Math 2584, Spring 2018

You are allowed a double-sided, 3 inch by 5 inch card of notes.

You are responsible for all of the formulas you will need, except for the following Laplace transforms, which will be written on the last page of the exam.

$$
\begin{aligned}
\mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st} f(t)\,dt \\
\mathcal{L}\{1\} &= \frac{1}{s}, && s > 0, \\
\mathcal{L}\{t\} &= \frac{1}{s^2}, && s > 0, \\
\mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}}, && s > 0, && n \geq 0 \\
\mathcal{L}\{e^{at}\} &= \frac{1}{s-a}, && s > a \\
\mathcal{L}\{\cos(kt)\} &= \frac{s}{s^2+k^2}, && s > 0 \\
\mathcal{L}\{\sin(kt)\} &= \frac{k}{s^2+k^2}, && s > 0 \\
\text{If } \mathcal{L}\{f(t)\} &= F(s) \text{ then } \mathcal{L}\{e^{at} f(t)\} = F(s-a), \\
\mathcal{L}^{-1}\{F(s)\} &= e^{at}\,\mathcal{L}^{-1}\{F(s+a)\}, \\
\mathcal{L}\{\mathcal{U}(t-c)\} &= \frac{e^{-cs}}{s}, && s > 0, && c \geq 0 \\
\mathcal{L}\{\delta(t-c)\} &= e^{-cs}, && s > 0, && c \geq 0 \\
\mathcal{L}\{\mathcal{U}(t-c) f(t)\} &= e^{-cs}\,\mathcal{L}\{f(t+c)\}, && && c \geq 0 \\
\mathcal{L}\{\mathcal{U}(t-c) g(t-c)\} &= e^{-cs}\,\mathcal{L}\{g(t)\}, && && c \geq 0 \\
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= s\,\mathcal{L}\{y(t)\} - y(0). \\
\mathcal{L}\{t\,f(t)\} &= -\frac{d}{ds}\,\mathcal{L}\{f(t)\}, \\
\mathcal{L}\left\{\int_0^t f(r)\,g(t-r)\,dr\right\} &= \mathcal{L}\left\{\int_0^t f(t-r)\,g(r)\,dr\right\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}.
\end{aligned}
$$

(AB 1) Find the Laplace transforms of the following functions. You may use the table on the front of the exam.

(a) $f(t) = t\,e^t \sin t$

(b) $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$

(c) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$

(d) $f(t) = \begin{cases} 0, & t < 1, \\ t - 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$

(e) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

(f) $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

(g) $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi \end{cases}$

(h) $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi \end{cases}$

(i) $f(t) = t^2 \sin 5t$

(AB 2) For each of the following problems, find y.

(a) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$

(b) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$

(c) $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$

(d) $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$

(e) $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$

(AB 3) Sketch the graph of $y = t^2 - t^2\,\mathcal{U}(t-1) + (2-t)\,\mathcal{U}(t-1)$.

(AB 4) Solve the following initial-value problems using the Laplace transform. Do you expect the graphs of y, $\frac{dy}{dt}$, or $\frac{d^2y}{dt^2}$ to show any corners or jump discontinuities? If so, at what values of t?

(a) $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \leq t < 4, \\ 0, & 4 \leq t \end{cases}$, $y(0) = 2$, $y'(0) = 0$

(b) $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \leq t < 2\pi, \\ 0, & 2\pi \leq t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

(c) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \leq t < 10, \\ 0, & 10 \leq t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

(d) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \leq t < 2, \\ 4, & 2 \leq t \end{cases}$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = 2$.

(e) $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \leq t < \pi, \\ 0, & \pi \leq t \end{cases}$, $y(0) = 1$, $y'(0) = 0$

(f) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \leq t < 2, \\ 3, & 2 \leq t \end{cases}$, $y(0) = 2$, $y'(0) = 1$

(g) $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

(h) $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$

(i) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$

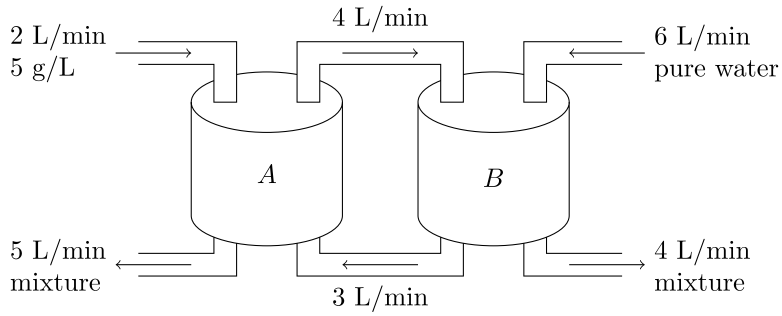
(j) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\,\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$.

