Math 2584, Spring 2018

You are allowed a double-sided, 8.5 inch by 11 inch page of notes.

You are responsible for all of the formulas you will need, except for the following Laplace transforms, which will be written on the last page of the exam.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0,$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \qquad s > 0,$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad s > 0, \qquad n \geq 0$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \qquad s > a$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}, \qquad s > 0$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}, \qquad s > 0$$

$$\text{If } \mathcal{L}\{f(t)\} = F(s) \text{ then } \mathcal{L}\{e^{at} f(t)\} = F(s-a),$$

$$\mathcal{L}^{-1}\{F(s)\} = e^{at}\,\mathcal{L}^{-1}\{F(s+a)\},$$

$$\mathcal{L}\{\mathcal{U}(t-c)\} = \frac{e^{-cs}}{s}, \qquad s > 0, \qquad c \geq 0$$

$$\mathcal{L}\{\delta(t-c)\} = e^{-cs}, \qquad s > 0, \qquad c \geq 0$$

$$\mathcal{L}\{\mathcal{U}(t-c) f(t)\} = e^{-cs}\,\mathcal{L}\{f(t+c)\}, \qquad c \geq 0$$

$$\mathcal{L}\{\mathcal{U}(t-c) g(t-c)\} = e^{-cs}\,\mathcal{L}\{g(t)\}, \qquad c \geq 0$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s\,\mathcal{L}\{y(t)\} - y(0).$$

$$\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}\,\mathcal{L}\{f(t)\},$$

$$\mathcal{L}\left\{\int_0^t f(r)\,g(t-r)\,dr\right\} = \mathcal{L}\left\{\int_0^t f(t-r)\,g(r)\,dr\right\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}.$$

(AB 1) Is $y = e^t$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 2) Is $y = e^{2t}$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 3) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 1$?

(AB 4) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 2$?

(AB 5) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 1$?

(AB 6) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 2$?

(AB 7) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + \arctan(t)\, y = e^t$, $y(3) = 7$.

(b) $\frac{1}{1+t^2}\frac{dy}{dt} - t^5 y = \cos(6t)$, $y(2) = -1$, $y'(2) = 3$.

(c) $\frac{d^2y}{dt^2} - 5\sin(t)\, y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.

(d) $e^t \frac{d^2y}{dt^2} + 3(t-4)\frac{dy}{dt} + 4t^6 y = 2$, $y(3) = 1$, $y'(3) = -1$.

(e) $\frac{d^2y}{dt^2} + \cos(t)\frac{dy}{dt} + 3\ln(1+t^2)\, y = 0$, $y(2) = 3$.

(f) $\frac{d^3y}{dt^3} - t^7\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\, y = t^3$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$, $y'''(1) = 3$.

(g) $(1+t^2)\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = t^3$, $y(3) = 9$, $y'(3) = 7$, $y''(3) = 5$.

(h) $\frac{d^3y}{dt^3} - e^t\frac{d^2y}{dt^2} + e^{2t}\frac{dy}{dt} + e^{3t}y = e^{4t}$, $y(-1) = 1$, $y'(-1) = 3$.

(i) $(2+\sin t)\frac{d^3y}{dt^3} + \cos t\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\, y = t^3$, $y(7) = 2$.

(AB 8) A lake initially has a population of 600 trout. The birth rate of trout is proportional to the number of trout living in the lake. Fishermen are allowed to harvest 30 trout/year from the lake. Write a differential equation for the fish population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 9) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write a differential equation for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 10) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C. Write a differential equation for the temperature of the cup of coffee. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 11) Suppose that an isolated town has 300 households. In 1920, two families install telephones in their homes. Write a differential equation for the number of telephones in the town if the rate at which families buy telephones is jointly proportional to the number of households with telephones and the number of households without telephones. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 12) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Write a differential equation for the amount of salt in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 13) I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write a differential equation for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 14) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write a differential equation for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 15) According to the Stefan-Boltzmann law, a black body in a dark vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write a differential equation for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

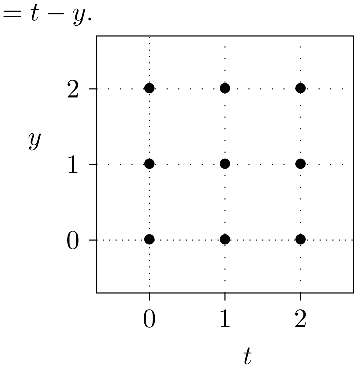
(AB 16) According to the Stefan-Boltzmann law, a black body in a vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). If there is some ambient light in the vacuum, then the black body absorbs energy at a rate proportional to the effective temperature of the light. The proportionality constant is the same in both cases; that is, if the object's temperature is equal to the effective temperature of the light, then its temperature will not change.

Suppose that a planet in space is currently at a temperature of 400 K. The effective temperature of its surroundings is 3 K. Write a differential equation for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 17) A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write a differential equation for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

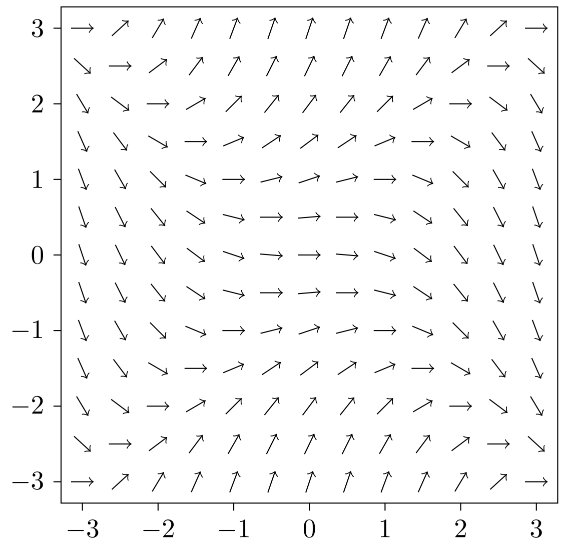
(AB 18) A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write a differential equation for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 19) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.



(AB 20) Consider the differential equation $\frac{dy}{dx} = (y^2 - x^2)/3$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

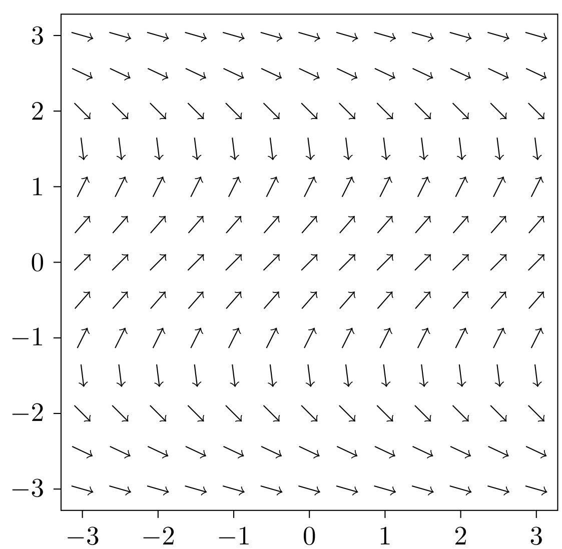
$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-2) = 1.$$



(AB 21) Consider the differential equation $\frac{dy}{dx} = \frac{2}{2-y^2}$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{2}{2 - y^2}, \quad y(1) = 0.$$

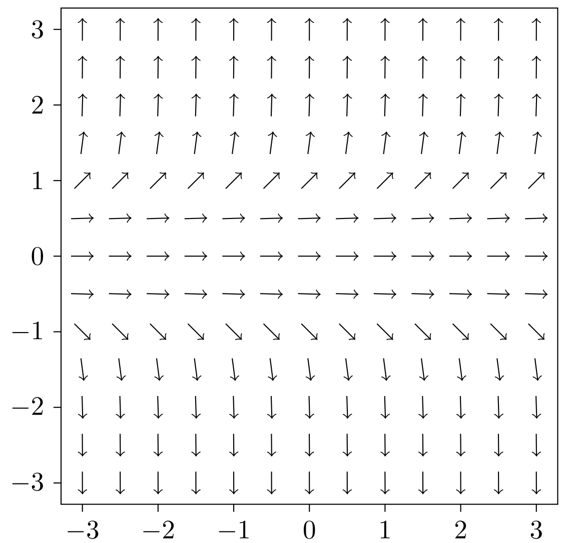
Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 22) Consider the differential equation $\frac{dy}{dx} = y^5$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = y^5, \quad y(1) = -1.$$

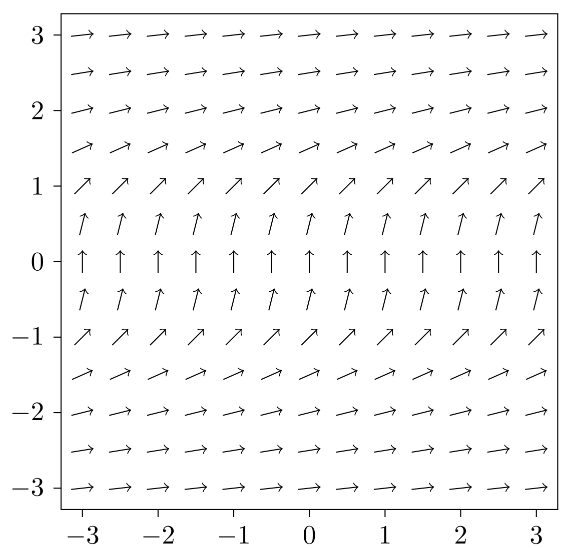
Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 23) Consider the differential equation $\frac{dy}{dx} = 1/y^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{1}{y^2}, \quad y(3) = 2.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 24) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y-2)(y+1)^2$. By hand, sketch some typical solutions.

(AB 25) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 26) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = y^2(y-2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 27) I owe my bank a debt of B dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

(a) Formulate a differential equation for the amount of money I owe.

(b) Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

(AB 28) A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

(a) Write the differential equation for the amount of salt in the tank.

(b) Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

(AB 29) A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where v is her velocity in meters per second.

(a) Write a differential equation for her velocity. Assume her velocity is always downwards.

(b) Find the (negative) critical points of this differential equation. Be sure to include units.

(c) What is the real-world meaning of these critical points?

(AB 30) Suppose that the population y of catfish in a certain lake, absent human intervention, satisfies the logistic equation $\frac{dy}{dt} = ry(1 - y/K)$, where r and K are constants and t denotes time.

(a) Assuming that $r > 0$ and $K > 0$, find the critical points of this equation and classify them as to stability. What is the long term behavior of the population?

(b) Suppose that humans intervene by harvesting fish at a rate proportional to the fish population. Write a new differential equation for the fish population. Find the critical points of your new equation and classify them as to stability. What is the long term behavior of the population?

(c) Suppose that humans intervene by harvesting fish at a constant rate. Write a new differential equation for the fish population. Find the critical points of your new equation and classify them as to stability. What is the long term behavior of the population?

(AB 31) For each of the following differential equations, determine whether it is linear, separable, exact, homogeneous, a Bernoulli equation, or of the form $\frac{dy}{dt} = f(At+By+C)$. Then solve the differential equation.

(a) $\frac{dy}{dt} = \frac{t+\cos t}{\sin y - y}$
(b) $\ln y + y + t + \left(\frac{t}{y} + t\right)\frac{dy}{dt} = 0$
(c) $(t^2+1)\frac{dy}{dt} = ty - t^2 - 1$
(d) $t^2\frac{dy}{dt} = y^2 + t^2 - ty$
(e) $4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5)\frac{dy}{dt} = 0$
(f) $\frac{dy}{dt} = -y\tan 2t - y^3\cos 2t$
(g) $(t+y)\frac{dy}{dt} = 5y - 3t$
(h) $\frac{dy}{dt} = \frac{1}{3t+2y+7}$.
(i) $\frac{dy}{dt} = -y^3\cos(2t)$
(j) $4ty\frac{dy}{dt} = 3y^2 - 2t^2$
(k) $t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$
(l) $\frac{dy}{dt} = \csc^2(y-t)$.
(m) $\frac{dy}{dt} = 8y - y^8$
(n) $t\frac{dy}{dt} = -\cos t - 3y$
(o) $\frac{dy}{dt} = \cot(y/t) + y/t$

(AB 32) For each of the following differential equations, determine whether it is linear, separable, homogeneous, Bernoulli, or a function of a linear term. Then solve the given initial-value problem.

