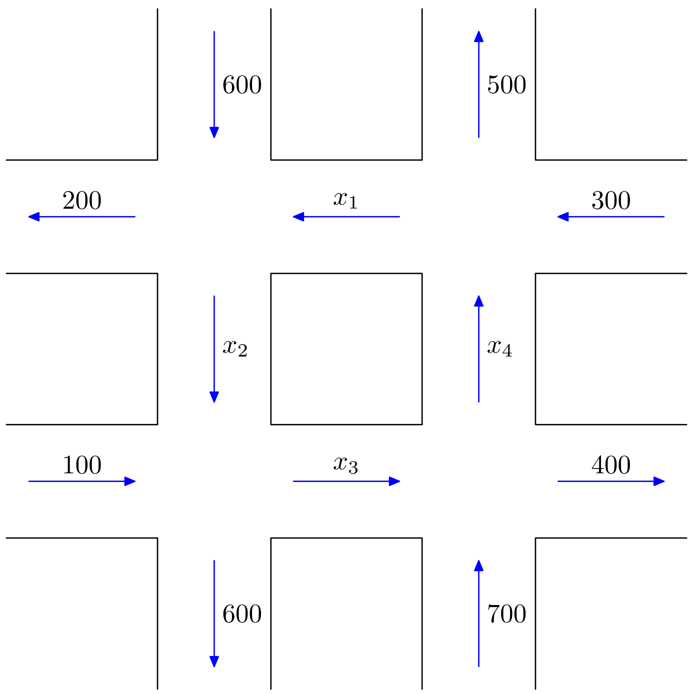
(AB 2) Write the augmented matrix for the system

$$\begin{cases} 3x_1 + 2x_2 - 5x_3 = 7, \\ 6x_1 + 4x_2 - 10x_3 = 14, \\ 2x_1 + x_2 - x_3 = 4. \end{cases}$$

Use Gauss-Jordan reduction to find a row equivalent matrix in reduced row echelon form. Indicate whether the system is consistent. If it is, find the solution set.

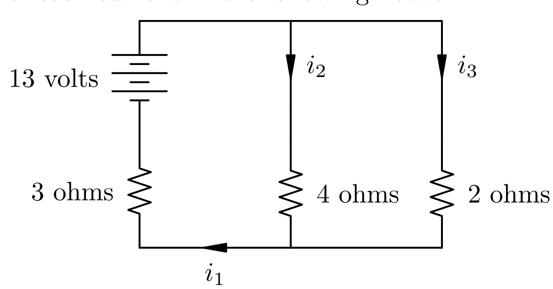
(AB 3) Do **one** of Parts (a), (b), or (c).

(a) Determine all possible values of x_1, x_2, x_3, and x_4 in the following traffic flow diagram.



(b) Ammonia reacts with oxygen to form nitric oxide and water. The chemical equation for this reaction is of the form $x_1\mathrm{NH}_3 + x_2\mathrm{O}_2 \to x_3\mathrm{NO} + x_4\mathrm{H}_2\mathrm{O}$. Determine values of x_1, x_2, x_3 and x_4 to balance the equation.

(c) Determine the amount of each current in the following network.



(AB 4)

(a) Find nonzero 2×2 matrices A and B such that $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(b) Find nonzero 2×2 matrices A, B and C such that $AB = CB$ and $A \neq C$. *Hint*: What is $AB - CB$?

(AB 5) Let $A = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$.

(a) Find A^{-1}.

(b) Find A^T.

(c) Find $(A^T)^{-1}$.

(AB 6) Find the inverses to the following elementary matrices.

(a) $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(b) $B = \begin{pmatrix} -5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(c) $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(AB 7)

(a) Solve $\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 1 & -3 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ by using Gauss-Jordan reduction to put the augmented matrix in reduced row echelon form.

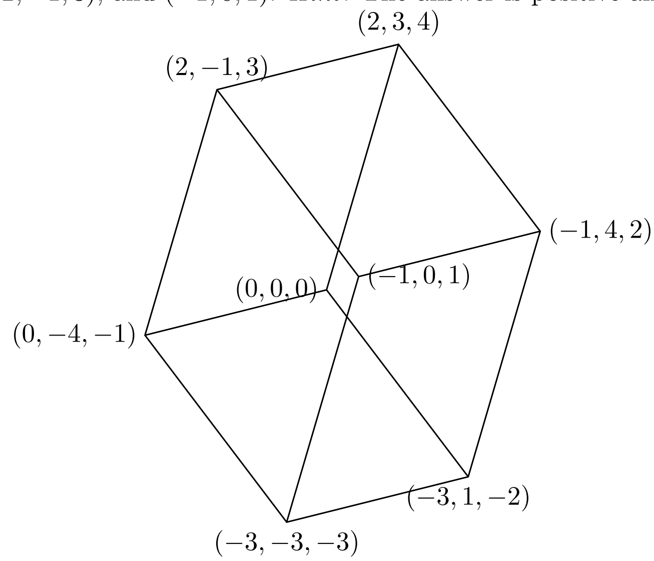
(b) Is $\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 1 & -3 \end{pmatrix}$ invertible? Why or why not? Do not use the determinant to answer this question. *Hint*: Try to answer without trying to find the inverse.