(k) $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$.

(l) $\frac{d^2y}{dt^2} + 4y = -2\delta(t-4\pi)$, $y(0) = 1/2$, $y'(0) = 0$

(m) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + \mathcal{U}(t-2)$, $y(0) = 1$, $y'(0) = 0$
(n) $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 5\delta(t-4)$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 2$.

(AB 5) Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



Write the differential equations and initial conditions that describe the amount of salt in each tank.

(AB 6) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 7) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 8) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 9) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Answer key

(Answer 1)

(a) $\mathcal{L}\{t\,e^t \sin t\} = \frac{2s-2}{(s^2-2s+2)^2}$.

(b) If $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-3s}e^3}{s-1}$.

(c) If $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$ then $\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{1}{s} + \frac{2}{s^3}\right)$.

(d) If $f(t) = \begin{cases} 0, & t < 1, \\ t-2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2}$.

(e) If $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{5}{s-2} - \frac{5e^6e^{-3s}}{s-2}$.

(f) If $f(t) = \begin{cases} 7t^2e^{-t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{14}{(s+1)^3} - \frac{14e^{-3s-3}}{(s+1)^3} - \frac{42e^{-3s-3}}{(s+1)^2} - \frac{63e^{-3s-3}}{s+1}$.

(g) If $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{s}{s^2+9} + e^{-\pi s}\frac{s}{s^2+9} - e^{-\pi s}\frac{3}{s^2+9}$.

(h) If $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s/2}\frac{1}{s^2+1} + e^{-\pi s}\frac{s}{s^2+1}$.

(i) $\mathcal{L}\{t^2 \sin 5t\} = \frac{40s^2 - 10(s^2+25)}{(s^2+25)^3}$.

(Answer 2)

(a) If $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$, then $y = 2\,\mathcal{U}(t-2)\,e^{t-2}\,\cos(t-2) + \mathcal{U}(t-2)\,e^{t-2}\,\sin(t-2)$.

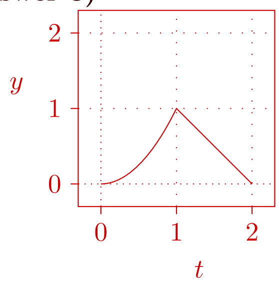
(b) If $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$, then $y = \frac{1}{2}\,\mathcal{U}(t-1)\,e^{3(t-1)} + \frac{1}{2}\,\mathcal{U}(t-1)\,e^{t-1}$.

(c) If $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$, then $y = \int_0^t \frac{1}{3}\sin 3r \cos(3t-3r)\,dr = \frac{1}{6}t\sin(3t)$.

(d) If $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$, then $y = e^{-2t}\int_0^t \sin(2r)\sin(2t-2r)\,dr = \frac{1}{4}e^{-2t}\sin(2t) - \frac{1}{2}e^{-2t}t\cos(2t)$.

(e) If $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$, then $y = \int_0^t \frac{e^{3r}+e^{-3r}}{2}\sqrt{t-r}\,dr$.

(Answer 3)



(Answer 4)

(a) If $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \le t < 4, \\ 0, & 4 \le t \end{cases}$, $y(0) = 2$, then

$$y(t) = \frac{2}{3} - \frac{2}{3}e^{-3t} - \frac{2}{3}\mathcal{U}(t-4) + \frac{2}{3}\mathcal{U}(t-4)e^{12-3t}.$$

The graph of y has a corner at $t = 4$. The graph of $\frac{dy}{dt}$ has a jump at $t = 4$.

(b) If $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = (1/6)(1 - \mathcal{U}(t-2\pi))(2\sin t - \sin 2t).$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2\pi$.

(c) If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - \mathcal{U}(t-10)\left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right].$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 10$. The graph of $\frac{d^2y}{dt^2}$ has a jump discontinuity at $t = 10$.

(d) If $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \le t < 2, \\ 4, & 2 \le t \end{cases}$, $y(0) = 3$, $y'(0) = 2$, $y''(0) = 1$, then

$$y(t) = 3e^{-t} + 5te^{-t} + 4t^2e^{-t} + \mathcal{U}(t-2)(4 - 4e^{2-t} - 4te^{2-t} - 2(t-2)^2e^{2-t}).$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2$. The graph of $\frac{d^3y}{dt^3}$ has a jump discontinuity at $t = 2$.