(a) $\cos(t+y^3) + 2t + 3y^2\cos(t+y^3)\frac{dy}{dt} = 0$, $y(\pi/2) = 0$
(b) $2ty\frac{dy}{dt} = 4t^2 - y^2$, $y(1) = 3$.
(c) $t\frac{dy}{dt} = -1 - y^2$, $y(1) = 1$
(d) $\frac{dy}{dt} = (2y + 2t - 5)^2$, $y(0) = 3$.
(e) $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$.
(f) $\frac{dy}{dt} = -3y - \sin t\, e^{-3t}$, $y(0) = 2$

(AB 33) Solve the initial-value problem $\frac{dy}{dt} = \frac{t-5}{y}$, $y(0) = 3$ and determine the range of t-values in which the solution is valid.

(AB 34) Solve the initial-value problem $\frac{dy}{dt} = y^2$, $y(0) = 1/4$ and determine the range of t-values in which the solution is valid.

(AB 35) The function $y_1(t) = e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2} - (1+2t)\frac{dy}{dt} + (t+1)y = 0$. Find the general solution to this differential equation.

(AB 36) The function $y_1(t) = t$ is a solution to the differential equation $t^2\frac{d^2y}{dt^2} - (t^2+2t)\frac{dy}{dt} + (t+2)y = 0$. Find the general solution to this differential equation.

(AB 37) The function $y_1(x) = x^3$ is a solution to the differential equation $x^2\frac{d^2y}{dx^2} + (x^2\tan x - 6x)\frac{dy}{dx} + (12 - 3x\tan x)y = 0$. Find the general solution to this differential equation on the interval $0 < x < \pi/2$.

(AB 38) Find the general solution to the following differential equations.

(a) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$.
(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.
(c) $\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 24y = 0$.
(d) $\frac{d^3y}{dt^3} + y = 0$.
(e) $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0$.

(AB 39) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 2y = 0$, $y(0) = 3$, $y'(0) = 2$.

(b) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$, $y(0) = 1$, $y'(0) = 4$.

(AB 40) An object weighing 5 lbs stretches a spring 4 inches. It is attached to a viscous damper with damping constant 16 $\text{lb} \cdot \text{sec/ft}$. The object is pulled down an additional 2 inches and is released with initial velocity 3 ft/sec upwards.

Write the differential equation and initial conditions that describe the position of the object.

(AB 41) A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant β. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20\ \text{N} \cdot \text{s/m}$, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30\ \text{N} \cdot \text{s/m}$, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(AB 42) A 3-kg object is attached to a spring with constant k and to a viscous damper with damping constant 42 N·sec/m. The object is set in motion from its equilibrium position with initial velocity 5 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object.

If $k = 100$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $k = 200$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(AB 43) A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant β. The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

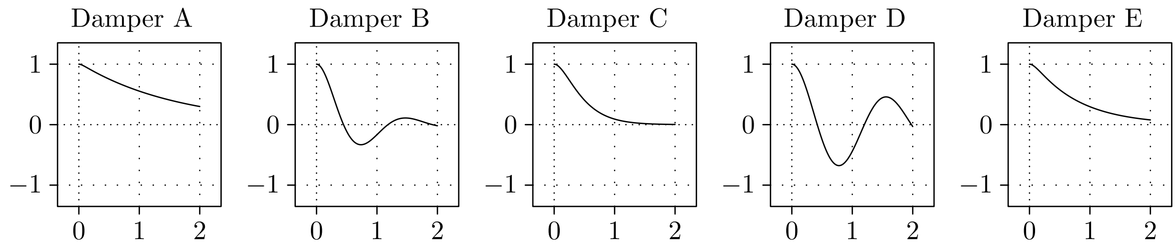
Write the differential equation and initial conditions that describe the position of the object. Then find the value of β for which the system is critically damped. Be sure to include units for β.

(AB 44) An object of mass m is attached to a spring with constant 80 N/m and to a viscous damper with damping constant 20 N·s/m. The object is pulled down to 5cm below its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of m for which the system is critically damped. Be sure to include units for m.

(AB 45) Five objects, each with mass 3 kg, are attached to five springs, each with constant 48 N/m. Five dampers with unknown constants are attached to the objects. In each case, the object is pulled down to a distance 1 cm below the equilibrium position and released from rest. You are given that the system is critically damped in exactly one of the five cases.

Here are the graphs of the objects' positions with respect to time:



(a) For which damper is the system critically damped?
(b) For which dampers is the system overdamped?
(c) For which dampers is the system underdamped?
(d) Which damper has the highest damping constant? Which damper has the lowest damping constant?

(AB 46) Find the general solution to the equation $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3}\ln t$ on the interval $0 < t < \infty$.

(AB 47) Find the general solution to the equation $4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 8e^{-t/2}\arctan t$.

(AB 48) Find the general solution to the equation $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$.

(AB 49) Solve the initial-value problem $9\frac{d^2y}{dt^2} + y = \sec^2(t/3)$, $y(0) = 4$, $y'(0) = 2$, on the interval $-3\pi/2 < t < 3\pi/2$.

(AB 50) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - 2x = 0$, $t > 0$, is $x(t) = C_1t^2 + C_2t^{-1}$. Solve the initial-value problem $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(AB 51) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - t\frac{dx}{dt} - 3x = 0$, $t > 0$, is $x(t) = C_1t^3 + C_2t^{-1}$. Find the general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

(AB 52) Find the general solution to the following differential equations.
(a) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 3e^{4t}$.
(b) $16\frac{d^2y}{dt^2} - y = e^{t/4}\sin t$.
(c) $\frac{d^2y}{dt^2} + 49y = 3t\sin 7t$.
(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$.
(e) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = t\sin(3t)$.
(f) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$.
(g) $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 6t^2 + \cos(2t)$.
(h) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = t^2e^{3t}$.
(i) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}$.
(j) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t)$.
(k) $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t)$.
(l) $\frac{d^2y}{dt^2} + 9y = 5\sin(3t)$.
(m) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$.
(n) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$.

(o) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$.
(p) $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$.
(q) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$.

(AB 53) Solve the following initial-value problems. Express your answers in terms of real functions.
(a) $16\frac{d^2y}{dt^2} - y = 3e^t$, $y(0) = 1$, $y'(0) = 0$.
(b) $\frac{d^2y}{dt^2} + 49y = \sin 7t$, $y(\pi) = 3$, $y'(\pi) = 4$.
(c) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 25t^2$, $y(2) = 0$, $y'(2) = 3$.
(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$.

(AB 54) A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 $\text{N}\cdot\text{s/m}$ and is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(20t)$ N upwards.

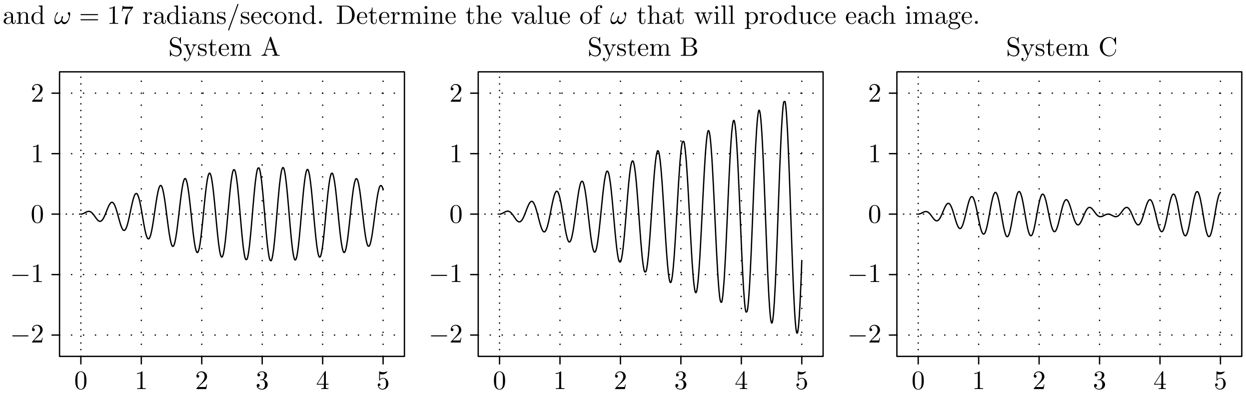
Write the differential equation and initial conditions that describe the position of the object.

(AB 55) An object weighing 4 lbs stretches a spring 2 inches. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(\omega t)$ pounds, directed upwards. There is no damping.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of ω for which resonance occurs. Be sure to include units for ω.

(AB 56) A 4-kg object is suspended from a spring. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $7\sin(\omega t)$ newtons, directed upwards.
(a) Write the differential equation and initial conditions that describe the position of the object.
(b) It is observed that resonance occurs when $\omega = 20$ rad/sec. What is the constant of the spring? Be sure to include units.

(AB 57) A 1-kg object is suspended from a spring with constant 225 N/m. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $15\cos(\omega t)$ pounds, directed upwards. Illustrated are the object's position for three different values of ω. You are given that the three values are $\omega = 15$ radians/second, $\omega = 16$ radians/second, and $\omega = 17$ radians/second. Determine the value of ω that will produce each image.



(AB 58) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ (**not** the table on the front of the exam), find the Laplace transforms of the following functions.
(a) $f(t) = e^{-11t}$
(b) $f(t) = t$
(c) $f(t) = \begin{cases} 3e^t, & 0 < t < 4, \\ 0, & 4 \le t. \end{cases}$

(AB 59) Find the Laplace transforms of the following functions. You may use the table on the front of the exam.

(a) $f(t) = t^4 + 5t^2 + 4$
(b) $f(t) = (t + 2)^3$
(c) $f(t) = 9e^{4t+7}$
(d) $f(t) = -e^{3(t-2)}$
(e) $f(t) = (e^t + 1)^2$
(f) $f(t) = 8\sin(3t) - 4\cos(3t)$
(g) $f(t) = t^2 e^{5t}$
(h) $f(t) = 7e^{3t}\cos 4t$
(i) $f(t) = 4e^{-t}\sin 5t$
(j) $f(t) = t\,e^t \sin t$
(k) $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$
(l) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$
(m) $f(t) = \begin{cases} 0, & t < 1, \\ t - 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$
(n) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$
(o) $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$
(p) $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi \end{cases}$
(q) $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi \end{cases}$
(r) $f(t) = t^2 \sin 5t$

(AB 60) For each of the following problems, find y.

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$
(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$
(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$
(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$
(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$
(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$
(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$
(h) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$
(i) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$
(j) $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$
(k) $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$
(l) $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$

(AB 61) Sketch the graph of $y = t^2 - t^2\,\mathcal{U}(t-1) + (2-t)\,\mathcal{U}(t-1)$.

(AB 62) Solve the following initial-value problems using the Laplace transform. Do you expect the graphs of y, $\frac{dy}{dt}$, or $\frac{d^2y}{dt^2}$ to show any corners or jump discontinuities? If so, at what values of t?

(a) $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$

(b) $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$

(c) $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

(e) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

(f) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$

(g) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$

(h) $\frac{d^2y}{dt^2} - 4y = e^t \sin(3t)$, $y(0) = 0$, $y'(0) = 0$

(i) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$

(j) $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \le t < 4, \\ 0, & 4 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 0$

(k) $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

(l) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

(m) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \le t < 2, \\ 4, & 2 \le t \end{cases}$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = 2$.

(n) $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t \end{cases}$, $y(0) = 1$, $y'(0) = 0$

(o) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$

(p) $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

(q) $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$

(r) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$

(s) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$.

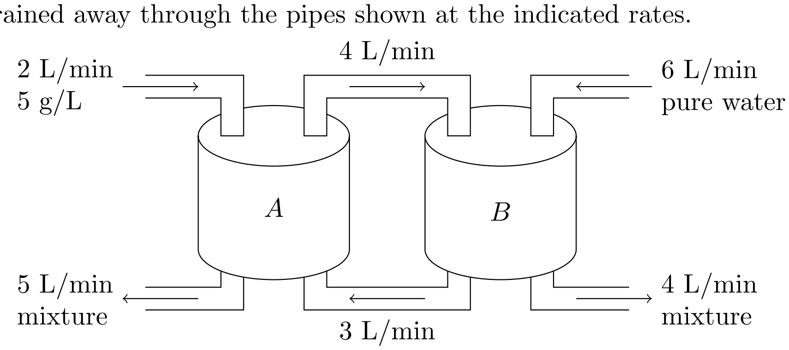
(t) $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$.

(u) $\frac{d^2y}{dt^2} + 4y = -2\delta(t-4\pi)$, $y(0) = 1/2$, $y'(0) = 0$

(v) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + \mathcal{U}(t-2)$, $y(0) = 1$, $y'(0) = 0$

(w) $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 5\delta(t-4)$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 2$.

