1.1. Partial differential equations

[Definition: Partial derivative] We say that $u_{x_i}(x) = \partial_{x_i} u(x) = \dfrac{\partial u}{\partial x_i}(x) = \lim_{h\to 0} \dfrac{u(x+h\vec{e}_i) - u(x)}{h}$ provided u is defined in a neighborhood of x and this limit exists.

[Definition: Multiindex notation] If $\alpha \in (\mathbb{N}_0)^n$ is a vector in $\mathbb{R}^n$ all of whose entries are nonnegative integers, then α is called a multiindex of order $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

We let $D^\alpha u(x) = \dfrac{\partial^{|\alpha|} u(x)}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_n}^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_n}^{\alpha_n} u(x)$.

We let $D^k u(x) = (D^\alpha u(x))_{|\alpha|=k}$.

(Problem 10) Let q_k be the number of multiindices of order k. Find an exact formula for q_k. Which is larger, q_k or n^k?

(Problem 11) Let $x, z \in \mathbb{R}^n$. Show that $\dfrac{d}{dt} u(x+tz) = z \cdot Du(x+tz)$.

[Definition: Partial differential equation] An expression of the form

$$F\big(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x\big) = 0, \tag{$*$}$$

where u is the unknown and $F : \mathbb{R}^{q_k} \times \mathbb{R}^{q_{k-1}} \times \mathbb{R}^n \times \mathbb{R} \times \Omega \mapsto \mathbb{R}$ is a given function, is called a partial differential equation.

[Definition: Order] The order of the differential equation $(*)$ is the number k.

[Definition: Systems] An expression of the form

$$\vec{F}\big(D^k \vec{u}(x), D^{k-1} \vec{u}(x), \ldots, D\vec{u}(x), \vec{u}(x), x\big) = 0, \tag{$**$}$$

where $\vec{u} : \Omega \mapsto \mathbb{R}^m$ is the unknown and where $\vec{F} : \mathbb{R}^{mq_k} \times \mathbb{R}^{mq_{k-1}} \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \mapsto \mathbb{R}^m$ is a given function, is called a system of partial differential equations.

[Definition: Classification] A partial differential equation is linear homogeneous if it is of the form $\displaystyle\sum_{|\alpha|\leq k} A_\alpha(x) D^\alpha u(x) = 0$.

It is linear if it is of the form $\displaystyle\sum_{|\alpha|\leq k} A_\alpha(x) D^\alpha u(x) = f(x)$.

It is semilinear if it is of the form $\displaystyle\sum_{|\alpha|=k} A_\alpha(x) D^\alpha u(x) = F\big(D^{k-1}u(x), \ldots, D^2 u(x), Du(x), u(x), x\big)$.

It is quasilinear if it is of the form $\displaystyle\sum_{|\alpha|=k} A_\alpha\big(D^{k-1}u(x), \ldots, u(x), x\big) D^\alpha u(x) = F\big(D^{k-1}u(x), \ldots, u(x), x\big)$.

It is fully nonlinear if it is of the form $(*)$, and F is nonlinear in at least one of the components of $D^k u$.

(Problem 20) In the case of ordinary differential equations we usually classify equations only into linear and nonlinear equations. Why do we have the additional categories semilinear and quasilinear in the case of partial differential equations?

1.2. Examples

(Problem 30)

(*a*) Give an example (with a name or specific application) of a partial differential equation of order 1.
(*b*) Give an example of a partial differential equation of order 2.
(*c*) Give an example of a partial differential equation of order at least 3.
(*d*) Give an example of a linear homogeneous partial differential equation.
(*e*) Give an example of a linear nonhomogeneous partial differential equation.
(*f*) Give an example of a partial differential equation that is semilinear but not linear.
(*g*) Give an example of a partial differential equation that is quasilinear but not semilinear.
(*h*) Give an example of a partial differential equation that is fully nonlinear.
(*i*) Give an example of a linear system of at least 2 differential equations.
(*j*) Give an example of a nonlinear system of at least 2 differential equations.

1.3. Strategies for studying PDE

[**Definition: Classical solution**] Suppose that $u : \Omega \mapsto \mathbb{R}$ is continuous and has continuous derivatives of order up to k at each point $x \in \Omega$. If $(*)$ is true for each $x \in \Omega$, we say that u is a classical solution to $(*)$.

(Problem 40*)
(a) Let $a : \mathbb{R} \mapsto \mathbb{R}$ be a C^1 function with $a(x) > 0$ for all $x \in \mathbb{R}$. Solve the ordinary differential equation $\dfrac{d}{dx}\left(a(x)\dfrac{d}{dx}u(x)\right) = 0$, where a is continuous and $a(x) > 0$ for all x.
(b) If $a(x)$ is not continuous, is your formula for $u(x)$ still a classical solution?

[**Definition: Boundary value problem**] We say that
$$\begin{cases} F(x, u(x), \dots, D^k u(x)) = 0 \text{ for all } x \text{ in } \Omega \subsetneq \mathbb{R}^n, \\ f_1(x, \dots, D^k u(x)) = f_2(x, \dots, D^k u(x)) = \ldots = f_m(x, \dots, D^k u(x)) = 0 \text{ for all } x \text{ in } E \subseteq \partial\Omega \end{cases}$$
is a boundary value problem.

[**Definition: Well posedness**] We say that a given problem is well posed if
(a) There is a (possibly weak) solution to the problem.
(b) The solution is unique.
(c) The solution depends continuously on the data given in the problem.

(Problem 31) Give an example of a boundary value problem (possibly one-dimensional) that is well posed.

(Problem 32) Give an example of a boundary value problem that has infinitely many solutions.

(Problem 33) Give an example of a boundary value problem that has no solutions.

(Problem 50*) Consider the (algebraic) equation $x^5 + x = a$, where a is a given real number.
(a) Prove that there is a unique real solution to this equation.
(b) Is x a continuous function of a?
(c) Can you write a formula for x in terms of a using only the standard algebraic functions (addition, subtraction, multiplication, division, exponentiation, and taking nth roots)? What is the name of the mathematician who first answered this question?

2.1. Transport equation

[**Definition: The transport equation**] The transport equation is
$$u_t + \vec{b} \cdot Du = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty)$$
where $\vec{b}$ is a constant vector.

(Problem 60) Suppose that $g : \mathbb{R}^n \mapsto \mathbb{R}$ is a C^1 function. Show that $u(x,t) = g(x - t\vec{b})$ is a solution to the transport equation.

(Problem 70) Show that $u(x,t) = g(x - t\vec{b})$ is the only solution to the initial value problem
$$\begin{cases} u_t + \vec{b} \cdot Du = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

(Problem 80) Let $g : \mathbb{R} \mapsto \mathbb{R}$ denote the triangle wave $g(x) = \min(x - \lfloor x \rfloor, \lceil x \rceil - x)$. Is $u(x,t) = g(x - tb)$ a classical solution the the wave equation on $\mathbb{R} \times (0, \infty)$? Is it a weak solution?

(Problem 90) Solve the nonhomogeneous transport equation
$$\begin{cases} u_t + \vec{b} \cdot Du = f & \text{in } \mathbb{R}^n \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$
What must be true of f and g in order for u to be a classical solution?

2.2. Laplace's equation

[Definition: The Laplace operator and related problems] The Laplacian in $\mathbb{R}^n$ is given by $\Delta = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2} + \ldots + \dfrac{\partial^2}{\partial x_n^2}$.

Laplace's equation is given by $-\Delta u = 0$.

Poisson's equation is given by $-\Delta u = f$.

A C^2 solution to Laplace's equation is called a harmonic function.

(Problem 100) Give two examples of places where the Laplace operator is used in physics, chemistry, biology, or other areas of mathematics.

(Problem 110) Let Ω be a bounded open set such that $\partial\Omega$ is C^1. Suppose that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Find a formula for $\displaystyle\int_\Omega \Delta u$ involving only u and Du.

(Problem 120) Let u, Ω be as in Problem 110 and let $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

(a) Find a formula for $\displaystyle\int_\Omega Du \cdot Dw$ involving no derivatives of u (only derivatives of w).

(b) Find a formula for $\displaystyle\int_\Omega \Delta u\, w$ involving only u, Du, and w and its derivatives.

2.2.1. Fundamental solution

(Problem 130) Let $u(x) = v(|x|)$, where $v : (0,\infty) \mapsto \mathbb{R}$ is a C^2 function. Find all v such that u is harmonic in $\mathbb{R}^n$. Find all v such that u is harmonic in $\mathbb{R}^n \setminus \{0\}$.

(Problem 140) Find a function $\Phi : \mathbb{R}^n \setminus \{0\} \mapsto 0$ that satisfies the following conditions:

(a) Φ is radial, that is, $\Phi(x) = v(|x|)$ for some $v : (0,\infty) \mapsto \mathbb{R}$.

(b) $\Delta\Phi = 0$ in $\mathbb{R}^n \setminus \{0\}$.

(c) $\displaystyle\int_{\partial B(0,1)} \vec{n} \cdot D\Phi\, dS = -1$, where $\vec{n}$ is the unit outward normal.

(Problem 150) What is $\displaystyle\int_{\partial B(0,r)} \vec{n} \cdot D\Phi\, dS$ for $r \neq 1$?

(Problem 160) Let $K \subset \mathbb{R}^n$ be compact. Suppose that $G \in C^1(\Omega)$ for some open set $\Omega \supset K$. Show that $\dfrac{G(x + h\vec{e}_j) - G(x)}{h} \to G_{x_j}(x)$ as $h \to 0$ uniformly for all $x \in K$.

(Problem 161) Let $K \subset \mathbb{R}^n$ be compact and let $\Omega \subset \mathbb{R}^n$ be open. Let $F \in L^1(K)$ and let $G : \Omega \times K \mapsto \mathbb{R}$ be such that the function $g_j(x,y) = \partial_{x_j} G(x,y)$ satisfies $g_j \in C(\Omega \times K)$ for any $1 \leq j \leq n$. Use the Lebesgue dominated convergence theorem to show that, if $x \in \Omega$, then

$$\frac{\partial}{\partial x_i} \int_K F(y)\, G(x,y)\, dy = \int_K F(y)\, \frac{\partial}{\partial x_i} G(x,y)\, dy.$$

(Problem 170) Let $K \subset \mathbb{R}^n$ be compact with nonempty interior. Suppose that $G \in C^1(\mathbb{R}^n)$ and that $G(x) = 0$ for all $x \notin K$. Suppose that $F \in L^1(E)$ for any compact set $E \subset \mathbb{R}^n$. Use the Lebesgue dominated convergence theorem to show that

$$\frac{\partial}{\partial x_i} \int_K F(x-y)\, G(y)\, dy = \int_K F(x-y)\, \frac{\partial}{\partial y_i} G(y)\, dy.$$

(Problem 180) Let $f \in C^2_c(\mathbb{R}^n)$ (that is, $f \in C^2(\mathbb{R}^n)$ and f is compactly supported). Let

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy.$$

Show that $u \in C^2(\mathbb{R}^n)$.

