Math 3083, Spring 2019

(Problem 1) Let $A = \begin{pmatrix} 6 & 5 & 4 \\ 1 & 2 & 3 \\ 7 & 9 & 11 \end{pmatrix}$.

(a) Find a basis for the row space of A.
(b) Find a basis for the column space of A.
(c) Find a basis for the null space of A.

(Problem 2) Let $A = \begin{pmatrix} 3 & 6 & 9 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

(a) Find a basis for the row space of A.
(b) Find a basis for the column space of A.
(c) Find a basis for the null space of A.

(Problem 3) Find the dimension of Span $\left(\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \right)$.

(Problem 4) Find the dimension of Span $\left(\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right)$.

(Problem 5) How many solutions are there to the system $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$?

(Problem 6) How many solutions are there to the system $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$?

(Problem 7) How many solutions are there to the system $\begin{pmatrix} 3 & 6 \\ 2 & 5 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$?

(Problem 8) In each of parts (a)–(f), one of statements (i)–(vi) is true. Determine which statement is true in each part.

(i) For every vector $\vec{b}$ in $\mathbb{R}^5$, the system $A\vec{x} = \vec{b}$ is inconsistent.
(ii) For every vector $\vec{b}$ in $\mathbb{R}^5$, there is exactly one solution $\vec{x}$ to $A\vec{x} = \vec{b}$.
(iii) For every vector $\vec{b}$ in $\mathbb{R}^5$, there are infinitely many solutions $\vec{x}$ to $A\vec{x} = \vec{b}$.
(iv) There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ has a unique solution.
(v) There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions.
(vi) There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ has a unique solution, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions.

(a) A is a 5×5 matrix of rank 5.
(b) A is a 5×7 matrix of rank 5.
(c) A is a 5×3 matrix of rank 3.
(d) A is a 5×4 matrix of rank 2.
(e) A is a 5×5 matrix of rank 4.
(f) A is a 5×7 matrix of rank 3.

(Problem 9) Is $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - x_2 \\ x_3 + 3x_1 \end{pmatrix}$ a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$?

(Problem 10) Is $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 x_2 \\ x_3 + 3x_1 \end{pmatrix}$ a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$?

(Problem 11) Is $L(A) = \det A$ a linear transformation $L : \mathbb{R}^{3\times 3} \to \mathbb{R}$?

(Problem 12) Is $L(p(x)) = p(0) + xp(1) + x^2 p'(0)$ a linear transformation $L : P_3 \to P_3$?

(Problem 13) Is $L(p(x)) = \int_1^2 p(x)\,dx + xp'(x)$ a linear transformation $L : P_3 \to P_3$?

(Problem 14) Let $L(f(x)) = f(2) + (x-2)f'(2) + \frac{1}{2}(x-2)^2 f''(2)$.
 (a) Is L a linear transformation $L : C[1,3] \to P_3$?
 (b) Is L a linear transformation $L : C^2[1,3] \to P_2$?
 (c) Is L a linear transformation $L : C^2[1,3] \to P_3$?

(Problem 15) Is $L(f(x)) = f(x^2)$ a linear transformation $L : C[0,1] \to C[0,1]$?

(Problem 16) Is $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \frac{x_1^2}{x_2}$ a linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}$?

(Problem 17) Find the kernel and range of the linear transformation $L : P_3 \to P_4$ given by $L(p(x)) = x^2 p'(x)$.

(Problem 18) Find the kernel and range of the linear transformation $L : \mathbb{R}^3 \mapsto \mathbb{R}^2$ given by $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - x_2 \\ x_3 \end{pmatrix}$.

(Problem 19) Find the kernel and range of the linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}^3$ given by $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 3x_1 + 2x_2 \\ 5x_1 - 3x_2 \\ x_1 + x_2 \end{pmatrix}$.

(Problem 20) Find the matrix A that represents the linear transformation $L : \mathbb{R}^3 \mapsto \mathbb{R}^3$, where L projects each vector $\vec{x}$ onto the vector $(3,2,1)^T$.

(Problem 21) Find the matrix A that represents the linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$, where L rotates each vector $\pi/4$ radians counterclockwise and then doubles its length.

(Problem 22) Find the matrix A that represents the linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$, where L reflects each vector about the line $x_1 = x_2$.

(Problem 23) Let $L : P_3 \to \mathbb{R}^3$ be given by $L(p(x)) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}$. Find the matrix A that represents L with respect to the bases $\mathcal{E} = [1, x, x^2]$ and $\mathcal{F} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$.

