Math 3083, Spring 2019

(Problem 1) Use back substitution to solve the system

$$\begin{cases} 3x_1 - 2x_2 + \ \ x_3 = 5, \\ \qquad - 5x_2 - 2x_3 = 1, \\ \qquad\qquad\quad 3x_3 = 7. \end{cases}$$

(Problem 2) In each of the following systems, interpret each equation as a line in the plane. For each system, graph the lines and determine geometrically the number of solutions.

$(a)\ \begin{cases} 3x_1 - 2x_2 = 6, \\ -6x_1 + 4x_2 = -12. \end{cases} \qquad (b)\ \begin{cases} 3x_1 - 2x_2 = 6, \\ -4x_1 + 2x_2 = 8. \end{cases} \qquad (c)\ \begin{cases} 4x_1 - 2x_2 = 8, \\ -2x_1 + \ \ x_2 = 6. \end{cases}$

(Problem 3) Solve each of the following systems by writing the augmented matrix and finding a row equivalent matrix in strict upper triangular or reduced row echelon form. Indicate whether the system is consistent. If it is, find the solution set.

$(a)\ \begin{cases} 4x_1 + \ \ x_2 - 5x_3 = -9, \\ 2x_1 - 5x_2 + 3x_3 = 1, \\ 3x_1 + 2x_2 - 2x_3 = 1. \end{cases} \qquad (b)\ \begin{cases} 2x_1 + 6x_2 - 3x_3 = 2, \\ 3x_1 + 9x_2 + \ \ x_3 = 14, \\ \ \ x_1 + 3x_2 - 2x_3 = 0. \end{cases} \qquad (c)\ \begin{cases} 2x_1 - \ \ x_2 - 4x_3 = 2, \\ 4x_1 + 3x_2 + 2x_3 = 25, \\ 3x_1 + 4x_2 + 5x_3 = 10. \end{cases}$

(Problem 4) Which of the following matrices are in reduced row echelon form?

$(a)\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$(b)\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$(c)\ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$
$(d)\ \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}$
$(e)\ \begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix}$
$(f)\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$(g)\ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$(h)\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$(i)\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix}$
$(j)\ \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 5 \end{pmatrix}$

$(k)\ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$
$(l)\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
$(m)\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$(n)\ \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$
$(o)\ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 4 \end{pmatrix}$
$(p)\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$(q)\ \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$
$(r)\ \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$(s)\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3 & 6 \end{pmatrix}$
$(t)\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(Problem 5) Which of the following matrices are in reduced row echelon form?

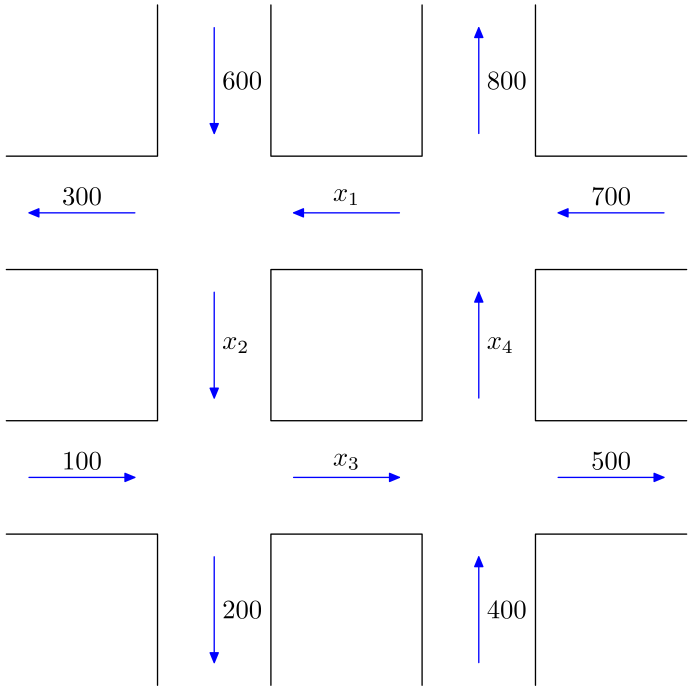
$(a)\ \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$
$(b)\ \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$(c)\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$(d)\ \begin{pmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$(e)\ \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 6 \end{pmatrix}$
$(f)\ \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$

$(g)\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$(h)\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$(i)\ \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$(j)\ \begin{pmatrix} 1 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

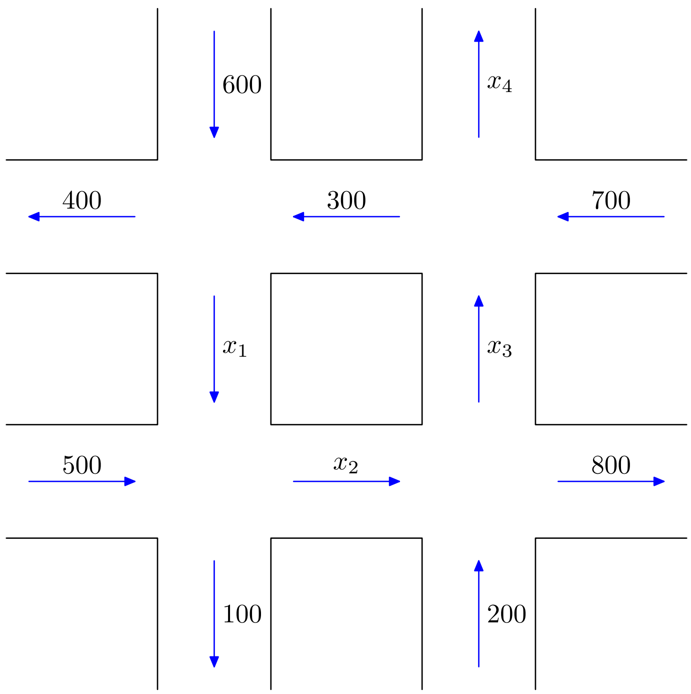
$(k)\ \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$
$(l)\ \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$
$(m)\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$(n)\ \begin{pmatrix} 2 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$
$(o)\ \begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$
$(p)\ \begin{pmatrix} 1 & 3 & 7 \\ 0 & 0 & 0 \end{pmatrix}$

$(q)\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$(r)\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$(s)\ \begin{pmatrix} 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$(t)\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(Problem 6) Determine all possible values of x_1, x_2, x_3, and x_4 in the following traffic flow diagram.

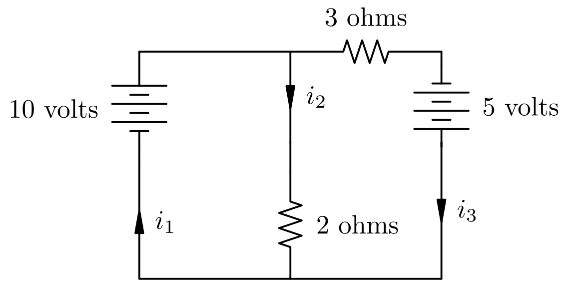


(Problem 7) Determine all possible values of x_1, x_2, x_3, and x_4 in the following traffic flow diagram.



(Problem 8) The solution set to Problem 6 had a free variable. Problem 7 has a unique solution. Why is that?

(Problem 9) Determine the values of the currents i_1, i_2, and i_3 in the following circuit diagram.



(Problem 10) Carbon monoxide reacts with hydrogen to produce octane and water. The chemical equation for this reaction is of the form $x_1\mathrm{CO} + x_2\mathrm{H}_2 \to x_3\mathrm{C}_8\mathrm{H}_{18} + x_4\mathrm{H}_2\mathrm{O}$. Determine (nonzero integer) values of x_1, x_2, x_3 and x_4 to balance the equation.

(Problem 11) Find the following products or state that they are not meaningful.

(a) $\begin{pmatrix} 6 & 2 & 1 \\ 3 & 4 & 8 \\ 0 & 7 & 5 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 2 \\ -2 & 3 \end{pmatrix}$.

(b) $\begin{pmatrix} 0 & -1 \\ 1 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 6 & 2 & 1 \\ 3 & 4 & 8 \\ 0 & 7 & 5 \end{pmatrix}$.

(c) $\begin{pmatrix} 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$.

(d) $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 3 & 4 & 1 \end{pmatrix}$.

(Problem 12) Write the system $\begin{cases} 4x_1 + x_2 - 5x_3 = -9, \\ 2x_1 - 5x_2 + 3x_3 = 1, \\ 3x_1 + 2x_2 - 2x_3 = 1. \end{cases}$ as a matrix equation.

(Problem 13) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix}$. Write $\vec{b} = \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix}$ as a linear combination of the columns of A. Then solve $A\vec{x} = \vec{b}$.

(Problem 14) Find a nonzero matrix A such that $A^2 = O$.

(Problem 15) Find a nonzero 2×2 matrix A such that $A\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(Problem 16) Compute the inverses of the following matrices.

(a) $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 4 \\ 2 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}$

(Problem 17) Let $A = \begin{pmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}$. Find A^2.

(Problem 18) Find an elementary matrix E such that $\begin{pmatrix} 5 & 7 & 9 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix} = E\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Problem 19) Find an elementary matrix E such that $\begin{pmatrix} 1 & 2 & 3 \\ -10 & -13 & -10 \\ 7 & 9 & 8 \end{pmatrix} = E\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Problem 20) Find an elementary matrix E such that $\begin{pmatrix} 4 & 8 & 12 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Problem 21) Find an elementary matrix E such that $\begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 9 & 8 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Problem 22) Find an elementary matrix E such that $\begin{pmatrix} 1 & 2 & 3 \\ -8 & -10 & -12 \\ 7 & 9 & 8 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Problem 23) Find an elementary matrix E such that $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 21 & 27 & 24 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 9 & 8 \end{pmatrix}$.

(Problem 24) Find an elementary matrix E such that $\begin{pmatrix} 7 & 9 & 8 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Problem 25) Find an elementary matrix E such that $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 10 & 15 & 17 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Problem 26) Find an elementary matrix E such that $\begin{pmatrix} 1 & 2 & 3 \\ 7 & 9 & 8 \\ 4 & 5 & 6 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Problem 27)

(a) Find three elementary matrices E_3, E_2 and E_1 such that $E_3 E_2 E_1 \begin{pmatrix} 2 & 5 & 4 \\ 6 & 2 & 1 \\ -4 & 3 & 5 \end{pmatrix}$ is upper triangular.

(b) Find a LU factorization of $M = \begin{pmatrix} 2 & 5 & 4 \\ 6 & 2 & 1 \\ -4 & 3 & 5 \end{pmatrix}$. That is, find an upper triangular matrix U and a lower triangular matrix L, where the diagonal entries of L are all 1s, such that $LU = M$.

(Problem 28) Compute the inverses of the following matrices.

(a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(Problem 29) Compute the inverses of the following matrices.

(a) $\begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 0 & 3 \\ 1 & 3 & -2 \\ 2 & 5 & -3 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 8 & 4 \\ 4 & 0 & 3 \\ 1 & 5 & 2 \end{pmatrix}$

(Problem 30) Use your answers to Problem 29 to find the inverses of the following matrices.

(a) $\begin{pmatrix} 1 & 4 & 7 \\ 3 & 5 & 8 \\ 2 & 6 & 9 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 5 \\ 3 & -2 & -3 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 4 & 1 \\ 8 & 0 & 5 \\ 4 & 3 & 2 \end{pmatrix}$

(Problem 31) Use your answers to Problem 29 to solve the following equations.

(a) $\begin{pmatrix} 2 & 8 & 4 \\ 4 & 0 & 3 \\ 1 & 5 & 2 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 0 & 3 \\ 1 & 3 & -2 \\ 2 & 5 & -3 \end{pmatrix} X = \begin{pmatrix} 7 & 2 \\ 3 & 5 \\ 2 & 1 \end{pmatrix}$

(Problem 32) Let $A = \begin{pmatrix} 0 & 2 & -1 & 2 & 1 \\ 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & 2 & 0 & -3 & 0 \\ -1 & 1 & 0 & 3 & 0 \end{pmatrix}$.

(a) Find $A \begin{pmatrix} 3 \\ 0 \\ 3 \\ 1 \\ 1 \end{pmatrix}$.

(b) Is A singular? How do you know?

(Problem 33) Let $A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 1 & 0 & -2 & 0 & 2 \\ 1 & 0 & 2 & -1 & 0 \\ 1 & 2 & 0 & 3 & 0 \\ -1 & 0 & -2 & 1 & 0 \end{pmatrix}$.

(a) Find $A \begin{pmatrix} -3 \\ 5 \\ 8 \\ 4 \\ 10 \end{pmatrix}$.

(b) Find $A \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$.

(c) Is A singular? How do you know?

(Problem 34) You are given that $A = \begin{pmatrix} 3 & 5 & 7 & 2 & 1 \\ 1 & 5 & -2 & 3 & 2 \\ 1 & 7 & 2 & -1 & 2 \\ 2 & 1 & 6 & 3 & 0 \\ -1 & 9 & -2 & 4 & 3 \end{pmatrix}$ is invertible. Is there a solution to

$\begin{pmatrix} 3 & 5 & 7 & 2 & 1 \\ 1 & 5 & -2 & 3 & 2 \\ 1 & 7 & 2 & -1 & 2 \\ 2 & 1 & 6 & 3 & 0 \\ -1 & 9 & -2 & 4 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$? (You don't have to find $\vec{x}$.)

(Problem 35) You are given that there are no solutions to $\begin{pmatrix} 3 & 5 & 7 & 2 & 1 \\ 3 & 5 & -2 & 3 & 2 \\ 1 & 14 & 4 & -2 & 4 \\ 2 & 1 & 6 & 3 & 0 \\ -1 & 6 & -2 & 4 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$. Is $A = \begin{pmatrix} 3 & 5 & 7 & 2 & 1 \\ 3 & 5 & -2 & 3 & 2 \\ 1 & 14 & 4 & -2 & 4 \\ 2 & 1 & 6 & 3 & 0 \\ -1 & 6 & -2 & 4 & 3 \end{pmatrix}$ invertible? How do you know?

(Problem 36) You are given that $A = \begin{pmatrix} 2 & 1 & 6 & 3 & 0 \\ 3 & 5 & 7 & 2 & 1 \\ 1 & 7 & 2 & -1 & 2 \\ -1 & 9 & -2 & 4 & 3 \\ 0 & 14 & -4 & 7 & 5 \end{pmatrix}$ is invertible. Find a matrix in reduced row echelon form that is row equivalent to A.