(Problem 190) Let $u(x)$ be as in Problem 180. What is $\Delta u(x)$? Simplify your answer as much as possible. *Hint:* Break the region of integration into $B(x,\varepsilon)$ and $\mathbb{R}^n \setminus B(x,\varepsilon)$.

2.2.2. Mean-value formulas

(Problem 200) Suppose that u is harmonic in an open set $\Omega \subset \mathbb{R}^n$. Let $x \in \Omega$ and, if r is small enough that $\overline{B(x,r)} \subset \Omega$, let $\varphi(r) = \fint_{\partial B(x,r)} u(y)\,dS(y)$. Find $\varphi'(r)$. Simplify your answer as much as possible.

(Problem 210) Let u, r and Ω be as in Problem 200. Find $\fint_{\partial B(x,r)} u(y)\,dS(y)$ and $\fint_{B(x,r)} u(y)\,dy$.

(Problem 220) Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $\Delta u \geq 0$ in Ω. If $\overline{B(x,r)} \subset \Omega$, what can you say about $u(x)$ and $\fint_{\partial B(x,r)} u(y)\,dS(y)$?

(Problem 230) Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $u \in C^2(\Omega)$ and that $u(x) = \fint_{\partial B(x,r)} u(y)\,dS(y)$ whenever $\overline{B(x,r)} \subset \Omega$. What can you say about u?

2.2.3. Properties of harmonic functions

[**Definition: Maximum principle**] Let $\Omega \subset \mathbb{R}^n$ be a connected bounded open set. A class of functions $\mathcal{F}$ satisfies the maximum principle if, whenever $u \in \mathcal{F}$, we have that $\sup_{\overline{\Omega}} u = \sup_{\partial\Omega} u$.

[**Definition: Strong maximum principle**] A class of functions $\mathcal{F}$ satisfies the strong maximum principle if, whenever $u \in \mathcal{F}$, we have that either u is constant in $\overline{\Omega}$ or $u(P) < \sup_{\overline{\Omega}} u$ for all $P \in \Omega$.

(Problem 231) Suppose that a class $\mathcal{F}$ of functions satisfies the strong maximum principle. Show that $\mathcal{F}$ satisfies the maximum principle.

(Problem 240) Prove that the set of functions that are harmonic in Ω and continuous on $\overline{\Omega}$ satisfies the strong maximum principle by using the mean value property. *Hint*: Let u be such a function. Show that $\{x \in \Omega : u(x) = u(P)\}$ and $\{x \in \Omega : u(x) < u(P)\}$ are both open and use the definition of connectedness in terms of open sets.

(Problem 241) Prove that $\{u \in C^2(\Omega) \cap C(\overline{\Omega}) : -\Delta u \leq 0\}$ satisfies the strong maximum principle. *Hint*: Use Problem 220.

(Problem 250) Prove uniqueness of solutions to the Dirichlet problem. That is, suppose that Ω is a bounded open set and let $g \in C(\partial\Omega)$ and $f \in C(\Omega)$. Prove that there is at most one function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ that satisfies

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = g & \text{on } \partial\Omega. \end{cases}$$

(Problem 260) Give an example of a function $f \in C^\infty(\mathbb{R})$ (that is, has continuous derivatives of all orders on the real line), satisfies $f(x) = 0$ for all $x \leq 0$, and satisfies $f(x) > 0$ for all $x > 0$.

(Problem 270)

(a) Construct a radial function $\eta \in C^\infty(\mathbb{R}^n)$ that satisfies $\eta(x) = 0$ whenever $|x| \geq 1$, $\eta(x) > 0$ whenever $|x| < 1$, and $\int_{B(0,1)} \eta(x)\,dx = 1$.

(b) Let $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$. What is $\int_{\mathbb{R}^n} \eta_\varepsilon(x)\,dx$? Where is η_ε equal to zero? Where is η_ε positive?

(Problem 280) Let f_ε be a bounded function that is zero outside of $B(0,\varepsilon)$, and let g and h be locally integrable functions.

(a) Suppose that $g(x) = h(x)$ for all x in some open set Ω. Can you conclude that $f_\varepsilon * g(x) = f_\varepsilon * h(x)$ for any value of x?

(b) Suppose that $g(x)$ is defined not on all of $\mathbb{R}^n$ but only in some open set Ω. For what values of x can you assign a meaningful definition to $f_\varepsilon * g(x)$?

(Problem 290) Let f_ε be a bounded function that is zero outside of $B(0,\varepsilon)$, and let g be a function integrable in $B(x,\varepsilon+\delta)$ for some $\delta>0$.

(a) Suppose that f_ε is a C^k function. Find a formula for $\partial^\alpha(f_\varepsilon * g)(x)$ for any α with $|\alpha|\leq k$.

(b) Suppose that $f_\varepsilon(x)=\dfrac{1}{\varepsilon^n}f\left(\dfrac{x}{\varepsilon}\right)$ for some C^k function f that is zero outside of $B(0,1)$. Find a formula for $\partial^\alpha(f_\varepsilon * g)(x)$ for any $\varepsilon>0$ and for any α with $|\alpha|\leq k$.

(Problem 300) Suppose that u is harmonic in an open set Ω. Let η_ε be as in Problem 270. What can you say about $\eta_\varepsilon * u(x)$?

(Problem 310) Show that if u is harmonic in an open set Ω, then $u\in C^\infty(\Omega)$.

(Problem 320) Suppose that u is harmonic in a bounded open set Ω. Need u be continuous up to the boundary $\partial\Omega$? If yes, prove it; if not, provide a counterexample.

(Problem 330) Suppose that u is harmonic in an open set $\Omega\subset\mathbb{R}^n$. Show that $\partial_{x_j}u$ is also harmonic in Ω for any $1\leq j\leq n$. Conclude that $\partial^\alpha u$ is harmonic in Ω for any multiindex α.

(Problem 340) Suppose that u is harmonic in $B(x_0,r)$ and that $u\in L^1(B(x_0,r))$. Find a bound on $|u(x_0)|$ in terms of $\|u\|_{L^1(B(x_0,r))}$.

(Problem 350) Suppose that u is harmonic in an open set containing $\overline{B(x,\rho)}$. Find a formula for $\partial_{x_j}u(x)$ in terms of the values of u in $\overline{B(x,\rho)}$.

(Problem 360) Suppose that u satisfies the mean value property $u(x)=\displaystyle\fint_{\partial B(x,r)}u(y)\,dS(y)$ for all $0<r<\varepsilon$. (In particular, we assume that u is measurable and that $\displaystyle\int_{\partial B(x,r)}|u(y)|\,dS(y)<\infty$ for all $0<r<\varepsilon$.) What can you say about $\eta_\varepsilon * u(x)$, where η_ε is as in Problem 270?

(Problem 370) Suppose instead that $u\in L^1(B(x,\varepsilon))$ and that $u(x)=\displaystyle\fint_{B(x,r)}u(y)\,dy$ for all $0<r<\varepsilon$. What can you say about $\eta_\varepsilon * u(x)$? (*Hint*: Use the fact that η_ε is radial and nonincreasing.)

(Problem 380) Suppose that $u\in L^1_{loc}(\Omega)$ for some open set Ω and satisfies the mean value property $u(x)=\displaystyle\fint_{B(x,r)}u(y)\,dy$ whenever $\overline{B(x,r)}\subset\Omega$. Show that $u\in C^2(\Omega)$. Based on Problem 230, what else can you say about u?

(Problem 390) Let $u(x,y)=\begin{cases}\operatorname{Im}\exp(-1/(x+iy)^4), & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0),\end{cases}$ where $i=\sqrt{-1}$.

(a) Show that $\Delta u=0$ in $\mathbb{R}^2$ in some sense.

(b) Show that $u(0)=\displaystyle\fint_{\partial B(x,r)}u(y)\,dS(y)$ for all $r>0$.

(c) Show that u is not C^2 at $(0,0)$.

(Problem 400) Suppose that u is harmonic in $B(x_0,r)$ and that $u\in L^1(B(x_0,r))$.

(a) Find a bound on $|Du(x_0)|$ in terms of $\|u\|_{L^1(B(x_0,r))}$.

(b) Find a bound on $|D^\alpha u(x_0)|$ in terms of $|\alpha|$ and $\|u\|_{L^1(B(x_0,r))}$, where α is any multiindex.

(c) Let $0<\rho<r$. Find a bound on $\displaystyle\sup_{x\in B(x_0,\rho)}|D^\alpha u(x)|$ in terms of ρ, r, $|\alpha|$, and $\|u\|_{L^1(B(x_0,r))}$.

(Problem 410) Suppose that u is harmonic in $\mathbb{R}^n$ and that u is bounded. Show that $Du\equiv 0$. Conclude that u is constant.

(Problem 420) Suppose that u is harmonic in $\mathbb{R}^n$ and that $|u(x)|\leq C_1+C_2|x|^k$ for some integer k. Show that u is a polynomial of degree at most k.

(Problem 430) Suppose that $u\in C^2(\mathbb{R}^n)$, $n\geq 3$, is bounded and that $f=-\Delta u$ lies in $C^2_c(\mathbb{R}^n)$. Show that there is a constant c such that $u(x)=c+\displaystyle\int_{\mathbb{R}^n}\Phi(x-y)\,f(y)\,dy$, where Φ is as in Problem 140.

(Problem 431) Let $v(x) = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy$, where f is bounded and compactly supported. What is the behavior of v as $x \to \infty$?

(Problem 432) Is the result of Problem 430 still true if $n = 2$?

(Problem 433) Let $k \geq 1$ be an integer and let $x \in \mathbb{R}^n$. Show that

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha = (x_1 + x_2 + \ldots + x_n)^k,$$

where $\alpha! = \alpha_1!\alpha_2!\alpha_3!\ldots\alpha_n!$ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$.

(Problem 434) Let $k \geq 1$ be an integer. Let x, $x_0 \in \mathbb{R}^n$, and let $u \in C^k(\Omega)$ for some open set $\Omega \ni x_0$. Show that

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha\, D^\alpha u(x_0) = \sum_{j_1=1}^n \sum_{j_2=1}^n \cdots \sum_{j_k=1}^n x_{j_1} x_{j_2} \ldots x_{j_k} D_{j_1} D_{j_2} \ldots D_{j_k} u(x_0).$$

(Problem 440) Let u in $C^{k+1}(B(x_0, r))$. Write a formula for the multivariable Taylor series P_k for u of order k expanded around the point x_0.