(Problem 24) Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. Suppose that $L\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$, $L\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

(a) Find the matrix A that represents L with respect to the basis $\mathcal{U} = \left[\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix} \right]$ in $\mathbb{R}^2$ and with respect to the standard basis in $\mathbb{R}^3$.

(b) Find the matrix B that represents L with respect to the standard basis in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

(Problem 25) Let $L\,(\,x_1, x_2\,) = \begin{pmatrix} 3x_1 + 2x_2 \\ x_1 - x_2 \end{pmatrix}$.

(a) Find the matrix A that represents L with respect to the standard basis.

(b) Find the matrix B that represents L with respect to the basis $\mathcal{U} = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right]$.

(Problem 26) Let L be the linear operator on P_3 given by $L(p(x)) = x^2 p''(x) + p'(x)$.

(a) Find the matrix A representing L with respect to the basis $[1, x, x^2]$.

(b) Find the matrix B representing L with respect to the basis $[1, x-3, x^2 - 6x + 9]$.

(c) Find a matrix S such that $B = S^{-1}AS$.

(Problem 27) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by $L(\vec{x}) = A\vec{x}$, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(a) Let $\mathcal{U} = \left[\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right]$. Find the matrix U that represents the change of basis from $\mathcal{U}$ to the standard basis.

(b) Find $U^{-1}AU$.

(c) Let $\vec{x}$ satisfy $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the standard coordinates of $\vec{x}$. Then find $L(\vec{x})$.

(d) Recall that $\vec{x}$ satisfies $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Use the matrix you found in part (b) to find $[L(\vec{x})]_{\mathcal{U}}$.

(e) Use your answer to parts (a) and (d) to find $L(\vec{x})$. Does your answer agree with your answer to part (c)?

(Problem 28) What is the angle between the vectors $(3, 2, 4)$ and $(7, 2, 5)$?

(Problem 29) Find the vector projection of $(7, 1, 2)$ onto $(3, 5, 4)$.

(Problem 30) Find the point on the line $y = \frac{5}{7}x$ that is closest to the point $(6, 2)$.

(Problem 31) Find the equation of the plane passing through the point $(3, 5, 2)$ and normal to the vector $(1, 4, 3)$.

(Problem 32) Find the equation of the plane passing through the points $(1, 2, 3)$, $(5, 2, 4)$, and $(7, 1, 6)$.

(Problem 33) Find the distance from the point $(3, 1, 6)$ to the plane $\{(x, y, z) : 6x + 2y - 3z = 0\}$.

(Problem 34) Find the point on the plane $\{(x, y, z) : 6x - 6y - 7z = 0\}$ closest to the point $(9, 5, 1)$.

(Problem 35) You are given that $\|\vec{x}\| = 2$ and $\|\vec{y}\| = 5$. What can you say about $\langle \vec{x}, \vec{y} \rangle$?

(Problem 36) You are given that $\|\vec{x}\| = 3$ and $\langle \vec{x}, \vec{y} \rangle = -6$. What can you say about $\|\vec{y}\|$?

(Problem 37) Let $A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 5 & 7 \end{pmatrix}$. Find a basis for $N(A)$ and for $R(A^T)$.

(Problem 38) Let $S = \operatorname{Span}\left(\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}\right)$. Find a basis for $S^\perp$.

(Problem 39) Let $S = \operatorname{Span}\left(\begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}\right)$. Find a basis for $S^\perp$.

(Problem 40) Is there a matrix A with $\begin{pmatrix} 3 & 2 \end{pmatrix}$ in the row space of A and $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ in the null space of A? If so, provide an example.

(Problem 41) Is there a matrix A with $\begin{pmatrix} 3 & 2 \end{pmatrix}$ in the row space of A and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ in the null space of A? If so, provide an example.

(Problem 42) Find the least squares solution to the system $\begin{cases} 3x_1 + 2x_2 = 5, \\ 2x_1 - 5x_2 = 7, \\ x_1 + 8x_2 = 3. \end{cases}$

(Problem 43) Find the least squares solution to the system $A\vec{x} = \vec{b}$, where $A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ -9 \end{pmatrix}$.

(Problem 44) Suppose that you have data

x	1	2	3	4
y	-1	0	5	6

Find the linear function that best approximates this data. That is, find values of m and b such that $mx + b$ best approximates y.