(Problem 37) Find the determinants of the following matrices.

(a) $\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 6 & 8 \\ 3 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 9 & 7 & 2 \\ 4 & 1 & 3 \\ 0 & 2 & 5 \end{pmatrix}$

(d) $\begin{pmatrix} 18 & 14 & 4 \\ 8 & 2 & 6 \\ 0 & 4 & 10 \end{pmatrix}$

(e) $\begin{pmatrix} 3 & 5 & 2 \\ 4 & 1 & 2 \\ 2 & 9 & 2 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 \\ 7 & 5 & 8 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix}$. Use the cofactor expansion in Section 2.1.

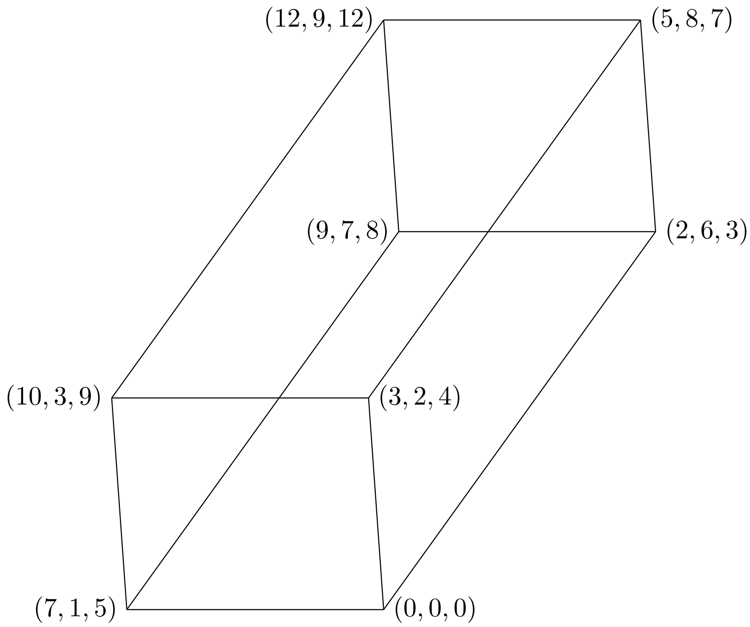
(g) $\begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 7 & 5 & 8 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix}$. Use the elimination method in Section 2.2.

(h) $\begin{pmatrix} 0 & 2 & 4 & 6 \\ 4 & 3 & 2 & 1 \\ 7 & 5 & 8 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix}$. Use the cofactor expansion in Section 2.1.

(i) $\begin{pmatrix} 0 & 2 & 4 & 6 \\ 4 & 5 & 6 & 7 \\ 7 & 5 & 8 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix}$. Use the elimination method in Section 2.2.

(j) A, where $A = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 4 & 7 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 3 & 7 & 0 \\ 5 & 1 & 5 \end{pmatrix}$.

(Problem 38) Find the volume of the following parallelepiped.



(Problem 39) Determine which of the following sets are subspaces of $\mathbb{R}^2$. In all cases, justify your answer.

(a) $\{(x_1, x_2)^T : 5x_1 + 3x_2 = 0\}$
(b) $\{(x_1, x_2)^T : 4x_1 - 2x_2 = 1\}$
(c) $\{(x_1, x_2)^T : x_1 x_2 = 1\}$
(d) $\{(x_1, x_2)^T : x_1^2 - x_2^2 = 0\}$

(Problem 40) Determine which of the following sets are subspaces of $\mathbb{R}^{2\times 2}$. In all cases, justify your answer.

(a) The set of all 2×2 symmetric matrices, that is, the set of 2×2 matrices A such that $A = A^T$.
(b) The set of all 2×2 matrices A that satisfy $A\begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} A$.
(c) The set of all 2×2 matrices A such that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

(Problem 41) Determine which of the following sets are subspaces of $C[-1, 1]$.

(a) The set of functions f in $C[-1, 1]$ such that $f(0) = f(1)$.
(b) The set of even functions f in $C[-1, 1]$.
(c) The set of functions f in $C[-1, 1]$ such that $f(0) = 1$.
(d) The set of functions f in $C[-1, 1]$ such that $f(0) = 0$ and $f(1) = 0$.
(e) The set of functions f in $C[-1, 1]$ such that $f(0) = 0$ or $f(1) = 0$.

(Problem 42) Which of the following sets are spanning sets for $\mathbb{R}^3$?

(a) $\left\{ \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$

(c) $\left\{ \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} \right\}$

(Problem 43) Is $\begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$ in Span $\left(\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right)$?

(Problem 44) Is $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ in Span $\left(\begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right)$?

(Problem 45) Find Span $\left(\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \\ 5 \end{pmatrix} \right)$.

(Problem 46) Determine the null space of each of the following matrices.

(a) $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 6 & 3 & 1 \\ 2 & 5 & 7 \\ 3 & 3 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \end{pmatrix}$

(Problem 47) Find a spanning set for the null space of each of the following matrices.

(a) $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} 6 & 3 & 1 \\ 2 & 5 & 7 \\ 3 & 3 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \end{pmatrix}$

(Problem 48) Compute the following products.

(a) $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix}$

(b) $\begin{pmatrix} 6 & 3 & 1 \\ 2 & 5 & 7 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

(Problem 49) Use the solutions to Problems 47 and 48 to find the solution sets for the following equations.

(a) $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 20 \\ 40 \end{pmatrix}$

(b) $\begin{pmatrix} 6 & 3 & 1 \\ 2 & 5 & 7 \\ 3 & 3 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 20 \\ 40 \end{pmatrix}$

(Problem 50) Which of the following sets of vectors are linearly independent?

(a) $\left\{\begin{pmatrix}2\\4\\8\end{pmatrix},\begin{pmatrix}3\\6\\12\end{pmatrix}\right\}$ (c) $\left\{\begin{pmatrix}3\\4\\7\end{pmatrix},\begin{pmatrix}3\\5\\6\end{pmatrix},\begin{pmatrix}3\\3\\8\end{pmatrix}\right\}$ (e) $\left\{\begin{pmatrix}3\\2\\4\end{pmatrix}\right\}$

(b) $\left\{\begin{pmatrix}3\\4\\7\end{pmatrix},\begin{pmatrix}3\\5\\6\end{pmatrix}\right\}$ (d) $\left\{\begin{pmatrix}3\\4\\7\end{pmatrix},\begin{pmatrix}3\\5\\6\end{pmatrix},\begin{pmatrix}3\\4\\8\end{pmatrix}\right\}$ (f) $\left\{\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}$

(Problem 51) Is the set $\{4, x^2, 3x^2-2\}$ linearly independent?

(Problem 52) Is the set $\{x^2, x^2-3x+2, x+3\}$ linearly independent?

(Problem 53) You are given that that the set $\left\{\begin{pmatrix}4\\3\\2\\1\end{pmatrix},\begin{pmatrix}0\\1\\2\\3\end{pmatrix},\begin{pmatrix}7\\5\\8\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\\-1\end{pmatrix}\right\}$ is linearly independent. Is the set $\left\{\begin{pmatrix}4\\3\\2\\1\end{pmatrix},\begin{pmatrix}7\\5\\8\\0\end{pmatrix},\begin{pmatrix}1\\2\\1\\-1\end{pmatrix}\right\}$ linearly independent?

(Problem 54) You are given that that the set $\left\{\begin{pmatrix}4\\3\\2\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\\3\\4\end{pmatrix},\begin{pmatrix}7\\5\\8\\0\\1\end{pmatrix},\begin{pmatrix}1\\2\\1\\-1\\3\end{pmatrix}\right\}$ is linearly independent and that $\begin{pmatrix}4\\3\\2\\1\\0\end{pmatrix}-\begin{pmatrix}0\\1\\2\\3\\4\end{pmatrix}+\begin{pmatrix}7\\5\\8\\0\\1\end{pmatrix}-\begin{pmatrix}1\\2\\1\\-1\\3\end{pmatrix}=\begin{pmatrix}10\\5\\7\\-1\\-6\end{pmatrix}$. How many choices of coefficients x_1, x_2, x_3, x_4 satisfy $x_1\begin{pmatrix}4\\3\\2\\1\\0\end{pmatrix}+x_2\begin{pmatrix}0\\1\\2\\3\\4\end{pmatrix}+x_3\begin{pmatrix}7\\5\\8\\0\\1\end{pmatrix}+x_4\begin{pmatrix}1\\2\\1\\-1\\3\end{pmatrix}=\begin{pmatrix}10\\5\\7\\-1\\-6\end{pmatrix}$?

(Problem 55) You are given that that the set $\left\{\begin{pmatrix}4\\3\\2\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\2\\3\\4\end{pmatrix},\begin{pmatrix}5\\5\\3\\0\\3\end{pmatrix},\begin{pmatrix}1\\2\\1\\-1\\3\end{pmatrix}\right\}$ is linearly dependent and that $\begin{pmatrix}4\\3\\2\\1\\0\end{pmatrix}-\begin{pmatrix}0\\1\\2\\3\\4\end{pmatrix}+\begin{pmatrix}5\\5\\3\\0\\3\end{pmatrix}-\begin{pmatrix}1\\2\\1\\-1\\3\end{pmatrix}=\begin{pmatrix}8\\5\\2\\-1\\-4\end{pmatrix}$. How many choices of coefficients x_1, x_2, x_3, x_4 satisfy $x_1\begin{pmatrix}4\\3\\2\\1\\0\end{pmatrix}+x_2\begin{pmatrix}0\\1\\2\\3\\4\end{pmatrix}+x_3\begin{pmatrix}5\\5\\3\\0\\3\end{pmatrix}+x_4\begin{pmatrix}1\\2\\1\\-1\\3\end{pmatrix}=\begin{pmatrix}8\\5\\2\\-1\\-4\end{pmatrix}$?

(Problem 56) Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 7 \\ 5 \\ 3 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) Find $3\vec{v}_1 + 2\vec{v}_2$.
(b) Find $-\vec{v}_1 + 18\vec{v}_3$.
(c) Are $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ linearly independent?

(Problem 57) Let $\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

(a) Show that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly independent.
(b) Do $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ form a basis for $\mathbb{R}^3$?

(Problem 58) Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$.

(a) Show that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ span $\mathbb{R}^3$.
(b) Do $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ form a basis for $\mathbb{R}^3$?

(Problem 59) Do $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}$ span $\mathbb{R}^4$?

(Problem 60) Are $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 5 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \end{pmatrix}$ linearly independent?

(Problem 61) Let $S = \{p(x) \text{ in } P_3 : p(2) = 0\}$. Find a basis for S.

(Problem 62) Find a basis for Span $\left\{ \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 10 \\ 3 \\ 9 \end{pmatrix} \right\}$.

(Problem 63) Let $p_1(x) = x^2 + 2x + 1$, $p_2(x) = 4x^2 + 8x + 4$, $p_3(x) = 2x^2 + 6x + 4$, $p_4(x) = 3x^2 + 9x + 6$, $p_5(x) = 5x^2 - 3x + 2$. Consider the set $\{p_1(x), p_2(x), p_3(x), p_4(x), p_5(x)\}$. This set spans P_3. Pare down this set to form a basis for P_3.

(Problem 64) The set $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 7 \\ 3 \end{pmatrix}, \begin{pmatrix} 9 \\ 2 \\ 5 \\ 6 \end{pmatrix} \right\}$ spans $\mathbb{R}^4$. Pare down this set to find a basis for $\mathbb{R}^4$ among these vectors.

(Problem 65) The vectors $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$ are linearly independent. Suppose that $\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\}$ forms a basis for $\mathbb{R}^3$. What must be true of x_1, x_2, and x_3?

(Problem 66) Let $\mathcal{U} = \left[\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]$.

(a) Find the transition matrix corresponding to the change of basis from $\mathcal{U}$ to the standard basis.
(b) Find the transition matrix corresponding to the change of basis from the standard basis to $\mathcal{U}$.
(c) Find the coordinates of the vector $\begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$ with respect to the ordered basis $\mathcal{U}$.

(Problem 67) Let $\mathcal{U} = \left[\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]$. Let $\mathcal{V} = \left[\begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix} \right]$.

(a) Find the transition matrix corresponding to the change of basis from $\mathcal{V}$ to $\mathcal{U}$.
(b) Suppose that $[\vec{x}]_{\mathcal{V}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Find $[\vec{x}]_{\mathcal{U}}$.

(Problem 68) Let $\mathcal{U} = [x^2 + 2x + 1, x^2 + 4x + 4, x^2 + 6x + 9]$ be a basis for P_3 and let $\mathcal{S} = [1, x, x^2]$ be the standard basis.

(a) Find the transition matrix corresponding to the change of basis from $\mathcal{U}$ to $\mathcal{S}$.
(b) Find the transition matrix corresponding to the change of basis from $\mathcal{S}$ to $\mathcal{U}$.
(c) Find the coordinates of $p(x) = 5 + 2x + 9x^2$ with respect to the basis $\mathcal{U}$.

(Problem 69) Let $A = \begin{pmatrix} 6 & 5 & 4 \\ 1 & 2 & 3 \\ 7 & 9 & 11 \end{pmatrix}$.

(a) Find a basis for the row space of A.
(b) Find a basis for the column space of A.
(c) Find a basis for the null space of A.

(Problem 70) Let $A = \begin{pmatrix} 3 & 6 & 9 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

(a) Find a basis for the row space of A.
(b) Find a basis for the column space of A.
(c) Find a basis for the null space of A.

(Problem 71) Find the dimension of Span $\left(\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \right)$.

(Problem 72) Find the dimension of Span $\left(\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right)$.

(Problem 73) How many solutions are there to the system $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$?

(Problem 74) How many solutions are there to the system $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$?

(Problem 75) How many solutions are there to the system $\begin{pmatrix} 3 & 6 \\ 2 & 5 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$?

(Problem 76) In each of parts (a)–(f), one of statements (i)–(vi) is true. Determine which statement is true in each part.

(i) For every vector $\vec{b}$ in $\mathbb{R}^5$, the system $A\vec{x} = \vec{b}$ is inconsistent.
(ii) For every vector $\vec{b}$ in $\mathbb{R}^5$, there is exactly one solution $\vec{x}$ to $A\vec{x} = \vec{b}$.
(iii) For every vector $\vec{b}$ in $\mathbb{R}^5$, there are infinitely many solutions $\vec{x}$ to $A\vec{x} = \vec{b}$.
(iv) There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ has a unique solution.
(v) There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions.
(vi) There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ has a unique solution, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions.