(Problem 450) Let u be as in Problem 440 and let $x \in B(x_0, r)$. Find an estimate for $u(x) - P_k(x)$.

(Problem 460) Let u be harmonic and integrable in $B(x_0, r)$. Let P_k be as in Problem 440. Find a bound on $\sup_{x \in B(x_0,\rho)} |u(x) - P_k(x)|$ for any $0 < \rho < r$.

(Problem 470) Let u and P_k be as in Problem 460. Show that if ρ is small enough, then $\lim_{k\to\infty} \sup_{x \in B(x_0,\rho)} |u(x) - P_k(x)| = 0$.

(Problem 480) Suppose that u is harmonic and nonnegative in $B(x, r)$. Let $y \in B(x, r/2)$. Show that $u(y) \leq 2^n u(x)$.

(Problem 490) Let Ω be an open set and let $K \subset \Omega$ be compact and connected. Suppose that u is harmonic and nonnegative in Ω. Show that there exists a constant C such that

$$\frac{1}{C} u(x) \leq u(y) \leq C u(x)$$

for all x, $y \in K$. What does C depend on?

2.2.4. Green's function

(Problem 500) Suppose that Ω is a bounded open set with C^1 boundary. Let $u \in C^2(\overline{\Omega})$ and let $x \in \mathbb{R}^n$, $x \notin \partial\Omega$. Use the divergence theorem to rewrite $\int_\Omega \Phi(x-y)\, \Delta u(y)\, dy$, where Φ is as in Problem 140.

(Problem 510) Let Ω, Φ, x, and u be as in Problem 500. Let $\varphi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy $\Delta\varphi = 0$ in Ω. Rewrite $\int_\Omega (\Phi(x-y) - \varphi(y))\, \Delta u(y)\, dy$.

(Problem 520) Let Ω be as in Problem 500. Suppose that φ_x solves the boundary value problem

$$\begin{cases} \Delta\varphi_x = 0 & \text{in } \Omega, \\ \varphi_x(z) = \Phi(z-x) & \text{for all } z \in \partial\Omega. \end{cases} \tag{520}$$

What is $\int_\Omega (\Phi(x-y) - \varphi_x(y))\, \Delta u(y)\, dy$?

(Problem 530) Let Ω be as in Problem 500. Suppose that for all $x \in \Omega$ we know the function φ_x given in Problem 520. Suppose that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solves

$$\begin{cases} \Delta u = -f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega \end{cases}$$

where $f \in C^2(\overline{\Omega})$ and where $g \in C^1(\partial\Omega)$. Write a formula for $u(x)$ in terms of f, g, Φ and φ_x.

(Problem 540) Let $G(x,y) = \Phi(x-y) - \varphi_x(y)$. Show that $G(x,y) = G(y,x)$ by letting $v(z) = G(x,z)$, $w(z) = G(y,z)$, and computing $\displaystyle\int_{\Omega\setminus B(x,\varepsilon)\setminus B(y,\varepsilon)} Dv(z)\cdot Dw(z)\,dz$ for small $\varepsilon > 0$ and letting $\varepsilon \to 0$.

(Problem 541) Let Ω be as in Problem 500. Show that $G(x,y) > 0$ for all x, $y \in \Omega$.

(Problem 542) Let Ω be as in Problem 500. Let $f \in C^2_c(\Omega)$. Let
$$u(x) = \int_\Omega G(x,y)\,f(y)\,dy.$$
Show that $-\Delta u = f$ in Ω.

(Problem 543) Let Ω be as in Problem 500. Let $x_0 \in \partial\Omega$, let $\{x_j\}_{j=1}^\infty \subset \Omega$ with $\lim_{j\to\infty} x_j = x_0$, and let $K \subset \Omega$ be compact. Show that $\lim_{j\to\infty} \sup_{y\in K} G(x_j, y) = 0$.

(Problem 544) Let Ω be as in Problem 500. Let x_0 and $\{x_j\}_{j=1}^\infty$ be as in Problem 543. Let f and u be as in Problem 542. Show that $\lim_{j\to\infty} u(x_j) = 0$.

(Problem 545) Let u be as in Problem 542. Show that $u \in C(\overline{\Omega})$ and that $u = 0$ on $\partial\Omega$.

(Problem 550) Let $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$ be the upper half space. Choose some $x \in \Omega$ and let $\bar{x} = (x_1, x_2, \dots, x_{n-1}, -x_n)$. Show that $\phi_x(y) = \Phi(y - \bar{x})$, solves the problem (520).

(Problem 551) Let $x \in \mathbb{R}^n_+$ and let $y \in \partial\mathbb{R}^n_+$. What is $\vec{n}\cdot D_y G(x,y)$, where $G(x,y) = \Phi(x-y) - \phi_x(y) = \Phi(x-y) - \Phi(y-\bar{x})$?

(Problem 552) Let $x \in \mathbb{R}^n_+$. What is $\displaystyle\int_{\partial\mathbb{R}^n_+} \vec{n}\cdot D_y G(x,y)\,dy$?

(Problem 560) Let $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ and let
$$u(x) = \frac{2x_n}{\omega_{n-1}} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\,dy$$
for all $x \in \mathbb{R}^n_+$. Show that $u \in L^\infty(\mathbb{R}^n_+)$.

(Problem 570) Let u be as in Problem 560. Show that u is harmonic in $\mathbb{R}^n_+$.

(Problem 580) Let u be as in Problem 560. Show that $\displaystyle\lim_{x\to x_0,\, x\in\mathbb{R}^n_+} u(x) = g(x_0)$ for all $x_0 \in \mathbb{R}^n_+$.

(Problem 590) Let u be as in Problem 560. Is u the only solution to the Dirichlet problem
$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+, \\ u = g & \text{on } \partial\mathbb{R}^n_+? \end{cases}$$

(Problem 600) Let $\widetilde{x} = \dfrac{x}{|x|^2}$. When does $x = \widetilde{x}$? If $x \in B(0,1)$, where is $\widetilde{x}$?

(Problem 610) Suppose that $y \in \partial B(0,1)$. Show that $\Phi(y-x) = \Phi(|x|(y-\widetilde{x}))$. Find a formula for the Green's function in the unit ball $B(0,1)$.

(Problem 620) Let $B(x_0, r)$ be a ball. What is the Green's function of this ball?

2.2.5. Energy methods

(Problem 630) Prove uniqueness of solutions to the Neumann problem. That is, suppose that Ω is a bounded open set with C^1 boundary and let $g \in C(\partial\Omega)$ and $f \in C(\Omega)$. Let u_1, $u_2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy
$$\begin{cases} -\Delta u_j = f & \text{in } \Omega, \\ \vec{n}\cdot Du_j = g & \text{on } \partial\Omega \end{cases}$$
where $f \in C^2(\overline{\Omega})$ and $g \in C(\partial\Omega)$. What can you say about u_1 and u_2?

(Problem 640) Use the methods of Problem 630 to give an alternate solution to Problem 250.

(Problem 650) Let $f \in C^2(\overline{\Omega})$. Define $I[w] = \displaystyle\int_\Omega \frac{1}{2}|Dw|^2 - wf$. Suppose that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Let $\varphi \in C^2_c(\Omega)$. Compute
$$\frac{d}{d\varepsilon} I[u+\varepsilon\varphi]\bigg|_{\varepsilon=0}.$$

(Problem 660) Let f and $I[w]$ be as in Problem 650. Let $g \in C^1(\partial\Omega)$ and let $\mathcal{A} = \{w \in C^2(\Omega) \cap C^1(\overline{\Omega}) : w = g \text{ on } \partial\Omega\}$. Suppose that $u \in \mathcal{A}$ and that $I[u] \leq I[w]$ for all $w \in \mathcal{A}$. What can you say about u?

(Problem 670) Let f and $I[w]$ be as in Problem 650, and let g and $\mathcal{A}$ be as in Problem 660. Suppose that u solves the boundary value problem
$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = g & \text{on } \partial\Omega. \end{cases}$$
Let $w \in \mathcal{A}$. Show that $I[u] \leq I[w]$. *Hint*: Write $I[w] - I[u]$ as an integral and then replace f by $-\Delta u$ throughout.

2.3. Heat equation

[Definition: The heat equation] The heat equation is given by
$$\frac{\partial}{\partial t} u(x,t) = \Delta u(x,t)$$
where the Laplacian Δ is taken in the x-variables only.

(Problem 680) Why is it called the heat equation?

[Definition: The nonhomogeneous heat equation] The nonhomogeneous heat equation is given by
$$\frac{\partial}{\partial t} u(x,t) = \Delta u(x,t) + f(x,t)$$
where f is a given function.

2.3.1. Fundamental solution

(Problem 690) Let $\varphi \in L^1(\mathbb{R}^n)$ and let $u(x,t) = \dfrac{1}{t^\alpha}\varphi\left(\dfrac{x}{t^\beta}\right)$ for some real numbers α and β. Find a condition on α and β so that $\displaystyle\int_{\mathbb{R}^n} u(x,t)\,dx$ is a constant.

(Problem 700) Find a function $\Phi : \mathbb{R}^n \times (0,\infty) \mapsto \mathbb{R}$ that satisfies the following conditions.

- $u(x,t) = \Phi(x,t)$ is a solution to the heat equation in $\mathbb{R}^n \times (0,\infty)$.
- $\Phi(x,t) = \dfrac{1}{t^\alpha}\varphi\left(\dfrac{x}{t^\beta}\right)$ for some $\varphi : \mathbb{R}^n \mapsto \mathbb{R}$ and some constants α and β that satisfy the conditions of Problem 690.
- φ is radial.

(Problem 701) Find a function Φ that satisfies the conditions of Problem 700 and in addition satisfies $\displaystyle\int \Phi(x,t)\,dx = 1$ for all $t > 0$.

(Problem 710) Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Let
$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\,g(y)\,dy.$$
Show that u is smooth in $\mathbb{R}^n \times (0,\infty)$.

(Problem 720) Let u be as in Problem 710. Show that u is a solution to the heat equation in $\mathbb{R}^n \times (0,\infty)$.

(Problem 730) Let $x_0 \in \mathbb{R}^n$. Let u be as in Problem 710. Show that $\displaystyle\lim_{(x,t)\to(x_0,0),\, t>0} u(x,t) = g(x_0)$.