Answer key

(Problem 1) Let $A = \begin{pmatrix} 6 & 5 & 4 \\ 1 & 2 & 3 \\ 7 & 9 & 11 \end{pmatrix}$.

(a) Find a basis for the row space of A.
$\{\begin{pmatrix} 1 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \end{pmatrix}\}$.

(b) Find a basis for the column space of A.
$\left\{\begin{pmatrix} 6 \\ 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 9 \end{pmatrix}\right\}$.

(c) Find a basis for the null space of A.
$\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\}$.

(Problem 2) Let $A = \begin{pmatrix} 3 & 6 & 9 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

(a) Find a basis for the row space of A.
$\{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\}$.

(b) Find a basis for the column space of A.
$\left\{\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}\right\}$.

(c) Find a basis for the null space of A.
There are many possible answers, including $\left\{\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}\right\}$.

(Problem 3) Find the dimension of Span $\left(\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}\right)$.

(Answer 3) 2.

(Problem 4) Find the dimension of Span $\left(\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}\right)$.

(Answer 4) 3.

(Problem 5) How many solutions are there to the system $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 9 \\ 6 \end{pmatrix}$?

(Answer 5) Infinitely many.

(Problem 6) How many solutions are there to the system $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$?

(Answer 6) None.

(Problem 7) How many solutions are there to the system $\begin{pmatrix} 3 & 6 \\ 2 & 5 \end{pmatrix} \vec{x} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$?

(Answer 7) Exactly one.

(Problem 8) In each of parts (a)–(f), one of statements (i)–(vi) is true. Determine which statement is true in each part.

(a) A is a 5×5 matrix of rank 5.
(ii); for every vector $\vec{b}$ in $\mathbb{R}^5$, there is exactly one solution $\vec{x}$ to $A\vec{x} = \vec{b}$.

(b) A is a 5×7 matrix of rank 5.
(iii); For every vector $\vec{b}$ in $\mathbb{R}^5$, there are infinitely many solutions $\vec{x}$ to $A\vec{x} = \vec{b}$.

(c) A is a 5×3 matrix of rank 3.
(iv); There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ has a unique solution.

(d) A is a 5×4 matrix of rank 2.
(v); There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions.

(e) A is a 5×5 matrix of rank 4.
(v); There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions.

(f) A is a 5×7 matrix of rank 3.
(v); There are some vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, and other vectors $\vec{b}$ in $\mathbb{R}^5$ such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions.

(Problem 9) Is $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - x_2 \\ x_3 + 3x_1 \end{pmatrix}$ a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$?

(Answer 9) Yes.

(Problem 10) Is $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 x_2 \\ x_3 + 3x_1 \end{pmatrix}$ a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^2$?

(Answer 10) No.

(Problem 11) Is $L(A) = \det A$ a linear transformation $L : \mathbb{R}^{3\times 3} \to \mathbb{R}$?

(Answer 11) No.

(Problem 12) Is $L(p(x)) = p(0) + xp(1) + x^2 p'(0)$ a linear transformation $L : P_3 \to P_3$?

(Answer 12) Yes.

(Problem 13) Is $L(p(x)) = \int_1^2 p(x)\,dx + xp'(x)$ a linear transformation $L : P_3 \to P_3$?

(Answer 13) Yes.

(Problem 14) Let $L(f(x)) = f(2) + (x-2)f'(2) + \frac{1}{2}(x-2)^2 f''(2)$.

(a) Is L a linear transformation $L : C[1,3] \to P_3$?
No; there are some $f(x)$ in $C[1,3]$, such as $f(x) = |x-2|$, such that $L(f(x))$ is not defined.

(b) Is L a linear transformation $L : C^2[1,3] \to P_2$?
No; there are some $f(x)$ in $C[1,3]$, such as $f(x) = x^2$, such that $L(f(x))$ is not in P_2.

(c) Is L a linear transformation $L : C^2[1,3] \to P_3$?
Yes.

(Problem 15) Is $L(f(x)) = f(x^2)$ a linear transformation $L : C[0,1] \to C[0,1]$?

(Answer 15) Yes.

(Problem 16) Is $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \dfrac{x_1^2}{x_2}$ a linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}$?

(Answer 16) No.

(Problem 17) Find the kernel and range of the linear transformation $L : P_3 \to P_4$ given by $L(p(x)) = x^2 p'(x)$.