(a) A is a 5×5 matrix of rank 5.
(b) A is a 5×7 matrix of rank 5.
(c) A is a 5×3 matrix of rank 3.
(d) A is a 5×4 matrix of rank 2.
(e) A is a 5×5 matrix of rank 4.
(f) A is a 5×7 matrix of rank 3.

(Problem 77) Is $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - x_2 \\ x_3 + 3x_1 \end{pmatrix}$ a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$?

(Problem 78) Is $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 x_2 \\ x_3 + 3x_1 \end{pmatrix}$ a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$?

(Problem 79) Is $L(A) = \det A$ a linear transformation $L : \mathbb{R}^{3\times 3} \to \mathbb{R}$?

(Problem 80) Is $L(p(x)) = p(0) + xp(1) + x^2 p'(0)$ a linear transformation $L : P_3 \to P_3$?

(Problem 81) Is $L(p(x)) = \int_1^2 p(x)\,dx + xp'(x)$ a linear transformation $L : P_3 \to P_3$?

(Problem 82) Let $L(f(x)) = f(2) + (x-2)f'(2) + \frac{1}{2}(x-2)^2 f''(2)$.
(a) Is L a linear transformation $L : C[1,3] \to P_3$?
(b) Is L a linear transformation $L : C^2[1,3] \to P_2$?
(c) Is L a linear transformation $L : C^2[1,3] \to P_3$?

(Problem 83) Is $L(f(x)) = f(x^2)$ a linear transformation $L : C[0,1] \to C[0,1]$?

(Problem 84) Is $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \frac{x_1^2}{x_2}$ a linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}$?

(Problem 85) Find the kernel and range of the linear transformation $L : P_3 \to P_4$ given by $L(p(x)) = x^2 p'(x)$.

(Problem 86) Find the kernel and range of the linear transformation $L : \mathbb{R}^3 \mapsto \mathbb{R}^2$ given by $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - x_2 \\ x_3 \end{pmatrix}$.

(Problem 87) Find the kernel and range of the linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}^3$ given by $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 3x_1 + 2x_2 \\ 5x_1 - 3x_2 \\ x_1 + x_2 \end{pmatrix}$.

(Problem 88) Find the matrix A that represents the linear transformation $L : \mathbb{R}^3 \mapsto \mathbb{R}^3$, where L projects each vector $\vec{x}$ onto the vector $(3, 2, 1)^T$.

(Problem 89) Find the matrix A that represents the linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$, where L rotates each vector $\pi/4$ radians counterclockwise and then doubles its length.

(Problem 90) Find the matrix A that represents the linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$, where L reflects each vector about the line $x_1 = x_2$.

(Problem 91) Let $L : P_3 \to \mathbb{R}^3$ be given by $L(p(x)) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}$. Find the matrix A that represents L with respect to the bases $\mathcal{E} = [1, x, x^2]$ and $\mathcal{F} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$.

(Problem 92) Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. Suppose that $L\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$, $L\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

(a) Find the matrix A that represents L with respect to the basis $\mathcal{U} = \left[\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right]$ in $\mathbb{R}^2$ and with respect to the standard basis in $\mathbb{R}^3$.

(b) Find the matrix B that represents L with respect to the standard basis in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

(Problem 93) Let $L(x_1, x_2) = \begin{pmatrix} 3x_1 + 2x_2 \\ x_1 - x_2 \end{pmatrix}$.

(a) Find the matrix A that represents L with respect to the standard basis.

(b) Find the matrix B that represents L with respect to the basis $\mathcal{U} = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right]$.

(Problem 94) Let L be the linear operator on P_3 given by $L(p(x)) = x^2 p''(x) + p'(x)$.

(a) Find the matrix A representing L with respect to the basis $[1, x, x^2]$.

(b) Find the matrix B representing L with respect to the basis $[1, x-3, x^2 - 6x + 9]$.

(c) Find a matrix S such that $B = S^{-1}AS$.

(Problem 95) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by $L(\vec{x}) = A\vec{x}$, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(a) Let $\mathcal{U} = \left[\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right]$. Find the matrix U that represents the change of basis from $\mathcal{U}$ to the standard basis.

(b) Find $U^{-1}AU$.

(c) Let $\vec{x}$ satisfy $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the standard coordinates of $\vec{x}$. Then find $L(\vec{x})$.

(d) Recall that $\vec{x}$ satisfies $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Use the matrix you found in part 95 to find $[L(\vec{x})]_{\mathcal{U}}$.

(e) Use your answer to parts 95 and 95 to find $L(\vec{x})$. Does your answer agree with your answer to part 95?

(Problem 96) What is the angle between the vectors $(3, 2, 4)$ and $(7, 2, 5)$?

(Problem 97) Find the vector projection of $(7, 1, 2)$ onto $(3, 5, 4)$.

(Problem 98) Find the point on the line $y = \frac{5}{7}x$ that is closest to the point $(6, 2)$.

(Problem 99) Find the equation of the plane passing through the point $(3, 5, 2)$ and normal to the vector $(1, 4, 3)$.

(Problem 100) Find the equation of the plane passing through the points $(1, 2, 3)$, $(5, 2, 4)$, and $(7, 1, 6)$.

(Problem 101) Find the distance from the point $(3, 1, 6)$ to the plane $\{(x, y, z) : 6x + 2y - 3z = 0\}$.

(Problem 102) Find the point on the plane $\{(x, y, z) : 6x - 6y - 7z = 0\}$ closest to the point $(9, 5, 1)$.

(Problem 103) You are given that $\|\vec{x}\| = 2$ and $\|\vec{y}\| = 5$. What can you say about $\langle \vec{x}, \vec{y} \rangle$?

(Problem 104) You are given that $\|\vec{x}\| = 3$ and $\langle \vec{x}, \vec{y} \rangle = -6$. What can you say about $\|\vec{y}\|$?

(Problem 105) Let $A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 5 & 7 \end{pmatrix}$. Find a basis for $N(A)$ and for $R(A^T)$.

(Problem 106) Let $S = \text{Span}\left(\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}\right)$. Find a basis for $S^\perp$.

(Problem 107) Let $S = \text{Span}\left(\begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}\right)$. Find a basis for $S^\perp$.

(Problem 108) Is there a matrix A with $\begin{pmatrix} 3 & 2 & 5 \end{pmatrix}$ in the row space of A and $\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$ in the null space of A? If so, provide an example. If not, explain why not.

(Problem 109) Is there a matrix A with $\begin{pmatrix} 3 & 2 & 5 \end{pmatrix}$ in the row space of A and $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ in the null space of A? If so, provide an example. If not, explain why not.

(Problem 110) Find the least squares solution to the system $\begin{cases} 3x_1 + 2x_2 = 5, \\ 2x_1 - 5x_2 = 7, \\ x_1 + 8x_2 = 3. \end{cases}$

(Problem 111) Find the least squares solution to the system $A\vec{x} = \vec{b}$, where $A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ -9 \end{pmatrix}$.

(Problem 112) Suppose that you have data $\begin{array}{c|c|c|c|c} x & 1 & 2 & 3 & 4 \\ \hline y & -1 & 0 & 5 & 6 \end{array}$ Find the linear function that best approximates this data. That is, find values of m and b such that $mx+b$ best approximates y.

(Problem 113) Let $V = P_3$ with the inner product $\langle p(x), q(x)\rangle = p(1)q(1) + p(2)q(2) + p(3)q(3)$. Find $\langle x^2, 3x+2\rangle$.

(Problem 114) Let $V = C[0,1]$ with the inner product $\langle f(x), g(x)\rangle = \displaystyle\int_0^1 f(x)\,g(x)\,dx$. Find $\langle e^x, e^{3x}\rangle$.

(Problem 115) Let $V = \mathbb{R}^2$ with norm $\left\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\|_1 = |x_1| + |x_2|$. Find $\left\|\begin{pmatrix} 5 \\ -12 \end{pmatrix}\right\|_1$.

(Problem 116) Let $V = \mathbb{R}^2$. Define $\left\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\|_\infty = \max(|x_1|, |x_2|)$. Is this a norm on $\mathbb{R}^2$?

(Problem 117) Let $V = \mathbb{R}^2$. Define $\left\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\|_{1/2} = (\sqrt{|x_1|} + \sqrt{|x_2|})^2$. Is this a norm on $\mathbb{R}^2$?

(Problem 118) Is $\left\{\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}, \begin{pmatrix} -2/3 \\ 2/3 \\ -1/3 \end{pmatrix}\right\}$ an orthonormal basis for $\mathbb{R}^3$?

(Problem 119) Is $\left\{\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}, \begin{pmatrix} -2/3 \\ -2/3 \\ 1/3 \end{pmatrix}\right\}$ an orthonormal basis for $\mathbb{R}^3$?

(Problem 120) Is $\left\{\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/5 \\ 1/5 \\ -2/5 \end{pmatrix}, \begin{pmatrix} -2/7 \\ 2/7 \\ -1/7 \end{pmatrix}\right\}$ an orthonormal basis for $\mathbb{R}^3$?

(Problem 121) Let $V = P_3$ with the inner product $\langle p(x), q(x)\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. Is $\left\{\dfrac{1}{\sqrt{3}}, \dfrac{x}{\sqrt{2}}, \dfrac{3x^2-2}{\sqrt{6}}\right\}$ an orthonormal basis?

(Problem 122) Let $V = P_3$ with the inner product $\langle p(x), q(x)\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. Is $\left\{\dfrac{1}{\sqrt{3}}, \dfrac{x}{\sqrt{2}}, \dfrac{2x^2-3}{\sqrt{6}}\right\}$ an orthonormal basis?

(Problem 123) Let $V = P_3$ with the inner product $\langle p(x), q(x)\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. Is $\left\{\dfrac{1}{\sqrt{3}}, \dfrac{x}{\sqrt{3}}, \dfrac{3x^2-2}{\sqrt{6}}\right\}$ an orthonormal basis?

(Problem 124) Let $V = C[0,1]$ with the inner product $\langle f(x), g(x)\rangle = \displaystyle\int_0^1 f(x)\,g(x)\,dx$. Is $\left\{1, 2\sqrt{3}\,x - \sqrt{3}\right\}$ an orthonormal set?

(Problem 125) Let $V = C[0,1]$ with the inner product $\langle f(x), g(x)\rangle = \displaystyle\int_0^1 f(x)\,g(x)\,dx$. Is $\{1, 2x-1\}$ an orthonormal set?

(Problem 126) Let $V = C[0,1]$ with the inner product $\langle f(x), g(x)\rangle = \int_0^1 f(x)\,g(x)\,dx$. Is $\left\{1, \sqrt{3}\,x\right\}$ an orthonormal set?

(Problem 127) Let $\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix}$. Find the orthogonal projection of $\vec{x}$ onto $S = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \\ -2 \end{pmatrix}\right)$.

(Problem 128) Let $\vec{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 5 \end{pmatrix}$. Find the orthogonal projection of $\vec{x}$ onto $S = \operatorname{Span}\left(\begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \end{pmatrix}\right)$.

(Problem 129) $\mathcal{U} = \left[\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/15 \\ 10/15 \\ -11/15 \end{pmatrix}, \begin{pmatrix} -14/15 \\ 5/15 \\ 2/15 \end{pmatrix}\right]$ is an orthonormal basis for $\mathbb{R}^3$. Let $\vec{x} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$. Find $[\vec{x}]_{\mathcal{U}}$.

(Problem 130) Let $V = P_3$ with the inner product $\langle p(x), q(x)\rangle = p(-2)q(-2) + p(0)q(0) + p(2)q(2)$. Then $\mathcal{U} = \left\{\frac{1}{\sqrt{3}}, \frac{x}{\sqrt{8}}, \frac{3x^2-8}{4\sqrt{3}}\right\}$ is orthonormal basis. Find $[x^2]_{\mathcal{U}}$.

(Problem 131) Let $[\vec{u}, \vec{v}]$ be an orthonormal basis for $\mathbb{R}^2$. Suppose that $\|\vec{x}\| = 3$ and that $\langle \vec{x}, \vec{u}\rangle = 2$. Find $|\langle \vec{x}, \vec{v}\rangle|$.

(Problem 132) Let $\mathcal{U} = [\vec{u}, \vec{v}, \vec{w}]$ be an orthonormal basis for $\mathbb{R}^3$. Suppose that $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. If $\langle \vec{x}, \vec{u}\rangle = 3$, $\|\vec{x}\| = 7$, and $\vec{x}$ is orthogonal to $\vec{w}$, what can you say about a, b, and c?

(Problem 133) Let $V = C[0,\pi]$ with inner product $\langle f(x), g(x)\rangle = \frac{2}{\pi}\int_0^\pi f(x)\,g(x)\,dx$. You are given that $\{\sin x, \sin 2x, \sin 3x\}$ is an orthonormal set in V. Find $\langle 7\sin x - 3\sin 2x, 4\sin 2x + 3\sin 3x\rangle$.

(Problem 134) Suppose that $\vec{x}$ and $\vec{y}$ are two vectors in $\mathbb{R}^3$ and that the angle θ between $\vec{x}$ and $\vec{y}$ is $\pi/7$. Let Q be a 3×3 orthogonal matrix. What is the angle between $Q\vec{x}$ and $Q\vec{y}$?

(Problem 135) You are given that Q is a 3×3 matrix and that $\left\langle Q\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, Q\begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}\right\rangle = 22$. Is it possible that Q is orthogonal? How do you know?

(Problem 136) Let $V = C[0,1]$ with inner product $\langle f(x), g(x) = \int_0^1 f(x)\,g(x)\,dx$. You are given that $S = \{1, 2x-1, 6x^2-6x+1\}$ is an orthogonal set. Find the best least squares approximation to x^3 by an element of $\operatorname{Span} S$.

(Problem 137) A careless person computes that the eigenvalues of $A = \begin{pmatrix} 3 & 1 & 2 & 4 & 6 \\ 5 & 1 & 3 & 4 & 2 \\ 7 & 6 & 2 & 1 & 3 \\ 2 & 5 & -1 & -3 & 1 \\ 1 & 2 & 0 & 4 & 2 \end{pmatrix}$ are 0, 1, 2, 3, and 4. Explain why you may be sure that the careless person is wrong.