(AB 8)

(a) Find three elementary matrices E_1, E_2 and E_3, all lower triangular and of type III, such that $E_1E_2E_3 \begin{pmatrix} 3 & 1 & 4 \\ 9 & 1 & 7 \\ -3 & -5 & -3 \end{pmatrix}$ is an upper triangular matrix.

(b) Find a LU factorization of $M = \begin{pmatrix} 3 & 1 & 4 \\ 9 & 1 & 7 \\ -3 & -5 & -3 \end{pmatrix}$. That is, find an upper triangular matrix U and a lower triangular matrix L, where the diagonal entries of L are all 1s, such that $LU = M$.

(AB 9) Find the volume of the parallelepiped with vertices at $(0,0,0)$, $(-3,1,-2)$, $(2,3,4)$, $(0,-4,-1)$, $(-3,-3,-3)$, $(-1,4,2)$, $(2,-1,3)$, and $(-1,0,1)$. *Hint*: The answer is positive and is less than 30.



(AB 10) Find the determinant of the matrix $\begin{pmatrix} 1 & 2 & -1 & 2 \\ 2 & 4 & -2 & 3 \\ 0 & 3 & -1 & 2 \\ 5 & 7 & -3 & 0 \end{pmatrix}$ using the elimination method.

(AB 11) Let V be the set of all 2×2 matrices and let U be the set of all singular 2×2 matrices. Show that U is not a subspace of V by either:

(a) Showing that the zero matrix O is not in U,
(b) Giving an example of a scalar α and a matrix A in U such that αA is not in U, or
(c) Giving an example of two matrices A and B, both in U, such that $A + B$ is not in U.

(AB 12) Find Span $\left(\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix} \right)$.

(AB 13) Let $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ -3 & 1 & -6 \end{pmatrix}$.

(a) Find the nullspace $N(A)$.
(b) Find a spanning set for $N(A)$. (Your spanning set should have exactly one vector in it.)
(c) You are given that $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ -3 & 1 & -6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 26 \\ -19 \end{pmatrix}$. Use your answer to part (b) to find the solution set for $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ -3 & 1 & -6 \end{pmatrix} \vec{x} = \begin{pmatrix} 13 \\ 26 \\ -19 \end{pmatrix}$. *Hint*: Use Theorem 3.2.2.

(AB 14) Let $\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix}$.

(a) Find $5\vec{v}_1 + \vec{v}_3$.
(b) Find $9\vec{v}_2 + 2\vec{v}_3$.
(c) Using Theorem 3.3.2, determine whether $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are linearly independent. Explain your reasoning.

(AB 15) Let $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$. Find a basis for $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$. What is the dimension of $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$?

(AB 16) Let $p_1(x) = x^2 + 2x + 1$, $p_2(x) = 2x^2 + 6x + 4$, $p_3(x) = 3x^2 + 9x + 6$, $p_4(x) = 4x^2 + 8x + 4$, $p_5(x) = 5x^2 - 3x + 2$. Consider the set $\{p_1(x), p_2(x), p_3(x), p_4(x), p_5(x)\}$. This set spans P_3. Pare down this set to form a basis for P_3.

(AB 17) Let $\mathcal{U} = [x^2, (x-1)^2, (x+1)^2]$. Then $\mathcal{U}$ is a basis for P_3.

- (a) Let $p(x) = 9 + 2x + 12x^2$. Find $[p(x)]_{\mathcal{U}}$, that is, the coordinates of $p(x)$ with respect to the basis $\mathcal{U}$.
- (b) Find the transition matrix representing the change in coordinates from $[1, x, x^2]$ to $\mathcal{U}$. *Note*: If you prefer to do part (b) first and then part (a), you may do so.

(AB 18) Let $A = \begin{pmatrix} 4 & 3 & 6 & 1 \\ 3 & 1 & 7 & 5 \\ 1 & 0 & 3 & 2 \\ 2 & 2 & 2 & 4 \end{pmatrix}$.

- (a) Find a basis for the row space of A.
- (b) Find a basis for the null space of A.
- (c) Find a basis for the column space of A.

(AB 19) In each of parts (a)–(d), one of statements (i)–(vi) is true. Determine which statement is true in each part.

- (i) For every vector $\vec{b}$ in $\mathbb{R}^6$, the system $A\vec{x} = \vec{b}$ is inconsistent.
- (ii) For every vector $\vec{b}$ in $\mathbb{R}^6$, there is exactly one solution $\vec{x}$ to $A\vec{x} = \vec{b}$.
- (iii) For every vector $\vec{b}$ in $\mathbb{R}^6$, there are infinitely many solutions $\vec{x}$ to $A\vec{x} = \vec{b}$.
- (iv) There are some vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ has a unique solution.
- (v) There are some vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions.
- (vi) There are some vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ has a unique solution, and other vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions.

- (a) A is a 6×7 matrix of rank 6.
- (b) A is a 6×5 matrix of rank 5.
- (c) A is a 6×7 matrix of rank 5.
- (d) A is a 6×6 matrix of rank 6.

(AB 20) Let $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \sqrt[3]{x_1\, x_2\, x_3}$.

- (a) Is it the case that $L(\alpha\,\vec{x}) = \alpha L(\vec{x})$ for all $\vec{x}$ in $\mathbb{R}^3$ and all α in $\mathbb{R}$? If so, prove it; if not, find a counterexample.
- (b) Is it the case that $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$ for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^3$? If so, prove it; if not, find a counterexample.
- (c) Is L a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}$?