(Answer 17) $\ker(L) = \{\alpha : \alpha \text{ is a real number}\}$, $L(P_3) = \{\alpha x^2 + \beta x^3\}$.

(Problem 18) Find the kernel and range of the linear transformation $L : \mathbb{R}^3 \mapsto \mathbb{R}^2$ given by $L\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} x_1 - x_2 \\ x_3 \end{pmatrix}$.

(Answer 18) $\ker(L) = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)$, $L(\mathbb{R}^3) = \mathbb{R}^2$.

(Problem 19) Find the kernel and range of the linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}^3$ given by $L\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 3x_1 + 2x_2 \\ 5x_1 - 3x_2 \\ x_1 + x_2 \end{pmatrix}$.

(Answer 19) $\ker(L) = \{\vec{0}\}$, $L(\mathbb{R}^2) = \operatorname{Span}\left(\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}\right) = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : 8x_1 - x_2 - 19x_3 = 0\right\}$.

(Problem 20) Find the matrix A that represents the linear transformation $L : \mathbb{R}^3 \mapsto \mathbb{R}^3$, where L projects each vector $\vec{x}$ onto the vector $(3,2,1)^T$.

(Answer 20) $A = \begin{pmatrix} 9/14 & 3/7 & 3/14 \\ 3/7 & 2/7 & 1/7 \\ 3/14 & 1/7 & 1/14 \end{pmatrix}$.

(Problem 21) Find the matrix A that represents the linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$, where L rotates each vector $\pi/4$ radians counterclockwise and then doubles its length.

(Problem 22) Find the matrix A that represents the linear transformation $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$, where L reflects each vector about the line $x_1 = x_2$.

(Problem 23) Let $L : P_3 \to \mathbb{R}^3$ be given by $L(p(x)) = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}$. Find the matrix A that represents L with respect to the bases $\mathcal{E} = [1, x, x^2]$ and $\mathcal{F} = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]$.

(Answer 23) $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$.

(Problem 24) Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. Suppose that $L\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}$, $L\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

(a) Find the matrix A that represents L with respect to the basis $\mathcal{U} = \left[\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}\right]$ in $\mathbb{R}^2$ and with respect to the standard basis in $\mathbb{R}^3$.

$\begin{pmatrix} 1 & 2 \\ 5 & 3 \\ 7 & 4 \end{pmatrix}$.

(b) Find the matrix B that represents L with respect to the standard basis in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

(Problem 25) Let $L\,(\,x_1, x_2\,) = \begin{pmatrix} 3x_1 + 2x_2 \\ x_1 - x_2 \end{pmatrix}$.

(a) Find the matrix A that represents L with respect to the standard basis.

$A = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}$.

(b) Find the matrix B that represents L with respect to the basis $\mathcal{U} = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right]$.

$B = \begin{pmatrix} -3 & -2 \\ 5 & 5 \end{pmatrix}$.

(Problem 26) Let L be the linear operator on P_3 given by $L(p(x)) = x^2 p''(x) + p'(x)$.

(a) Find the matrix A representing L with respect to the basis $[1, x, x^2]$.

$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

(b) Find the matrix B representing L with respect to the basis $[1, x - 3, x^2 - 6x + 9]$.

$A = \begin{pmatrix} 0 & 1 & -18 \\ 0 & 0 & 14 \\ 0 & 0 & 2 \end{pmatrix}$.

(c) Find a matrix S such that $B = S^{-1}AS$.

$S = \begin{pmatrix} 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix}$.

(Problem 27) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator given by $L(\vec{x}) = A\vec{x}$, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(a) Let $\mathcal{U} = \left[\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right]$. Find the matrix U that represents the change of basis from $\mathcal{U}$ to the standard basis.
$U = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}$.

(b) Find $U^{-1}AU$.
$U^{-1}AU = \begin{pmatrix} 66/13 & 34/13 \\ 8/13 & -1/13 \end{pmatrix}$.

(c) Let $\vec{x}$ satisfy $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Find the standard coordinates of $\vec{x}$. Then find $L(\vec{x})$.
$\vec{x} = \begin{pmatrix} 12 \\ 17 \end{pmatrix}$; $L(\vec{x}) = (\,46 \quad 104\,)$.

(d) Recall that $\vec{x}$ satisfies $[\vec{x}]_{\mathcal{U}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Use the matrix you found in part 44 to find $[L(\vec{x})]_{\mathcal{U}}$.
$[L(\vec{x})]_{\mathcal{U}} = \begin{pmatrix} 266/13 \\ 22/13 \end{pmatrix}$.