(Problem 138) A careless person computes that the eigenvalues of $A = \begin{pmatrix} 3 & 1 & 2 & 4 & 6 \\ 5 & 1 & 3 & 4 & 2 \\ 8 & 2 & 5 & 8 & 8 \\ 2 & 5 & -1 & -3 & 1 \\ 1 & 2 & 0 & 4 & 2 \end{pmatrix}$ are 0, -1, 2, 3, and 4. Is there an easy way to be sure that the careless person is wrong? (Row reducing or computing the determinant of a 5×5 matrix is not considered easy.)

(Problem 139) Find *one* eigenvalue of the matrix $A = \begin{pmatrix} 7 & 5 & 1 & 3 & 2 \\ 14 & 10 & 2 & 6 & 4 \\ 2 & 4 & 1 & 3 & 5 \\ 9 & 9 & 2 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

(Problem 140) Find *one* eigenvalue of the matrix $A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$.

(Problem 141) You are given that the eigenvalues of $A = \begin{pmatrix} 522 & -1522 & 698 & 428 & -891 \\ 516 & -1512 & 694 & 426 & -885 \\ 343 & -1008 & 463 & 284 & -589 \\ 713 & -2090 & 959 & 590 & -1223 \\ 32 & -92 & 42 & 26 & -53 \end{pmatrix}$ are 0, 1, 2, 3, and 4. What are the eigenvalues of A^3? What can you say about the eigenvectors of A and A^3?

(Problem 142) For each of the following matrices, either find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$ or state that A is defective.

(a) $\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & -1 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 3 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} -82 & 60 \\ -112 & 82 \end{pmatrix}$

(d) $\begin{pmatrix} 4 & -1 & -2 \\ -4 & 0 & 4 \\ 4 & -2 & -2 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 4 \end{pmatrix}$

(f) $\begin{pmatrix} 4 & 3 & -3 \\ -5 & 4 & 5 \\ -5 & 3 & 6 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

(h) $\begin{pmatrix} 6 & -1 & -3 \\ -4 & 0 & 4 \\ 6 & -2 & -3 \end{pmatrix}$

(Problem 143) Use your answer to Problem 142 to find $\begin{pmatrix} -82 & 60 \\ -112 & 82 \end{pmatrix}^7$.

(Problem 144) Use your answer to Problem 142 to find $\begin{pmatrix} 4 & -1 & -2 \\ -4 & 0 & 4 \\ 4 & -2 & -2 \end{pmatrix}^4$.

(Problem 145) Use your answer to Problem 142 to find a matrix B such that $B^2 = \begin{pmatrix} 4 & 3 & -3 \\ -5 & 4 & 5 \\ -5 & 3 & 6 \end{pmatrix}$.

(Problem 146) Let A be a 3×3 matrix. You are given that $A\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$, that $A\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and that 2 is an eigenvalue of A^T. Find the eigenspace of A^T corresponding to the eigenvalue 2.

(Problem 147) Let $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$. Find an orthogonal matrix Q and a diagonal matrix D such that $QDQ^T = A$ or state that it cannot be done.

(Problem 148) Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$. Find an orthogonal matrix Q and a diagonal matrix D such that $QDQ^T = A$ or state that it cannot be done.

(Problem 149) Let $A = \begin{pmatrix} 7 & 1 & -2 \\ 1 & 7 & -2 \\ -2 & -2 & 10 \end{pmatrix}$. Find an orthogonal matrix Q and a diagonal matrix D such that $QDQ^T = A$ or state that it cannot be done.

(Problem 150) Let $A = \begin{pmatrix} 7 & 1 & -2 \\ -1 & 7 & -2 \\ 2 & 2 & 10 \end{pmatrix}$. Find an orthogonal matrix Q and a diagonal matrix D such that $QDQ^T = A$ or state that it cannot be done.

(Problem 151) Find an orthonormal basis for Span $\left(\begin{pmatrix} 4 \\ 6 \\ 6 \\ 9 \end{pmatrix}, \begin{pmatrix} 17 \\ 4 \\ 20 \\ 22 \end{pmatrix}, \begin{pmatrix} 30 \\ 2 \\ 34 \\ 31 \end{pmatrix}\right)$.

(Problem 152) Find an orthonormal basis for Span $\left(\begin{pmatrix} 0 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 17 \\ -3 \\ 0 \\ 6 \end{pmatrix}\right)$.

(Problem 153) Find an orthonormal basis for P_4 under the inner product $\langle p(x), q(x)\rangle = p(0)q(0)+p(1)q(1)+p(2)q(2)+p(3)q(3)$.

(Problem 154) Find an orthonormal basis for P_3 under the inner product $\langle p(x), q(x)\rangle = \int_0^1 p(x)\,q(x)\,dx$.

Answer key

(Problem 1) Use back substitution to solve the system

$$\begin{cases} 3x_1 - 2x_2 + \ \ x_3 = 5, \\ \qquad - 5x_2 - 2x_3 = 1, \\ \qquad\qquad\quad 3x_3 = 7. \end{cases}$$

(Answer 1) $x_3 = 7/3$, $x_2 = -17/15$, $x_1 = 2/15$.

(Problem 2) In each of the following systems, interpret each equation as a line in the plane. For each system, graph the lines and determine geometrically the number of solutions.

(a) $\begin{cases} 3x_1 - 2x_2 = 6, \\ -6x_1 + 4x_2 = -12. \end{cases}$

There are infinitely many solutions.

(b) $\begin{cases} 3x_1 - 2x_2 = 6, \\ -4x_1 + 2x_2 = 8. \end{cases}$

There is exactly one solution.

(c) $\begin{cases} 4x_1 - 2x_2 = 8, \\ -2x_1 + \ \ x_2 = 6. \end{cases}$

There are no solutions.

(Problem 3) Solve each of the following systems by writing the augmented matrix and finding a row equivalent matrix in strict upper triangular or reduced row echelon form. Indicate whether the system is consistent. If it is, find the solution set.

(a) $\begin{cases} 4x_1 + \ \ x_2 - 5x_3 = -9, \\ 2x_1 - 5x_2 + 3x_3 = 1, \\ 3x_1 + 2x_2 - 2x_3 = 1. \end{cases}$

The augmented matrix is $\left(\begin{array}{ccc|c} 4 & 1 & -5 & -9 \\ 2 & -5 & 3 & 1 \\ 3 & 2 & -2 & 1 \end{array}\right)$.

The system is consistent. The only solution is $x_1 = 1$, $x_2 = 2$, $x_3 = 3$.

(b) $\begin{cases} 2x_1 + 6x_2 - 3x_3 = 2, \\ 3x_1 + 9x_2 + \ \ x_3 = 14, \\ \ \ x_1 + 3x_2 - 2x_3 = 0. \end{cases}$

The augmented matrix is $\left(\begin{array}{ccc|c} 2 & 6 & -3 & 2 \\ 3 & 9 & 1 & 14 \\ 1 & 3 & -2 & 0 \end{array}\right)$. It is row equivalent to $\left(\begin{array}{ccc|c} 1 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

The system is consistent. The solution set is $\{(4 - 3\alpha, \alpha, 2)\}$ or $\{(x_1, x_2, 2) : x_1 = 4 - 3x_2\}$.

(c) $\begin{cases} 2x_1 - \ \ x_2 - 4x_3 = 2, \\ 4x_1 + 3x_2 + 2x_3 = 25, \\ 3x_1 + 4x_2 + 5x_3 = 10. \end{cases}$

The augmented matrix is $\left(\begin{array}{ccc|c} 2 & -1 & -4 & 2 \\ 4 & 3 & 2 & 25 \\ 3 & 4 & 5 & 10 \end{array}\right)$. It is row equivalent to $\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right)$.

The system is inconsistent.

(Problem 4) Which of the following matrices are in reduced row echelon form?

(a) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
Yes.

(b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
Yes.

(c) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$
Yes.

(d) $\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}$
No.

(e) $\begin{pmatrix} 3 & 7 \\ 0 & 0 \end{pmatrix}$
No.

(f) $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
No.

(g) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
No.

(h) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Yes.

(i) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{pmatrix}$
No.

(j) $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 5 \end{pmatrix}$
No.

(k) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$
No.

(l) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
Yes.

(m) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
Yes.

(n) $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$
No.

(o) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 4 \end{pmatrix}$
No.

(p) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Yes.

(q) $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{pmatrix}$
Yes.

(r) $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Yes.

(s) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3 & 6 \end{pmatrix}$

No.

(t) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Yes.

(Problem 5) Which of the following matrices are in reduced row echelon form?

(a) $\begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

Yes.

(b) $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

No.

(c) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Yes.

(d) $\begin{pmatrix} 1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Yes.

(e) $\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 6 \end{pmatrix}$

Yes.

(f) $\begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$

No.

(g) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

No.

(h) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Yes.

(i) $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Yes.

(j) $\begin{pmatrix} 1 & 3 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Yes.

(k) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$

No.

(l) $\begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

No.

(m) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Yes.

(n) $\begin{pmatrix} 2 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

No.

(o) $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

No.

(p) $\begin{pmatrix} 1 & 3 & 7 \\ 0 & 0 & 0 \end{pmatrix}$

Yes.

(q) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Yes.

(r) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

No.

(s) $\begin{pmatrix} 0 & 3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

No.

(t) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

No.

(Problem 6) Determine all possible values of x_1, x_2, x_3, and x_4 in the following traffic flow diagram.

(Answer 6) $\{(\alpha-100, \alpha+200, \alpha+100, \alpha)\}$ or $\{(x_1, x_2, x_3, x_4) : x_1 = x_4-100, x_2 = x_4+200, x_3 = x_4+100\}$.

(Problem 7) Determine all possible values of x_1, x_2, x_3, and x_4 in the following traffic flow diagram.

(Answer 7) $x_1 = 500$, $x_2 = 900$, $x_3 = 300$, $x_4 = 700$.

(Problem 8) The solution set to Problem 6 had a free variable. Problem 7 has a unique solution. Why is that?

(Problem 9) Determine the values of the currents i_1, i_2, and i_3 in the following circuit diagram.

(Answer 9) $i_1 = 10$ amperes, $i_2 = 5$ amperes, $i_3 = 5$ amperes.

(Problem 10) Carbon monoxide reacts with hydrogen to produce octane and water. The chemical equation for this reaction is of the form $x_1\mathrm{CO} + x_2\mathrm{H}_2 \to x_3\mathrm{C}_8\mathrm{H}_{18} + x_4\mathrm{H}_2\mathrm{O}$. Determine (nonzero integer) values of x_1, x_2, x_3 and x_4 to balance the equation.

(Answer 10) $x_1 = 8$, $x_2 = 17$, $x_3 = 1$, $x_4 = 8$, or any multiple of those numbers.

(Problem 11) Find the following products or state that they are not meaningful.

(a) $\begin{pmatrix} 6 & 2 & 1 \\ 3 & 4 & 8 \\ 0 & 7 & 5 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \\ -2 & 3 \end{pmatrix}$.

$\begin{pmatrix} 6 & 2 & 1 \\ 3 & 4 & 8 \\ 0 & 7 & 5 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -12 & 29 \\ -3 & 29 \end{pmatrix}$.

(b) $\begin{pmatrix} 0 & -1 \\ 1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 6 & 2 & 1 \\ 3 & 4 & 8 \\ 0 & 7 & 5 \end{pmatrix}$.

The product is not meaningful.

(c) $\begin{pmatrix} 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$.

$\begin{pmatrix} 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \end{pmatrix}$.

(d) $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 3 & 4 & 1 \end{pmatrix}$.

$\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 6 & 8 & 2 \\ 9 & 12 & 3 \end{pmatrix}$

(Problem 12) Write the system $\begin{cases} 4x_1 + x_2 - 5x_3 = -9, \\ 2x_1 - 5x_2 + 3x_3 = 1, \\ 3x_1 + 2x_2 - 2x_3 = 1. \end{cases}$ as a matrix equation.

(Answer 12) $\begin{pmatrix} 4 & 1 & -5 \\ 2 & -5 & 3 \\ 3 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -9 \\ 1 \\ 1 \end{pmatrix}$.

(Problem 13) Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix}$. Write $\vec{b} = \begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix}$ as a linear combination of the columns of A. Then solve $A\vec{x} = \vec{b}$.

(Answer 13) $\vec{b} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 4 \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$. The solution to $A\vec{x} = \vec{b}$ is $\vec{x} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

(Problem 14) Find a nonzero matrix A such that $A^2 = O$.

(Answer 14) There are many possible answers, including $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}$.

(Problem 15) Find a nonzero 2×2 matrix A such that $A \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(Answer 15) There are many possible answers, including $A = \begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ -2 & 3 \end{pmatrix}$, $\begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix}$.

(Problem 16) Compute the inverses of the following matrices.

(a) $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

$$\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}.$$

(b) $\begin{pmatrix} 0 & 4 \\ 2 & 1 \end{pmatrix}$

$$\begin{pmatrix} 0 & 4 \\ 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1/8 & 1/2 \\ 1/4 & 0 \end{pmatrix}.$$

(c) $\begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}$

$$\begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 1/3 & -2/3 \\ 0 & 1/2 \end{pmatrix}.$$

(Problem 17) Let $A = \begin{pmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}$. Find A^2.

(Answer 17) $A^2 = A$.

(Problem 18) Find an elementary matrix E such that $\begin{pmatrix} 5 & 7 & 9 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Answer 18) $E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(Problem 19) Find an elementary matrix E such that $\begin{pmatrix} 1 & 2 & 3 \\ -10 & -13 & -10 \\ 7 & 9 & 8 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Answer 19) $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$.

(Problem 20) Find an elementary matrix E such that $\begin{pmatrix} 4 & 8 & 12 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Answer 20) $E = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(Problem 21) Find an elementary matrix E such that $\begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 9 & 8 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Answer 21) $E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(Problem 22) Find an elementary matrix E such that $\begin{pmatrix} 1 & 2 & 3 \\ -8 & -10 & -12 \\ 7 & 9 & 8 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Answer 22) $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(Problem 23) Find an elementary matrix E such that $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 21 & 27 & 24 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 9 & 8 \end{pmatrix}$.

(Answer 23) $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(Problem 24) Find an elementary matrix E such that $\begin{pmatrix} 7 & 9 & 8 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Answer 24) $E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

(Problem 25) Find an elementary matrix E such that $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 10 & 15 & 17 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Answer 25) $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$.