(AB 21) Let $L : P_3 \to P_3$ be given by $L(p(x)) = p(x) - p(0) - x\,p'(0)$. Find the kernel and range of L.

(AB 22) Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. You are given that $L\begin{pmatrix}3\\2\\7\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix}$, $L\begin{pmatrix}2\\1\\6\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$, $L\begin{pmatrix}5\\4\\8\end{pmatrix} = \begin{pmatrix}4\\6\end{pmatrix}$.

(a) Find the matrix A that represents L with respect to the basis $\mathcal{U} = \left[\begin{pmatrix}3\\2\\7\end{pmatrix}, \begin{pmatrix}2\\1\\6\end{pmatrix}, \begin{pmatrix}5\\4\\8\end{pmatrix}\right]$ and the standard basis $\mathcal{F} = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]$.

(b) Find the matrix B that represents L with respect to the standard basis in both $\mathbb{R}^3$ and $\mathbb{R}^2$.

(AB 23)

(a) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator that doubles the length of each vector $\vec{x}$ and then rotates it $\pi/2$ radians in the clockwise direction. Find the matrix A that represents L with respect to the standard basis.

(b) Let $\mathcal{U} = \left[\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}4\\2\end{pmatrix}\right]$. Find the matrix U that represents the change of basis from $\mathcal{U}$ to the standard basis.

(c) Find $U^{-1}AU$.

(d) Let $\vec{x}$ satisfy $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix}-5\\7\end{pmatrix}$. Find the standard coordinates of $\vec{x}$. Then find $L(\vec{x})$.

(e) Recall that $\vec{x}$ satisfies $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix}-5\\7\end{pmatrix}$. Use the matrix you found in part (c) to find $[L(\vec{x})]_{\mathcal{U}}$.

(f) Use your answer to parts (b) and (e) to find $L(\vec{x})$. Does your answer agree with your answer to part (d)?

(AB 24) Let P be the plane $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : x + 4y - 8z = 0\right\}$.

(a) Find the distance from the point $\begin{pmatrix}3\\7\\5\end{pmatrix}$ to the plane P.

(b) Find the point on the plane P that is closest to the point $\begin{pmatrix}3\\7\\5\end{pmatrix}$.

(AB 25) Let $S = \left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} : 3x_1 - 5x_2 + 4x_3 = 0\right\}$.

(a) Find a spanning set for S.

(b) Find $S^\perp$.

(AB 26) Find the least squares solution to the system $A\vec{x} = \vec{b}$, where $A = \begin{pmatrix}1 & 2\\2 & 3\\3 & 4\end{pmatrix}$ and where $\vec{b} = \begin{pmatrix}7\\11\\9\end{pmatrix}$.

(AB 27) Let $V = C[0,\pi]$ with the inner product $\langle f(x), g(x)\rangle = \int_0^\pi f(x)\,g(x)\,dx$.

(a) Let m and n be two distinct positive integers. Find $\langle \cos(mx), \cos(nx)\rangle$.

(b) Let m be a positive integer. Find $\|\cos(mx)\|$.

(AB 28) Let $V = P_3$ with the inner product $\langle p(x), q(x)\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$. You are given that $\mathcal{U} = \left[\dfrac{1}{\sqrt{3}}, \dfrac{x-1}{\sqrt{2}}, \dfrac{3x^2-6x+1}{\sqrt{6}}\right]$ is an orthonormal basis for P_3 under this inner product.

(a) Let $p(x) = \sqrt{6}x^2$. Find $[p(x)]_{\mathcal{U}}$.

(b) Find $\|p(x)\|^2$ by computing $\langle p(x), p(x)\rangle$.

(c) Find the norm of $[p(x)]_{\mathcal{U}}$. (In this problem, treat $[p(x)]_{\mathcal{U}}$ as a column vector and find its norm using the usual inner product in $\mathbb{R}^3$).

(AB 29) Let $S = \operatorname{Span}\left(\begin{pmatrix}1\\2\\2\\4\end{pmatrix}, \begin{pmatrix}4\\-4\\-8\\5\end{pmatrix}\right)$ and let $\vec{x} = \begin{pmatrix}13\\-6\\-7\\22\end{pmatrix}$. Find the projection $\vec{p}$ of $\vec{x}$ onto S. That is, find a vector $\vec{p}$ in S such that $\vec{x} - \vec{p}$ is orthogonal to S.

(AB 30) Let $V = C[0,\pi]$ with the inner product $\langle f(x), g(x)\rangle = \dfrac{2}{\pi}\int_0^\pi f(x)\,g(x)\,dx$. You are given that $\left\{\dfrac{1}{\sqrt{2}}, \cos x, \cos 3x\right\}$ is an orthonormal set in V.

Find the best least squares approximation to the function $f(x) = x$ on $[0,\pi]$ by a function in $S = \operatorname{Span}\left(\dfrac{1}{\sqrt{2}}, \cos x, \cos 3x\right)$.

