Math 2584, Fall 2017

You are allowed a double-sided, 3 inch by 5 inch card of notes.

(AB 1) Find the general solution to the following differential equations.

(a) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0.$
(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0.$
(c) $\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 24y = 0.$
(d) $\frac{d^3y}{dt^3} + y = 0.$
(e) $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0.$
(f) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 3e^{4t}.$
(g) $16\frac{d^2y}{dt^2} - y = e^{t/4}\sin t.$
(h) $\frac{d^2y}{dt^2} + 49y = 3t\sin 7t.$
(i) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}.$
(j) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = t\sin(3t).$
(k) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}.$
(l) $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 6t^2 + \cos(2t).$
(m) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = t^2e^{3t}.$
(n) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}.$
(o) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t).$
(p) $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t).$
(q) $\frac{d^2y}{dt^2} + 9y = 5\sin(3t).$
(r) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2.$
(s) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t.$
(t) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t).$
(u) $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2.$
(v) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t.$

(AB 2) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $16\frac{d^2y}{dt^2} - y = 3e^t$, $y(0) = 1$, $y'(0) = 0$.
(b) $\frac{d^2y}{dt^2} + 49y = \sin 7t$, $y(\pi) = 3$, $y'(\pi) = 4$.
(c) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 25t^2$, $y(2) = 0$, $y'(2) = 3$.
(d) $9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 2y = 0$, $y(0) = 3$, $y'(0) = 2$.
(e) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$, $y(0) = 1$, $y'(0) = 4$.
(f) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$.

(AB 3) Find the general solution to the equation $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^t \ln t$ on the interval $0 < t < \infty$.

(AB 4) Find the general solution to the equation $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-2t}\arctan t$.

(AB 5) Find the general solution to the equation $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sin(e^t)$.

(AB 6) Solve the initial-value problem $\frac{d^2y}{dt^2} + 9y = 9\sec^2 3t$, $y(0) = 4$, $y'(0) = 6$, on the interval $-\pi/6 < t < \pi/6$.

(AB 7) The general solution to the differential equation $t^2 \frac{d^2y}{dt^2} - 2y = 0$, $t > 0$, is $y(t) = C_1 t^2 + C_2 t^{-1}$. Solve the initial-value problem $t^2 \frac{d^2y}{dt^2} - 2y = 7t^3$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(AB 8) The general solution to the differential equation $t^2 \frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 0$, $t > 0$, is $y(t) = C_1 t^3 + C_2 t^{-1}$. Find the general solution to the differential equation $t^2 \frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

(AB 9) An object weighing 5 lbs stretches a spring 4 inches. It is attached to a viscous damper with damping constant 16 $\text{lb} \cdot \text{sec/ft}$. The object is pulled down an additional 2 inches and is released with initial velocity 3 ft/sec upwards.

Write the differential equation and initial conditions that describe the position of the object.

(AB 10) A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant β. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20\ \text{N} \cdot \text{s/m}$, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30\ \text{N} \cdot \text{s/m}$, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(AB 11) A 3-kg object is attached to a spring with constant k and to a viscous damper with damping constant 42 N·sec/m. The object is set in motion from its equilibrium position with initial velocity 5 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object.

If $k = 100$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $k = 200$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(AB 12) A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant β. The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of β for which the system is critically damped. Be sure to include units for β.

(AB 13) An object of mass m is attached to a spring with constant 80 N/m and to a viscous damper with damping constant 20 N·s/m. The object is pulled down to 5cm below its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of m for which the system is critically damped. Be sure to include units for m.

(AB 14) A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 $\text{N} \cdot \text{s/m}$ and is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(20t)$ N upwards.

Write the differential equation and initial conditions that describe the position of the object.

(AB 15) An object weighing 4 lbs stretches a spring 2 inches. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(\omega t)$ pounds, directed upwards. There is no damping.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of ω for which resonance occurs. Be sure to include units for ω.

(AB 16) A 4-kg object is suspended from a spring. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $7\sin(\omega t)$ pounds, directed upwards.

(a) Write the differential equation and initial conditions that describe the position of the object.

(b) It is observed that resonance occurs when $\omega = 20$ rad/sec. What is the constant of the spring? Be sure to include units.

Answer key

(Answer 1)

(a) If $\frac{d^2y}{dt^2}+12\frac{dy}{dt}+85y=0$, then $y=C_1e^{-6t}\cos(7t)+C_2e^{-6t}\sin(7t)$.

(b) If $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+2y=0$, then $y=C_1e^{(-2+\sqrt{2})t}+C_2e^{(-2-\sqrt{2})t}$.