(Problem 26) Find an elementary matrix E such that $\begin{pmatrix} 1 & 2 & 3 \\ 7 & 9 & 8 \\ 4 & 5 & 6 \end{pmatrix} = E \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 8 \end{pmatrix}$.

(Answer 26) $E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

(Problem 27)

(a) Find three elementary matrices E_3, E_2 and E_1 such that $E_3E_2E_1\begin{pmatrix} 2 & 5 & 4 \\ 6 & 2 & 1 \\ -4 & 3 & 5 \end{pmatrix}$ is upper triangular.

One possible answer is $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

Another possible answer is $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

(b) Find a LU factorization of $M = \begin{pmatrix} 2 & 5 & 4 \\ 6 & 2 & 1 \\ -4 & 3 & 5 \end{pmatrix}$. That is, find an upper triangular matrix U and a lower triangular matrix L, where the diagonal entries of L are all 1s, such that $LU = M$.

$U = \begin{pmatrix} 2 & 5 & 4 \\ 0 & -13 & -11 \\ 0 & 0 & 2 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -1 & 1 \end{pmatrix}$.

(Problem 28) Compute the inverses of the following matrices.

(a) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/7 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(Problem 29) Compute the inverses of the following matrices.

(a) $\begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$$\begin{pmatrix} 1 & 3 & 2 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} -0.333333333333 & -1.22222222222 & 0.888888888889 \\ 0.666666666667 & -0.555555555556 & 0.222222222222 \\ -0.333333333333 & 1.44444444444 & -0.777777777778 \end{pmatrix}$$

(b) $\begin{pmatrix} 4 & 0 & 3 \\ 1 & 3 & -2 \\ 2 & 5 & -3 \end{pmatrix}$

$$\begin{pmatrix} 4 & 0 & 3 \\ 1 & 3 & -2 \\ 2 & 5 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 15 & -9 \\ -1 & -18 & 11 \\ -1 & -20 & 12 \end{pmatrix}$$

(c) $\begin{pmatrix} 2 & 8 & 4 \\ 4 & 0 & 3 \\ 1 & 5 & 2 \end{pmatrix}$

$$\begin{pmatrix} 2 & 8 & 4 \\ 4 & 0 & 3 \\ 1 & 5 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -1.5 & 0.4 & 2.4 \\ -0.5 & 0.0 & 1.0 \\ 2.0 & -0.2 & -3.2 \end{pmatrix}$$

(Problem 30) Use your answers to Problem 29 to find the inverses of the following matrices.

(a) $\begin{pmatrix} 1 & 4 & 7 \\ 3 & 5 & 8 \\ 2 & 6 & 9 \end{pmatrix}$

$$\begin{pmatrix} 1 & 4 & 7 \\ 3 & 5 & 8 \\ 2 & 6 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} -0.333333333333 & 0.666666666667 & -0.333333333333 \\ -1.22222222222 & -0.555555555556 & 1.44444444444 \\ 0.888888888889 & 0.222222222222 & -0.777777777778 \end{pmatrix}$$

(b) $\begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 5 \\ 3 & -2 & -3 \end{pmatrix}$

$$\begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 5 \\ 3 & -2 & -3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ 15 & -18 & -20 \\ -9 & 11 & 12 \end{pmatrix}$$

(c) $\begin{pmatrix} 2 & 4 & 1 \\ 8 & 0 & 5 \\ 4 & 3 & 2 \end{pmatrix}$

$$\begin{pmatrix} 2 & 4 & 1 \\ 8 & 0 & 5 \\ 4 & 3 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -1.5 & -0.5 & 2.0 \\ 0.4 & 0.0 & -0.2 \\ 2.4 & 1.0 & -3.2 \end{pmatrix}$$

(Problem 31) Use your answers to Problem 29 to solve the following equations.

(a) $\begin{pmatrix} 2 & 8 & 4 \\ 4 & 0 & 3 \\ 1 & 5 & 2 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$

$\vec{x} = \begin{pmatrix} 8.7 \\ 3.5 \\ -10.4 \end{pmatrix}.$

(b) $\begin{pmatrix} 4 & 0 & 3 \\ 1 & 3 & -2 \\ 2 & 5 & -3 \end{pmatrix} X = \begin{pmatrix} 7 & 2 \\ 3 & 5 \\ 2 & 1 \end{pmatrix}$

$X = \begin{pmatrix} 34 & 68 \\ -39 & -81 \\ -43 & -90 \end{pmatrix}.$

(Problem 32) Let $A = \begin{pmatrix} 0 & 2 & -1 & 2 & 1 \\ 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & 2 & 0 & -3 & 0 \\ -1 & 1 & 0 & 3 & 0 \end{pmatrix}.$

(a) Find $A \begin{pmatrix} 3 \\ 0 \\ 3 \\ 1 \\ 1 \end{pmatrix}.$

$$\begin{pmatrix} 0 & 2 & -1 & 2 & 1 \\ 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & 2 & 0 & -3 & 0 \\ -1 & 1 & 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(b) Is A singular? How do you know?

Yes; if $A\vec{x} = \vec{0}$ has a nonzero solution $\vec{x}$ then A is singular.

(Problem 33) Let $A = \begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 1 & 0 & -2 & 0 & 2 \\ 1 & 0 & 2 & -1 & 0 \\ 1 & 2 & 0 & 3 & 0 \\ -1 & 0 & -2 & 1 & 0 \end{pmatrix}$.

(a) Find $A \begin{pmatrix} -3 \\ 5 \\ 8 \\ 4 \\ 10 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 1 & 0 & -2 & 0 & 2 \\ 1 & 0 & 2 & -1 & 0 \\ 1 & 2 & 0 & 3 & 0 \\ -1 & 0 & -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ 5 \\ 8 \\ 4 \\ 10 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 9 \\ 19 \\ -9 \end{pmatrix}.$$

(b) Find $A \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 0 & -1 & 2 & 1 \\ 1 & 0 & -2 & 0 & 2 \\ 1 & 0 & 2 & -1 & 0 \\ 1 & 2 & 0 & 3 & 0 \\ -1 & 0 & -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ 9 \\ 19 \\ -9 \end{pmatrix}.$$

(c) Is A singular? How do you know?

Yes; if there is a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ has more than one solution, then A is singular.

(Problem 34) You are given that $A = \begin{pmatrix} 3 & 5 & 7 & 2 & 1 \\ 1 & 5 & -2 & 3 & 2 \\ 1 & 7 & 2 & -1 & 2 \\ 2 & 1 & 6 & 3 & 0 \\ -1 & 9 & -2 & 4 & 3 \end{pmatrix}$ is invertible. Is there a solution to $\begin{pmatrix} 3 & 5 & 7 & 2 & 1 \\ 1 & 5 & -2 & 3 & 2 \\ 1 & 7 & 2 & -1 & 2 \\ 2 & 1 & 6 & 3 & 0 \\ -1 & 9 & -2 & 4 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$? (You don't have to find $\vec{x}$.)

(Answer 34) Yes; $\vec{x} = \begin{pmatrix} 3 & 5 & 7 & 2 & 1 \\ 1 & 5 & -2 & 3 & 2 \\ 1 & 7 & 2 & -1 & 2 \\ 2 & 1 & 6 & 3 & 0 \\ -1 & 9 & -2 & 4 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$.

(Problem 35) You are given that there are no solutions to $\begin{pmatrix} 3 & 5 & 7 & 2 & 1 \\ 3 & 5 & -2 & 3 & 2 \\ 1 & 14 & 4 & -2 & 4 \\ 2 & 1 & 6 & 3 & 0 \\ -1 & 6 & -2 & 4 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$. Is $A = \begin{pmatrix} 3 & 5 & 7 & 2 & 1 \\ 3 & 5 & -2 & 3 & 2 \\ 1 & 14 & 4 & -2 & 4 \\ 2 & 1 & 6 & 3 & 0 \\ -1 & 6 & -2 & 4 & 3 \end{pmatrix}$ invertible? How do you know?

(Answer 35) No; if A were invertible, then there would be a solution $A^{-1}\vec{b}$ to $A\vec{x} = \vec{b}$ for every $\vec{b}$ of appropriate length.

(Problem 36) You are given that $A = \begin{pmatrix} 2 & 1 & 6 & 3 & 0 \\ 3 & 5 & 7 & 2 & 1 \\ 1 & 7 & 2 & -1 & 2 \\ -1 & 9 & -2 & 4 & 3 \\ 0 & 14 & -4 & 7 & 5 \end{pmatrix}$ is invertible. Find a matrix in reduced row echelon form that is row equivalent to A.

(Problem 37) Find the determinants of the following matrices.

(a) $\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$

2

(b) $\begin{pmatrix} 6 & 8 \\ 3 & 4 \end{pmatrix}$

0

(c) $\begin{pmatrix} 9 & 7 & 2 \\ 4 & 1 & 3 \\ 0 & 2 & 5 \end{pmatrix}$

$$\det \begin{pmatrix} 9 & 7 & 2 \\ 4 & 1 & 3 \\ 0 & 2 & 5 \end{pmatrix} = -133$$

(d) $\begin{pmatrix} 18 & 14 & 4 \\ 8 & 2 & 6 \\ 0 & 4 & 10 \end{pmatrix}$

$$\det \begin{pmatrix} 18 & 14 & 4 \\ 8 & 2 & 6 \\ 0 & 4 & 10 \end{pmatrix} = 8 \det \begin{pmatrix} 9 & 7 & 2 \\ 4 & 1 & 3 \\ 0 & 2 & 5 \end{pmatrix} = -1064$$

(e) $\begin{pmatrix} 3 & 5 & 2 \\ 4 & 1 & 2 \\ 2 & 9 & 2 \end{pmatrix}$

0

(f) $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 \\ 7 & 5 & 8 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix}$. Use the cofactor expansion in Section 2.1.

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(g) $\begin{pmatrix} 4 & 3 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 7 & 5 & 8 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix}$. Use the elimination method in Section 2.2.

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(h) $\begin{pmatrix} 0 & 2 & 4 & 6 \\ 4 & 3 & 2 & 1 \\ 7 & 5 & 8 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix}$. Use the cofactor expansion in Section 2.1.

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(i) $\begin{pmatrix} 0 & 2 & 4 & 6 \\ 4 & 5 & 6 & 7 \\ 7 & 5 & 8 & 0 \\ 1 & 2 & 1 & -1 \end{pmatrix}$. Use the elimination method in Section 2.2.

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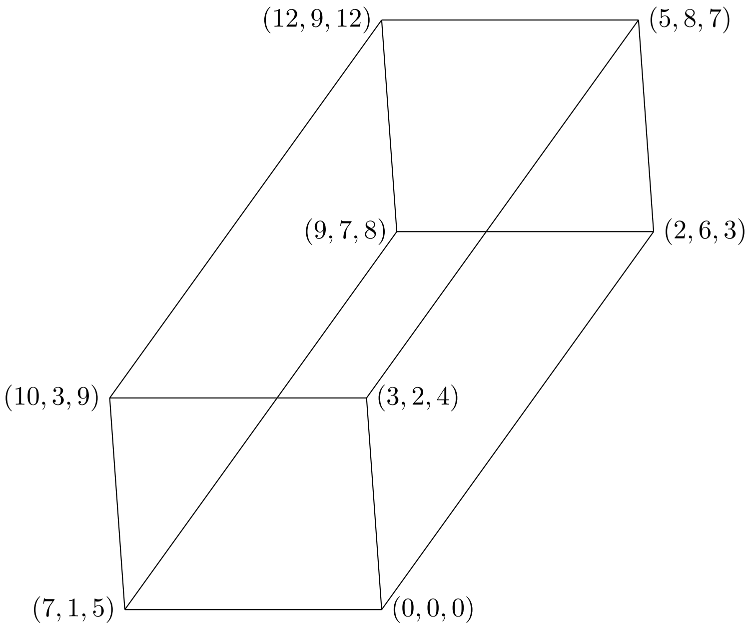
(j) A, where $A = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 4 & 7 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 3 & 7 & 0 \\ 5 & 1 & 5 \end{pmatrix}$.

$(3 \times 4 \times 1) \times (2 \times 7 \times 5) = 840.$

(Problem 38) Find the volume of the following parallelepiped.



(Answer 38) 57

(Problem 39) Determine which of the following sets are subspaces of $\mathbb{R}^2$. In all cases, justify your answer.

(a) $\{(x_1, x_2)^T : 5x_1 + 3x_2 = 0\}$
$\{(x_1, x_2)^T : 5x_1 + 3x_2 = 0\}$ is a subspace.

(b) $\{(x_1, x_2)^T : 4x_1 - 2x_2 = 1\}$
$\{(x_1, x_2)^T : 4x_1 - 2x_2 = 1\}$ is not a subspace.

(c) $\{(x_1, x_2)^T : x_1 x_2 = 1\}$
$\{(x_1, x_2)^T : x_1 x_2 = 1\}$ is not a subspace.

(d) $\{(x_1, x_2)^T : x_1^2 - x_2^2 = 0\}$
$\{(x_1, x_2)^T : x_1^2 - x_2^2 = 0\}$ is not subspace.

(Problem 40) Determine which of the following sets are subspaces of $\mathbb{R}^{2\times 2}$. In all cases, justify your answer.

(a) The set of all 2×2 symmetric matrices, that is, the set of 2×2 matrices A such that $A = A^T$.
This is a subspace.

(b) The set of all 2×2 matrices A that satisfy $A \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} A$.
This is a subspace.

(c) The set of all 2×2 matrices A such that $A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
This is not a subspace.

(Problem 41) Determine which of the following sets are subspaces of $C[-1,1]$.

(a) The set of functions f in $C[-1,1]$ such that $f(0) = f(1)$.
This is a subspace.

(b) The set of even functions f in $C[-1,1]$.
This is a subspace.

(c) The set of functions f in $C[-1,1]$ such that $f(0) = 1$.
This is not a subspace.

(d) The set of functions f in $C[-1,1]$ such that $f(0) = 0$ and $f(1) = 0$.
This is a subspace.

(e) The set of functions f in $C[-1,1]$ such that $f(0) = 0$ or $f(1) = 0$.
This is not a subspace.

(Problem 42) Which of the following sets are spanning sets for $\mathbb{R}^3$?

(a) $\left\{ \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$
This is a spanning set.

(b) $\left\{ \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\}$
This is a spanning set.

(c) $\left\{ \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} \right\}$
This is not a spanning set.