On a single set of axes, sketch the graph of $f(x)$ and your approximation on the interval $[0,\pi]$. A good online graphing calculator is Desmos: `https://www.desmos.com/calculator/wjvtttzcnj`

(AB 31) Find the eigenvalues and eigenspaces of the matrix $\begin{pmatrix}6 & -1 & -3\\-4 & 0 & 4\\6 & -2 & -3\end{pmatrix}$. *Hint*: One way to simplify a term of the form $a + bi$, where a and b are real numbers, is to multiply it by $a - bi$.

(AB 32) Find an invertible matrix S and a diagonal matrix D such that $SDS^{-1} = \begin{pmatrix}1 & 6 & -12\\14 & 18 & -42\\8 & 12 & -27\end{pmatrix}$.

Answer key

(AB 1) Solve the system

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 = 20, \\ -5x_1 + 1x_2 + 2x_3 = -7, \\ 3x_1 + 4x_2 - 5x_3 = -24. \end{cases}$$

(Answer 1) $(x_1, x_2, x_3) = (3, -2, 5)$.

(AB 2) Write the augmented matrix for the system

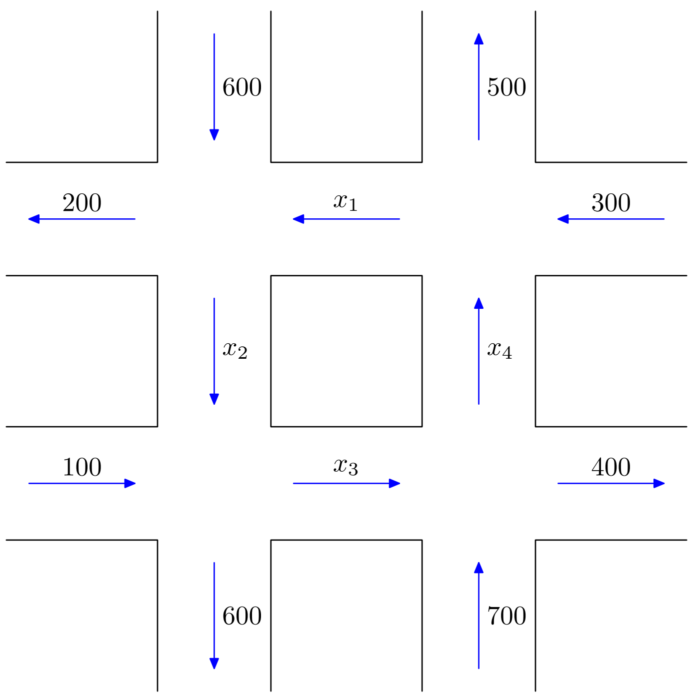
$$\begin{cases} 3x_1 + 2x_2 - 5x_3 = 7, \\ 6x_1 + 4x_2 - 10x_3 = 14, \\ 2x_1 + x_2 - x_3 = 4. \end{cases}$$

Use Gauss-Jordan reduction to find a row equivalent matrix in reduced row echelon form. Indicate whether the system is consistent. If it is, find the solution set.

(Answer 2) The solution set is $\{(1-3\alpha, 2+7\alpha, \alpha)\}$; it may also be written as $\{(x_1, x_2, x_3) : x_1 = 1-3x_3, x_2 = 2 + 7x_3\}$.

(AB 3) Do **one** of Parts (a), (b), or (c).

(a) Determine all possible values of x_1, x_2, x_3, and x_4 in the following traffic flow diagram.

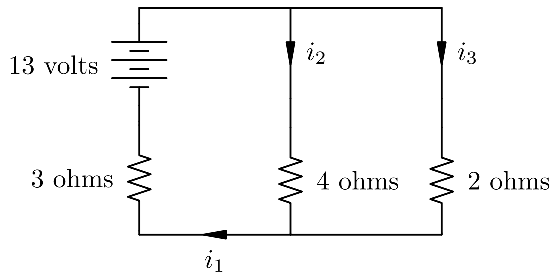


$x_1 = x_4 - 200$, $x_2 = x_4 + 200$, $x_3 = x_4 - 300$.

(b) Ammonia reacts with oxygen to form nitric oxide and water. The chemical equation for this reaction is of the form $x_1 NH_3 + x_2 O_2 \to x_3 NO + x_4 H_2O$. Determine values of x_1, x_2, x_3 and x_4 to balance the equation.

$x_1 = 4$, $x_2 = 5$, $x_3 = 4$, and $x_4 = 6$, or any fixed constant multiple of those values.

(c) Determine the amount of each current in the following network.



$i_1 = 3$ amperes, $i_2 = 1$ ampere, $i_3 = 2$ amperes.

(AB 5) Let $A = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$.

(a) Find A^{-1}.

$A^{-1} = \begin{pmatrix} 3/2 & -5/2 \\ -1 & 2 \end{pmatrix}$.

(b) Find A^T.

$A^T = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$.

(c) Find $(A^T)^{-1}$.

$(A^T)^{-1} = \begin{pmatrix} 3/2 & -1 \\ -5/2 & 2 \end{pmatrix}$.

(AB 6) Find the inverses to the following elementary matrices.

(a) $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$A^{-1} = A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(b) $B = \begin{pmatrix} -5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$B^{-1} = \begin{pmatrix} -1/5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(c) $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$C^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(AB 7)

(a) Solve $\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 1 & -3 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ by using Gauss-Jordan reduction to put the augmented matrix in reduced row echelon form.