(c) If $\frac{d^3y}{dt^3}-6\frac{d^2y}{dt^2}+4\frac{dy}{dt}-24y=0$, then $y=C_1e^{6t}+C_2\cos 2t+C_3\sin 2t$.

(d) If $\frac{d^3y}{dt^3}+y=0$, then $y=C_1e^{-t}+C_2e^{t/2}\cos\frac{\sqrt{3}}{2}t+C_3e^{t/2}\sin\frac{\sqrt{3}}{2}t$.

(e) If $\frac{d^4y}{dt^4}-8\frac{d^2y}{dt^2}+16y=0$, then $y=C_1e^{2t}+C_2te^{2t}+C_3e^{-2t}+C_4te^{-2t}$.

(f) The general solution to $6\frac{d^2y}{dt^2}+5\frac{dy}{dt}+y=0$ is $y_g=C_1e^{-t/2}+C_2e^{-t/3}$. To solve $6\frac{d^2y}{dt^2}+5\frac{dy}{dt}+y=2e^{4t}$ we make the guess $y_p=Ae^{4t}$. The solution is $y=C_1e^{-t/2}+C_2e^{-t/3}+\frac{1}{39}e^{4t}$.

(g) The general solution to $16\frac{d^2y}{dt^2}-y=0$ is $y_g=C_1e^{t/4}+C_2e^{-t/4}$. To solve $16\frac{d^2y}{dt^2}-y=e^{t/4}\sin t$ we make the guess $y_p=Ae^{t/4}\sin t+Be^{t/4}\cos t$. The solution is $y=C_1e^{t/4}+C_2e^{-t/4}-(1/20)e^{t/4}\sin t-(1/40)e^{t/4}\cos t$.

(h) The general solution to $\frac{d^2y}{dt^2}+49y=0$ is $y_g=C_1\sin 7t+C_2\cos 7t$. To solve $\frac{d^2y}{dt^2}+49y=3t\sin 7t$ we make the guess $y_p=At^2\sin 7t+Bt\sin 7t+Ct^2\cos 7t+Dt\cos 7t$. The solution is $y=C_1\sin 7t+C_2\cos 7t-(3/28)t^2\cos 7t+(3/4)t\sin 7t$.

(i) The general solution to $\frac{d^2y}{dt^2}-4\frac{dy}{dt}=0$ is $y_g=C_1+C_2e^{4t}$. To solve $\frac{d^2y}{dt^2}-4\frac{dy}{dt}=4e^{3t}$ we make the guess $y_p=Ae^{3t}$. The solution is $y=C_1+C_2e^{4t}-(4/3)e^{3t}$.

(j) The general solution to $\frac{d^2y}{dt^2}+12\frac{dy}{dt}+85y=0$ is $y_g=C_1e^{-6t}\cos(7t)+C_2e^{-6t}\sin(7t)$. To solve $\frac{d^2y}{dt^2}+12\frac{dy}{dt}+85y=t\sin(3t)$ we make the guess $y_p=At\sin(3t)+Bt\cos(3t)+C\sin(3t)+D\cos(3t)$. The solution is

$$y=C_1e^{-6t}\cos(7t)+C_2e^{-6t}\sin(7t)+\frac{19}{1768}t\sin(3t)-\frac{9}{1768}t\cos(3t)-\frac{1353}{781456}\sin(3t)+\frac{606}{781456}\cos(3t).$$

(k) The general solution to $\frac{d^2y}{dt^2}+3\frac{dy}{dt}-10y=0$ is $y_g=C_1e^{2t}+C_2e^{-5t}$. To solve $\frac{d^2y}{dt^2}+3\frac{dy}{dt}-10y=7e^{-5t}$ we make the guess $y_p=Ate^{-5t}$. The solution is $y=C_1e^{2t}+C_2e^{-5t}-(1/7)te^{-5t}$.

(l) The general solution to $16\frac{d^2y}{dt^2}-24\frac{dy}{dt}+9y=0$ is $y_g=C_1e^{(3/4)t}+C_2te^{(3/4)t}$. To solve $16\frac{d^2y}{dt^2}-24\frac{dy}{dt}+9y=6t^2+\cos(2t)$ we make the guess $y_p=At^2+Bt+C+D\cos(2t)+E\sin(2t)$. The solution is $y=C_1e^{(3/4)t}+C_2te^{(3/4)t}+(2/3)t^2+(32/9)t+64/9-(48/5329)\cos(2t)-(55/5329)\sin(2t)$.