(Problem 43) Is $\begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$ in Span $\left(\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right)$?

(Answer 43) Yes, $\begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$ is in Span $\left(\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right)$.

(Problem 44) Is $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ in Span $\left(\begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right)$?

(Answer 44) No, $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ is not in Span $\left(\begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right)$.

(Problem 45) Find Span $\left(\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \\ 5 \end{pmatrix} \right)$.

(Answer 45) $\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 + x_2 - 2x_3 = 0 \right\}$.

(Problem 46) Determine the null space of each of the following matrices.

(a) $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$

$$N\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : 3x_1 + 2x_2 = 0\right\}.$$

(b) $\begin{pmatrix} 6 & 3 & 1 \\ 2 & 5 & 7 \\ 3 & 3 & 3 \end{pmatrix}$

$$N\begin{pmatrix} 6 & 3 & 1 \\ 2 & 5 & 7 \\ 3 & 3 & 3 \end{pmatrix} = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 = \frac{2}{3}x_3,\ x_2 = -\frac{5}{3}x_3\right\}.$$

(c) $\begin{pmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \end{pmatrix}$

$$N\begin{pmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \end{pmatrix} = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 4x_1 + 3x_2 + 2x_3 = 0\right\}.$$

(Problem 47) Find a spanning set for the null space of each of the following matrices.

(a) $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$

$$N\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} = \mathrm{Span}\left(\begin{pmatrix} 2 \\ -3 \end{pmatrix}\right).$$

(b) $\begin{pmatrix} 6 & 3 & 1 \\ 2 & 5 & 7 \\ 3 & 3 & 3 \end{pmatrix}$

$$N\begin{pmatrix} 6 & 3 & 1 \\ 2 & 5 & 7 \\ 3 & 3 & 3 \end{pmatrix} = \mathrm{Span}\left(\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}\right).$$

(c) $\begin{pmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \end{pmatrix}$

$$N\begin{pmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \end{pmatrix} = \mathrm{Span}\left(\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}\right).$$ (There are many other solutions.)

(Problem 48) Compute the following products.

(a) $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 7 \end{pmatrix}$

$$\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 20 \\ 40 \end{pmatrix}.$$

(b) $\begin{pmatrix} 6 & 3 & 1 \\ 2 & 5 & 7 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$\begin{pmatrix} 6 & 3 & 1 \\ 2 & 5 & 7 \\ 3 & 3 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}.$$

(c) $\begin{pmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 40 \end{pmatrix}$$

(Problem 49) Use the solutions to Problems 47 and 48 to find the solution sets for the following equations.

(a) $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 20 \\ 40 \end{pmatrix}$

$\left\{ \begin{pmatrix} 2 \\ 7 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ -3 \end{pmatrix} \right\}.$

(b) $\begin{pmatrix} 6 & 3 & 1 \\ 2 & 5 & 7 \\ 3 & 3 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}$

$\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} \right\}.$

(c) $\begin{pmatrix} 4 & 3 & 2 \\ 8 & 6 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 20 \\ 40 \end{pmatrix}$

$\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} \right\}.$

(Problem 50) Which of the following sets of vectors are linearly independent?

(a) $\left\{ \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 12 \end{pmatrix} \right\}$

The set $\left\{ \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 12 \end{pmatrix} \right\}$ is linearly dependent.

(b) $\left\{ \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} \right\}$

The set $\left\{ \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} \right\}$ is linearly independent.

(c) $\left\{ \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix} \right\}$

The set $\left\{ \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix} \right\}$ is linearly dependent.

(d) $\left\{ \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 8 \end{pmatrix} \right\}$

The set $\left\{ \begin{pmatrix} 3 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 8 \end{pmatrix} \right\}$ is linearly independent.

(e) $\left\{ \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \right\}$

The set $\left\{ \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} \right\}$ is linearly independent.

(f) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

The set $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ is not linearly independent.

(Problem 51) Is the set $\{4, x^2, 3x^2 - 2\}$ linearly independent?

(Answer 51) No.

(Problem 52) Is the set $\{x^2, x^2 - 3x + 2, x + 3\}$ linearly independent?

(Answer 52) Yes.

(Problem 53) You are given that that the set $\left\{ \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix} \right\}$ is linearly independent. Is the set $\left\{ \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix} \right\}$ linearly independent?

(Answer 53) Yes.

(Problem 54) You are given that that the set $\left\{ \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 8 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \\ 3 \end{pmatrix} \right\}$ is linearly independent and that $\begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 7 \\ 5 \\ 8 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 7 \\ -1 \\ -6 \end{pmatrix}$. How many choices of coefficients x_1, x_2, x_3, x_4 satisfy $x_1 \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ 5 \\ 8 \\ 0 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \\ 7 \\ -1 \\ -6 \end{pmatrix}$?

(Answer 54) Only one: $x_1 = x_3 = 1$, $x_2 = x_4 = -1$.

(Problem 55) You are given that that the set $\left\{ \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 3 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \\ 3 \end{pmatrix} \right\}$ is linearly dependent and that $\begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \\ 3 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 2 \\ -1 \\ -4 \end{pmatrix}$. How many choices of coefficients x_1, x_2, x_3, x_4 satisfy $x_1 \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + x_3 \begin{pmatrix} 5 \\ 5 \\ 3 \\ 0 \\ 3 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 5 \\ 2 \\ -1 \\ -4 \end{pmatrix}$?

(Answer 55) Infinitely many.

(Problem 56) Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 7 \\ 5 \\ 3 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) Find $3\vec{v}_1 + 2\vec{v}_2$.

$3\vec{v}_1 + 2\vec{v}_2 = \begin{pmatrix} 17 \\ 16 \\ 15 \end{pmatrix}$.

(b) Find $-\vec{v}_1 + 18\vec{v}_3$.

$-\vec{v}_1 + 18\vec{v}_3 = \begin{pmatrix} 17 \\ 16 \\ 15 \end{pmatrix}$.

(c) Are $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ linearly independent?

No; if they were linearly independent then there would be a unique choice of x_1, x_2, x_3 such that $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \begin{pmatrix} 17 \\ 16 \\ 15 \end{pmatrix}$, but there are at least two choices, $x_1 = 3, x_2 = 2, x_3 = 0$ and $x_1 = -1, x_2 = 0, x_3 = 18$.

(Problem 57) Let $\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 7 \\ 0 \\ 2 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

(a) Show that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are linearly independent.

(b) Do $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ form a basis for $\mathbb{R}^3$?

Yes; $\dim \mathbb{R}^3 = 3$, so any three linearly independent vectors form a basis for $\mathbb{R}^3$.

(Problem 58) Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 5 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$.

(a) Show that $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ span $\mathbb{R}^3$.

(b) Do $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ form a basis for $\mathbb{R}^3$?

Yes; $\dim \mathbb{R}^3 = 3$, so any three vectors that span $\mathbb{R}^3$ form a basis for $\mathbb{R}^3$.

(Problem 59) Do $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 7 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}$ span $\mathbb{R}^4$?

(Answer 59) No; $\dim \mathbb{R}^4 = 4$, and so no spanning set can contain fewer than 4 vectors.

(Problem 60) Are $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\begin{pmatrix} 7 \\ 2 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} 8 \\ 5 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 4 \\ 2 \end{pmatrix}$ linearly independent?

(Answer 60) No; $\dim \mathbb{R}^4 = 4$, and so no linearly independent set of vectors in $\mathbb{R}^4$ can contain more than 4 vectors.

(Problem 61) Let $S = \{p(x) \text{ in } P_3 : p(2) = 0\}$. Find a basis for S.

(Answer 61) There are many possible answers; $\{x - 2, (x - 2)^2\}$ is an example.

(Problem 62) Find a basis for Span $\left\{\begin{pmatrix}3\\2\\4\end{pmatrix}, \begin{pmatrix}6\\4\\8\end{pmatrix}, \begin{pmatrix}7\\1\\5\end{pmatrix}, \begin{pmatrix}10\\3\\9\end{pmatrix}\right\}$.

(Answer 62) $\left\{\begin{pmatrix}3\\2\\4\end{pmatrix}, \begin{pmatrix}7\\1\\5\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}3\\2\\4\end{pmatrix}, \begin{pmatrix}10\\3\\9\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}6\\4\\8\end{pmatrix}, \begin{pmatrix}7\\1\\5\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}6\\4\\8\end{pmatrix}, \begin{pmatrix}10\\3\\9\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}7\\1\\5\end{pmatrix}, \begin{pmatrix}10\\3\\9\end{pmatrix}\right\}$, or $\left\{\begin{pmatrix}13\\6\\0\end{pmatrix}, \begin{pmatrix}11\\0\\6\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}13\\6\\0\end{pmatrix}, \begin{pmatrix}0\\11\\-13\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}11\\0\\6\end{pmatrix}, \begin{pmatrix}0\\11\\-13\end{pmatrix}\right\}$.

(Problem 63) Let $p_1(x) = x^2 + 2x + 1$, $p_2(x) = 4x^2 + 8x + 4$, $p_3(x) = 2x^2 + 6x + 4$, $p_4(x) = 3x^2 + 9x + 6$, $p_5(x) = 5x^2 - 3x + 2$. Consider the set $\{p_1(x), p_2(x), p_3(x), p_4(x), p_5(x)\}$. This set spans P_3. Pare down this set to form a basis for P_3.

(Answer 63) The following four solutions are all valid:

- $\{p_1(x), p_3(x), p_5(x)\}$.
- $\{p_1(x), p_4(x), p_5(x)\}$.
- $\{p_2(x), p_3(x), p_5(x)\}$.
- $\{p_2(x), p_4(x), p_5(x)\}$.

(Problem 64) The set $\left\{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}1\\2\\1\\2\end{pmatrix}, \begin{pmatrix}3\\6\\5\\8\end{pmatrix}, \begin{pmatrix}5\\2\\7\\3\end{pmatrix}, \begin{pmatrix}9\\2\\5\\6\end{pmatrix}\right\}$ spans $\mathbb{R}^4$. Pare down this set to find a basis for $\mathbb{R}^4$ among these vectors.

(Answer 64) $\left\{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}1\\2\\1\\2\end{pmatrix}, \begin{pmatrix}5\\2\\7\\3\end{pmatrix}, \begin{pmatrix}9\\2\\5\\6\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}1\\2\\3\\4\end{pmatrix}, \begin{pmatrix}3\\6\\5\\8\end{pmatrix}, \begin{pmatrix}5\\2\\7\\3\end{pmatrix}, \begin{pmatrix}9\\2\\5\\6\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}1\\2\\1\\2\end{pmatrix}, \begin{pmatrix}3\\6\\5\\8\end{pmatrix}, \begin{pmatrix}5\\2\\7\\3\end{pmatrix}, \begin{pmatrix}9\\2\\5\\6\end{pmatrix}\right\}$.

(Problem 65) The vectors $\begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}5\\4\\2\end{pmatrix}$ are linearly independent. Suppose that $\left\{\begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}5\\4\\2\end{pmatrix}, \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\right\}$ forms a basis for $\mathbb{R}^3$. What must be true of x_1, x_2, and x_3?

(Answer 65) $x_2 - 2x_3 \neq 0$.

(Problem 66) Let $\mathcal{U} = \left[\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]$.

(a) Find the transition matrix corresponding to the change of basis from $\mathcal{U}$ to the standard basis.

$U = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 6 & 0 \\ 5 & 6 & 1 \end{pmatrix}$.

(b) Find the transition matrix corresponding to the change of basis from the standard basis to $\mathcal{U}$.

$U^{-1} = \begin{pmatrix} 3 & 2 & -3 \\ -1 & -1/2 & 1 \\ -9 & -7 & 10 \end{pmatrix}$.

(c) Find the coordinates of the vector $\begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$ with respect to the ordered basis $\mathcal{U}$.

$\begin{pmatrix} -10 \\ 3.5 \\ 36 \end{pmatrix}$

(Problem 67) Let $\mathcal{U} = \left[\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]$. Let $\mathcal{V} = \left[\begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix} \right]$.

(a) Find the transition matrix corresponding to the change of basis from $\mathcal{V}$ to $\mathcal{U}$.

$\begin{pmatrix} 1 & 9 & 20 \\ 0.5 & -2.5 & -6.5 \\ -5 & -28 & -60 \end{pmatrix}$.

(b) Suppose that $[\vec{x}]_{\mathcal{V}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Find $[\vec{x}]_{\mathcal{U}}$.

$[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} 79 \\ -24 \\ -241 \end{pmatrix}$

(Problem 68) Let $\mathcal{U} = [x^2 + 2x + 1, x^2 + 4x + 4, x^2 + 6x + 9]$ be a basis for P_3 and let $\mathcal{S} = [1, x, x^2]$ be the standard basis.

(a) Find the transition matrix corresponding to the change of basis from $\mathcal{U}$ to $\mathcal{S}$.

$U = \begin{pmatrix} 1 & 4 & 9 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{pmatrix}$.

(b) Find the transition matrix corresponding to the change of basis from $\mathcal{S}$ to $\mathcal{U}$.

$U^{-1} = \begin{pmatrix} 0.5 & -1 & 0.5 \\ -1.25 & 2 & -0.75 \\ 3 & -3 & 1 \end{pmatrix}$.

(c) Find the coordinates of $p(x) = 5 + 2x + 9x^2$ with respect to the basis $\mathcal{U}$.

$[p(x)]_{\mathcal{U}} = \begin{pmatrix} 5 \\ -9 \\ 18 \end{pmatrix}$

(Problem 69) Let $A = \begin{pmatrix} 6 & 5 & 4 \\ 1 & 2 & 3 \\ 7 & 9 & 11 \end{pmatrix}$.

(a) Find a basis for the row space of A.
$\{(1 \quad 0 \quad -1), (0 \quad 1 \quad 2)\}$.

(b) Find a basis for the column space of A.
$\left\{ \begin{pmatrix} 6 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 9 \end{pmatrix} \right\}$.

(c) Find a basis for the null space of A.
$\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$.

(Problem 70) Let $A = \begin{pmatrix} 3 & 6 & 9 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

(a) Find a basis for the row space of A.
$\{(1 \quad 2 \quad 3)\}$.

(b) Find a basis for the column space of A.
$\left\{ \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right\}$.

(c) Find a basis for the null space of A.
There are many possible answers, including $\left\{ \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} \right\}$.