(e) If $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t \end{cases}$, $y(0) = 1$, $y'(0) = 0$, then

$$\begin{aligned} y(t) = {} & e^{-t/2}\cos t + \frac{1}{2}e^{-t/2}\sin t \\ & + \mathcal{U}(t-\pi)\left(-\frac{16}{17}\cos(t-\pi) + \frac{4}{17}\sin(t-\pi)\right) \\ & + \mathcal{U}(t-\pi)\left(\frac{4}{17}e^{-(t-\pi)/2}\sin(t-\pi) + \frac{16}{17}e^{-(t-\pi)/2}\cos(t-\pi)\right). \end{aligned}$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = \pi$.

(f) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$, then

$$y(t) = 2e^{-2t} + 5te^{-2t} + \frac{3}{4}u_2(t) - \frac{3}{4}e^{-2t+4}u_2(t) - \frac{3}{2}(t-2)e^{-2t+4}u_2(t).$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 2$. The graph of $\frac{d^2y}{dt^2}$ has a jump at $t = 2$.

(g) If $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t r\sin 3r \sin(3t - 3r)\, dr$.

(h) If $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t \sin 3r \sin(3t - 3r)\, dr = \frac{1}{6}\sin 3t - \frac{1}{2}t\cos 3t$.

(i) If $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$, then $y(t) = 7e^{-2t} - 5e^{-3t} + \int_0^t (e^{-2r} - e^{-3r})\sqrt{t - r + 1}\, dr$.

(j) If $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$, then

$$y = 6e^{-t/3} - 6e^{-t/2} + 4\mathcal{U}(t-2) - 12\mathcal{U}(t-2)e^{-(t-2)/3} + 8\mathcal{U}(t-2)e^{-(t-2)/2}.$$

The graph of $y'(t)$ has a corner at $t = 2$, and the graph of $y''(t)$ has a jump at $t = 2$.

(k) If $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$, then $y(t) = 3e^{-9t} + 7\mathcal{U}(t-2)e^{-9t+18}$. The graph of $y(t)$ has a jump at $t = 2$.

(l) If $\frac{d^2y}{dt^2} + 4y = \delta(t - 4\pi)$, $y(0) = 1/2$, $y'(0) = 0$, then

$$y = \frac{1}{2}\cos(2t) - \mathcal{U}(t - 4\pi)\sin(2t).$$

The graph of $y(t)$ has a corner at $t = 4\pi$, and graph of $y'(t)$ has a jump at $t = 4\pi$.

(m) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + \mathcal{U}(t-2)$, $y(0) = 1$, $y'(0) = 0$, then

$$y = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} + \frac{1}{2}\mathcal{U}(t-1)e^{-t+1} - \frac{1}{2}\mathcal{U}(t-1)e^{-3t+3} + \frac{1}{3}\mathcal{U}(t-2) - \frac{1}{2}e^{-t+2}\mathcal{U}(t-2) + \frac{1}{6}\mathcal{U}(t-2)e^{-3t+6}.$$

The graph of $y(t)$ has a corner at $t = 1$. The graph of $y'(t)$ has a corner at $t = 2$, and a jump at $t = 1$. $y''(t)$ has an impulse at $t = 1$, and a jump at $t = 2$.

(n) If $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - \frac{dy}{dt} + 2y = 5\delta(t-4)$, $y(0) = 3$, $y'(0) = 0$, $y''(0) = 0$, then

$$y = \frac{3}{5}e^{2t} + \frac{4}{5}\cos t - \frac{2}{5}\sin t + \mathcal{U}(t-4)(e^{2t} - \cos t - 2\sin t)$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 4$, and the graph of $\frac{d^2y}{dt^2}$ has a jump at $t = 4$.

(Answer 5) Let t denote time (in minutes).

Let x denote the amount of salt (in grams) in tank A.

Let y denote the amount of salt (in grams) in tank B.

Then $x(0) = 3000$ and $y(0) = 2000$.

If $t < 50$, then

$$\frac{dx}{dt} = -\frac{9x}{200 - 4t} + \frac{3y}{300 + 3t}, \qquad \frac{dy}{dt} = \frac{4x}{200 - 4t} - \frac{7y}{300 + 3t}.$$

(Answer 6)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t}\begin{pmatrix} 4t - 5 \\ -2t + 3 \end{pmatrix}.$$

(Answer 7)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-20t} \begin{pmatrix} 3/20 \\ 9/10 \end{pmatrix} + e^{10t} \begin{pmatrix} -3/20 \\ 1/10 \end{pmatrix}.$$

(Answer 8)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 15t + 1 \\ (15/4)t + 4 \end{pmatrix}.$$

(Answer 9)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} + e^{-6t} \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix}.$$