(AB 63) Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



Write the differential equations and initial conditions that describe the amount of salt in each tank.

(AB 64) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 65) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 66) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 67) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 68) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 69) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 70) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 71) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 72) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, x(0) = 3, \quad y(0) = 2, \quad z(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 73) Find the solution to the initial-value problem

$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, x(0) = 4, \quad y(0) = 3, \quad z(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 74) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y - 15, \qquad \frac{dy}{dt} = -5x + 6y - 8, \qquad x(0) = 2, \quad y(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 75) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 76) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y + e^{2t}\cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 0, \quad y(0) = 0.$$

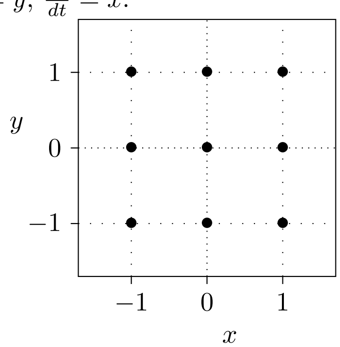
Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 77) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 78) Here is a grid. Draw a small phase plane (vector field) with nine arrows for the autonomous system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = x$.

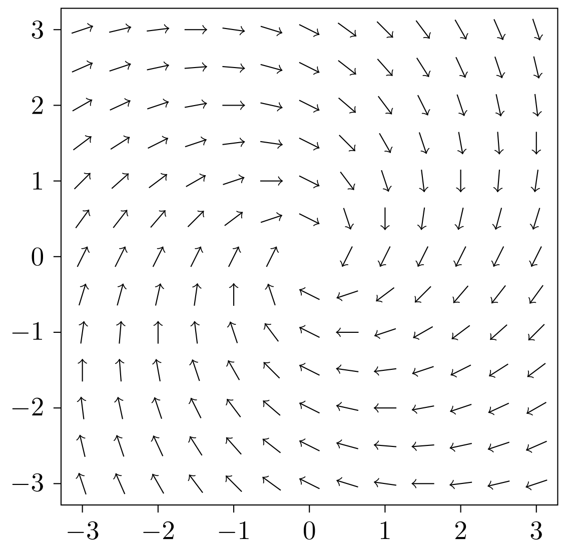


(AB 79) Here is the phase plane for the system

$$\frac{dx}{dt} = 2y - x, \quad \frac{dy}{dt} = -2x - y$$

Sketch the solution to the initial value problem

$$\frac{dx}{dt} = 2y - x, \quad \frac{dy}{dt} = -2x - y, \quad x(0) = 2, \quad y(0) = 1.$$



(AB 80) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2.6 & 1.8 \\ -1.2 & 1.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

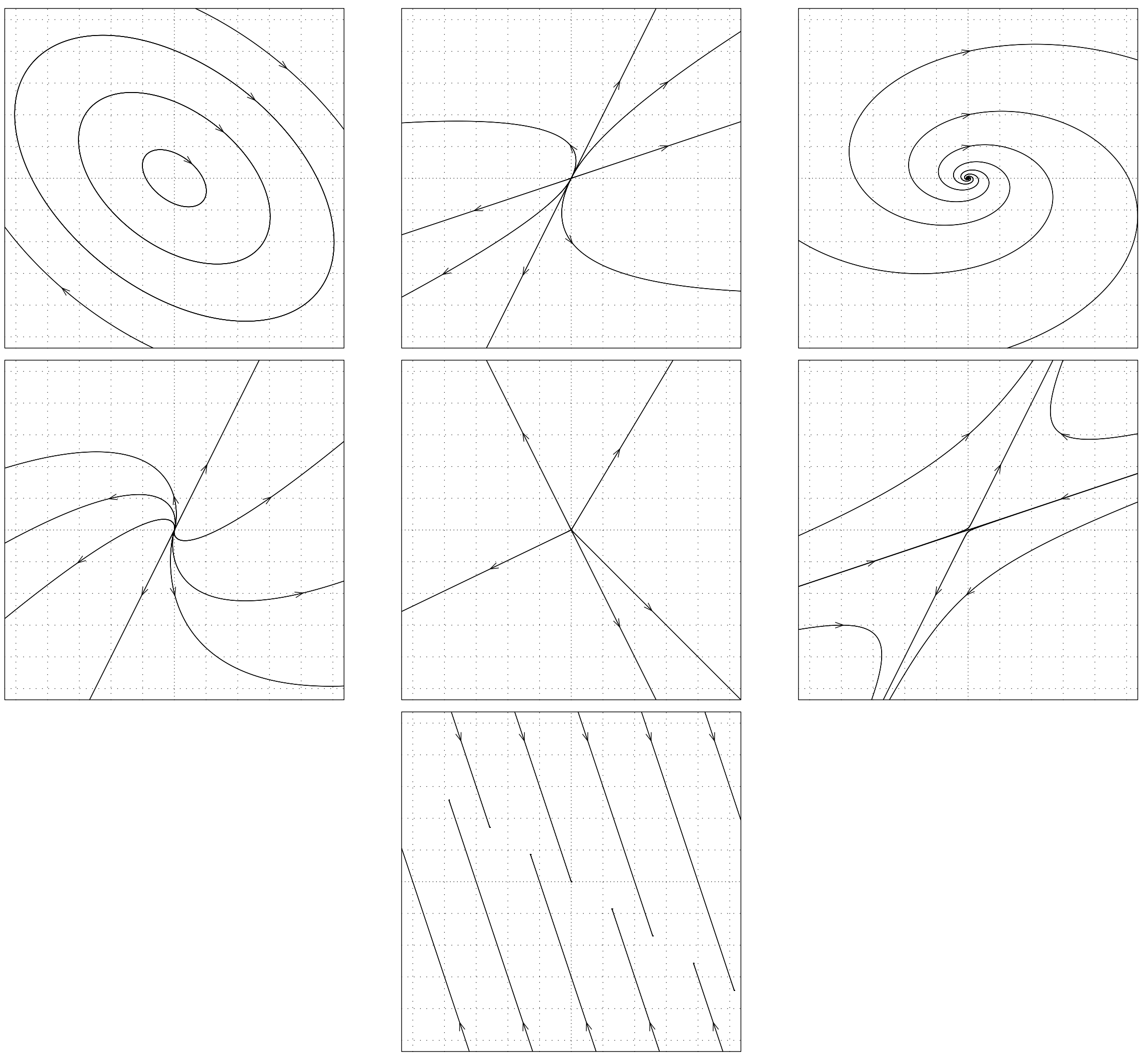
(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4/3 & -1/6 \\ 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} t+3 \\ 2t \end{pmatrix}$

(d) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4.5 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 3\sin 3t \\ 2\cos 3t \end{pmatrix} + C_2 e^{t} \begin{pmatrix} -3\cos 3t \\ 2\sin 3t \end{pmatrix}$

(e) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 5\sin 4t \\ 4\cos 4t - 2\sin 4t \end{pmatrix} + C_2 \begin{pmatrix} 5\cos 4t \\ -4\sin 4t - 2\cos 4t \end{pmatrix}$

(f) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

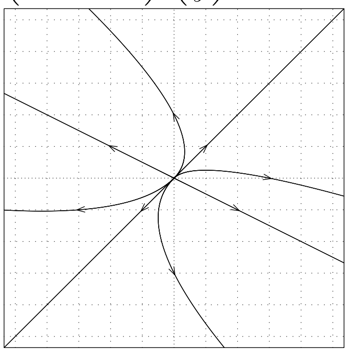
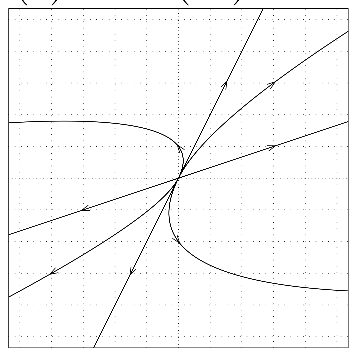
(g) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -6 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 3 \\ -2 \end{pmatrix} + C_2 e^{-7t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$



(AB 81) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

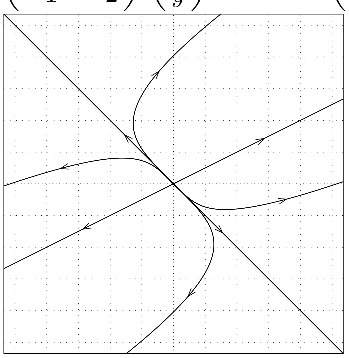
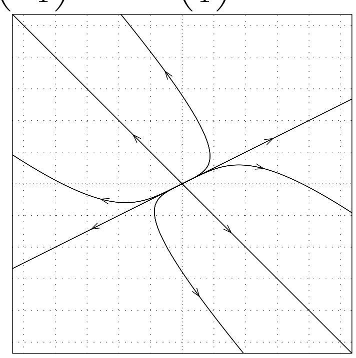
(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

(AB 82) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

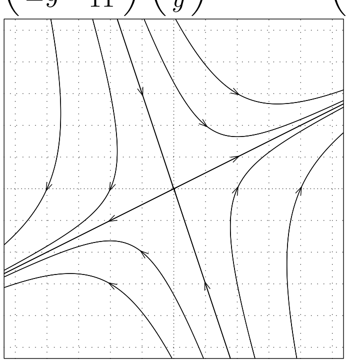
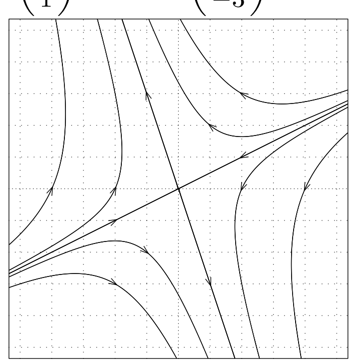
(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(AB 83) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 9 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-14t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

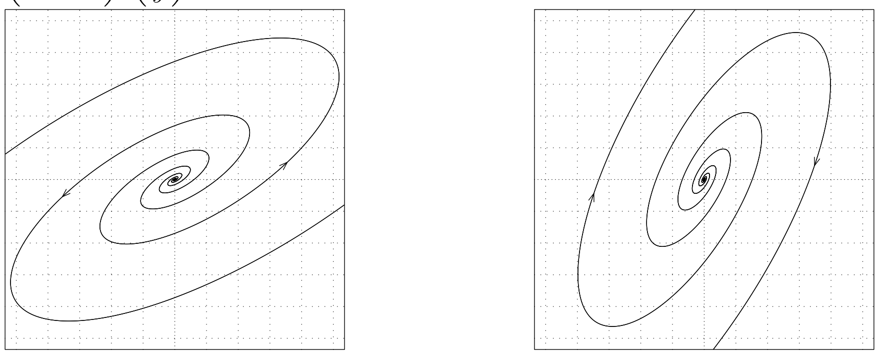
(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -4 & -6 \\ -9 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{14t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

(AB 84) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -8 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, eigenvalues $r = 1 \pm 4i$

(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, eigenvalues $r = 1 \pm 4i$



(AB 85) The equation $\cos x + (\sin x + 1 + y)\frac{dy}{dx} = 0$ is not exact. Find an integrating factor $\mu(y)$ such that $\mu(y)\cos x + \mu(y)(\sin x + 1 + y)\frac{dy}{dx} = 0$ is exact. Then solve the equation.

(AB 86) The equation $5y^3 + 7x^2y^2 + (3xy^2 + 2x^3y)\frac{dy}{dx} = 0$ is not exact. Find an integrating factor $\mu(x)$ such that $\mu(x)(5y^3 + 7x^2y^2) + \mu(x)(3xy^2 + 2x^3y)\frac{dy}{dx} = 0$ is exact. Then solve the equation.

(AB 87) Find the shape of the trajectories of solutions to the system of differential equations $\frac{dx}{dt} = 3x - 4y$, $\frac{dy}{dt} = 4x - 3y$.

(AB 88) Find the shape of the trajectories of solutions to the system of differential equations $\frac{dx}{dt} = 3y - 2xy$, $\frac{dy}{dt} = 4xy - 3y$.

(AB 89) Find the shape of the trajectories of solutions to the system of differential equations $\frac{dx}{dt} = 3x - 4xy$, $\frac{dy}{dt} = 5xy - 2y$.

(AB 90) A chain with density 4 pounds/foot that is initially lying on the floor and is pulled upwards by a constant force of 28 pounds on one end satisfies the initial value problem

$$\frac{1}{8}x\frac{d^2x}{dt^2} + \frac{1}{8}\left(\frac{dx}{dt}\right)^2 = 28 - 4x, \quad x(0) = 0$$

where t denotes time (in seconds) and where x denotes the length of the chain that has been lifted off the ground (in feet).