(Problem 71) Find the dimension of Span $\left(\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \right)$.

(Answer 71) 2.

(Problem 72) Find the dimension of Span $\left(\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right)$.

(Answer 72) 3.

(Problem 73) How many solutions are there to the system $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$?

(Answer 73) Infinitely many.

(Problem 74) How many solutions are there to the system $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$?

(Answer 74) None.

(Problem 75) How many solutions are there to the system $\begin{pmatrix} 3 & 6 \\ 2 & 5 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$?

(Answer 75) Exactly one.

(Problem 76) In each of parts (a)–(f), one of statements (i)–(vi) is true. Determine which statement is true in each part.

(a) A is a 5×5 matrix of rank 5.
(ii); for every vector $\vec{b}$ in $\mathbb{R}^5$, there is exactly one solution $\vec{x}$ to $A\vec{x}=\vec{b}$.

(b) A is a 5×7 matrix of rank 5.
(iii); For every vector $\vec{b}$ in $\mathbb{R}^5$, there are infinitely many solutions $\vec{x}$ to $A\vec{x}=\vec{b}$.

(c) A is a 5×3 matrix of rank 3.
(iv); There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x}=\vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x}=\vec{b}$ has a unique solution.

(d) A is a 5×4 matrix of rank 2.
(v); There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x}=\vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x}=\vec{b}$ has infinitely many solutions.

(e) A is a 5×5 matrix of rank 4.
(v); There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x}=\vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x}=\vec{b}$ has infinitely many solutions.

(f) A is a 5×7 matrix of rank 3.
(v); There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x}=\vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x}=\vec{b}$ has infinitely many solutions.

(Problem 77) Is $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3\end{pmatrix}\right) = \begin{pmatrix} x_1-x_2 \\ x_3+3x_1\end{pmatrix}$ a linear transformation $L:\mathbb{R}^3\to\mathbb{R}^2$?

(Answer 77) Yes.

(Problem 78) Is $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3\end{pmatrix}\right) = \begin{pmatrix} x_1x_2 \\ x_3+3x_1\end{pmatrix}$ a linear transformation $L:\mathbb{R}^3\to\mathbb{R}^2$?

(Answer 78) No.

(Problem 79) Is $L(A)=\det A$ a linear transformation $L:\mathbb{R}^{3\times 3}\to\mathbb{R}$?

(Answer 79) No.

(Problem 80) Is $L(p(x))=p(0)+xp(1)+x^2p'(0)$ a linear transformation $L:P_3\to P_3$?

(Answer 80) Yes.

(Problem 81) Is $L(p(x))=\displaystyle\int_1^2 p(x)\,dx+xp'(x)$ a linear transformation $L:P_3\to P_3$?

(Answer 81) Yes.

(Problem 82) Let $L(f(x))=f(2)+(x-2)f'(2)+\dfrac{1}{2}(x-2)^2f''(2)$.

(a) Is L a linear transformation $L:C[1,3]\to P_3$?
No; there are some $f(x)$ in $C[1,3]$, such as $f(x)=|x-2|$, such that $L(f(x))$ is not defined.

(b) Is L a linear transformation $L:C^2[1,3]\to P_2$?
No; there are some $f(x)$ in $C[1,3]$, such as $f(x)=x^2$, such that $L(f(x))$ is not in P_2.

(c) Is L a linear transformation $L:C^2[1,3]\to P_3$?
Yes.

(Problem 83) Is $L(f(x)) = f(x^2)$ a linear transformation $L : C[0,1] \to C[0,1]$?

(Answer 83) Yes.

(Problem 84) Is $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \dfrac{x_1^2}{x_2}$ a linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}$?

(Answer 84) No.

(Problem 85) Find the kernel and range of the linear transformation $L : P_3 \to P_4$ given by $L(p(x)) = x^2 p'(x)$.

(Answer 85) $\ker(L) = \{\alpha : \alpha \text{ is a real number}\}$, $L(P_3) = \{\alpha x^2 + \beta x^3\}$.

(Problem 86) Find the kernel and range of the linear transformation $L : \mathbb{R}^3 \mapsto \mathbb{R}^2$ given by $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - x_2 \\ x_3 \end{pmatrix}$.

(Answer 86) $\ker(L) = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)$, $L(\mathbb{R}^3) = \mathbb{R}^2$.

(Problem 87) Find the kernel and range of the linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}^3$ given by $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 3x_1 + 2x_2 \\ 5x_1 - 3x_2 \\ x_1 + x_2 \end{pmatrix}$.

(Answer 87) $\ker(L) = \{\vec{0}\}$, $L(\mathbb{R}^2) = \operatorname{Span}\left(\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right) = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 8x_1 - x_2 - 19x_3 = 0\right\}$.

(Problem 88) Find the matrix A that represents the linear transformation $L : \mathbb{R}^3 \mapsto \mathbb{R}^3$, where L projects each vector $\vec{x}$ onto the vector $(3, 2, 1)^T$.

(Answer 88) $A = \begin{pmatrix} 9/14 & 3/7 & 3/14 \\ 3/7 & 2/7 & 1/7 \\ 3/14 & 1/7 & 1/14 \end{pmatrix}$.

(Problem 89) Find the matrix A that represents the linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$, where L rotates each vector $\pi/4$ radians counterclockwise and then doubles its length.

(Problem 90) Find the matrix A that represents the linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$, where L reflects each vector about the line $x_1 = x_2$.

(Problem 91) Let $L : P_3 \to \mathbb{R}^3$ be given by $L(p(x)) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}$. Find the matrix A that represents L with respect to the bases $\mathcal{E} = [1, x, x^2]$ and $\mathcal{F} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$.

(Answer 91) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$.

(Problem 92) Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. Suppose that $L\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$, $L\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

(a) Find the matrix A that represents L with respect to the basis $\mathcal{U} = \left[\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right]$ in $\mathbb{R}^2$ and with respect to the standard basis in $\mathbb{R}^3$.

$\begin{pmatrix} 1 & 2 \\ 5 & 3 \\ 7 & 4 \end{pmatrix}$.

(b) Find the matrix B that represents L with respect to the standard basis in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

(Problem 93) Let $L(x_1, x_2) = \begin{pmatrix} 3x_1 + 2x_2 \\ x_1 - x_2 \end{pmatrix}$.

(a) Find the matrix A that represents L with respect to the standard basis.

$A = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}$.

(b) Find the matrix B that represents L with respect to the basis $\mathcal{U} = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right]$.

$B = \begin{pmatrix} -3 & -2 \\ 5 & 5 \end{pmatrix}$.

(Problem 94) Let L be the linear operator on P_3 given by $L(p(x)) = x^2 p''(x) + p'(x)$.

(a) Find the matrix A representing L with respect to the basis $[1, x, x^2]$.

$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

(b) Find the matrix B representing L with respect to the basis $[1, x-3, x^2 - 6x + 9]$.

$A = \begin{pmatrix} 0 & 1 & -18 \\ 0 & 0 & 14 \\ 0 & 0 & 2 \end{pmatrix}$.

(c) Find a matrix S such that $B = S^{-1}AS$.

$S = \begin{pmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix}$.

(Problem 95) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by $L(\vec{x}) = A\vec{x}$, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(a) Let $\mathcal{U} = \left[\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right]$. Find the matrix U that represents the change of basis from $\mathcal{U}$ to the standard basis.

$U = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}$.

(b) Find $U^{-1}AU$.

$U^{-1}AU = \begin{pmatrix} 66/13 & 34/13 \\ 8/13 & -1/13 \end{pmatrix}$.

(c) Let $\vec{x}$ satisfy $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the standard coordinates of $\vec{x}$. Then find $L(\vec{x})$.

$\vec{x} = \begin{pmatrix} 12 \\ 17 \end{pmatrix}$; $L(\vec{x}) = (\,46 \quad 104\,)$.

(d) Recall that $\vec{x}$ satisfies $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Use the matrix you found in part 95 to find $[L(\vec{x})]_{\mathcal{U}}$.

$[L(\vec{x})]_{\mathcal{U}} = \begin{pmatrix} 266/13 \\ 22/13 \end{pmatrix}$.

(e) Use your answer to parts 95 and 95 to find $L(\vec{x})$. Does your answer agree with your answer to part 95?

$L(\vec{x}) = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 266/13 \\ 22/13 \end{pmatrix} = \begin{pmatrix} 598/13 \\ 1352/13 \end{pmatrix} = \begin{pmatrix} 46 \\ 104 \end{pmatrix}$. Yes, they do agree.

(Problem 96) What is the angle between the vectors $(3, 2, 4)$ and $(7, 2, 5)$?

(Problem 97) Find the vector projection of $(7, 1, 2)$ onto $(3, 5, 4)$.

(Problem 98) Find the point on the line $y = \frac{5}{7}x$ that is closest to the point $(6, 2)$.

(Problem 99) Find the equation of the plane passing through the point $(3, 5, 2)$ and normal to the vector $(1, 4, 3)$.

(Answer 99) $x + 4y + 3z = 29$.

(Problem 100) Find the equation of the plane passing through the points $(1, 2, 3)$, $(5, 2, 4)$, and $(7, 1, 6)$.

(Answer 100) $\{(x, y, z) : x - 6y - 4z = -23\}$.

(Problem 101) Find the distance from the point $(3, 1, 6)$ to the plane $\{(x, y, z) : 6x + 2y - 3z = 0\}$.

(Problem 102) Find the point on the plane $\{(x, y, z) : 6x - 6y - 7z = 0\}$ closest to the point $(9, 5, 1)$.

(Problem 103) You are given that $\|\vec{x}\| = 2$ and $\|\vec{y}\| = 5$. What can you say about $\langle \vec{x}, \vec{y} \rangle$?

(Answer 103) $-10 \leq \langle \vec{x}, \vec{y} \rangle \leq 10$.

(Problem 104) You are given that $\|\vec{x}\| = 3$ and $\langle \vec{x}, \vec{y} \rangle = -6$. What can you say about $\|\vec{y}\|$?

(Answer 104) $\|\vec{y}\| \geq 2$.

(Problem 105) Let $A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 5 & 7 \end{pmatrix}$. Find a basis for $N(A)$ and for $R(A^T)$.

(Answer 105) $N(A) = \operatorname{Span}\left(\begin{pmatrix} 6 \\ 17 \\ -13 \end{pmatrix}\right)$; $R(A^T) = \operatorname{Span}\left(\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}\right)$.

(Problem 106) Let $S = \operatorname{Span}\left(\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}\right)$. Find a basis for $S^\perp$.

(Answer 106) There are many possible answers, including $\left\{\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}\right\}$.

(Problem 107) Let $S = \operatorname{Span}\left(\begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}\right)$. Find a basis for $S^\perp$.

(Answer 107) There are many possible answers, including $\left\{\begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ -2 \end{pmatrix}\right\}$.

(Problem 108) Is there a matrix A with $\begin{pmatrix} 3 & 2 & 5 \end{pmatrix}$ in the row space of A and $\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$ in the null space of A? If so, provide an example. If not, explain why not.

(Answer 108) Yes; there are many such A, including $A = \begin{pmatrix} 3 & 2 & 5 \end{pmatrix}$, $A = \begin{pmatrix} 3 & 2 & 5 \\ 6 & 4 & 10 \end{pmatrix}$, and $A = \begin{pmatrix} 6 & 4 & 10 \\ -9 & -6 & -15 \\ 12 & 8 & 20 \end{pmatrix}$.

(Problem 109) Is there a matrix A with $\begin{pmatrix} 3 & 2 & 5 \end{pmatrix}$ in the row space of A and $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ in the null space of A? If so, provide an example. If not, explain why not.

(Answer 109) This is not possible. $\left\langle \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \right\rangle = 16$. If $\begin{pmatrix} 3 & 2 & 5 \end{pmatrix}$ is in the row space of A, then $\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$ is in the column space of A^T, and the null space of A and the column space of A^T are orthogonal; but $\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ are not orthogonal, and so if $\begin{pmatrix} 3 & 2 & 5 \end{pmatrix}$ is in the row space then $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ is not in the null space.

(Problem 110) Find the least squares solution to the system $\begin{cases} 3x_1 + 2x_2 = 5, \\ 2x_1 - 5x_2 = 7, \\ x_1 + 8x_2 = 3. \end{cases}$

(Problem 111) Find the least squares solution to the system $A\vec{x} = \vec{b}$, where $A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ -9 \end{pmatrix}$.

(Problem 112) Suppose that you have data $\begin{array}{c|c|c|c|c} x & 1 & 2 & 3 & 4 \\ \hline y & -1 & 0 & 5 & 6 \end{array}$ Find the linear function that best approximates this data. That is, find values of m and b such that $mx + b$ best approximates y.

(Problem 113) Let $V = P_3$ with the inner product $\langle p(x), q(x)\rangle = p(1)q(1) + p(2)q(2) + p(3)q(3)$. Find $\langle x^2, 3x + 2\rangle$.

(Problem 114) Let $V = C[0, 1]$ with the inner product $\langle f(x), g(x)\rangle = \displaystyle\int_0^1 f(x)\, g(x)\, dx$. Find $\langle e^x, e^{3x}\rangle$.

(Answer 114) $\dfrac{e^4 - 1}{4}$.

(Problem 115) Let $V = \mathbb{R}^2$ with norm $\left\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\|_1 = |x_1| + |x_2|$. Find $\left\|\begin{pmatrix} 5 \\ -12 \end{pmatrix}\right\|_1$.

(Answer 115) 17.

(Problem 116) Let $V = \mathbb{R}^2$. Define $\left\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\|_\infty = \max(|x_1|, |x_2|)$. Is this a norm on $\mathbb{R}^2$?

(Answer 116) Yes.

(Problem 117) Let $V = \mathbb{R}^2$. Define $\left\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\|_{1/2} = (\sqrt{|x_1|} + \sqrt{|x_2|})^2$. Is this a norm on $\mathbb{R}^2$?

(Answer 117) No. The triangle inequality fails: $\left\|\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\|_{1/2} = 4 > 1+1 = \left\|\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\|_{1/2} + \left\|\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\|_{1/2}$.

(Problem 118) Is $\left\{\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}, \begin{pmatrix} -2/3 \\ 2/3 \\ -1/3 \end{pmatrix}\right\}$ an orthonormal basis for $\mathbb{R}^3$?

(Answer 118) Yes.