The solution set is $\{(\frac{5}{3}\alpha, -\frac{1}{3}\alpha, \alpha)\}$; it may also be written as $\{(x_1, x_2, x_3) : x_1 = \frac{5}{3}x_3, x_2 = -\frac{1}{3}x_3\}$.

(b) Is $\begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 1 & -3 \end{pmatrix}$ invertible? Why or why not? Do not use the determinant to answer this question.

Hint: Try to answer without trying to find the inverse.

No; if the matrix was invertible, then there would be no more than one solution to the system.

(Alternative solution) No; the matrix is row equivalent to $\begin{pmatrix} 1 & 0 & 5/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{pmatrix}$, which is clearly not invertible.

(AB 8)

(a) Find three elementary matrices E_1, E_2 and E_3, all lower triangular and of type III, such that $E_1 E_2 E_3 \begin{pmatrix} 3 & 1 & 4 \\ 9 & 1 & 7 \\ -3 & -5 & -3 \end{pmatrix}$ is an upper triangular matrix.

The following answers are all correct:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

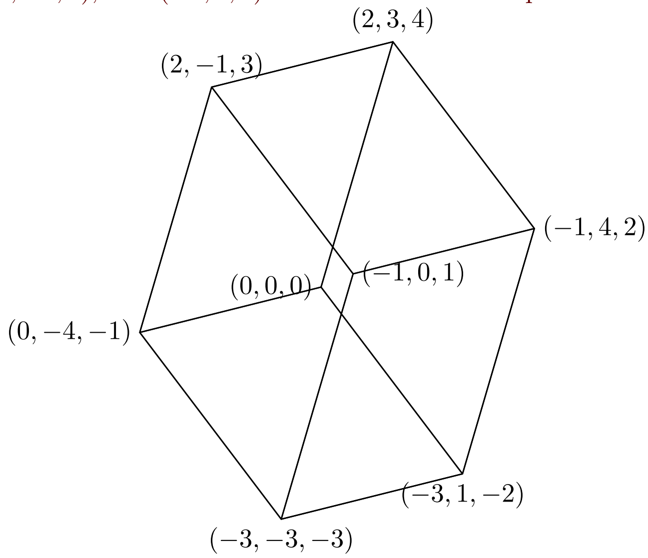
$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2\frac{1}{3} & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix}.$$

(b) Find a LU factorization of $M = \begin{pmatrix} 3 & 1 & 4 \\ 9 & 1 & 7 \\ -3 & -5 & -3 \end{pmatrix}$. That is, find an upper triangular matrix U and a lower triangular matrix L, where the diagonal entries of L are all 1s, such that $LU = M$.

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}, U = \begin{pmatrix} 3 & 1 & 4 \\ 0 & -2 & -5 \\ 0 & 0 & 11 \end{pmatrix}.$$

(AB 9) Find the volume of the parallelepiped with vertices at $(0,0,0)$, $(-3,1,-2)$, $(2,3,4)$, $(0,-4,-1)$, $(-3,-3,-3)$, $(-1,4,2)$, $(2,-1,3)$, and $(-1,0,1)$. *Hint*: The answer is positive and is less than 30.



(Answer 9) $\begin{vmatrix} -3 & 1 & -2 \\ 2 & 3 & 4 \\ 0 & -4 & -1 \end{vmatrix} = 21.$

(AB 10) Find the determinant of the matrix $\begin{pmatrix} 1 & 2 & -1 & 2 \\ 2 & 4 & -2 & 3 \\ 0 & 3 & -1 & 2 \\ 5 & 7 & -3 & 0 \end{pmatrix}$ using the elimination method.

(Answer 10) The best method is to row reduce to an upper triangular matrix, keeping track of the changes your row reduction steps make to the determinant.

For example, you might compute $\det \begin{pmatrix} 1 & 2 & -1 & 2 \\ 2 & 4 & -2 & 3 \\ 0 & 3 & -1 & 2 \\ 5 & 7 & -3 & 0 \end{pmatrix} = -\det \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & -3 & 2 & -10 \\ 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -3.$

(AB 11) Let V be the set of all 2×2 matrices and let U be the set of all singular 2×2 matrices. Show that U is not a subspace of V by either:

(a) Showing that the zero matrix O is not in U,

(b) Giving an example of a scalar α and a matrix A in U such that αA is not in U, or

(c) Giving an example of two matrices A and B, both in U, such that $A + B$ is not in U.

(Answer 11) Parts (a) and (b) are impossible. There are many possible answers to part (c); one such answer is $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(AB 12) Find Span $\left(\begin{pmatrix}0\\3\\2\end{pmatrix}, \begin{pmatrix}5\\4\\1\end{pmatrix}\right)$.

(Answer 12) Span $\left(\begin{pmatrix}0\\3\\2\end{pmatrix}, \begin{pmatrix}5\\4\\1\end{pmatrix}\right) = \left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} : x_1 - 2x_2 + 3x_3 = 0\right\}$.

(AB 13) Let $A = \begin{pmatrix}1 & 3 & 2\\2 & 6 & 4\\-3 & 1 & -6\end{pmatrix}$.

(a) Find the nullspace $N(A)$.

$N(A) = \left\{\begin{pmatrix}x_1\\0\\x_3\end{pmatrix} : x_1 = -2x_3\right\} = \left\{\begin{pmatrix}-2x_3\\0\\x_3\end{pmatrix}\right\} = \left\{\alpha\begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}$.