(a) Rewrite the problem in the form of a system of two first order equations. Use the chain's velocity v as the second dependent variable.

(b) Use the chain rule (phase plane method) to write a first order differential equation for v in terms of x.

(c) Rewrite the equation without any fractions and with a zero on the right hand side. Then find an integrating factor $\mu(x)$ such that the equation becomes exact when multiplied by $\mu(x)$.

(d) Solve the exact equation.

(e) Find v as a function of x. What is $v(0)$?

(AB 91) Suppose that $\frac{d^2x}{dt^2} = 18x^3$, $x(0) = 1$, $x'(0) = 3$. Let $v = \frac{dx}{dt}$. Find a formula for v in terms of x. Then find a formula for x in terms of t.

(AB 92) Suppose that a rocket of mass $m = 1000$ kg is launched straight up from the surface of the earth with initial velocity 10 km/sec. The radius of the earth is 6,371 km. When the rocket is r meters from the center of the earth, it experiences a force due to gravity of magnitude GMm/r^2, where $GM = 3.98 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the rocket's position.
(b) Find the velocity of the rocket as a function of position.
(c) How far away from the earth is the rocket when it stops moving and starts to fall back?

(AB 93) A 5-kg toolbox is dropped (from rest) out of a spaceship at an altitude of 10,000 km above the surface of the earth. The radius of the earth is 6,371 km. When the toolbox is r meters from the center of the earth, it experiences a force due to gravity of magnitude GMm/r^2, where $GM = 3.98 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the toolbox's position.
(b) Find the velocity of the rocket as a function of position.
(c) How fast is the toolbox moving when it strikes the earth?

(AB 94) A particle of mass $m = 3$ kg a distance r from an infinitely long string experiences a force due to gravity of magnitude Gm/r, where $G = 2000$ meters^2/second^2, directed directly toward the string. Suppose that the particle is initially 1000 meters from the string and takes off with initial velocity 200 meters/second directly away from the string.

(a) Formulate the initial value problem for the particle's position.
(b) Find the velocity of the particle as a function of position.
(c) How far away from the string is the particle when it stops moving and starts to fall back?

(AB 95) A charged particle of mass $m = 20$ g a distance r meters from an electric dipole experiences a force of $3/r^3$ newtons, directed directly toward the dipole. Suppose that the particle is initially 3 meters from the dipole and is set in motion with initial velocity 5 meters/second away from the dipole.

(a) Formulate the initial value problem for the particle's position.
(b) Find the velocity of the particle as a function of position.
(c) What is the limiting velocity of the particle?

(AB 96) Suppose that a bob of mass 300g hangs from a pendulum of length 15cm. The pendulum is set in motion from its equilibrium point with an initial velocity of 3 meters/second.

If θ denotes the angle between the pendulum and the vertical, then the pendulum satisfies the equation of motion $m\frac{d^2\theta}{dt^2} = -\frac{mg}{\ell}\sin\theta$, where m is the mass of the pendulum bob, ℓ is the length of the pendulum and g is the acceleration of gravity (which you may take to be 9.8 meters/second^2).

Find a formula for $\omega = \frac{d\theta}{dt}$ in terms of θ.

Answer key

(Answer 1) No, $y = e^t$ is not a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(Answer 2) Yes, $y = e^{2t}$ is a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(Answer 3) No.

(Answer 4) No.

(Answer 5) Yes.

(Answer 6) No.

(Answer 7)
- (a) We expect a unique solution.
- (b) We do not expect any solutions.
- (c) We do not expect any solutions.
- (d) We expect a unique solution.
- (e) We expect an infinite family of solutions.
- (f) We do not expect any solutions.
- (g) We expect a unique solution.
- (h) We expect an infinite family of solutions.
- (i) We expect an infinite family of solutions.

(Answer 8) Independent variable: $t =$ time (in years).
Dependent variable: $P =$ Number of trout in the lake
Initial condition: $P(0) = 600$.
Parameters: $\alpha =$ birth rate (in 1/years).
Differential equation: $\frac{dP}{dt} = \alpha P - 30$.

(Answer 9) Independent variable: $t =$ time (in years).
Dependent variable: $P =$ Number of birds on the island.
Parameters: $\alpha =$ birth rate parameter (in 1/years); $\beta =$ death rate parameter (in 1/(bird·years)).
Initial condition: $P(0) = 5$.
Differential equation: $\frac{dP}{dt} = \alpha P - \beta P^2$.

(Answer 10) Independent variable: $t =$ time (in seconds).
Dependent variable: $T =$ Temperature of the cup (in degrees Celsius)
Initial condition: $T(0) = 95$.
Differential equation: $\frac{dT}{dt} = -\alpha(T - 25)$, where α is a positive parameter (constant of proportionality) with units of 1/seconds.

(Answer 11) Independent variable: $t =$ time (in years).
Dependent variable: $T =$ Number of telephones installed in the town.
Initial condition: $T(1920) = 2$.
Differential equation: $\frac{dT}{dt} = \alpha T(300 - T)$, where α is a positive parameter (constant of proportionality) with units of $1/\text{year} \cdot \text{telephone}$.

(Answer 12) Independent variable: t = time (in minutes).
Dependent variable: Q = amount of dissolved salt (in kilograms).
Initial condition: $Q(0) = 3$.
Differential equation: $\frac{dQ}{dt} = 0.01 - \frac{3Q}{600-t}$.

(Answer 13) Independent variable: t = time (in years).
Dependent variable: B = balance of my loan (in dollars).
Initial condition: $B(0) = 300{,}000$.
Differential equation: $\frac{dB}{dt} = 0.04B - 12 \cdot 1600$

(Answer 14) Independent variable: t = time (in minutes).
Dependent variables:
h = depth of water in the hole (in centimeters)
V = volume of water in the hole (in cubic centimeters); notice that $V = \frac{1}{3}\pi(5h)^2 h$
Initial condition: $V(0) = 0$.
Differential equation: $\frac{dV}{dt} = 1 - \alpha\pi(5h)^2 = 1 - 25\alpha\pi(3V/25\pi)^{2/3}$, where α is a proportionality constant with units of cm/s.

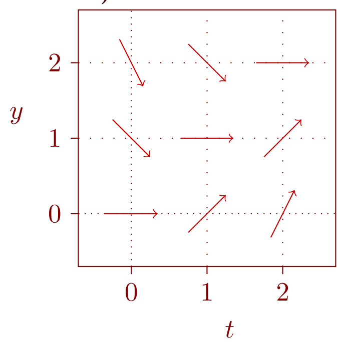
(Answer 15) Independent variable: t = time (in seconds).
Dependent variable: T = object's temperature (in kelvins)
Parameter: σ = proportionality constant (in 1/(seconds·kelvin^3))
Initial condition: $T(0) = 400$.
Differential equation: $\frac{dT}{dt} = -\sigma T^4$.

(Answer 16) Independent variable: t = time (in seconds).
Dependent variable: T = object's temperature (in kelvins)
Parameter: σ = proportionality constant (in 1/(seconds·kelvin^3))
Initial condition: $T(0) = 400$.
Differential equation: $\frac{dT}{dt} = -\sigma T^4 + 81\sigma$.

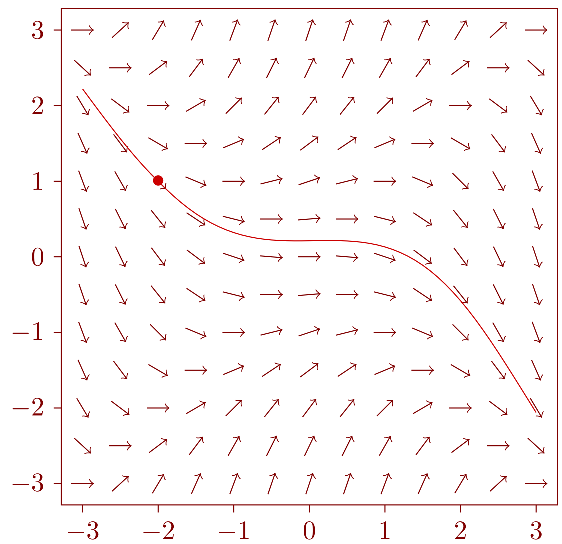
(Answer 17) Independent variable: t = time (in seconds).
Dependent variable: v = velocity of the ball (in meters/second, where a positive velocity denotes upward motion).
Parameters: α = proportionality constant of the drag force (in newton·seconds/meter)
m = mass of the ball (in kilograms)
Initial condition: $v(0) = 10$.
Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/m)v$.

(Answer 18) Independent variable: t = time (in seconds).
Dependent variable: v = velocity of the ball (in meters/second, where a positive velocity denotes upward motion).
Parameter: α = proportionality constant of the drag force (in newton·seconds/meter).
Initial condition: $v(0) = 20$.
Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/0.3)v|v|$, or $\frac{dv}{dt} = \begin{cases} -9.8 - (\alpha/0.3)v^2, & v \geq 0, \\ -9.8 + (\alpha/0.3)v^2, & v < 0. \end{cases}$

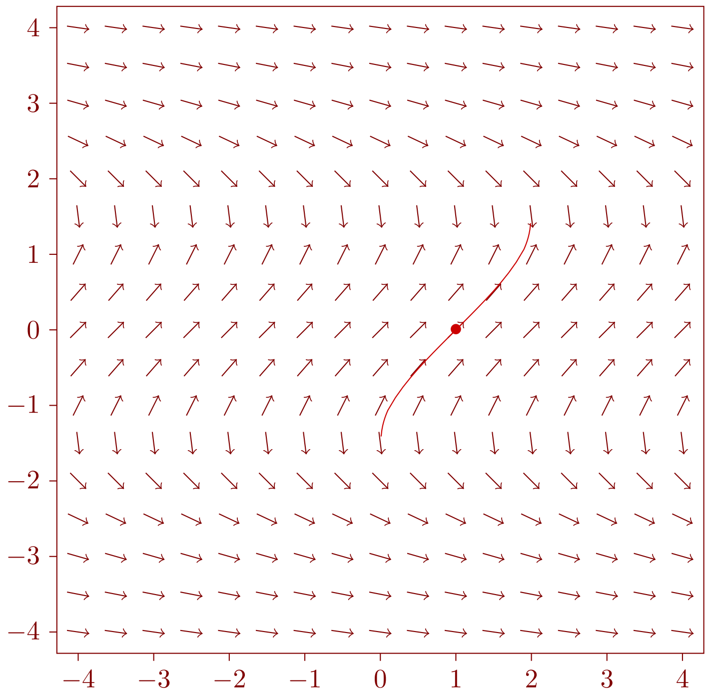
(Answer 19) Here is the direction field for the differential equation $\frac{dy}{dt} = t - y$.



(Answer 20)

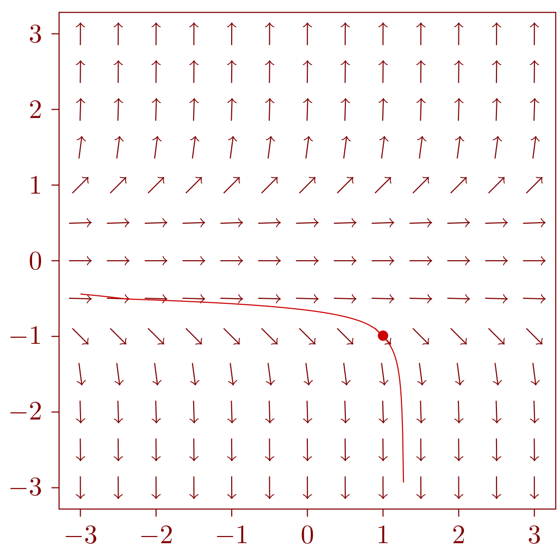


(Answer 21)



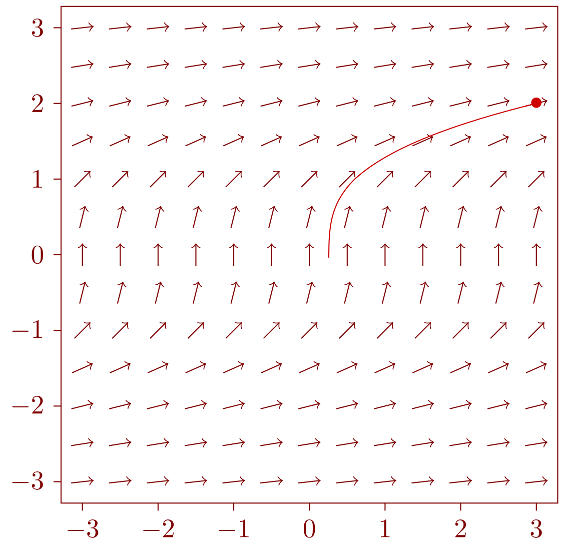
The domain of definition of the solution appears to be $0 < t < 2$.