(Problem 740) Let $g_1,\ g_2 \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and let $u_j(x,t) = \displaystyle\int_{\mathbb{R}^n} \Phi(x-y,t)\, g_j(y)\, dy$. Suppose that $g_1(x) \leq g_2(x)$ for all $x \in \mathbb{R}^n$ and that $g_1(x_0) < g_2(x_0)$ for at least one $x_0 \in \mathbb{R}^n$. Show that $u_1(x,t) < u_2(x,t)$ for all $x \in \mathbb{R}^n$ and all $t > 0$.

(Problem 750) Let $f : \mathbb{R}^n \times (0,\infty) \mapsto \mathbb{R}$ be bounded and C^2. For each $S > 0$, let u_S solve

$$\begin{cases} \dfrac{\partial}{\partial t} u_S(x,t) = \Delta u(x,t) + f(x,t), & x \in \mathbb{R}^n,\ 0 < t < S, \\ \dfrac{\partial}{\partial t} u_S(x,t) = \Delta u(x,t), & x \in \mathbb{R}^n,\ t > S, \\ u_S(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases}$$

Fix some $S > 0$ and let $v_h(x,t) = \dfrac{u_{S+h}(x,t) - u_S(x,t)}{h}$. Write an initial value problem that $v_h(x,t)$ satisfies.

(Problem 751) Make a reasonable guess as to the values of $\dfrac{d}{dS} u_S(x,t) = \displaystyle\lim_{h\to 0} \frac{u_{S+h}(x,t) - u_S(x,t)}{h}$ for (a) $0 < t < S$, and (b) $t > S$.

(Problem 760) Based on Problems 710 and 751, make a guess as to a formula for the solution to the problem

$$\begin{cases} \dfrac{\partial}{\partial t} u(x,t) = \Delta u(x,t) + f(x,t), & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases}$$

(Problem 770) Let $g(x) \equiv 0$. Let $f(x,t) \in C^2_1(\mathbb{R}^n \times [0,\infty))$ have compact support (that is, $D_{x_i} D_{x_j} D_t f$ exists and is continuous whenever $1 \leq i \leq n$, $1 \leq j \leq n$). Let u be given by the formula you found in Problem 760. Show that $u \in C^2_1(\mathbb{R}^n \times [0,\infty))$.

(Problem 790) Let u and f be as in Problem 770. Show that $\displaystyle\lim_{t\to 0,\, t>0} u(x,t) = 0$, uniformly for all $x \in \mathbb{R}^n$.

(Problem 780) Let u and f be as in Problem 770. Show that $u_t(x,t) - \Delta u(x,t) = f(x,t)$ for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

2.3.2. Mean-value formula

[**Definition: Parabolic cylinder**] Let $\Omega \subset \mathbb{R}^n$ be an open set. We define the parabolic cylinder $\Omega_T = \Omega \times (0,T]$.

[**Definition: Parabolic boundary**] The parabolic boundary of Ω_T is $\Gamma_T = \overline{\Omega_T} \setminus \Omega_T = \overline{\Omega} \times \{0\} \cup \partial\Omega \times [0,T]$.

(Problem 800) Let $I \subset \mathbb{R}$ be an open interval. Draw I_T and Γ_T in $\mathbb{R}^2$.

[**Definition: Parabolic ball**] Let $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $r > 0$. Define

$$E(x,t;r) = \left\{ (y,s) \in \mathbb{R}^n \times \mathbb{R} \middle| s \leq t, \Phi(x-y,t-s) \geq \frac{1}{r^n}. \right\}$$

(Problem 810) What is the relationship between $E(x,t;r)$ and $E(0,0;1)$?

(Problem 820) Fix some x, t, r, and s. Describe the set $\{y \in \mathbb{R}^n : (y,s) \in E(x,t;r)\}$.

(Problem 830) Sketch $E(x,t;r)$.

(Problem 840) Show that $E(x,t;r) \subset \overline{B(x, r\sqrt{n/2\pi e})} \times [t - r^2/4\pi, t]$.

(Problem 841) Show that if $0 < r < R$, then $E(x,t;r) \setminus \{(x,t)\} \subset E(x,t;R)^O$, where $E(x,t;R)^O$ denotes the interior of $E(x,t;R)$.

(Problem 842) Show that if $r > 0$, then the ellipsoid $\left\{ (y,s) : \dfrac{r^2}{8n\pi} |y-x|^2 + (t - r^2/8\pi - s)^2 \leq \left(\dfrac{r^2}{8\pi}\right)^2 \right\}$ is contained in $E(x,t;r)$. Where is the point (x,t) in relation to this ellipsoid?

(Problem 850) Let $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $r > 0$. Show that $\displaystyle\frac{1}{4r^n}\int_{(y,s)\in E(x,t;r)} \frac{|x-y|^2}{(t-s)^2}\,dy\,ds$ is independent of x, t, and r.

(Problem 851*) Let $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $r > 0$. Show that $\displaystyle\frac{1}{4r^n}\int_{(y,s)\in E(x,t;r)} \frac{|x-y|^2}{(t-s)^2}\,dy\,ds = 1$.

(Problem 860) Let $\psi(y,s) = -\dfrac{n}{2}\log(-4\pi s) + \dfrac{|y|^2}{4s} + n\log r$. Show that $\psi(y,s) = 0$ whenever $(y,s) \in \partial E(0,0;r)$ and that $\psi(y,s) > 0$ whenever $(y,s) \in E(0,0;r)$.

(Problem 870) Find $\partial_{y_i}\psi(y,s)$ and $\partial_s\psi(y,s)$.

(Problem 871) Suppose that u solves the heat equation in a neighborhood of $E(0,0;R)$. Find

$$\frac{d}{dr}\left(\frac{1}{4r^n}\int_{(y,s)\in E(0,0;r)} u(y,s)\frac{|y|^2}{s^2}\,dy\,ds\right)$$

for $0 < r < R$.

(Problem 872) Show that the expression you found in Problem 871 is equal to zero. *Hint*: Write $D_s u(y,s)\dfrac{|y|^2}{2s} = D_s u(y,s)(y \cdot D\psi(y,s))$.

(Problem 880) Show that if u solves the heat equation in a neighborhood of $E(x,t;r)$, then

$$u(x,t) = \frac{1}{4r^n}\int_{(y,s)\in E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}\,dy\,ds.$$

2.3.3.a. Properties of solutions: Strong maximum principle, uniqueness

(Problem 900) Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ solves the heat equation in Ω_T, and that for some $(x_0,t_0) \in \Omega_T$ we have that $u(x_0,t_0) \geq u(x,t)$ for all $(x,t) \in \Omega_T$. Show that u is constant in Ω_{t_0}.

(Problem 910) Need u be constant in $\Omega_T \setminus \Omega_{t_0}$?

(Problem 920) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose that $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ solves the heat equation in Ω_T. Show that $\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u$.

(Problem 930) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $f \in C(\overline{\Omega_T})$, $g \in C(\Gamma_T)$. Show that there is at most one function $u \in C_1^2(\Omega_T) \cap C(\overline{\Omega_T})$ that satisfies

$$\begin{cases} D_t u - \Delta u = f & \text{in } \Omega_T, \\ u = g & \text{on } \Gamma_T. \end{cases} \tag{930}$$

(Problem 940*) Let $u(x,t) : \mathbb{R} \times [0,\infty) \mapsto \mathbb{R}$ be given by $u(x,t) = -1 + \displaystyle\sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!}F^{(2j)}(t)$, where $F(t) = e^{-1/t}$. Show that the series converges absolutely for any fixed x and t.

(Problem 950*) Let u be as in Problem 930. Show that u is a solution to the heat equation in $\mathbb{R} \times (0,\infty)$. (You may assume that in this case, derivatives commute with the infinite sum.)

(Problem 960*) Let u be as in Problem 930. Show that $u(x,0) = 0$ for all $x \in \mathbb{R}$. (You may assume that in this case, limits commute with the infinite sum.)

(Problem 970) Let $S > 0$. Let $v(x,t) = \dfrac{1}{(S-t)^{n/2}}e^{|x|^2/4(S-t)}$. Show that $v(x,t)$ is a solution to the heat equation in $\mathbb{R}^n \times (-\infty, S)$.

(Problem 980) Let $T > 0$. Suppose that $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ satisfies

$$\begin{cases} D_t u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u = g & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Suppose further that there exist constants A and a such that $|u(x,t)| \leq Ae^{a|x|^2}$ for all $x \in \mathbb{R}^n$ and all $0 \leq t \leq T$. Let $\delta > 0$. Show that if S is small enough, then $u(x,t) - \delta v(x,t) \leq \max\big(0, \sup_{y\in\mathbb{R}^n} g(y)\big)$ for all $x \in \mathbb{R}^n$ and all $0 < t < S$.

(Problem 990) Let u be as in Problem 980. Show that $u(x,t) \leq \max\big(0, \sup_{y\in\mathbb{R}^n} g(y)\big)$ for all $x \in \mathbb{R}^n$ and all $0 < t < T$.

(Problem 1000) Let $T > 0$, let $f \in C(\mathbb{R}^n \times [0,T])$, $g \in C(\mathbb{R}^n)$. Show that there is at most one function $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ that satisfies

$$\begin{cases} D_t u - \Delta u = f & \text{in } \mathbb{R}^n \times (0,T), \\ u = g & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

and that satisfies $|u(x,t)| \leq Ae^{a|x|^2}$ for some constants A and a.

2.3.3.bc. Properties of solutions: regularity and local estimates for solutions of the heat equation

(Problem 1010) Extend Φ to the lower half space by

$$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Show that Φ is smooth in $\mathbb{R}^{n+1} \setminus \{0\}$.

(Problem 1020) Let η_ε be a standard smooth mollifier in $\mathbb{R}^n$ (not $\mathbb{R}^{n+1}$), as in Problem 270. Let $u_\varepsilon(x,t) = \int_{|y|<\varepsilon} u(x-y,t)\, \eta_\varepsilon(y)\, dy$. Show that if u is a solution to the heat equation in Ω_T, then u_ε is a solution to the heat equation in $(\Omega^\varepsilon)_T$, where $\Omega^\varepsilon = \{y \in \Omega : \overline{B(y,\varepsilon)} \subset \Omega\}$.

(Problem 1030) Let u_ε be as in Problem 1020. Show that if $u \in C^2_1(\Omega_T)$, then $D_x u_\varepsilon \in C^2_1(\Omega_T)$.

(Problem 1040) Suppose that $D_t u - \Delta u = 0$ in some region Ω_T. Let ζ be smooth and compactly supported in Ω_T. Find a formula for $D_t(u\zeta) - \Delta(u\zeta)$.