(m) The general solution to $\frac{d^2y}{dt^2}-6\frac{dy}{dt}+25y=0$ is $y_g=C_1e^{3t}\cos(4t)+C_2e^{3t}\sin(4t)$. To solve $\frac{d^2y}{dt^2}-6\frac{dy}{dt}+25y=t^2e^{3t}$ we make the guess $y_p=At^2e^{3t}+Bte^{3t}+Ce^{3t}$. The solution is $y=C_1e^{3t}\cos(4t)+C_2e^{3t}\sin(4t)+(1/16)t^2e^{3t}-(1/128)e^{3t}$.

(n) The general solution to $\frac{d^2y}{dt^2}+10\frac{dy}{dt}+25y=0$ is $y_g=C_1e^{-5t}+C_2te^{-5t}$. To solve $\frac{d^2y}{dt^2}+10\frac{dy}{dt}+25y=3e^{-5t}$ we make the guess $y_p=At^2e^{-5t}$. The solution is $y=C_1e^{-5t}+C_2te^{-5t}+(3/2)t^2e^{-5t}$.

(o) The general solution to $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=0$ is $y_g=C_1e^{-2t}+C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=3\cos(2t)$, we make the guess $y_p=A\cos(2t)+B\sin(2t)$. The solution is $y=C_1e^{-2t}+C_2e^{-3t}+\frac{15}{52}\sin(2t)+\frac{3}{52}\cos(2t)$.

(p) The general solution to $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=0$ is $y_g=C_1e^{-3t}+C_2te^{-3t}$. To solve $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=5\sin(4t)$, we make the guess $y_p=A\cos(4t)+B\sin(4t)$.

(q) The general solution to $\frac{d^2y}{dt^2}+9y=0$ is $y_g=C_1\cos(3t)+C_2\sin(3t)$. To solve $\frac{d^2y}{dt^2}+9y=5\sin(3t)$, we make the guess $y_p=C_1t\cos(3t)+C_2t\sin(3t)$. The solution is $y=C_1\cos(3t)+C_2\sin(3t)-\frac{5}{6}t\cos(3t)$.

(r) The general solution to $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=0$ is $y_g=C_1e^{-t}+C_2te^{-t}$. To solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=2t^2$, we make the guess $y_p=At^2+Bt+C$. The solution is $y=C_1e^{-t}+C_2te^{-t}+2t^2-8t+12$.

(s) The general solution to $\frac{d^2y}{dt^2}+2\frac{dy}{dt}=0$ is $y_g=C_1+C_2e^{-2t}$. To solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}=3t$, we make the guess $y_p=At^2+Bt$. The solution is $y=C_1+C_2e^{-2t}+\frac{3}{4}t^2-\frac{3}{4}t$.

(t) The general solution to $\frac{d^2y}{dt^2}-7\frac{dy}{dt}+12y=0$ is $y_g=C_1e^{3t}+C_2e^{4t}$. To solve $\frac{d^2y}{dt^2}-7\frac{dy}{dt}+12y=5t+\cos(2t)$, we make the guess $y_p=At+B+C\cos(2t)+D\sin(2t)$.

(u) The general solution to $\frac{d^2y}{dt^2}-9y=0$ is $y_g=C_1e^{3t}+C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2}-9y=2e^t+e^{-t}+5t+2$, we make the guess $y_p=Ae^t+Be^{-t}+Ct+D$.

(v) The general solution to $\frac{d^2y}{dt^2}-4\frac{dy}{dt}+4y=0$ is $y_g=C_1e^{2t}+C_2te^{2t}$. To solve $\frac{d^2y}{dt^2}-4\frac{dy}{dt}+4y=3e^{2t}+5\cos t$, we make the guess $y_p=At^2e^{2t}+B\cos t+C\sin t$.

(Answer 2)

(a) If $16\frac{d^2y}{dt^2}-y=3e^t$, $y(0)=1$, $y'(0)=0$, then $y(t)=\frac{1}{5}e^t+\frac{4}{5}e^{-t/4}$.

(b) If $\frac{d^2y}{dt^2}+49y=\sin 7t$, $y(\pi)=3$, $y'(\pi)=4$, then $y(t)=-\frac{1}{14}t\cos(7t)+\frac{\pi-42}{14}\cos(7t)-\frac{55}{98}\sin(7t)$.

(c) If $\frac{d^2y}{dt^2}+3\frac{dy}{dt}-10y=25t^2$, $y(2)=0$, $y'(2)=0$, then $y(t)=-\frac{5}{2}t^2-\frac{3}{2}t+\frac{19}{20}+\frac{11}{60}e^{2t-4}-\frac{356}{30}e^{-5t+10}$.