(Answer 22)



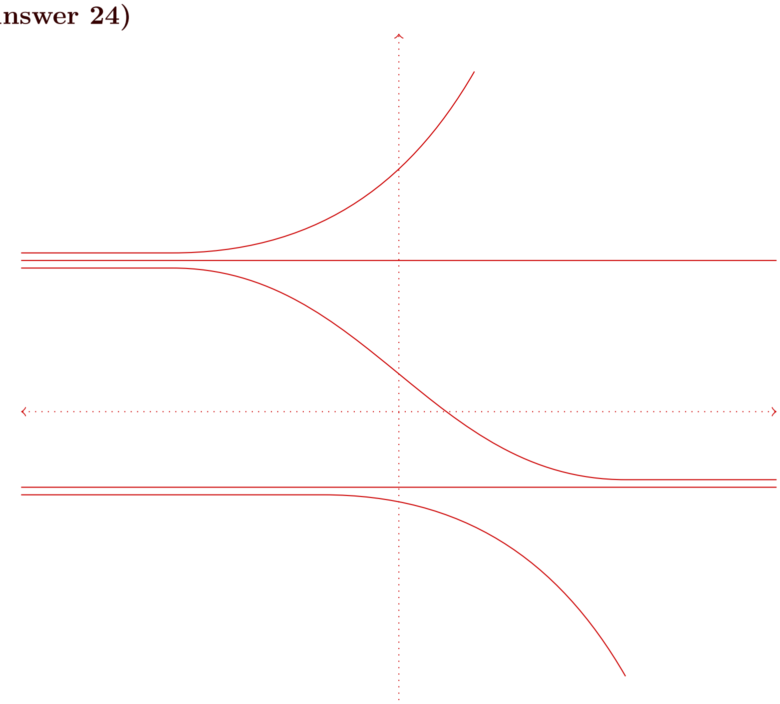
The domain of definition of the solution appears to be approximately $t < 1.3$.

(Answer 23)



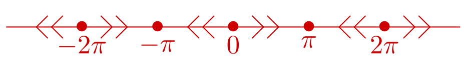
The domain of definition of the solution appears to be $\frac{1}{4} < t$.

(Answer 24)



(Answer 25)

Critical points: $y = k\pi$ for any integer k.



If k is even then $y = k\pi$ is unstable.
If k is odd then $y = k\pi$ is stable.

(Answer 26)

Critical points: $y = 0$ and $y = 2$.

$y = 0$ is semistable. $y = 2$ is unstable.

(Answer 27)

(a) $\frac{dB}{dt} = 0.05B - 19200$, where t denotes time in years.

(b) The critical point is $B = \$384{,}000$. It is unstable. If my initial debt is less than \$384,000, then I will eventually pay it off (have a debt of zero dollars), but if my initial debt is greater than \$384,000, my debt will grow exponentially. The critical point $B = 384{,}000$ corresponds to the balance that will allow me to make interest-only payments on my debt.

(Answer 28)

(a) $\frac{dQ}{dt} = 10 - Q/300$, where Q denotes the amount of salt in grams and t denotes time in minutes.

(b) The critical point is $Q = 3000$. It is stable. No matter the initial amount of salt in the tank, as time passes the amount of salt will approach 3000 g or 3 kg.

(Answer 29)

(a) If $v \leq 0$ then $70\frac{dv}{dt} = -70 * 9.8 + 2v^2$. (If $v > 0$ then $70\frac{dv}{dt} = -70 * 9.8 - 2v^2$.)

(b) $v = -\sqrt{343}$ meters/second.

(c) As $t \to \infty$, her velocity will approach the stable critical point of $v = -\sqrt{343}$ meters/second.

(Answer 30)

(a) The critical points are $y = 0$ (unstable) and $y = K$ (stable). If any positive number of fish are present, then eventually the population will approach a level of $y = K$.

(b) $\frac{dy}{dt} = ry(1 - y/K) - Ey$, where E is a proportionality constant. The critical points are $y = 0$ and $y = K - KE/r$.

If $E < r$, then $y = 0$ is unstable and $y = K - KE/r$ is stable, and the population will approach a level of $y = K - KE/r$.

If $E > r$, then $y = -KE/r + K$ is unstable (but has no physical significance) and 0 is stable. If $E = r$, then $y = 0$ is the only critical point and is semistable. If $E \leq r$, then the population will eventually tend towards extinction.

(c) $\frac{dy}{dt} = ry(1 - y/K) - h$, where h is the harvesting rate. If $h < Kr/4$, then the critical points are $y = K/2 - \sqrt{(K/2)^2 - Kh/r}$ (unstable) and $y = K/2 + \sqrt{(K/2)^2 - Kh/r}$ (stable). Notice that both critical points are positive (i.e., correspond to the physically meaningful case of at least zero fish in the lake.) If $h = Kr/4$, then there is one critical point at $y = K/2$; it is semistable. Finally, if $h > Kr/4$, then there are no critical points and the fish population will decrease until it goes extinct.

(Answer 31)

(a) $\frac{dy}{dt} = \frac{t+\cos t}{\sin y - y}$ is separable and has solution $\frac{1}{2}y^2 + \cos y = -\frac{1}{2}t^2 - \sin t + C$.

(b) $\ln y + y + t + \left(\frac{t}{y} + t\right)\frac{dy}{dt} = 0$ is exact and has solution $t \ln y + ty + \frac{1}{2}t^2 = C$.

(c) $(t^2 + 1)\frac{dy}{dt} = ty - t^2 - 1$ is linear and has solution $y = -\sqrt{t^2+1}\ln(t + \sqrt{t^2+1}) + C\sqrt{t^2+1}$.

(d) $t^2\frac{dy}{dt} = y^2 + t^2 - ty$ is homogeneous and has solution $y = \frac{t}{C - \ln|t|} + t$.

(e) $4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5)\frac{dy}{dt} = 0$ is exact and has solution $2t^2y^3 + \frac{3}{2}t^4 + 3y + \frac{1}{6}y^6 = C$.

(f) Let $v = y^{-2}$. Then $\frac{dv}{dt} = 2v\tan 2t + 2\cos 2t$, so $v = \frac{1}{2}\sin(2t) + t\sec(2t) + C\sec(2t)$ and $y = \frac{1}{\sqrt{\frac{1}{2}\sin(2t) + t\sec(2t) + C\sec(2t)}}$.

(g) $(t + y)\frac{dy}{dt} = 5y - 3t$ is homogeneous and has solution $(y - 3t)^2 = C(y - t)$.

(h) If $\frac{dy}{dt} = \frac{1}{3t+2y+7}$, then $\frac{3t+2y+7}{3} - \frac{2}{9}\ln|3t + 2y + 7 + 2/3| = \ln|t| + C$.

(i) $\frac{dy}{dt} = -y^3\cos(2t)$ is separable and has solution $y = \pm\frac{1}{\sqrt{\sin(2t)+C}}$.

(j) $4ty\frac{dy}{dt} = 3y^2 - 2t^2$ is homogeneous and has solution $2\ln(y^2/t^2 + 2) = -\ln|t| + C$.

(k) Let $v = y^6$. Then $t\frac{dv}{dt} = 18v - 6t^2$, so $v = Ct^{18} + \frac{3}{8}t^2$ and $y = \sqrt[6]{Ct^{18} + \frac{3}{8}t^2}$.
(l) If $\frac{dy}{dt} = \csc^2(y - t)$, then $\tan(y - t) - y = C$.
(m) Make the substitution $v = y^{-7}$. Then $\frac{dv}{dt} = -56v + 7$, so $v = Ce^{-56t} + \frac{1}{8}$ and $y = \frac{1}{\sqrt[7]{Ce^{-56t}+1/8}}$.
(n) $t\frac{dy}{dt} = -\cos t - 3y$ is linear and has solution $y = \frac{1}{t}\sin t + \frac{2}{t^2}\cos t - \frac{2}{t^3}\sin t + \frac{C}{t^3}$.
(o) $\frac{dy}{dt} = \cot(y/t) + y/t$ is homogeneous and has solution $\sec(y/t) = Ct$.

(Answer 32)
(a) If $\cos(t + y^3) + 2t + 3y^2\cos(t + y^3)\frac{dy}{dt} = 0$, $y(\pi/2) = 0$, then $\sin(t + y^3) + t^2 = 1 + \pi^2/4$.
(b) If $2ty\frac{dy}{dt} = 4t^2 - y^2$, $y(1) = 3$, then $y = \sqrt{\frac{4}{3}t^2 + \frac{23}{3t}}$.
(c) If $t\frac{dy}{dt} = -1 - y^2$, $y(1) = 1$, then $y = \tan(\pi/4 - \ln t)$.
(d) If $\frac{dy}{dt} = (2y + 2t - 5)^2$, $y(0) = 3$, then $y = \frac{1}{2}\tan(2t + \pi/4) + \frac{5}{2} - t$.
(e) If $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$, then $y = \sqrt[3]{340e^{6t} + 3}$.
(f) If $\frac{dy}{dt} = -3y - \sin t\, e^{-3t}$, $y(0) = 2$, then $y = e^{-3t}\cos t + e^{-3t}$.

(Answer 33) $y = \sqrt{(t-5)^2 - 16} = \sqrt{(t-1)(t-9)} = \sqrt{t^2 - 10t + 9}$. The solution is valid for all $t < 1$.

(Answer 34) $y = \frac{1}{4-t}$. The solution is valid for all $t < 4$.

(Answer 35) $y(t) = C_1\, e^t + C_2\, t^2 e^t$.

(Answer 36) $y(t) = C_1\, t + C_2\, te^t$.

(Answer 37) $y(x) = C_1\, x^3 + C_2\, x^3 \sin x$.

(Answer 38)
(a) If $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$, then $y = C_1e^{-6t}\cos(7t) + C_2e^{-6t}\sin(7t)$.
(b) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$, then $y = C_1e^{(-2+\sqrt{2})t} + C_2e^{(-2-\sqrt{2})t}$.
(c) If $\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 24y = 0$, then $y = C_1e^{6t} + C_2\cos 2t + C_3\sin 2t$.
(d) If $\frac{d^3y}{dt^3} + y = 0$, then $y = C_1e^{-t} + C_2e^{t/2}\cos\frac{\sqrt{3}}{2}t + C_3e^{t/2}\sin\frac{\sqrt{3}}{2}t$.
(e) If $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0$, then $y = C_1e^{2t} + C_2te^{2t} + C_3e^{-2t} + C_4te^{-2t}$.

(Answer 39)
(a) If $9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 2y = 0$, $y(0) = 3$, $y'(0) = 2$, then $y = 3e^{-t/3}\cos(t/3) + 9e^{-t/3}\sin(t/3)$.
(b) If $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$, $y(0) = 1$, $y'(0) = 4$, then $y = e^{-5t} + 9te^{-5t}$.

(Answer 40) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then

$$\frac{5}{32}\frac{d^2x}{dt^2} + 16\frac{dx}{dt} + 15x = 0, \qquad x(0) = -\frac{1}{6}, \quad x'(0) = 3.$$

(Answer 41) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$2\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.1, \quad x'(0) = 0.$$

If $\beta = 20$ N $\cdot$ s/m, then the system is underdamped, and we do expect to see decaying oscillations.
If $\beta = 30$ N $\cdot$ s/m, then the system overdamped, and we do not expect to see decaying oscillations.

(Answer 42) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$3\frac{d^2x}{dt^2} + 42\frac{dx}{dt} + kx = 0, \qquad x(0) = 0, \quad x'(0) = -5.$$

If $k = 100$ N/m, then the system is overdamped, and we do not expect to see decaying oscillations. If $k = 200$ N/m, then the system underdamped, and we do expect to see decaying oscillations.

(Answer 43) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$4\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 70x = 0, \qquad x(0) = 0.05, \quad x'(0) = -3.$$

Critical damping occurs when $\beta = 4\sqrt{70}$ $\mathrm{N \cdot s/m}$.

(Answer 44) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$m\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.05, \quad x'(0) = -3.$$

Critical damping occurs when $m = \frac{5}{4}$ kg.

(Answer 45)
- (a) Damper C.
- (b) Dampers E and A.
- (c) Dampers B and D.
- (d) Damper A has the highest damping constant. Damper D has the lowest damping constant.