(e) Use your answer to parts 44 and 44 to find $L(\vec{x})$. Does your answer agree with your answer to part 44?

$L(\vec{x}) = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 266/13 \\ 22/13 \end{pmatrix} = \begin{pmatrix} 598/13 \\ 1352/13 \end{pmatrix} = \begin{pmatrix} 46 \\ 104 \end{pmatrix}$. Yes, they do agree.

(Problem 28) What is the angle between the vectors $(3, 2, 4)$ and $(7, 2, 5)$?

(Problem 29) Find the vector projection of $(7, 1, 2)$ onto $(3, 5, 4)$.

(Problem 30) Find the point on the line $y = \frac{5}{7}x$ that is closest to the point $(6, 2)$.

(Problem 31) Find the equation of the plane passing through the point $(3, 5, 2)$ and normal to the vector $(1, 4, 3)$.

(Answer 31) $x + 4y + 3z = 29$.

(Problem 32) Find the equation of the plane passing through the points $(1, 2, 3)$, $(5, 2, 4)$, and $(7, 1, 6)$.

(Answer 32) $\{(x, y, z) : x - 6y - 4z = -23\}$.

(Problem 33) Find the distance from the point $(3, 1, 6)$ to the plane $\{(x, y, z) : 6x + 2y - 3z = 0\}$.

(Problem 34) Find the point on the plane $\{(x, y, z) : 6x - 6y - 7z = 0\}$ closest to the point $(9, 5, 1)$.

(Problem 35) You are given that $\|\vec{x}\| = 2$ and $\|\vec{y}\| = 5$. What can you say about $\langle \vec{x}, \vec{y} \rangle$?

(Answer 35) $-10 \leq \langle \vec{x}, \vec{y} \rangle \leq 10$.

(Problem 36) You are given that $\|\vec{x}\| = 3$ and $\langle \vec{x}, \vec{y} \rangle = -6$. What can you say about $\|\vec{y}\|$?

(Answer 36) $\|\vec{y}\| \geq 2$.

(Problem 37) Let $A = \begin{pmatrix} 3 & 2 & 4 \\ 1 & 5 & 7 \end{pmatrix}$. Find a basis for $N(A)$ and for $R(A^T)$.

(Answer 37) $N(A) = \text{Span}\left(\begin{pmatrix} 6 \\ 17 \\ -13 \end{pmatrix}\right)$; $R(A^T) = \text{Span}\left(\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}\right)$.

(Problem 38) Let $S = \text{Span}\left(\begin{pmatrix} 3 \\ 1 \\ 5 \end{pmatrix}\right)$. Find a basis for $S^\perp$.

(Answer 38) There are many possible answers, including $\left\{\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}\right\}$.

(Problem 39) Let $S = \text{Span}\left(\begin{pmatrix} 3 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}\right)$. Find a basis for $S^\perp$.

(Answer 39) There are many possible answers, including $\left\{\begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ -2 \end{pmatrix}\right\}$.

(Problem 40) Is there a matrix A with $\begin{pmatrix} 3 & 2 \end{pmatrix}$ in the row space of A and $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ in the null space of A? If so, provide an example.

(Answer 40) Yes; there are many such A, including $A = \begin{pmatrix} 3 & 2 \end{pmatrix}$, $A = \begin{pmatrix} 3 & 2 \\ 4 & 6 \end{pmatrix}$, $A = \begin{pmatrix} 6 & 4 \\ -9 & -6 \\ 12 & 8 \end{pmatrix}$.

(Problem 41) Is there a matrix A with $\begin{pmatrix} 3 & 2 \end{pmatrix}$ in the row space of A and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ in the null space of A? If so, provide an example.

(Answer 41) This is not possible.

(Problem 42) Find the least squares solution to the system $\begin{cases} 3x_1 + 2x_2 = 5, \\ 2x_1 - 5x_2 = 7, \\ x_1 + 8x_2 = 3. \end{cases}$

(Problem 43) Find the least squares solution to the system $A\vec{x} = \vec{b}$, where $A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 1 \\ -9 \end{pmatrix}$.

(Problem 44) Suppose that you have data

x	1	2	3	4
y	-1	0	5	6

Find the linear function that best approximates this data. That is, find values of m and b such that $mx + b$ best approximates y.