(d) If $9\frac{d^2y}{dt^2}+6\frac{dy}{dt}+2y=0$, $y(0)=3$, $y'(0)=2$, then $y=3e^{-t/3}\cos(t/3)+9e^{-t/3}\sin(t/3)$.

(e) If $\frac{d^2y}{dt^2}+10\frac{dy}{dt}+25y=0$, $y(0)=1$, $y'(0)=4$, then $y=e^{-5t}+9te^{-5t}$.

(f) If $\frac{d^2y}{dt^2}-4\frac{dy}{dt}=4e^{3t}$, $y(0)=1$, $y'(0)=3$, then $y(t)=-\frac{4}{3}e^{3t}+\frac{7}{4}e^{4t}+\frac{7}{12}$.

(Answer 3) If $\frac{d^2y}{dt^2}-2\frac{dy}{dt}+y=e^t\ln t$, then $y(t)=c_1\,e^t+c_2\,t\,e^t+\frac{1}{2}t^2e^t\ln t-\frac{3}{4}t^2e^t$ for all $t>0$.

(Answer 4) If $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=e^{-2t}\arctan t$, then $y(t)=c_1\,e^{-2t}+c_2\,t\,e^{-2t}+\frac{1}{2}\big(t^2\arctan t-t\ln(1+t^2)-t+\arctan t\big)$.

(Answer 5) If $\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=\sin(e^t)$, then $y=c_1e^{-t}+c_2e^{-2t}-e^{-2t}\sin(e^t)$.

(Answer 6) If $\frac{d^2y}{dt^2}+9y=9\sec^2 3t$, $y(0)=4$, $y'(0)=6$, then

$$y(t)=5\cos(3t)+2\sin(3t)+(\sin(3t))\ln(\tan(3t)+\sec(3t))-1$$

for all $-\pi/6<t<\pi/6$.

(Answer 7) If $t^2\frac{d^2y}{dt^2}-2y=7t^3$, $y(1)=1$, $y'(1)=2$, then $y(t)=\frac{7}{4}t^3-\frac{4}{3}t^2+\frac{7}{12}t^{-1}$ for all $0<t<\infty$.

(Answer 8) If $t^2\frac{d^2y}{dt^2}-t\frac{dy}{dt}-3y=6t^{-1}$, then $y(t)=-\frac{3}{2}t^{-1}\ln t+C_1t^3+C_2t^{-1}$ for all $t>0$.

(Answer 9) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then

$$\frac{5}{32}\frac{d^2x}{dt^2}+16\frac{dx}{dt}+15x=0,\qquad x(0)=-\frac{1}{6},\quad x'(0)=3.$$

(Answer 10) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$2\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+80x=0,\qquad x(0)=-0.1,\quad x'(0)=0.$$

If $\beta=20$ N$\cdot$s/m, then the system is underdamped, and we do expect to see decaying oscillations.
If $\beta=30$ N$\cdot$s/m, then the system overdamped, and we do not expect to see decaying oscillations.

(Answer 11) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$3\frac{d^2x}{dt^2} + 42\frac{dx}{dt} + kx = 0, \qquad x(0) = 0, \quad x'(0) = -5.$$

If $k = 100$ N/m, then the system is overdamped, and we do not expect to see decaying oscillations.
If $k = 200$ N/m, then the system underdamped, and we do expect to see decaying oscillations.

(Answer 12) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$4\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 70x = 0, \qquad x(0) = 0.05, \quad x'(0) = -3.$$

Critical damping occurs when $\beta = 4\sqrt{70}$ N $\cdot$ s/m.

(Answer 13) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$m\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.05, \quad x'(0) = -3.$$

Critical damping occurs when $m = \frac{5}{4}$ kg.

(Answer 14) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters).
Then

$$3\frac{d^2x}{dt^2} + 27\frac{dx}{dt} + 588x = 3\cos(20t), \qquad u(0) = 0, \quad u'(0) = 0.$$

(Answer 15) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then

$$\frac{4}{32}\frac{d^2x}{dt^2} + 24x = 3\cos(\omega t), \qquad x(0) = 0, \quad x'(0) = 0.$$

Resonance occurs when $\omega = 8\sqrt{3}$ s^{-1}.

(Answer 16)

(a) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Let k denote the constant of the spring (in N·s/m). Then

$$4\frac{d^2x}{dt^2} + kx = 7\sin(\omega t), \qquad x(0) = 0, \quad x'(0) = 0.$$

(b) The spring constant is $k = 1600$ N·s/m.