(Answer 46) If $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3}\ln t$, then $y(t) = c_1\,e^{t/3} + c_2\,t\,e^{t/3} + \frac{1}{2}t^2e^{t/3}\ln t - \frac{3}{4}t^2e^{t/3}$ for all $t > 0$.

(Answer 47) If $4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 8e^{-t/2}\arctan t$, then $y(t) = c_1\,e^{-t/2} + c_2\,t\,e^{-t/2} + e^{-t/2}\big(t^2\arctan t + t - \arctan t - t\ln(1+t^2)\big)$.

(Answer 48) If $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$, then $y = c_1e^{-t} + c_2e^{-2t} - 2e^{-t}\sin(e^{t/2})$.

(Answer 49) If $9\frac{d^2y}{dt^2} + y = \sec^2(t/3)$, $y(0) = 4$, $y'(0) = 2$, then

$$y(t) = 5\cos(t/3) + 6\sin(t/3) + (\sin(t/3))\ln(\tan(t/3) + \sec(t/3)) - 1$$

for all $-3\pi/2 < t < 3\pi/2$.

(Answer 50) If $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$, then $y(t) = -4\sqrt{t} + 3t^2 + \frac{2}{t}$ for all $0 < t < \infty$.

(Answer 51) If $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$, then $y(t) = -\frac{3}{2}t^{-1}\ln t + C_1t^3 + C_2t^{-1}$ for all $t > 0$.

(Answer 52)

(a) The general solution to $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 0$ is $y_g = C_1e^{-t/2} + C_2e^{-t/3}$. To solve $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 2e^{4t}$ we make the guess $y_p = Ae^{4t}$. The solution is $y = C_1e^{-t/2} + C_2e^{-t/3} + \frac{1}{39}e^{4t}$.

(b) The general solution to $16\frac{d^2y}{dt^2} - y = 0$ is $y_g = C_1e^{t/4} + C_2e^{-t/4}$. To solve $16\frac{d^2y}{dt^2} - y = e^{t/4}\sin t$ we make the guess $y_p = Ae^{t/4}\sin t + Be^{t/4}\cos t$. The solution is $y = C_1e^{t/4} + C_2e^{-t/4} - (1/20)e^{t/4}\sin t - (1/40)e^{t/4}\cos t$.

(c) The general solution to $\frac{d^2y}{dt^2} + 49y = 0$ is $y_g = C_1\sin 7t + C_2\cos 7t$. To solve $\frac{d^2y}{dt^2} + 49y = 3t\sin 7t$ we make the guess $y_p = At^2\sin 7t + Bt\sin 7t + Ct^2\cos 7t + Dt\cos 7t$. The solution is $y = C_1\sin 7t + C_2\cos 7t - (3/28)t^2\cos 7t + (3/4)t\sin 7t$.

(d) The general solution to $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 0$ is $y_g = C_1 + C_2e^{4t}$. To solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$ we make the guess $y_p = Ae^{3t}$. The solution is $y = C_1 + C_2e^{4t} - (4/3)e^{3t}$.

(e) The general solution to $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$ is $y_g = C_1e^{-6t}\cos(7t) + C_2e^{-6t}\sin(7t)$. To solve $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = t\sin(3t)$ we make the guess $y_p = At\sin(3t) + Bt\cos(3t) + C\sin(3t) + D\cos(3t)$. The solution is

$$y = C_1e^{-6t}\cos(7t) + C_2e^{-6t}\sin(7t) + \frac{19}{1768}t\sin(3t) - \frac{9}{1768}t\cos(3t) - \frac{1353}{781456}\sin(3t) + \frac{606}{781456}\cos(3t).$$

(f) The general solution to $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 0$ is $y_g = C_1e^{2t} + C_2e^{-5t}$. To solve $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$ we make the guess $y_p = Ate^{-5t}$. The solution is $y = C_1e^{2t} + C_2e^{-5t} - (1/7)te^{-5t}$.

(g) The general solution to $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 0$ is $y_g = C_1e^{(3/4)t} + C_2te^{(3/4)t}$. To solve $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 6t^2 + \cos(2t)$ we make the guess $y_p = At^2 + Bt + C + D\cos(2t) + E\sin(2t)$. The solution is $y = C_1e^{(3/4)t} + C_2te^{(3/4)t} + (2/3)t^2 + (32/9)t + 64/9 - (48/5329)\cos(2t) - (55/5329)\sin(2t)$.

(h) The general solution to $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = 0$ is $y_g = C_1e^{3t}\cos(4t) + C_2e^{3t}\sin(4t)$. To solve $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = t^2e^{3t}$ we make the guess $y_p = At^2e^{3t} + Bte^{3t} + Ce^{3t}$. The solution is $y = C_1e^{3t}\cos(4t) + C_2e^{3t}\sin(4t) + (1/16)t^2e^{3t} - (1/128)e^{3t}$.

(i) The general solution to $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$ is $y_g = C_1e^{-5t} + C_2te^{-5t}$. To solve $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}$ we make the guess $y_p = At^2e^{-5t}$. The solution is $y = C_1e^{-5t} + C_2te^{-5t} + (3/2)t^2e^{-5t}$.

(j) The general solution to $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$ is $y_g = C_1e^{-2t} + C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t)$, we make the guess $y_p = A\cos(2t) + B\sin(2t)$. The solution is $y = C_1e^{-2t} + C_2e^{-3t} + \frac{15}{52}\sin(2t) + \frac{3}{52}\cos(2t)$.

(k) The general solution to $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$ is $y_g = C_1e^{-3t} + C_2te^{-3t}$. To solve $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t)$, we make the guess $y_p = A\cos(4t) + B\sin(4t)$.

(l) The general solution to $\frac{d^2y}{dt^2} + 9y = 0$ is $y_g = C_1\cos(3t) + C_2\sin(3t)$. To solve $\frac{d^2y}{dt^2} + 9y = 5\sin(3t)$, we make the guess $y_p = C_1t\cos(3t) + C_2t\sin(3t)$. The solution is $y = C_1\cos(3t) + C_2\sin(3t) - \frac{5}{6}t\cos(3t)$.

(m) The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ is $y_g = C_1e^{-t} + C_2te^{-t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$, we make the guess $y_p = At^2 + Bt + C$. The solution is $y = C_1e^{-t} + C_2te^{-t} + 2t^2 - 8t + 12$.

(n) The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 0$ is $y_g = C_1 + C_2e^{-2t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$, we make the guess $y_p = At^2 + Bt$. The solution is $y = C_1 + C_2e^{-2t} + \frac{3}{4}t^2 - \frac{3}{4}t$.

(o) The general solution to $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 0$ is $y_g = C_1e^{3t} + C_2e^{4t}$. To solve $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$, we make the guess $y_p = At + B + C\cos(2t) + D\sin(2t)$.

(p) The general solution to $\frac{d^2y}{dt^2} - 9y = 0$ is $y_g = C_1e^{3t} + C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$, we make the guess $y_p = Ae^t + Be^{-t} + Ct + D$.

(q) The general solution to $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$ is $y_g = C_1e^{2t} + C_2te^{2t}$. To solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$, we make the guess $y_p = At^2e^{2t} + B\cos t + C\sin t$.

(Answer 53)

(a) If $16\frac{d^2y}{dt^2} - y = 3e^t$, $y(0) = 1$, $y'(0) = 0$, then $y(t) = \frac{1}{5}e^t + \frac{4}{5}e^{-t/4}$.

(b) If $\frac{d^2y}{dt^2} + 49y = \sin 7t$, $y(\pi) = 3$, $y'(\pi) = 4$, then $y(t) = -\frac{1}{14}t\cos(7t) + \frac{\pi-42}{14}\cos(7t) - \frac{55}{98}\sin(7t)$.

(c) If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 25t^2$, $y(2) = 0$, $y'(2) = 0$, then $y(t) = -\frac{5}{2}t^2 - \frac{3}{2}t + \frac{19}{20} + \frac{11}{60}e^{2t-4} - \frac{356}{30}e^{-5t+10}$.

(d) If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$, then $y(t) = -\frac{4}{3}e^{3t} + \frac{7}{4}e^{4t} + \frac{7}{12}$.

(Answer 54) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters).

Then
$$3\frac{d^2x}{dt^2} + 27\frac{dx}{dt} + 588x = 3\cos(20t), \qquad u(0) = 0, \quad u'(0) = 0.$$

(Answer 55) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then
$$\frac{4}{32}\frac{d^2x}{dt^2} + 24x = 3\cos(\omega t), \qquad x(0) = 0, \quad x'(0) = 0.$$
Resonance occurs when $\omega = 8\sqrt{3}$ s^{-1}.

(Answer 56)

(a) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Let k denote the constant of the spring (in N·s/m). Then
$$4\frac{d^2x}{dt^2} + kx = 7\sin(\omega t), \qquad x(0) = 0, \quad x'(0) = 0.$$

(b) The spring constant is $k = 1600$ N·s/m.

(Answer 57) $\omega = 15$ radians/second in Picture B. $\omega = 16$ radians/second in Picture A. $\omega = 17$ radians/second in Picture C.

(Answer 58)

(a) $\mathcal{L}\{t\} = \frac{1}{s^2}$.

(b) $\mathcal{L}\{e^{-11t}\} = \frac{1}{s+11}$.

(c) $\mathcal{L}\{f(t)\} = \frac{3-3e^{4-4s}}{s-1}$.

(Answer 59)

(a) $\mathcal{L}\{t^4 + 5t^2 + 4\} = \frac{96}{s^5} + \frac{10}{s^3} + \frac{4}{s}$.

(b) $\mathcal{L}\{(t+2)^3\} = \mathcal{L}\{t^3 + 6t^2 + 12t + 8\} = \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{8}{s}$.

(c) $\mathcal{L}\{9e^{4t+7}\} = \mathcal{L}\{9e^7 e^{4t}\} = \frac{9e^7}{s-4}$.

(d) $\mathcal{L}\{-e^{3(t-2)}\} = \mathcal{L}\{-e^{-6}e^{3t}\} = -\frac{e^{-6}}{s-3}$.

(e) $\mathcal{L}\{(e^t + 1)^2\} = \mathcal{L}\{e^{2t} + 2e^t + 1\} = \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s}$.

(f) $\mathcal{L}\{8\sin(3t) - 4\cos(3t)\} = \frac{24}{s^2+9} - \frac{4s}{s^2+9}$.

(g) $\mathcal{L}\{t^2 e^{5t}\} = \frac{2}{(s-5)^3}$.

(h) $\mathcal{L}\{7e^{3t}\cos 4t\} = \frac{7s-21}{(s-3)^2+16}$.

(i) $\mathcal{L}\{4e^{-t}\sin 5t\} = \frac{20}{(s+1)^2+25}$.

(j) $\mathcal{L}\{t\,e^t \sin t\} = \frac{2s-2}{(s^2-2s+2)^2}$.

(k) If $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-3s}e^3}{s-1}$.

(l) If $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$ then $\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{1}{s} + \frac{2}{s^3}\right)$.

(m) If $f(t) = \begin{cases} 0, & t < 1, \\ t-2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2}$.

(n) If $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{5}{s-2} - \frac{5e^6 e^{-3s}}{s-2}$.

(o) If $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{14}{(s+1)^3} - \frac{14e^{-3s-3}}{(s+1)^3} - \frac{42e^{-3s-3}}{(s+1)^2} - \frac{63e^{-3s-3}}{s+1}$.

(p) If $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{s}{s^2+9} + e^{-\pi s}\frac{s}{s^2+9} - e^{-\pi s}\frac{3}{s^2+9}$.

(q) If $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s/2}\frac{1}{s^2+1} + e^{-\pi s}\frac{s}{s^2+1}$.

(r) $\mathcal{L}\{t^2 \sin 5t\} = \frac{40s^2 - 10(s^2+25)}{(s^2+25)^3}$.

(Answer 60)

(a) If $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$, then $y = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$.

(b) If $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$, then $y = \frac{5}{2}t^2 - \frac{7}{6}t^3$.

(c) If $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$, then $y = \frac{1}{2}t^2 e^{-t} + \frac{1}{6}t^3 e^{-t}$.

(d) If $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$, then $y = (1/4)e^{2t} + (7/4)e^{-2t}$.

(e) If $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$, then $y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{1}{9}t\,e^{3t}$.

(f) If $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$, then $y = \frac{1}{2} - \frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t)$.

(g) If $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$, then $y = \frac{1}{8}\cos t - \frac{1}{8}\cos 3t$.