(Problem 1041) Let $x_0 \in \mathbb{R}^n$ and let $t_0 \in \mathbb{R}$. Let ζ be smooth, supported in $B(0,2) \times (-2,2) \subset \mathbb{R}^n \times \mathbb{R}$ and equal to 1 in $B(0,1) \times (-1,1)$. Let $\zeta_r(x,t) = \zeta((x-x_0)/r, (t-t_0)/r^2)$. Where is ζ_r equal to zero? Where are all the derivatives of ζ_r equal to zero? Give an upper bound on $|D^k_x D^\ell_t \zeta_r(x,t)|$ in terms of ζ, r, k, and ℓ.

(Problem 1050) Suppose that $u \in C^2_1(\Omega_T)$ and that $D_t u - \Delta u = 0$ in Ω_T. Let u_ε be as in Problem 1020. Let $(x_0,t_0) \in \Omega_T$. Let $r > 0$ satisfy $r^2 < t_0/3$ and $B(x_0,3r) \subset \Omega$. Find a formula for $\zeta_r(x,t)u_\varepsilon(x,t)$ in terms of $u_\varepsilon(y,s)$ for (y,s) in Ω_T such that $D\zeta_r(y,s) \neq 0$.

(Problem 1060) Let u be as in Problem 1050. Find a formula for $u(x,t)$, for any $(x,t) \in B((x_0,t_0),r) \cap \Omega_T$, in terms of $u(y,s)$ for (y,s) in $\Omega_T \cap B((x_0,t_0),2r) \setminus B((x_0,t_0),r)$.

(Problem 1061) Let $C(x,t;r) = \{(y,s) : y \in \mathbb{R}^n,\ s \in \mathbb{R},\ |x-y| \leq r,\ t - r^2 \leq s \leq t\}$. Show that if $(y,s) \in C(x,t;2r) \setminus C(x,t;r)$, then $|D^k_x D^\ell_t \Phi(x-y,t-s)| \leq \dfrac{C_{k\ell}}{r^{n+k+2\ell}}$.

(Problem 1070) Suppose that $u \in C^2_1(C(x,t;r))$ satisfies the heat equation in $C(x,t;r)$. Show that if $k \geq 0$ and $\ell \geq 0$ are nonnegative integers, then there is a constant $C_{k\ell}$ depending only on k and ℓ such that

$$|D^k_x D^\ell_t u(x,t)| \leq \frac{C_{k\ell}}{r^{k+2\ell+n+2}} \int_{C(x,t;r)} |u|.$$

2.3.4.a. Energy methods: forward uniqueness

(Problem 1080) Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $T > 0$. Suppose that $w \in C^2_1(\Omega_T)$ is a (real-valued) solution to

$$\begin{cases} D_t w - \Delta w = 0 & \text{in } \Omega_T, \\ w = 0 & \text{on } \Gamma_T. \end{cases}$$

Let $e(t) = \displaystyle\int_\Omega w(x,t)^2\,dx$. Show that $e'(t) \leq 0$ for all $0 < t < T$.

(Problem 1090) Use Problem 1080 to provide another proof of Problem 930.

2.3.4.b. Energy methods: backward uniqueness

(Problem 1100) Suppose that $w \in C^2_1(\Omega_T)$ is a (real-valued) solution to

$$\begin{cases} D_t w - \Delta w = 0 & \text{in } \Omega_T, \\ w = 0 & \text{on } \partial\Omega \times [0,T], \\ w = 0 & \text{on } \Omega \times \{T\}. \end{cases}$$

The difference between this problem and that of Problem 1080 is that in this case, w is required to be 0 on $\Omega \times \{T\}$ rather than on $\Omega \times \{0\}$. Let $e(t)$ be as in Problem 1080. Show that $(e'(t))^2 \leq e(t)\, e''(t)$.

(Problem 1110) Let w and $e(t)$ be as in Problem 1100. Let $[a,b]$ be any interval in $(0,T)$ over which $e(t)$ is positive. Show that $f(t) = \log e(t)$ is convex in this interval. *Hint*: Recall the characterization from calculus of convexity of C^2 functions.

(Problem 1120) Let w and $e(t)$ be as in Problem 1100 and let (a,b) be as in Problem 1110. Show that if $a < t < b$, $e(t) > 0$ on $[a,b]$, and $\tau = \dfrac{b-t}{b-a}$, then $e(t) \leq e(a)^\tau e(b)^{1-\tau}$. Conclude that $e(t) = 0$ for all $0 < t \leq T$.

(Problem 1130) Show that if $\Omega \subset \mathbb{R}^n$ is a bounded open set and $T > 0$, and if f, g and h are continuous, then there is at most one solution $u \in C^2_1(\Omega \times (0,T)) \cap C(\overline{\Omega} \times (0,T])$ to the final-boundary value problem

$$\begin{cases} D_t u - \Delta u = f & \text{in } \Omega \times (0,T), \\ u = g & \text{on } \partial\Omega \times [0,T], \\ u = h & \text{on } \Omega \times \{T\}. \end{cases}$$

(Problem 1140) Give an example of a bounded final condition $h : \mathbb{R}^n \mapsto \mathbb{R}$ such that there is no bounded solution to the problem

$$\begin{cases} D_t u - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,T), \\ u = h & \text{on } \mathbb{R}^n \times \{T\}. \end{cases}$$

(Problem 1150) Let $h_n(x) = \sin(nx)$, where $n \geq 1$ is a positive integer. Find functions $u_n^+(x,t)$ and $u_n^-(x,t)$ that satisfy

$$\begin{cases} D_t u_n^+ - \Delta u_n^+ = 0 & \text{in } (0,\pi) \times (0,1), \\ u_n^+ = 0 & \text{on } \{0,\pi\} \times [0,T], \\ u_n^+ = h_n & \text{on } [0,\pi] \times \{0\}. \end{cases} \qquad \begin{cases} D_t u_n^- - \Delta u_n^- = 0 & \text{in } (0,\pi) \times (0,1), \\ u_n^- = 0 & \text{on } \{0,\pi\} \times [0,T], \\ u_n^- = h_n & \text{on } [0,\pi] \times \{1\}. \end{cases}$$

Hint: Let $u_n^\pm(x,t) = h_n(x) f(t)$ for some appropriate function $f(t)$. Are solutions to the forward problem better behaved in some sense than solutions to the reverse problem?

2.4. Wave equation

[Definition: The wave equation] The wave equation is given by

$$\frac{\partial^2}{\partial t^2} u(x,t) = \Delta u(x,t)$$

where the Laplacian Δ is taken in the x-variables only.

(Problem 1160) Suppose that $u(x,t)$ is a solution to the wave equation. Is $v(x,t) = u(x,-t)$ also a solution to the wave equation?

2.4.1. Solution by spherical means

(Problem 1170) Let $\vec{b} \in \mathbb{R}^n$ with $|\vec{b}| = 1$. Let $f \in C^2(\mathbb{R})$. Show that $u(x,t) = f(t - \vec{b} \cdot x)$ is a solution to the wave equation. How does this differ from solutions to the transport equation?

(Problem 1180) Give an example of a solution to the wave equation that is not C^3 (and thus not C^∞).

(Problem 1190) (The wave equation in one dimension.) Let $g \in C^2(\mathbb{R})$ and let $h \in C^1(\mathbb{R})$. Find a solution to the initial value problem

$$\begin{cases} D_{tt}^2 u - D_{xx}^2 u = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x), \quad u_t(x,t)\big|_{t=0} = h(x), & x \in \mathbb{R}. \end{cases} \tag{1190}$$

(Problem 1200) Let $v(x,t) = D_t u(x,t) - D_x u(x,t)$, where u is a C^2 solution to the one-dimensional wave equation. Show that v is a solution to the transport equation.

(Problem 1210) Use uniqueness of solutions to the transport equation to show that there is exactly one solution to the initial value problem (1190).

(Problem 1220) Find a solution to the initial value problem

$$\begin{cases} D_{tt}^2 u - D_{xx}^2 u = 0 & \text{in } (0,\infty) \times (0,\infty), \\ u(x,0) = g(x), \quad u_t(x,t)\big|_{t=0} = h(x), & x > 0, \\ u(0,t) = 0, & t > 0. \end{cases}$$

Is there only one solution?

(Problem 1230) Fix some $x \in \mathbb{R}^n$. Let $U(r,t) = r \fint_{\partial B(x,r)} u(y,t)\, dS(y)$, where u is a C^2 solution to the wave equation in $\mathbb{R}^n \times (0,\infty)$. Find an initial value problem that U satisfies in $(0,\infty) \times [0,\infty)$.

(Problem 1240) (The wave equation in three dimensions.) Find a solution to the initial value problem

$$\begin{cases} D_{tt}^2 u - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = g(x), \quad u_t(x,t)\big|_{t=0} = h(x), & x \in \mathbb{R}^3. \end{cases} \tag{1240}$$

Is your answer the only solution?

(Problem 1250) (The wave equation in two dimensions.) Find a solution to the initial value problem

$$\begin{cases} D_{tt}^2 u - \Delta u = 0 & \text{in } \mathbb{R}^2 \times (0,\infty), \\ u(x,0) = g(x), \quad u_t(x,0) = h(x), & x \in \mathbb{R}^2. \end{cases} \tag{1250}$$

Simplify your answer as much as possible. Is your answer the only solution?

(Problem 1251) Suppose that $g(x) = 0$ and $h(x) = 0$ for all x outside of $(-R,R) = B(0,R) \subset \mathbb{R}^1$. Let u be the solution to the initial value problem 1190. For what values of (x,t) can you be sure that $u(x,t) = 0$? Are there any regions where u is a (possibly nonzero) constant?

(Problem 1260) Suppose that $g(x) = 0$ and $h(x) = 0$ for all x outside of $B(0,R) \subset \mathbb{R}^3$. Let u be the solution to the initial value problem 1240. For what values of (x,t) can you be sure that $u(x,t) = 0$?

(Problem 1270) Suppose that $g(x) = 0$ and $h(x) = 0$ for all x outside of $B(0,R) \subset \mathbb{R}^2$. Let u be the solution to the initial value problem 1250. For what values of (x,t) can you be sure that $u(x,t) = 0$?

2.4.2. Nonhomogeneous problem

(Problem 1280) Let $u(x,t;s)$ solve the initial value problem

$$\begin{cases} D_{tt}^2 u(x,t;s) - \Delta u(x,t;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty), \\ \qquad u(x,s;s) = 0, \quad u_t(x,t;s)\big|_{t=s} = f(x,s), & x \in \mathbb{R}^n. \end{cases}$$

Let $u(x,t) = \displaystyle\int_0^t u(x,t;s)\,ds$. Show that if f is sufficiently smooth, then u is a solution to the inhomogeneous problem

$$\begin{cases} D_{tt}^2 u - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty), \\ \quad u(x,0) = 0, \quad u_t(x,0) = 0, & x \in \mathbb{R}^n. \end{cases} \tag{1280}$$

Is this u the only solution?