(b) Find a spanning set for $N(A)$. (Your spanning set should have exactly one vector in it.)

$N(A) = \text{Span}\left(\begin{pmatrix}-2\\0\\1\end{pmatrix}\right)$.

(c) You are given that $\begin{pmatrix}1 & 3 & 2\\2 & 6 & 4\\-3 & 1 & -6\end{pmatrix}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}13\\26\\-19\end{pmatrix}$. Use your answer to part (b) to find the solution set for $\begin{pmatrix}1 & 3 & 2\\2 & 6 & 4\\-3 & 1 & -6\end{pmatrix}\vec{x} = \begin{pmatrix}13\\26\\-19\end{pmatrix}$. *Hint*: Use Theorem 3.2.2.

The solution set is $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix} + \alpha\begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}$.

(AB 14) Let $\vec{v}_1 = \begin{pmatrix}3\\2\\4\end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix}1\\1\\2\end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix}6\\1\\2\end{pmatrix}$.

(a) Find $5\vec{v}_1 + \vec{v}_3$.

$5\begin{pmatrix}3\\2\\4\end{pmatrix} + \begin{pmatrix}6\\1\\2\end{pmatrix} = \begin{pmatrix}21\\11\\22\end{pmatrix}$.

(b) Find $9\vec{v}_2 + 2\vec{v}_3$.

$9\begin{pmatrix}1\\1\\2\end{pmatrix} + 2\begin{pmatrix}6\\1\\2\end{pmatrix} = \begin{pmatrix}21\\11\\22\end{pmatrix}$.

(c) Using Theorem 3.3.2, determine whether $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are linearly independent. Explain your reasoning.

No. $\vec{x} = \begin{pmatrix}21\\11\\22\end{pmatrix}$ is in Span$(\vec{v}_1, \vec{v}_2, \vec{v}_3)$, and there are at least two different ways to write $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$. Thus, by Theorem 3.3.2, the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ cannot be linearly independent.

(AB 15) Let $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$. Find a basis for $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$. What is the dimension of $\mathrm{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$?

(Answer 15) The dimension is 2. There are many possible bases, including:

- $\{\vec{v}_1, \vec{v}_2\}$.
- $\{\vec{v}_1, \vec{v}_3\}$.
- $\{\vec{v}_2, \vec{v}_3\}$.
- $\left\{\begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix}\right\}$.

(AB 16) Let $p_1(x) = x^2 + 2x + 1$, $p_2(x) = 2x^2 + 6x + 4$, $p_3(x) = 3x^2 + 9x + 6$, $p_4(x) = 4x^2 + 8x + 4$, $p_5(x) = 5x^2 - 3x + 2$. Consider the set $\{p_1(x), p_2(x), p_3(x), p_4(x), p_5(x)\}$. This set spans P_3. Pare down this set to form a basis for P_3.

(Answer 16) The following four solutions are all valid:

- $\{p_1(x), p_2(x), p_5(x)\}$.
- $\{p_1(x), p_3(x), p_5(x)\}$.
- $\{p_2(x), p_4(x), p_5(x)\}$.
- $\{p_3(x), p_4(x), p_5(x)\}$.

(AB 17) Let $\mathcal{U} = [x^2, (x-1)^2, (x+1)^2]$. Then $\mathcal{U}$ is a basis for P_3.

(a) Let $p(x) = 9 + 2x + 12x^2$. Find $[p(x)]_{\mathcal{U}}$, that is, the coordinates of $p(x)$ with respect to the basis $\mathcal{U}$.

$[p(x)]_{\mathcal{U}} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$; that is, $p(x) = 3x^2 + 4(x-1)^2 + 5(x+1)^2$.

(b) Find the transition matrix representing the change in coordinates from $[1, x, x^2]$ to $\mathcal{U}$. *Note*: If you prefer to do part (b) first and then part (a), you may do so.

$\begin{pmatrix} -1 & 0 & 1 \\ 1/2 & -1/4 & 0 \\ 1/2 & 1/4 & 0 \end{pmatrix}$.

(AB 18) Let $A = \begin{pmatrix} 4 & 3 & 6 & 1 \\ 3 & 1 & 7 & 5 \\ 1 & 0 & 3 & 2 \\ 2 & 2 & 2 & 4 \end{pmatrix}$.

(a) Find a basis for the row space of A.

There are many possible answers, including $\{(1 \;\; 0 \;\; 3 \;\; 0), (0 \;\; -1 \;\; 2 \;\; 0), (0 \;\; 0 \;\; 0 \;\; 1)\}$.

(b) Find a basis for the null space of A.

$\left\{\begin{pmatrix} -3 \\ 2 \\ 1 \\ 0 \end{pmatrix}\right\}$

(c) Find a basis for the column space of A.

$\left\{\begin{pmatrix} 4 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 2 \\ 4 \end{pmatrix}\right\}$ or $\left\{\begin{pmatrix} 4 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 2 \\ 4 \end{pmatrix}\right\}$ or $\left\{\begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ 7 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 2 \\ 4 \end{pmatrix}\right\}$

(AB 19) In each of parts (a)–(d), one of statements (i)–(vi) is true. Determine which statement is true in each part.