(h) If $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$, then $y = 2\,\mathcal{U}(t-2)\,e^{t-2}\,\cos(t-2) + \mathcal{U}(t-2)\,e^{t-2}\,\sin(t-2)$.

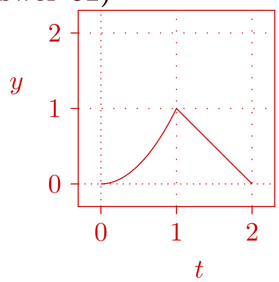
(i) If $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$, then $y = \frac{1}{2}\,\mathcal{U}(t-1)\,e^{3(t-1)} + \frac{1}{2}\,\mathcal{U}(t-1)\,e^{t-1}$.

(j) If $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$, then $y = \int_0^t \frac{1}{3}\sin 3r \cos(3t-3r)\,dr = \frac{1}{6}t\sin(3t)$.

(k) If $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$, then $y = e^{-2t}\int_0^t \sin(2r)\sin(2t-2r)\,dr = \frac{1}{4}e^{-2t}\sin(2t) - \frac{1}{2}e^{-2t}t\cos(2t)$.

(l) If $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$, then $y = \int_0^t \frac{e^{3r}+e^{-3r}}{2}\sqrt{t-r}\,dr$.

(Answer 61)



(Answer 62)

(a) If $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$, then $y(t) = -\frac{1}{30}\sin 3t - \frac{1}{90}\cos 3t + \frac{31}{30}e^{9t}$.

(b) If $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$, then $y = 3te^{2t} + 2e^{2t}$.

(c) If $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$, then $y = \frac{1}{5}t^3 - \frac{3}{25}t^2 + \frac{6}{125}t - \frac{6}{625} + \frac{1881}{625}e^{-5t}$.

(d) If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$, then $y(t) = e^{2t} - te^{2t}$.

(e) If $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$, then $y(t) = \frac{1}{5}(e^{-t} - e^t\cos t + 7e^t \sin t)$.

(f) If $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$, then $y(t) = 4e^{3t} - 2e^{4t} - te^{3t}$.

(g) If $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$, then $y(t) = -\frac{15}{8}e^{4t} + \frac{39}{8}e^{2t} - \frac{1}{4}te^{2t} - \frac{1}{2}t^2e^{2t} - t^3e^{2t}$.

(h) If $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{40}e^{2t} - \frac{1}{72}e^{-2t} - \frac{1}{90}e^t\cos(3t) - \frac{1}{45}e^t\sin(3t)$.

(i) If $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$, then $y(t) = \frac{1}{5}\cos 2t + \cos 3t + \frac{5}{3}\sin 3t$.

(j) If $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \le t < 4, \\ 0, & 4 \le t \end{cases}$, $y(0) = 2$, then

$$y(t) = \frac{2}{3} - \frac{2}{3}e^{-3t} - \frac{2}{3}\mathcal{U}(t-4) + \frac{2}{3}\mathcal{U}(t-4)e^{12-3t}.$$

The graph of y has a corner at $t = 4$. The graph of $\frac{dy}{dt}$ has a jump at $t = 4$.

(k) If $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = (1/6)(1 - \mathcal{U}(t-2\pi))(2\sin t - \sin 2t).$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2\pi$.

(l) If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - \mathcal{U}(t-10)\left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right].$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 10$. The graph of $\frac{d^2y}{dt^2}$ has a jump discontinuity at $t = 10$.

(m) If $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \le t < 2, \\ 4, & 2 \le t \end{cases}$, $y(0) = 3$, $y'(0) = 2$, $y''(0) = 1$, then

$$y(t) = 3e^{-t} + 5te^{-t} + 4t^2e^{-t} + \mathcal{U}(t-2)(4 - 4e^{2-t} - 4te^{2-t} - 2(t-2)^2e^{2-t}).$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2$. The graph of $\frac{d^3y}{dt^3}$ has a jump discontinuity at $t = 2$.

(n) If $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t \end{cases}$, $y(0) = 1$, $y'(0) = 0$, then

$$\begin{aligned} y(t) = {} & e^{-t/2}\cos t + \frac{1}{2}e^{-t/2}\sin t \\ & + \mathcal{U}(t-\pi)\left(-\frac{16}{17}\cos(t-\pi) + \frac{4}{17}\sin(t-\pi)\right) \\ & + \mathcal{U}(t-\pi)\left(\frac{4}{17}e^{-(t-\pi)/2}\sin(t-\pi) + \frac{16}{17}e^{-(t-\pi)/2}\cos(t-\pi)\right). \end{aligned}$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = \pi$.

(o) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$, then

$$y(t) = 2e^{-2t} + 5te^{-2t} + \frac{3}{4}u_2(t) - \frac{3}{4}e^{-2t+4}u_2(t) - \frac{3}{2}(t-2)e^{-2t+4}u_2(t).$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 2$. The graph of $\frac{d^2y}{dt^2}$ has a jump at $t = 2$.

(p) If $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t r\sin 3r \sin(3t - 3r)\, dr$.

(q) If $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t \sin 3r \sin(3t - 3r)\, dr = \frac{1}{6}\sin 3t - \frac{1}{2}t\cos 3t$.

(r) If $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$, then $y(t) = 7e^{-2t} - 5e^{-3t} + \int_0^t (e^{-2r} - e^{-3r})\sqrt{t-r+1}\, dr$.

(s) If $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$, then

$$y = 6e^{-t/3} - 6e^{-t/2} + 4\mathcal{U}(t-2) - 12\mathcal{U}(t-2)e^{-(t-2)/3} + 8\mathcal{U}(t-2)e^{-(t-2)/2}.$$

The graph of $y'(t)$ has a corner at $t = 2$, and the graph of $y''(t)$ has a jump at $t = 2$.

(t) If $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$, then $y(t) = 3e^{-9t} + 7\mathcal{U}(t-2)e^{-9t+18}$. The graph of $y(t)$ has a jump at $t = 2$.

(u) If $\frac{d^2y}{dt^2} + 4y = \delta(t - 4\pi)$, $y(0) = 1/2$, $y'(0) = 0$, then

$$y = \frac{1}{2}\cos(2t) - \mathcal{U}(t - 4\pi)\sin(2t).$$

The graph of $y(t)$ has a corner at $t = 4\pi$, and graph of $y'(t)$ has a jump at $t = 4\pi$.

(v) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + \mathcal{U}(t-2)$, $y(0) = 1$, $y'(0) = 0$, then

$$y = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} + \frac{1}{2}\mathcal{U}(t-1)e^{-t+1} - \frac{1}{2}\mathcal{U}(t-1)e^{-3t+3} + \frac{1}{3}\mathcal{U}(t-2) - \frac{1}{2}e^{-t+2}\mathcal{U}(t-2) + \frac{1}{6}\mathcal{U}(t-2)e^{-3t+6}.$$

The graph of $y(t)$ has a corner at $t = 1$. The graph of $y'(t)$ has a corner at $t = 2$, and a jump at $t = 1$. $y''(t)$ has an impulse at $t = 1$, and a jump at $t = 2$.

(w) If $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - \frac{dy}{dt} + 2y = 5\delta(t-4)$, $y(0) = 3$, $y'(0) = 0$, $y''(0) = 0$, then

$$y = \frac{3}{5}e^{2t} + \frac{4}{5}\cos t - \frac{2}{5}\sin t + \mathcal{U}(t-4)(e^{2t} - \cos t - 2\sin t)$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 4$, and the graph of $\frac{d^2y}{dt^2}$ has a jump at $t = 4$.

(Answer 63) Let t denote time (in minutes).
Let x denote the amount of salt (in grams) in tank A.
Let y denote the amount of salt (in grams) in tank B.
Then $x(0) = 3000$ and $y(0) = 2000$.
If $t < 50$, then

$$\frac{dx}{dt} = -\frac{9x}{200-4t} + \frac{3y}{300+3t}, \qquad \frac{dy}{dt} = \frac{4x}{200-4t} - \frac{7y}{300+3t}.$$

(Answer 64) If

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 4t - 5 \\ -2t + 3 \end{pmatrix}.$$

(Answer 65) If

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} -3\cos t + 16 \sin t \\ 2\cos t + 7\sin t \end{pmatrix}.$$

(Answer 66) If

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-20t} \begin{pmatrix} 3/20 \\ 9/10 \end{pmatrix} + e^{10t} \begin{pmatrix} -3/20 \\ 1/10 \end{pmatrix}.$$

(Answer 67) If

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 5\cos 12t + \sin 12t \\ 2\cos 12t + 2\sin 12t \end{pmatrix}.$$

(Answer 68) If

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos 3t - 11 \sin 3t \\ -3\cos 3t - (23/3)\sin 3t \end{pmatrix}.$$

(Answer 69) If

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 15t + 1 \\ (15/4)t + 4 \end{pmatrix}.$$

(Answer 70) If

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} + e^{-6t} \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix}.$$

(Answer 71) If

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{3t} \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} + e^{6t} \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}.$$

(Answer 72) If

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, x(0) = 3, \quad y(0) = 2, \quad z(0) = 1$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{3t} \begin{pmatrix} 2t^2 + 2t + 3 \\ 4t + 2 \\ 2t^2 + 2t + 1 \end{pmatrix}.$$

(Answer 73) If

$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, x(0) = 4, \quad y(0) = 3, \quad z(0) = 5$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 1 \\ 2t + 3 \\ 2t + 2 \end{pmatrix} + e^{-2t} \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}.$$

(Answer 74) If

$$\frac{dx}{dt} = -6x + 9y - 15, \qquad \frac{dy}{dt} = -5x + 6y - 8, \qquad x(0) = 2, \quad y(0) = 5$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 6\sin 3t \\ 4\sin 3t + 2\cos 3t \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

(Answer 75) If

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 0, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} (38/45)t^{10} + (1/9)t^9 \\ -(29/45)t^{10} \end{pmatrix}.$$

(Answer 76) If

$$\frac{dx}{dt} = 5y + e^{2t}\cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 0, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} -t\sin t + (1/2)t\cos t + (1/2)\sin t \\ -(1/2)t\sin t \end{pmatrix}.$$

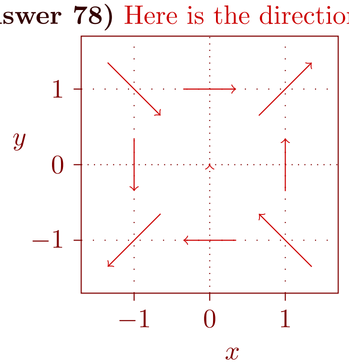
(Answer 77) If

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0$$

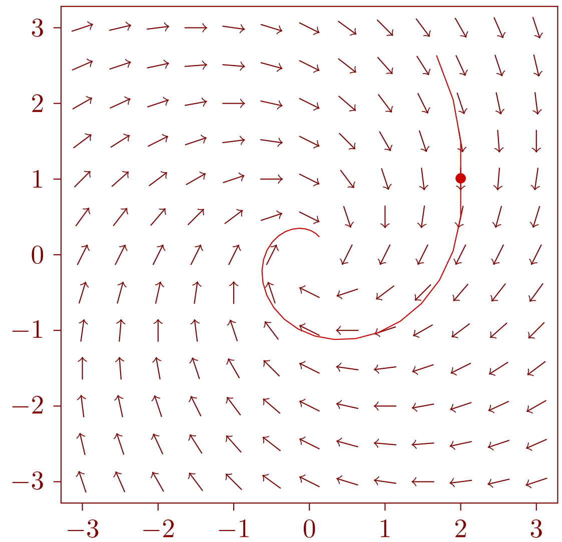
then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 4 - 4\cos(e^t) \\ 4\cos(e^t) - 4 \end{pmatrix} + e^{-2t} \begin{pmatrix} 3\sin(e^t) - 3e^t\cos(e^t) - 3\sin 1 + 3\cos 1 \\ 4e^t\cos(e^t) - 4\sin(e^t) - 4\cos 1 + 4\sin 1 \end{pmatrix}.$$

(Answer 78) Here is the direction field for the differential equation system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = x$.