2.4.3. Energy methods

(Problem 1290) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $u \in C^2(\Omega_T) \cap C^1(\overline{\Omega_T})$ satisfy

$$\begin{cases} D_{tt}^2 u - \Delta u = 0 & \text{in } \Omega_T, \\ \qquad u = 0 & \text{on } \partial\Omega \times [0,T], \\ \quad u(x,0) = g, \quad w_t(x,t)\big|_{t=0} = h, & x \in \Omega. \end{cases}$$

Let $E(t) = \dfrac{1}{2}\displaystyle\int_\Omega (D_t w(x,t))^2 + |D_x w(x,t)|^2\,dx$. Find $\dfrac{d}{dt}E(t)$. Do you have uniqueness of solutions to any initial value problem?

(Problem 1300) Let $u \in C^2(\mathbb{R}^n \times (0,\infty)) \cap C^1(\mathbb{R}^n \times [0,\infty))$ be a solution to the wave equation. Let $x_0 \in \mathbb{R}^n$ and $t_0 > 0$, and let $e(t) = \dfrac{1}{2}\displaystyle\int_{B(x_0,t_0-t)} (D_t u(x,t))^2 + |D_x u(x,t)|^2\,dx$, $0 \leq t \leq t_0$. Find a formula for $\dfrac{d}{dt}e(t)$ in terms of u and its derivatives on $\partial B(x_0, t_0 - t)$.

(Problem 1301) Let $e(t)$ be as in Problem 1300. Show that $\dfrac{d}{dt}e(t) < 0$ for all $0 < t < t_0$.

(Problem 1310) Suppose that $u \in C^2(\mathbb{R}^n \times (0,\infty)) \cap C^1(\mathbb{R}^n \times [0,\infty))$ is a solution to the wave equation. Suppose that $u_t(x,0) = 0$ and that $u(x,0) = C$ in $B(x_0,t_0)$. Show that $u(x_0,t_0) = C$, regardless of the values of $u(x,0)$ and $u_t(x,0)$ for $x \notin B(x_0,t_0)$.

Comparison of differential equations

Consider the transport, heat, wave, and Laplace's equations.

(Problem 1320) Which of these equations meaningfully have a "time" coordinate?

(Problem 1330) Which of these equations has a preferred direction for time to flow in (a meaningful notion of "future" and "past")?

(Problem 1340) Which of these equations has a finite speed of propagation?

(Problem 1350) In which of these equations do we have smoothing, that is, solutions which are smoother than initial and boundary data?

3.2. Characteristics

3.2.2. Examples

(Problem 1360) Suppose that $u \in C^1(\Omega) \cap C(\overline{\Omega})$ and that $\vec{b} \cdot Du(x) = 0$ for all $x \in \Omega$ for some constant vector $\vec{b} \neq 0$. Let $x_0 \in \mathbb{R}^n$. Show that u is constant in all connected subsets of $\{x_0 + s\vec{b} : s \in \mathbb{R}\} \cap \Omega$.

(Problem 1370) Suppose that $u \in C^1(\Omega) \cap C(\overline{\Omega})$ and that $\vec{b} \cdot Du(x) = f(x) + g(x)u$ for all $x \in \Omega$. Let $x_0 \in \overline{\Omega}$ and let ℓ be a connected component of $\{x_0 + s\vec{b} : s \in \mathbb{R}\} \cap \Omega$ with $x_0 \in \overline{\ell}$. If $x \in \ell$, find a formula for $u(x)$ in terms of $u(x_0)$, g, and f.

(Problem 1380) Suppose that $u \in C^1(\Omega) \cap C(\overline{\Omega})$ and that $\vec{b}(x) \cdot Du(x) = 0$ for all $x \in \Omega$. Let $x_0 \in \overline{\Omega}$. Find a (possibly differential or parametric) equation that describes a curve through x_0 on which u is constant.

(Problem 1390) Suppose that $u \in C^1(\Omega) \cap C(\overline{\Omega})$ and that $\vec{b}(x) \cdot Du(x) = f(x, u(x))$ for all $x \in \Omega$. Let $x_0 \in \overline{\Omega}$. Let the curve that you found in Problem 1380 be given by $\{\xi(s) : s \in I\}$ for some interval I and with $x_0 = \xi(0)$. Write an *ordinary* differential equation and initial condition that is satisfied by $z(s) = u(\xi(s))$.

(Problem 1400) Suppose that $u \in C^1(\Omega) \cap C(\overline{\Omega})$ and that $\vec{b}(x, u) \cdot Du(x) = 0$ in Ω. Let $x_0 \in \overline{\Omega}$. Find a (possibly differential or parametric) equation that describes a curve through x_0 on which u is constant.

(Problem 1410) Suppose that $u \in C^1(\Omega) \cap C(\overline{\Omega})$ and that $\vec{b}(x, u) \cdot Du = f(x, u)$ in Ω. Let $x_0 \in \overline{\Omega}$. Write a system of ordinary differential equations and initial conditions such that if z and ξ are the solutions to that system, then $\xi(0) = x_0$ and $u(\xi(s)) = z(s)$.

3.2.1. Derivation of characteristic ODE

(Problem 1420) Let $F : \mathbb{R}^n \times \mathbb{R} \times \Omega \mapsto \mathbb{R}$ be a C^1 function and let $u \in C^2(\Omega)$. Find $\dfrac{d}{dx_j} F(Du(x), u(x), x)$.

(Problem 1430) Let $\xi : (0, 1) \mapsto \Omega \subset \mathbb{R}^n$ be a smooth curve. Let $u \in C^2(\Omega)$ and let $p_j(s) = D_{x_j} u(x)\big|_{x=\xi(s)}$. Find a formula for $p_j'(s)$ in terms of u, ξ, and their derivatives.

(Problem 1440) Suppose that $u \in C^2(\Omega)$ is a solution to the fully nonlinear equation $F(Du, u, x) = 0$ in Ω. Find a formula for $\xi'(s)$ such that the sums involving second derivatives of u in Problems 1420 and 1430 are equal.

(Problem 1450) Let u and ξ be as in Problem 1440. Define $z(s) = u(\xi(s))$ and $p(s) = Du(\xi(s))$. Find a system of ordinary differential equations that p, z, and ξ must satisfy.

3.2.3. Boundary conditions

(Problem 1460) Let $\Omega \subset \mathbb{R}^n$ have a C^1 boundary and let $x_0 \in \partial\Omega$. Suppose that there are two functions g and u that are C^1 in a neighborhood of x_0 and such that $u = g$ on $\partial\Omega$. What can you say about $Du(x_0)$ and $Dg(x_0)$?

Picard-Lindelöf theorem. *Let $f : U \times (-a, a) \mapsto \mathbb{R}^m$ for some open set $U \subset \mathbb{R}^m$ and some $a > 0$, and suppose that f is C^1. Then for each $y_0 \in U$ there is an $\varepsilon > 0$ such that there is a unique classical solution to the problem $y'(s) = f(y, s)$, $y'(0) = y_0$, on the interval $(-\varepsilon, \varepsilon)$. Furthermore, the length ε and the solution $y(s)$ are continuous in the initial value y_0.*

(Problem 1470) Let $\Omega \subset \mathbb{R}^n$ be an open set and let Ω' be an open set with $\overline{\Omega} \subset \Omega' \subset \mathbb{R}^n$. Let $F \in C^1(\mathbb{R}^n \times \mathbb{R} \times \Omega')$. Show that for any $x_0 \in \overline{\Omega}$, any $z_0 \in \mathbb{R}$, and any $\vec{p}_0 \in \mathbb{R}^n$, there is a unique solution to the differential equations of Problem 1450 with initial conditions $\xi(0) = x_0$, $z(0) = z_0$, and $\vec{p}(0) = \vec{p}_0$.

(Problem 1480) Let Ω and F be as in Problem 1470. Suppose in addition that Ω has a C^1 boundary. Let U be an open set, let $\Gamma = U \cap \partial\Omega$, and let $g \in C^1(U)$. Consider the boundary value problem

$$\begin{cases} F(Du, u, x) = 0 \quad \text{in } \Omega, \quad u = g, \quad Du = \vec{p} \quad \text{on } \Gamma \end{cases}$$

where $\vec{p}$ is a continuous function. Give a *necessary* (not necessarily sufficient) condition on $\vec{p}$ for the problem to have a solution $u \in C^1(\overline{\Omega})$.

(Problem 1490) Let Ω, F, Γ, and g satisfy the conditions specified in Problem 1480. Give an example of (a) a choice of Ω, F, Γ, and g such that no $\vec{p}$ satisfies the conditions you found; (b) a choice such that two or more $\vec{p}$ satisfy the conditions you found.

(Problem 1500) Let Ω, F, Γ, g, and $\vec{p}$ be as in Problem 1480. What must be true in order to conclude that there is at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ to the boundary value problem?

(Problem 1501) Consider the boundary value problem

$$\begin{cases} u_t - u\, u_x = 0 \quad \text{in } \mathbb{R} \times (0, \infty), \quad u(x, 0) = \arctan x \quad \text{for all } x \in \mathbb{R}. \end{cases}$$

Find a formula for the characteristic through $(x, 0)$. Do characteristics ever cross?

(Problem 1510) Let $n \geq 2$ be an integer and let $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$. For what values of the (nonzero constant) vector $\vec{b}$ is there a unique solution to the following boundary value problem for all $g \in C^1(\partial\mathbb{R}^n_+)$?

$$\begin{cases} \vec{b} \cdot Du = 0 \quad \text{in } \mathbb{R}^n_+, \quad u = g \quad \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

(Problem 1520) Let Ω, F, U, g, and Γ be as in Problem 1480. Let $x_0 \in \Gamma$. Choose some $\vec{p}_0$ such that $F(\vec{p}_0, g(x_0), x_0) = 0$ and such that $\vec{p}_0 - Dg(x_0)$ is parallel to $\vec{n}(x_0)$. Our goal is to show that there is some neighborhood U of x_0 such that there exists a unique solution u to

$$\begin{cases} F(Du, u, x) = 0 & \text{in } \Omega \cap U, \\ \qquad\qquad u = g & \text{on } \Gamma \cap U, \quad Du(x_0) = \vec{p}_0. \end{cases} \tag{1520}$$

Define $f : U \times \mathbb{R} \mapsto \mathbb{R}$ by $f(x, s) = F(Dg(x) + s\,\vec{n}(x), g(x), x)$. Show that there is an $s_0 \in \mathbb{R}$ with $f(x_0, s_0) = 0$.