(i) For every vector $\vec{b}$ in $\mathbb{R}^6$, the system $A\vec{x} = \vec{b}$ is inconsistent.

(ii) For every vector $\vec{b}$ in $\mathbb{R}^6$, there is exactly one solution $\vec{x}$ to $A\vec{x} = \vec{b}$.

(iii) For every vector $\vec{b}$ in $\mathbb{R}^6$, there are infinitely many solutions $\vec{x}$ to $A\vec{x} = \vec{b}$.

(iv) There are some vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ has a unique solution.

(v) There are some vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions.

(vi) There are some vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ has a unique solution, and other vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions.

(a) A is a 6×7 matrix of rank 6.
(iii); For every vector $\vec{b}$ in $\mathbb{R}^6$, there are infinitely many solutions $\vec{x}$ to $A\vec{x} = \vec{b}$.

(b) A is a 6×5 matrix of rank 5.
(iv); There are some vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ has a unique solution.

(c) A is a 6×7 matrix of rank 5.
(v); There are some vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^6$ such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions.

(d) A is a 6×6 matrix of rank 6.
(ii); for every vector $\vec{b}$ in $\mathbb{R}^6$, there is exactly one solution $\vec{x}$ to $A\vec{x} = \vec{b}$.

(AB 20) Let $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \sqrt[3]{x_1\,x_2\,x_3}$.

(a) Is it the case that $L(\alpha\,\vec{x}) = \alpha L(\vec{x})$ for all $\vec{x}$ in $\mathbb{R}^3$ and all α in $\mathbb{R}$? If so, prove it; if not, find a counterexample.
Yes. $L(\alpha\,\vec{x}) = \sqrt[3]{(\alpha\,x_1)\,(\alpha\,x_2)\,(\alpha\,x_3)} = \sqrt[3]{\alpha^3 x_1\,x_2\,x_3} = \alpha\sqrt[3]{x_1\,x_2\,x_3} = \alpha\,L(\vec{x})$.

(b) Is it the case that $L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y})$ for all $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^3$? If so, prove it; if not, find a counterexample.
No. Let $\vec{x} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix}$ and let $\vec{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Then $L(\vec{x}) = 0$, $L(\vec{y}) = 1$, but $L(\vec{x} + \vec{y}) = 2$.

(c) Is L a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}$?
No.

(AB 21) Let $L : P_3 \to P_3$ be given by $L(p(x)) = p(x) - p(0) - x\,p'(0)$. Find the kernel and range of L.

(Answer 21) The kernel is $P_2 = \{\alpha x + \beta : \alpha,\, \beta \text{ are real}\}$. The range is $\{\gamma x^2 : \gamma \text{ is real}\}$.

(AB 22) Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation. You are given that $L\begin{pmatrix}3\\2\\7\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix}$, $L\begin{pmatrix}2\\1\\6\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$, $L\begin{pmatrix}5\\4\\8\end{pmatrix} = \begin{pmatrix}4\\6\end{pmatrix}$.

(a) Find the matrix A that represents L with respect to the basis $\mathcal{U} = \left[\begin{pmatrix}3\\2\\7\end{pmatrix}, \begin{pmatrix}2\\1\\6\end{pmatrix}, \begin{pmatrix}5\\4\\8\end{pmatrix}\right]$ and the standard basis $\mathcal{F} = \left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right]$.

$A = \begin{pmatrix}5 & 1 & 4\\2 & 3 & 6\end{pmatrix}$.

(b) Find the matrix B that represents L with respect to the standard basis in both $\mathbb{R}^3$ and $\mathbb{R}^2$.

$B = \begin{pmatrix}-48 & 43 & 9\\34 & -29 & -6\end{pmatrix}$.

(AB 23)

(a) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator that doubles the length of each vector $\vec{x}$ and then rotates it $\pi/2$ radians in the clockwise direction. Find the matrix A that represents L with respect to the standard basis.

$A = \begin{pmatrix}0 & 2\\-2 & 0\end{pmatrix}$.

(b) Let $\mathcal{U} = \left[\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}4\\2\end{pmatrix}\right]$. Find the matrix U that represents the change of basis from $\mathcal{U}$ to the standard basis.

$U = \begin{pmatrix}3 & 4\\1 & 2\end{pmatrix}$.

(c) Find $U^{-1}AU$.

$U^{-1}AU = \begin{pmatrix}1 & -2\\-0.5 & 1.5\end{pmatrix}\begin{pmatrix}0 & 2\\-2 & 0\end{pmatrix}\begin{pmatrix}3 & 4\\1 & 2\end{pmatrix} = \begin{pmatrix}14 & 20\\-10 & -14\end{pmatrix}$.

(d) Let $\vec{x}$ satisfy $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix}-5\\7\end{pmatrix}$. Find the standard coordinates of $\vec{x}$. Then find $L(\vec{x})$.

$\vec{x} = \begin{pmatrix}13\\9\end{pmatrix}$; $L(\vec{x}) = \begin{pmatrix}18\\-26\end{pmatrix}$.

(e) Recall that $\vec{x}$ satisfies $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix}-5\\7\end{pmatrix}$. Use the matrix you found in part (c) to find $[L(\vec{x})]_{\mathcal{U}}$.