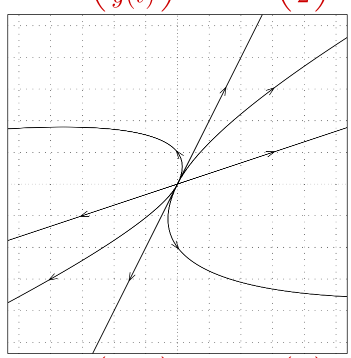


(Answer 79)

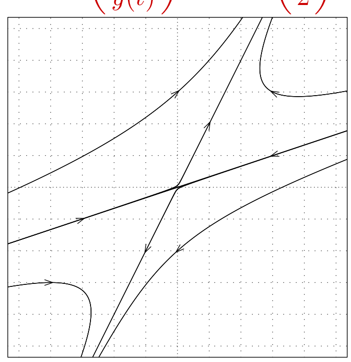


(Answer 80)

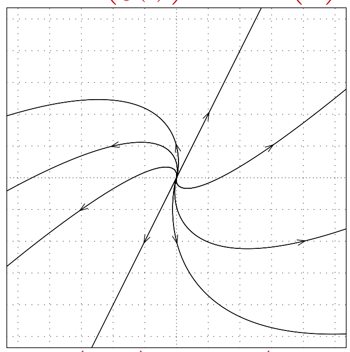
(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



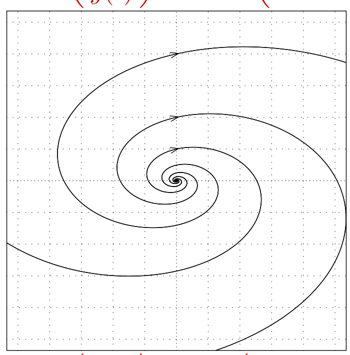
(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2.6 & 1.8 \\ -1.2 & 1.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



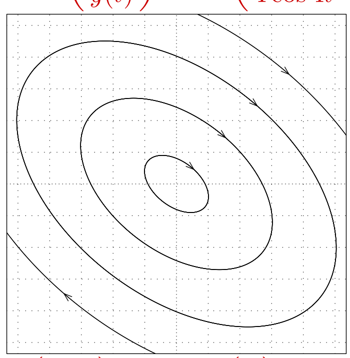
(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4/3 & -1/6 \\ 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^t \begin{pmatrix} t+3 \\ 2t \end{pmatrix}$



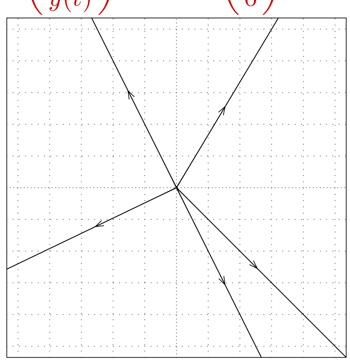
(d) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4.5 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 3\sin 3t \\ 2\cos 3t \end{pmatrix} + C_2 e^t \begin{pmatrix} -3\cos 3t \\ 2\sin 3t \end{pmatrix}$



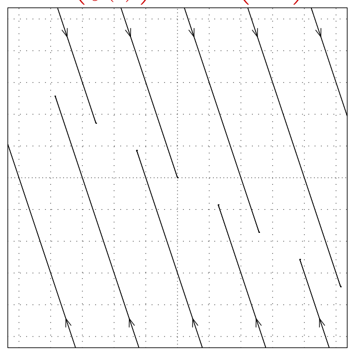
(e) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 5\sin 4t \\ 4\cos 4t - 2\sin 4t \end{pmatrix} + C_2 \begin{pmatrix} 5\cos 4t \\ -4\sin 4t - 2\cos 4t \end{pmatrix}$



(f) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

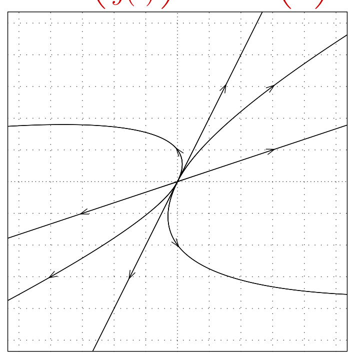


(g) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -6 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 3 \\ -2 \end{pmatrix} + C_2 e^{-7t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

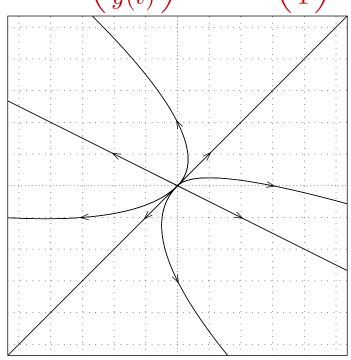


(Answer 81)

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

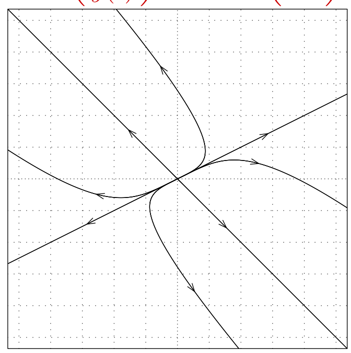


(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

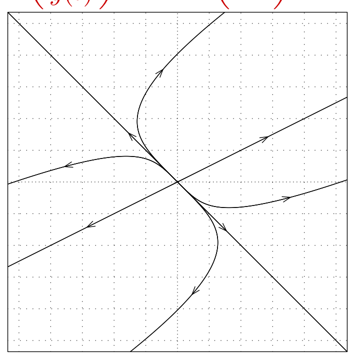


(Answer 82)

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

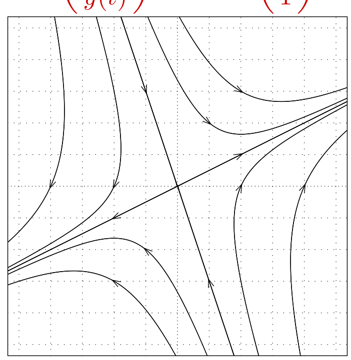


(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

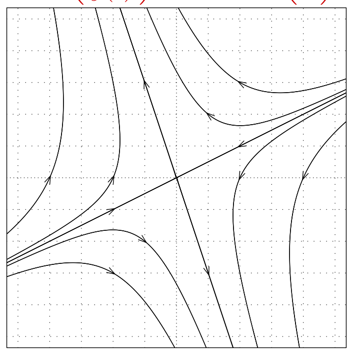


(Answer 83)

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 9 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-14t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

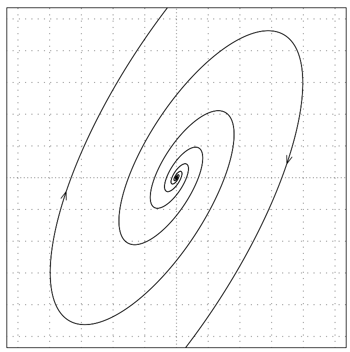


(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -4 & -6 \\ -9 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{14t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

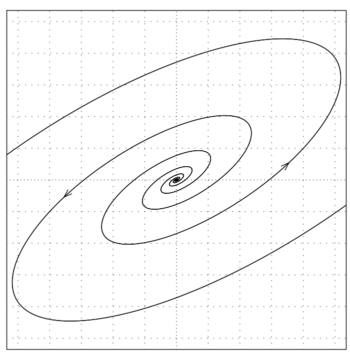


(Answer 84)

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -8 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, eigenvalues $r = 1 \pm 4i$



(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, eigenvalues $r = 1 \pm 4i$



(Answer 85) $\mu(y) = e^y$, and $e^y \sin x + ye^y = C$.

(Answer 86) $\mu(x) = x^4$, and $x^5 y^3 + x^7 y^2 = C$.

(Answer 87) The trajectories of solutions to $\frac{dx}{dt} = 3x - 4y$, $\frac{dy}{dt} = 4x - 3y$ satisfy $4y^2 - 6xy + 4x^2 = C$ for constants C.

(Answer 88) The trajectories of solutions to $\frac{dx}{dt} = 3y - 2xy$, $\frac{dy}{dt} = 4xy - 3y$ satisfy $y = -2x - \frac{3}{2} \ln |x - 3/2| + C$.

(Answer 89) The trajectories of solutions to $\frac{dx}{dt} = 3x - 4xy$, $\frac{dy}{dt} = 5xy - 2y$ satisfy $3 \ln |y| + 2 \ln |x| - 4y - 5x = C$.

(Answer 90)

(a) $\frac{dx}{dt} = v$, $\frac{1}{8}x\frac{dv}{dt} + \frac{1}{8}v^2 = 28 - 4x$, $x(0) = 0$.

(b) $\frac{1}{8}xv\frac{dv}{dx} + \frac{1}{8}v^2 = 28 - 4x$.

(c) $xv\frac{dv}{dx} + v^2 + 32x - 224 = 0$, $\mu(x) = x$.

(d) The solution to $x^2v\frac{dv}{dx} + xv^2 + 32x^2 - 224x = 0$ is $\frac{1}{2}x^2v^2 + \frac{32}{3}x^3 - 112x^2 = C$. Recalling that $x(0) = 0$, we see that $C = 0$ and so $\frac{1}{2}x^2v^2 + \frac{32}{3}x^3 - 112x^2 = 0$.

(e) $v = \sqrt{224 - \frac{64}{3}x}$. $v(0) = \sqrt{224}$.

(Answer 91) We have that $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ and also $\frac{dv}{dt} = \frac{d^2x}{dt^2}$. Thus $v\frac{dv}{dx} = 18x^3$, $v(1) = 3$. Solving, we see that $v = 3x^2$.

But then $\frac{dx}{dt} = 3x^2$, $x(0) = 1$, and so $x = \frac{1}{1-3t}$.

(Answer 92)

(a) The initial value problem is

$$1000\frac{d^2r}{dt^2} = -\frac{3.98 \times 10^{17}}{r^2}, \quad r(0) = 6{,}371{,}000, \quad r'(0) = 10000$$

where r denotes the distance to the center of the earth in meters and t denotes time in seconds.

(b) Let v be the rocket's velocity in meters/second. We have that

$$1000v\frac{dv}{dr} = -\frac{3.98 \times 10^{17}}{r^2}, \quad v(6{,}371{,}000) = 10000$$

and so

$$v^2 = \frac{7.96 \times 10^{14}}{r} - 2.494 \times 10^7.$$

(c) $v = 0$ when $r = 3.191 \times 10^7$ meters.

(Answer 93)

(a) The initial value problem is

$$5\frac{d^2r}{dt^2} = -\frac{3.98 \times 10^{17}}{r^2}, \quad r(0) = 16{,}371{,}000, \quad r'(0) = 0$$

where r denotes the distance to the center of the earth in meters and t denotes time in seconds.

(b) Let v be the rocket's velocity in meters/second. We have that

$$5v\frac{dv}{dr} = -\frac{3.98 \times 10^{17}}{r^2}, \quad v(16371000) = 0$$

and so

$$\frac{5}{2}v^2 = \frac{3.98 \times 10^{17}}{r} - 1.217 \times 10^8.$$

(c) When $r = 6{,}371{,}000$, then $v = -8743$ meters/second.

(Answer 94)

(a) The initial value problem is

$$3\frac{d^2r}{dt^2} = -\frac{6000}{r}, \quad r(0) = 1000, \quad r'(0) = 200$$

where r denotes the distance to the string in meters and t denotes time in seconds.

(b) Let v be the particle's velocity in meters/second. We have that

$$3v\frac{dv}{dr} = -\frac{6000}{r}, \quad v(1000) = 200$$

and so

$$v^2 = -4000\ln r + 4000\ln 1000 + 40000.$$

(c) $v = 0$ when $r = 1000e^{10}$ meters.

(Answer 95)

(*a*) The initial value problem is

$$0.02\frac{d^2r}{dt^2} = -\frac{3}{r^3}, \quad r(0) = 3, \quad r'(0) = 5$$

where r denotes the distance to the dipole in meters and t denotes time in seconds.

(*b*) Let v be the particle's velocity in meters/second. We have that

$$0.02v\frac{dv}{dr} = -\frac{3}{r^3}, \quad v(3) = 5$$

and so

$$v^2 = \frac{150}{r^2} + \frac{25}{3}.$$

(*c*) As $r \to \infty$, we see that v approaches $\frac{5}{\sqrt{3}}$ meters/second.

(Answer 96) Let θ be the angle between the pendulum and a vertical line (in radians), and let t denote time in seconds. Then

$$0.3\frac{d^2\theta}{dt^2} = -\frac{9.8}{0.5}\sin\theta, \quad \theta(0) = 0, \quad \theta'(0) = 20.$$

Let $\omega = \frac{d\theta}{dt}$ be the pendulum's angular velocity. We have that $\frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}$ and also $\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$. Thus $\omega\frac{d\omega}{d\theta} = -\frac{196}{3}\sin\theta$ and $\omega(0) = 20$. Solving, we see that $\frac{1}{2}\omega^2 = \frac{196}{3}\cos\theta + \frac{404}{3}$.