The Implicit Function Theorem. *Suppose that $f : U \times \mathbb{R} \mapsto \mathbb{R}$ is $C^k(U \times \mathbb{R})$ for some integer $k \geq 1$ and some open set U. Suppose that $f(x_0, s_0) = 0$ and that $D_s f(x_0, s_0) \neq 0$ for some $x_0 \in U$ and some $s_0 \in \mathbb{R}$. Then there is a connected neighborhood $V \subset U$ of x_0 and a unique function $g \in C^k(V)$ such that $g(x_0) = s_0$ and such that $f(x, g(x)) = 0$ for all $x \in V$.*

(Problem 1530) Let Ω, F, U, g, and Γ be as in Problem 1480, and let $\vec{p}_0$ be as in Problem 1520 (in particular, such a $\vec{p}_0$ exists). Suppose that $\vec{n}(x_0) \cdot D_p F(\vec{p}_0, g(x_0), x_0) \neq 0$. Use the Implicit Function Theorem to show that there is some neighborhood V of x_0 such that if $x \in \Gamma \cap V$, then there is some $s \in \mathbb{R}$ with $f(x, s) = 0$. Let $\vec{p}(x) = Dg(x) + s\,\vec{n}(x)$. What can you say about $\vec{p}(x)$?

(Problem 1540) Let Ω, F, g, and Γ be as in Problem 1480, let x_0 and $\vec{p}_0$ be as in Problem 1520, and let V and $\vec{p}$ be as in Problem 1530. For each $x \in \Gamma \cap V$, let $(\vec{p}_x, z_x, \xi_x)$ be the solution to the differential equations found in Problem 1450 with initial conditions $(\vec{p}(x), g(x), x)$. Show that $F(\vec{p}_x(s), z_x(s), \xi_x(s)) = 0$.

The Inverse Function Theorem. *Let $U \subset \mathbb{R}^n$ be an open set and suppose that $\vec{f} : U \mapsto \mathbb{R}^n$ is C^k for some $k \geq 1$. Let Df be the $n \times n$ matrix with $(Df)_{j,k} = D_{x_k} f_j$. Let $J\vec{f} = |\det D\vec{f}|$. If $J\vec{f}(x_0) \neq 0$, then there is a neighborhood W of x_0 and a neighborhood Ψ of $\vec{f}(x_0)$ such that the mapping $\vec{f} : W \mapsto \Psi$ is one-to-one, onto, and with a C^k inverse.*

(Problem 1550) Let Ω, F, g, and Γ be as in Problem 1480, let x_0 and $\vec{p}_0$ be as in Problem 1520, and let V and $\vec{p}$ be as in Problem 1530. Let $(\vec{p}_x, z_x, \xi_x)$ be as in Problem 1540. Assume that $z_x(s)$, $\xi_x(s)$ are defined for $s \in (-\varepsilon, \varepsilon)$, where ε does not depend on x, and are C^1 in x as well as in s.

Show that there are neighborhoods W, Ψ of x_0 and a number $r > 0$ such that if $y \in \Psi$, then there is a unique $s \in (-r, r)$ and a unique $x \in \Gamma \cap W$ such that $y = \xi_s(x)$.

Hint: There is an open neighborhood $\widetilde{U} \subset \mathbb{R}^{n-1}$ of 0 and a neighborhood $\widetilde{V}$ of x_0 such that there is a C^1 bijection $\varphi : \widetilde{U} \mapsto \widetilde{V} \cap \Gamma$ with $x_0 = \varphi(0)$. Let $f : \widetilde{U} \times (-\varepsilon, \varepsilon) \mapsto \mathbb{R}^n$ be given by $f(z, s) = \xi_s(\varphi(z))$ and use the Inverse Function Theorem.

(Problem 1560) Let u be given by $u(\xi_x(s)) = z_x(s)$. What is $u\big|_\Gamma$?

(Problem 1570) Show that $u \in C^1(\Psi \cap \overline{\Omega})$.

(Problem 1580) Show that $Du(\xi_x(s)) \cdot \xi_x'(s) = \vec{p}_x(s) \cdot \xi_x'(s)$.

(Problem 1590*) Show that $Du(\xi_x(s)) = \vec{p}_x(s)$.

(Problem 1600) Conclude that u solves the problem (1520).

4.3.1. Fourier transform

[Definition: The Fourier transform] Let $f \in L^1(\mathbb{R}^n)$. We let

$$\mathcal{F}f(y) = \widehat{f}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} f(x)\,dx, \qquad \check{f}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y} f(x)\,dx.$$

(Problem 1650) Show that if $f \in L^1(\mathbb{R}^n)$ then $\widehat{f} \in L^\infty(\mathbb{R}^n)$.

(Problem 1620) Find two formulas for $\check{u}(y)$ in terms of $\widehat{\ }$, complex conjugates, or changes of input.

(Problem 1730) Let α be a multiindex with $|\alpha| = k \geq 1$. Suppose that $u \in C^k(\mathbb{R}^n)$ and that $D^\beta u \in L^1(\mathbb{R}^n)$ whenever $|\beta| \leq k$. What is $\widehat{D^\alpha u}(y)$?

(Problem 1740) Suppose that $u \in L^1(\mathbb{R}^n)$ and that $|x|^k u \in L^1(\mathbb{R}^n)$. Let α be a multiindex of length k. What is $\widehat{x^\alpha u(x)}(y)$?

(Problem 1660) Show that if $f \in L^1(\mathbb{R}^n)$ then $\widehat{f}$ is uniformly continuous.

(Problem 1601) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be given by $f(x) = e^{-tx^2}$. Recall that $\displaystyle\int_{\mathbb{R}} f(x)\,dx = \sqrt{\pi/t}$. What is $\widehat{f}(0)$?

(Problem 1602) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be given by $f(x) = e^{-tx^2}$. Find $f'(x)$. Then find $\widehat{(f')}(y)$ in two different ways. What is $\widehat{f}(y)$?

(Problem 1610) Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be given by $f(x) = e^{-t|x|^2}$. What is $\widehat{f}(y)$? What is $\check{f}(y)$?

(Problem 1630) Show that if $f \in L^1(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ then $f * g \in L^1(\mathbb{R}^n)$. What is $\widehat{f * g}$?

(Problem 1640) Let $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and let $w(x) = u * v(x)$, where $v(x) = \overline{u(-x)}$. What is $w(0)$ in terms of u (not v)? What is $\widehat{w}(y)$ in terms of $\widehat{u}$ (not $\widehat{v}$)?

(Problem 1641) Suppose that $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is uniformly continuous. Let $v(x) = \overline{u(-x)}$. Show that $w = u * v$ is also continuous.

(Problem 1670) Suppose that $f,\, w \in L^1(\mathbb{R}^n)$. Show that $\displaystyle\int_{\mathbb{R}^n} f(x)\,\widehat{w}(x)\,dx = \int_{\mathbb{R}^n} \widehat{f}(y)\,w(y)\,dy$.

(Problem 1680) Apply Problem 1670 with f as in Problem 1610 and with w as in Problem 1640 and let $t \to 0^+$. Suppose that u is uniformly continuous. What can you conclude about u and $\widehat{u}$?

(Problem 1690) Explain how we may extend the Fourier transform to an operator defined on all of $L^2(\mathbb{R}^n)$ (and not only on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$).

(Problem 1691) Show that $\|u\|_{L^2(\mathbb{R}^n)} = \|\widehat{u}\|_{L^2(\mathbb{R}^n)}$ for all $u \in L^2(\mathbb{R}^n)$.

(Problem 1692) Show that the conclusion of Problem 1670 is valid for all $f,\, w \in L^2(\mathbb{R}^n)$.

(Problem 1700) Let $u,\, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ be uniformly continuous. Show that $\displaystyle\int_{\mathbb{R}^n} u\overline{v} = \int_{\mathbb{R}^n} \widehat{u}\overline{\widehat{v}}$.

(Problem 1720) Let $u,\, v \in L^2(\mathbb{R}^n)$. Show that $\displaystyle\int_{\mathbb{R}^n} (\widehat{\check{u}})v = \int_{\mathbb{R}^n} uv$. Conclude that $u = (\widehat{\check{u}})$.

(Problem 1750) Show that if $f \in C^1(\mathbb{R}^n)$ is compactly supported then $\displaystyle\lim_{|y|\to\infty} \widehat{f}(y) = 0$.

(Problem 1760) Show that if $f \in L^1(\mathbb{R}^n)$ then $\displaystyle\lim_{|y|\to\infty} \widehat{f}(y) = 0$.

(Problem 1770) Suppose that $1 < p < 2$. Show that if $f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ then $\widehat{f} \in L^q(\mathbb{R}^n)$, where $1/p + 1/q = 1$.

(Problem 1780) Give an example of a function $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $\widehat{f} \notin L^1(\mathbb{R}^n)$.

(Problem 1781) Let $\operatorname{Re} f$ and $\operatorname{Im} f$ denote the real and imaginary parts of a function. Show that if $f \in L^1(\mathbb{R}^n)$ is real-valued, then $\operatorname{Re}\widehat{f}$ and $\operatorname{Re}\check{f}$ are even and $\operatorname{Im}\widehat{f}$ and $\operatorname{Im}\check{f}$ are odd.

(Problem 1782) Let $\operatorname{Re} f$ and $\operatorname{Im} f$ denote the real and imaginary parts of a function. Show that if $f \in L^1(\mathbb{R}^n)$, if $\operatorname{Re} f$ is even, and if $\operatorname{Im} f$ is odd, then $\widehat{f}$ and $\check{f}$ are both real-valued.

(Problem 1790) Suppose that u solves the heat equation

$$\begin{cases} D_t u - \Delta u = f \quad \text{in } \mathbb{R}^n \times (0,\infty), \quad u = g \quad \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{1790}$$

Suppose in addition that $u(\,\cdot\,,t)$ is in $L^1(\mathbb{R}^n)$, uniformly in t. Let $\widehat{u}(y,t) = \dfrac{1}{(2\pi)^{n/2}} \displaystyle\int_{\mathbb{R}^n} e^{-ix\cdot y}\,u(x,t)\,dx$. What is $D_t\widehat{u}(x,t)$?

(Problem 1800) Let $\widehat{u}(y,t)$ be as in Problem 1790. Fix $y \in \mathbb{R}^n$. Find a differential equation and initial condition for $\widehat{u}(y,t)$.

(Problem 1810) Solve the initial value problem of Problem 1800 and find a formula for $\widehat{u}(y,t)$.