$[L(\vec{x})]_{\mathcal{U}} = U^{-1}AU[\vec{x}]_{\mathcal{U}} = \begin{pmatrix}14 & 20\\-10 & -14\end{pmatrix}\begin{pmatrix}-5\\7\end{pmatrix} = \begin{pmatrix}70\\-48\end{pmatrix}$.

(f) Use your answer to parts (b) and (e) to find $L(\vec{x})$. Does your answer agree with your answer to part (d)?

$L(\vec{x}) = U[L(\vec{x})]_{\mathcal{U}} = \begin{pmatrix}3 & 4\\1 & 2\end{pmatrix}\begin{pmatrix}70\\-48\end{pmatrix} = \begin{pmatrix}18\\-26\end{pmatrix}$. Yes.

(AB 24) Let P be the plane $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + 4y - 8z = 0 \right\}$.

(a) Find the distance from the point $\begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$ to the plane P.

The distance is the absolute value of the scalar projection of $\begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$ on $\begin{pmatrix} 1 \\ 4 \\ -8 \end{pmatrix}$, that is, 1.

(b) Find the point on the plane P that is closest to the point $\begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix}$.

$$\begin{pmatrix} 3 \\ 7 \\ 5 \end{pmatrix} - \frac{-9}{81} \begin{pmatrix} 1 \\ 4 \\ -8 \end{pmatrix} = \begin{pmatrix} 28/9 \\ 67/9 \\ 37/9 \end{pmatrix}.$$

(AB 25) Let $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 3x_1 - 5x_2 + 4x_3 = 0 \right\}$.

(a) Find a spanning set for S.

There are many possible answers, such as $\left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix} \right\}$, $\left\{ \begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix} \right\}$, and $\left\{ \begin{pmatrix} 0 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix} \right\}$.

(b) Find $S^\perp$.

$S^\perp = \operatorname{Span}\left(\begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} \right)$.

(AB 26) Find the least squares solution to the system $A\vec{x} = \vec{b}$, where $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{pmatrix}$ and where $\vec{b} = \begin{pmatrix} 7 \\ 11 \\ 9 \end{pmatrix}$.

(Answer 26) The least squares solution $\vec{x}$ satisfies $\begin{pmatrix} 14 & 20 \\ 20 & 29 \end{pmatrix} \vec{x} = \begin{pmatrix} 56 \\ 83 \end{pmatrix}$ and so $\vec{x} = \begin{pmatrix} -6 \\ 7 \end{pmatrix}$.

(AB 27) Let $V = C[0,\pi]$ with the inner product $\langle f(x), g(x) \rangle = \int_0^\pi f(x)\, g(x)\, dx$.

(a) Let m and n be two distinct positive integers. Find $\langle \cos(mx), \cos(nx) \rangle$.

$\langle \cos(mx), \cos(nx) \rangle = \int_0^\pi \cos(mx)\, \cos(nx)\, dx = \frac{1}{2} \int_0^\pi \cos(mx - nx) + \cos(mx + nx)\, dx = 0.$

(b) Let m be a positive integer. Find $\| \cos(mx) \|$.

$\| \cos(mx) \|^2 = \int_0^\pi \cos(mx)\, \cos(mx)\, dx = \frac{1}{2} \int_0^\pi 1 + \cos(2mx)\, dx = \frac{\pi}{2}$, so $\| \cos(mx) \| = \sqrt{\pi/2}$.

(AB 28) Let $V = P_3$ with the inner product $\langle p(x), q(x)\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$. You are given that $\mathcal{U} = \left[\frac{1}{\sqrt{3}}, \frac{x-1}{\sqrt{2}}, \frac{3x^2-6x+1}{\sqrt{6}}\right]$ is an orthonormal basis for P_3 under this inner product.

(a) Let $p(x) = \sqrt{6}x^2$. Find $[p(x)]_{\mathcal{U}}$.

$[p(x)]_{\mathcal{U}} = \begin{pmatrix} 5\sqrt{2} \\ 4\sqrt{3} \\ 2 \end{pmatrix}$.

(b) Find $\|p(x)\|^2$ by computing $\langle p(x), p(x)\rangle$.

$\langle p(x), p(x)\rangle = 0 \cdot 0 + \sqrt{6} \cdot \sqrt{6} + 4\sqrt{6} \cdot 4\sqrt{6} = 102$.

(c) Find the norm of $[p(x)]_{\mathcal{U}}$. (In this problem, treat $[p(x)]_{\mathcal{U}}$ as a column vector and find its norm using the usual inner product in $\mathbb{R}^3$).

$\sqrt{5\sqrt{2} \cdot 5\sqrt{2} + 4\sqrt{3} \cdot 4\sqrt{3} + 2 \cdot 2} = \sqrt{102}$.

(AB 29) Let $S = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ -4 \\ -8 \\ 5 \end{pmatrix}\right)$ and let $\vec{x} = \begin{pmatrix} 13 \\ -6 \\ -7 \\ 22 \end{pmatrix}$. Find the projection $\vec{p}$ of $\vec{x}$ onto S. That is, find a vector $\vec{p}$ in S such that $\vec{x} - \vec{p}$ is orthogonal to S.

(Answer 29) $\vec{p} = \begin{pmatrix} 11 \\ -2 \\ -10 \\ 22 \end{pmatrix}$.