(Problem 119) Is $\left\{\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}, \begin{pmatrix} -2/3 \\ -2/3 \\ 1/3 \end{pmatrix}\right\}$ an orthonormal basis for $\mathbb{R}^3$?

(Answer 119) No.

(Problem 120) Is $\left\{\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/5 \\ 1/5 \\ -2/5 \end{pmatrix}, \begin{pmatrix} -2/7 \\ 2/7 \\ -1/7 \end{pmatrix}\right\}$ an orthonormal basis for $\mathbb{R}^3$?

(Answer 120) No.

(Problem 121) Let $V = P_3$ with the inner product $\langle p(x), q(x)\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. Is $\left\{\frac{1}{\sqrt{3}}, \frac{x}{\sqrt{2}}, \frac{3x^2-2}{\sqrt{6}}\right\}$ an orthonormal basis?

(Answer 121) Yes.

(Problem 122) Let $V = P_3$ with the inner product $\langle p(x), q(x)\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. Is $\left\{\frac{1}{\sqrt{3}}, \frac{x}{\sqrt{2}}, \frac{2x^2-3}{\sqrt{6}}\right\}$ an orthonormal basis?

(Answer 122) No.

(Problem 123) Let $V = P_3$ with the inner product $\langle p(x), q(x)\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$. Is $\left\{\frac{1}{\sqrt{3}}, \frac{x}{\sqrt{3}}, \frac{3x^2-2}{\sqrt{6}}\right\}$ an orthonormal basis?

(Answer 123) No.

(Problem 124) Let $V = C[0,1]$ with the inner product $\langle f(x), g(x)\rangle = \int_0^1 f(x)\,g(x)\,dx$. Is $\left\{1, 2\sqrt{3}\,x - \sqrt{3}\right\}$ an orthonormal set?

(Answer 124) Yes.

(Problem 125) Let $V = C[0,1]$ with the inner product $\langle f(x), g(x)\rangle = \int_0^1 f(x)\,g(x)\,dx$. Is $\{1, 2x-1\}$ an orthonormal set?

(Answer 125) No.

(Problem 126) Let $V = C[0,1]$ with the inner product $\langle f(x), g(x)\rangle = \int_0^1 f(x)\,g(x)\,dx$. Is $\left\{1, \sqrt{3}\,x\right\}$ an orthonormal set?

(Answer 126) No.

(Problem 127) Let $\vec{x} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix}$. Find the orthogonal projection of $\vec{x}$ onto $S = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \\ -2 \end{pmatrix}\right)$.

(Problem 128) Let $\vec{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 5 \end{pmatrix}$. Find the orthogonal projection of $\vec{x}$ onto $S = \operatorname{Span}\left(\begin{pmatrix} 4 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \\ 2 \end{pmatrix}\right)$.

(Problem 129) $\mathcal{U} = \left[\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \begin{pmatrix} 2/15 \\ 10/15 \\ -11/15 \end{pmatrix}, \begin{pmatrix} -14/15 \\ 5/15 \\ 2/15 \end{pmatrix}\right]$ is an orthonormal basis for $\mathbb{R}^3$. Let $\vec{x} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$. Find $[\vec{x}]_{\mathcal{U}}$.

(Problem 130) Let $V = P_3$ with the inner product $\langle p(x), q(x)\rangle = p(-2)q(-2) + p(0)q(0) + p(2)q(2)$. Then $\mathcal{U} = \left\{\frac{1}{\sqrt{3}}, \frac{x}{\sqrt{8}}, \frac{3x^2-8}{4\sqrt{3}}\right\}$ is orthonormal basis. Find $[x^2]_{\mathcal{U}}$.

(Answer 130) $[x^2]_{\mathcal{U}} = \begin{pmatrix} 8/\sqrt{3} \\ 0 \\ 4/\sqrt{3} \end{pmatrix}$.

(Problem 131) Let $[\vec{u}, \vec{v}]$ be an orthonormal basis for $\mathbb{R}^2$. Suppose that $\|\vec{x}\| = 3$ and that $\langle \vec{x}, \vec{u} \rangle = 2$. Find $|\langle \vec{x}, \vec{v} \rangle|$.

(Answer 131) $\sqrt{5}$

(Problem 132) Let $\mathcal{U} = [\vec{u}, \vec{v}, \vec{w}]$ be an orthonormal basis for $\mathbb{R}^3$. Suppose that $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. If $\langle \vec{x}, \vec{u} \rangle = 3$, $\|\vec{x}\| = 7$, and $\vec{x}$ is orthogonal to $\vec{w}$, what can you say about a, b, and c?

(Answer 132) $a = 3$, $c = 0$, $b = \pm\sqrt{40}$.

(Problem 133) Let $V = C[0, \pi]$ with inner product $\langle f(x), g(x) \rangle = \dfrac{2}{\pi} \displaystyle\int_0^\pi f(x)\, g(x)\, dx$. You are given that $\{\sin x, \sin 2x, \sin 3x\}$ is an orthonormal set in V. Find $\langle 7 \sin x - 3 \sin 2x, 4 \sin 2x + 3 \sin 3x \rangle$.

(Answer 133) -12

(Problem 134) Suppose that $\vec{x}$ and $\vec{y}$ are two vectors in $\mathbb{R}^3$ and that the angle θ between $\vec{x}$ and $\vec{y}$ is $\pi/7$. Let Q be a 3×3 orthogonal matrix. What is the angle between $Q\vec{x}$ and $Q\vec{y}$?

(Answer 134) Also $\pi/7$.

(Problem 135) You are given that Q is a 3×3 matrix and that $\left\langle Q \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, Q \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} \right\rangle = 22$. Is it possible that Q is orthogonal? How do you know?

(Answer 135) No; if Q was orthogonal then we would have that $\left\langle Q \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, Q \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} \right\rangle = 21$.

(Problem 136) Let $V = C[0, 1]$ with inner product $\langle f(x), g(x) = \displaystyle\int_0^1 f(x)\, g(x)\, dx$. You are given that $S = \{1, 2x - 1, 6x^2 - 6x + 1\}$ is an orthogonal set. Find the best least squares approximation to x^3 by an element of $\operatorname{Span} S$.

(Problem 137) A careless person computes that the eigenvalues of $A = \begin{pmatrix} 3 & 1 & 2 & 4 & 6 \\ 5 & 1 & 3 & 4 & 2 \\ 7 & 6 & 2 & 1 & 3 \\ 2 & 5 & -1 & -3 & 1 \\ 1 & 2 & 0 & 4 & 2 \end{pmatrix}$ are 0, 1, 2, 3, and 4. Explain why you may be sure that the careless person is wrong.

(Answer 137) If a $n \times n$ matrix A has n distinct eigenvalues, then their sum is equal to the sum of the diagonal elements. But $0 + 1 + 2 + 3 + 4 = 10$ and $3 + 1 + 2 - 3 + 2 = 5$, and so the eigenvalues of A cannot be 0, 1, 2, 3, and 4.

(Problem 138) A careless person computes that the eigenvalues of $A = \begin{pmatrix} 3 & 1 & 2 & 4 & 6 \\ 5 & 1 & 3 & 4 & 2 \\ 8 & 2 & 5 & 8 & 8 \\ 2 & 5 & -1 & -3 & 1 \\ 1 & 2 & 0 & 4 & 2 \end{pmatrix}$ are 0, -1, 2, 3, and 4. Is there an easy way to be sure that the careless person is wrong? (Row reducing or computing the determinant of a 5×5 matrix is not considered easy.)

(Answer 138) Not that I can see.

(Problem 139) Find *one* eigenvalue of the matrix $A = \begin{pmatrix} 7 & 5 & 1 & 3 & 2 \\ 14 & 10 & 2 & 6 & 4 \\ 2 & 4 & 1 & 3 & 5 \\ 9 & 9 & 2 & 6 & 7 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

(Answer 139) A is singular, and so $\lambda = 0$ is an eigenvalue.

(Problem 140) Find *one* eigenvalue of the matrix $A = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}$.

(Answer 140) $A - I$ is singular, and so $\lambda = 1$ is an eigenvalue.

(Problem 141) You are given that the eigenvalues of $A = \begin{pmatrix} 522 & -1522 & 698 & 428 & -891 \\ 516 & -1512 & 694 & 426 & -885 \\ 343 & -1008 & 463 & 284 & -589 \\ 713 & -2090 & 959 & 590 & -1223 \\ 32 & -92 & 42 & 26 & -53 \end{pmatrix}$ are 0, 1, 2, 3, and 4. What are the eigenvalues of A^3? What can you say about the eigenvectors of A and A^3?

(Answer 141) Every eigenvector of A is an eigenvector of A^3. The eigenvalues of A^3 are 0, 1, 8, 27, and 64.

(Problem 142) For each of the following matrices, either find an invertible matrix S and a diagonal matrix D such that $A = SDS^{-1}$ or state that A is defective.

(a) $\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & -1 & 3 \end{pmatrix}$

This matrix is defective.

(b) $\begin{pmatrix} 0 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 3 & -1 \end{pmatrix}$

This matrix is defective.

(c) $\begin{pmatrix} -82 & 60 \\ -112 & 82 \end{pmatrix}$

$S = \begin{pmatrix} 5 & 3 \\ 7 & 4 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

(d) $\begin{pmatrix} 4 & -1 & -2 \\ -4 & 0 & 4 \\ 4 & -2 & -2 \end{pmatrix}$

$S = \begin{pmatrix} 1 & i & -i \\ 0 & 2 & 2 \\ 1 & 2i & -2i \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix}$

(e) $\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 4 \end{pmatrix}$

$S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(f) $\begin{pmatrix} 4 & 3 & -3 \\ -5 & 4 & 5 \\ -5 & 3 & 6 \end{pmatrix}$

$S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

$S = \begin{pmatrix} 1 & -i & i \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1+2i & 0 \\ 0 & 0 & 1-2i \end{pmatrix}$

(h) $\begin{pmatrix} 6 & -1 & -3 \\ -4 & 0 & 4 \\ 6 & -2 & -3 \end{pmatrix}$

$S = \begin{pmatrix} 1 & i & -i \\ 0 & 2 & 2 \\ 1 & 2i & -2i \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix}$

(Problem 143) Use your answer to Problem 142 to find $\begin{pmatrix} -82 & 60 \\ -112 & 82 \end{pmatrix}^7$.

(Answer 143) $\begin{pmatrix} -5248 & 3840 \\ -7168 & 5248 \end{pmatrix}$

(Problem 144) Use your answer to Problem 142 to find $\begin{pmatrix} 4 & -1 & -2 \\ -4 & 0 & 4 \\ 4 & -2 & -2 \end{pmatrix}^4$.

(Answer 144) $\begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{pmatrix}$

(Problem 145) Use your answer to Problem 142 to find a matrix B such that $B^2 = \begin{pmatrix} 4 & 3 & -3 \\ -5 & 4 & 5 \\ -5 & 3 & 6 \end{pmatrix}$.

(Problem 146) Let A be a 3×3 matrix. You are given that $A \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = 5 \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$, that $A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 7 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, and that 2 is an eigenvalue of A^T. Find the eigenspace of A^T corresponding to the eigenvalue 2.

(Answer 146) $\left\{ \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}^\perp = \mathrm{Span}\left(\begin{pmatrix} -2 \\ -5 \\ 4 \end{pmatrix} \right)$.

(Problem 147) Let $A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$. Find an orthogonal matrix Q and a diagonal matrix D such that $QDQ^T = A$ or state that it cannot be done.

(Answer 147) It cannot be done.

(Problem 148) Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$. Find an orthogonal matrix Q and a diagonal matrix D such that $QDQ^T = A$ or state that it cannot be done.

(Answer 148) There are many possible answers, including $D = \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix}$, $Q = \begin{pmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$.

(Problem 149) Let $A = \begin{pmatrix} 7 & 1 & -2 \\ 1 & 7 & -2 \\ -2 & -2 & 10 \end{pmatrix}$. Find an orthogonal matrix Q and a diagonal matrix D such that $QDQ^T = A$ or state that it cannot be done.

(Answer 149) There are many possible answers, including
$D = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$, $Q = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \end{pmatrix}$.

(Problem 150) Let $A = \begin{pmatrix} 7 & 1 & -2 \\ -1 & 7 & -2 \\ 2 & 2 & 10 \end{pmatrix}$. Find an orthogonal matrix Q and a diagonal matrix D such that $QDQ^T = A$ or state that it cannot be done.

(Answer 150) It cannot be done.

(Problem 151) Find an orthonormal basis for $\operatorname{Span}\left(\begin{pmatrix}4\\6\\6\\9\end{pmatrix}, \begin{pmatrix}17\\4\\20\\22\end{pmatrix}, \begin{pmatrix}30\\2\\34\\31\end{pmatrix}\right)$.

(Answer 151) There are many possible answers, including $\left[\begin{pmatrix}4/13\\6/13\\6/13\\9/13\end{pmatrix}, \begin{pmatrix}9/15\\-8/15\\8/15\\4/15\end{pmatrix}\right]$.

(Problem 152) Find an orthonormal basis for $\operatorname{Span}\left(\begin{pmatrix}0\\1\\2\\2\end{pmatrix}, \begin{pmatrix}4\\3\\3\\0\end{pmatrix}, \begin{pmatrix}17\\-3\\0\\6\end{pmatrix}\right)$.

(Answer 152) There are many possible answers, including $\left[\begin{pmatrix}0\\1/3\\2/3\\2/3\end{pmatrix}, \begin{pmatrix}4/5\\2/5\\1/5\\-2/5\end{pmatrix}, \begin{pmatrix}9/15\\-8/15\\-4/15\\8/15\end{pmatrix}\right]$.

(Problem 153) Find an orthonormal basis for P_4 under the inner product $\langle p(x), q(x)\rangle = p(0)q(0)+p(1)q(1)+p(2)q(2)+p(3)q(3)$.

(Answer 153) There are many possible answers, such as $\left[\frac{1}{2}, \frac{x-1.5}{\sqrt{5}}, \frac{x^2-3x+1}{2}, \frac{10x^3-45x^2+47x-3}{\sqrt{180}}\right]$.

(Problem 154) Find an orthonormal basis for P_3 under the inner product $\langle p(x), q(x)\rangle = \int_0^1 p(x)\,q(x)\,dx$.

(Answer 154) There are many possible answers, including $[1, 2\sqrt{3}x-\sqrt{3}, 6\sqrt{5}x^2-6\sqrt{5}x+\sqrt{5}]$.