(Problem 1811) Let $u(x,t)$ be the inverse Fourier transform (in the x-variables) of the function you found in Problem 1810. If f and g are real-valued, need u be real-valued?

(Problem 1812) Let $u(x,t)$ be the inverse Fourier transform (in the x-variables) of the function you found in Problem 1810. Suppose that $f(x,t) = 0$ and that $g \in L^2(\mathbb{R}^n)$. Is $u_t \in L^2(\mathbb{R}^n)$, where $u_t(x) = u(x,t)$? What can you say about $\|u_t\|_{L^2(\mathbb{R}^n)}$ and $\|g\|_{L^2(\mathbb{R}^n)}$?

(Problem 1813) Let u_t be as in the previous problem. Recall that $g \in L^2(\mathbb{R}^n)$. We do not assume any smoothness of g. Is $D_x^\alpha u_t \in L^2(\mathbb{R}^n)$, where α is a multiindex?

(Problem 1820) Find a formula for the function $u(x,t)$ of Problem 1790.

(Problem 1830) Let $f \in L^2(\mathbb{R}^n)$. Find $\widehat{u}$ if $-\Delta u + u = f$ in $\mathbb{R}^n$.

(Problem 1831) Let $u(x,t)$ be the inverse Fourier transform of the function you found in Problem 1830. If f is real-valued, need u be real-valued?

(Problem 1832) Let $u(x,t)$ be the inverse Fourier transform of the function you found in Problem 1830. If $f \in L^2(\mathbb{R}^n)$, need u be in $L^2(\mathbb{R}^n)$? What can you say about $\|f\|_{L^2(\mathbb{R}^n)}$ and $\|u\|_{L^2(\mathbb{R}^n)}$?

(Problem 1833) Let u be as in the previous problem. Recall that $f \in L^2(\mathbb{R}^n)$. We do not assume any smoothness of f. Is $Du \in L^2(\mathbb{R}^n)$? Is $D^2 u \in L^2(\mathbb{R}^n)$? Is $D^3 u \in L^2(\mathbb{R}^n)$?

(Problem 1840) Let $B(x) = 2^{-n/2} \displaystyle\int_0^\infty t^{-n/2} e^{-t-|x|^2/4t}\,dt$. Show that $B \in L^1(\mathbb{R}^n)$.

(Problem 1850) What is $\widehat{B}(y)$?

(Problem 1860) Use the Fourier transform to solve the differential equation $-\Delta u + u = f$ in $\mathbb{R}^n$, where $f \in L^2(\mathbb{R}^n)$.

(Problem 1870) Use the Fourier transform to solve the initial value problem for Schrödinger's equation
$$\left\{ iD_t u + \Delta_x u = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty), \quad u = g \quad \text{on } \mathbb{R}^n \times \{0\}. \right.$$

(Problem 1871) Let $u(x,t)$ be the function you found in Problem 1870. If g is real-valued, need u be real-valued?

(Problem 1872) Let $u(x,t)$ be the function you found in Problem 1870. Suppose that $g \in L^2(\mathbb{R}^n)$. Is $u_t \in L^2(\mathbb{R}^n)$, where $u_t(x) = u(x,t)$? What can you say about $\|u_t\|_{L^2(\mathbb{R}^n)}$ and $\|g\|_{L^2(\mathbb{R}^n)}$?

(Problem 1873) Let u_t be as in the previous problem. Recall that $g \in L^2(\mathbb{R}^n)$. We do not assume any smoothness of g. Need any derivatives of u lie in $L^2(\mathbb{R}^n)$?

(Problem 1880) Use the Fourier transform to solve the initial value problem for the wave equation
$$\left\{ D_{tt}^2 u - \Delta_x u = 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty), \quad u = g, \quad D_t u = h \quad \text{on } \mathbb{R}^n \times \{0\}. \right.$$

(Problem 1881) Let $u(x,t)$ be the function you found in Problem 1880. If g and h are real-valued, need u be real-valued?

(Problem 1882) Let $u(x,t)$ be the function you found in Problem 1880. Suppose that g, Dg, and h are all in $L^2(\mathbb{R}^n)$ and $h \in L^2(\mathbb{R}^n)$. Are $D_x u_t$ and $D_t u_t$ in $L^2(\mathbb{R}^n)$, where $u_t(x) = u(x,t)$? What can you say about $\|u_t\|_{L^2(\mathbb{R}^n)}$ and $\|g\|_{L^2(\mathbb{R}^n)}$?

(Problem 1890) Use the Fourier transform to solve the initial value problem for Airy's equation
$$\left\{ u_t - u_{xxx} = 0 \quad \text{in } \mathbb{R} \times (0,\infty), \quad u = g \quad \text{on } \mathbb{R} \times \{0\}. \right.$$

(Problem 1891) Let $u(x,t)$ be the function you found in Problem 1890. If g is real-valued, need u be real-valued?

(Problem 1892) Let $u(x,t)$ be the function you found in Problem 1890. Suppose that $g \in L^2(\mathbb{R}^n)$. Is $u_t \in L^2(\mathbb{R}^n)$, where $u_t(x) = u(x,t)$? What can you say about $\|u_t\|_{L^2(\mathbb{R}^n)}$ and $\|g\|_{L^2(\mathbb{R}^n)}$?

(Problem 1893) Let u_t be as in the previous problem. Recall that $g \in L^2(\mathbb{R}^n)$. We do not assume any smoothness of g. Need any derivatives of u lie in $L^2(\mathbb{R}^n)$?

(Problem 1830) Use the Fourier transform to solve the initial value problem for the Laplacian

$$\left\{ \Delta u = 0 \quad \text{in } \mathbb{R}^n_+, \quad u = g \quad \text{on } \partial\mathbb{R}^n_+. \right.$$

(Problem 1900) What classes of equations can be solved using the Fourier transform?

5.1. Hölder spaces

[**Definition: Hölder spaces**] If $0 < \gamma \leq 1$ and $\Omega \subset \mathbb{R}^n$, we define the Hölder seminorm by

$$[u]_{C^{0,\gamma}(\Omega)} = \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma}$$

and the Hölder norm by

$$\|u\|_{C^{0,\gamma}(\Omega)} = \|u\|_{L^\infty(\Omega)} + \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma}.$$

(Problem 1901) Show that if $[u]_{C^{0,\gamma}(\Omega)} < \infty$ then u is uniformly continuous on Ω. Is the converse true?

(Problem 1902) Suppose that Du is bounded in Ω. Is there any γ for which $[u]_{C^{0,\gamma}(\Omega)}$ is necessarily finite?

(Problem 1903) Let $\Omega = [-1, 1] \subset \mathbb{R}$. Give an example of a function u such that $[u]_{C^{0,1/2}(\Omega)} < \infty$ but such that u' is unbounded on Ω.

(Problem 1904) Suppose that Ω is bounded and that $0 < \gamma < \widetilde{\gamma} \leq 1$. Show that if $[u]_{C^{0,\widetilde{\gamma}}(\Omega)}$ is finite then so is $[u]_{C^{0,\gamma}(\Omega)}$. Can you bound $[u]_{C^{0,\gamma}(\Omega)}$?

(Problem 1910) Let $\Omega \subset \mathbb{R}^n$ and let $\overline{\Omega}$ be its closure. Show that if $[u]_{C^{0,\gamma}(\Omega)} < \infty$ then there is a unique extension $\tilde{u}$ defined on $\overline{\Omega}$ with $\tilde{u} = u$ in Ω and with $[u]_{C^{0,\gamma}(\Omega)} = [\tilde{u}]_{C^{0,\gamma}(\overline{\Omega})}$.

5.2.1. Weak derivatives

[**Definition: Test functions**] We let $C_c^\infty(\Omega)$ be the space of all infinitely differentiable functions $\varphi : \Omega \mapsto \mathbb{R}$ where $\varphi = 0$ outside of some compact set $K \subset \Omega$.

[**Definition: Locally integrable**] We let $L^1_{loc}(\Omega)$ be the space of all functions $u : \Omega \mapsto \mathbb{R}$ such that $\displaystyle\int_K |u| < \infty$ whenever $K \subset \Omega$ is compact.

[**Definition: Weak derivative**] Let α be a multiindex and let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $u \in L^1_{loc}(\Omega)$ and that there is a function $v \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega v\,\varphi = \int_\Omega (-1)^{|\alpha|} u\, D^\alpha \varphi \quad \text{for all } \varphi \in C_0^\infty(\Omega).$$

Then we say that v is the αth weak partial derivative of u and write $D^\alpha u = v$.

(Problem 1930) Suppose that $|\alpha| = k$ and that $u \in C^k(\Omega)$. Show that the classical derivative $D^\alpha u$ is a weak derivative of u.

Lebesgue's differentiation theorem. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in L^1_{loc}(\Omega)$. Then for almost every $x \in \Omega$ we have that* $\displaystyle\lim_{r\to 0^+} \fint_{B(x,r)} |f(y) - f(x)|\,dy = 0.$

(Problem 1940) Let $\Omega \subset \mathbb{R}^n$ be an open set.

(a) Let $w \in L^1_{loc}(\Omega)$. Suppose that $\int_\Omega w\varphi = 0$ for every $\varphi \in C_0^\infty(\Omega)$. Show that $w(x) = 0$ for almost every $x \in \Omega$.

(b) Show that if $u \in L^1_{loc}(\Omega)$, and if v and $\tilde{v}$ are both weak derivatives of u of order α, then $v = \tilde{v}$ except on a set of measure zero.

(Problem 1960) Let $u(x) = |x|$. Does u have a weak first derivative on $\mathbb{R}$? Does u have a weak second derivative on $\mathbb{R}$?

5.2.2. Definition of Sobolev spaces

[Definition: Sobolev spaces] Let $\Omega \subset \mathbb{R}^n$ be an open set. Then $W^{k,p}(\Omega)$ is the set of all $u \in L^p(\Omega)$ such that, if $|\alpha| \leq k$, then $D^\alpha u$ exists in Ω in the weak sense and lies in $L^p(\Omega)$.

[Definition: Sobolev norm] We say that $\|u\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leq k} \operatorname{ess\,sup}_\Omega |D^\alpha u|$. If $1 \leq p < \infty$, then we say that $\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p\right)^{1/p}$.

(Problem 1970) Suppose that $\|u\|_{W^{k,p}(\Omega)} = 0$. What must be true about u?

(Problem 1980) Suppose that $p \geq 1$ and that $u \in L^p(\Omega)$. Is it necessarily the case that $u \in L^1_{loc}(\Omega)$?

(Problem 1990) Let $r > 0$ and let $u(x) = |x|^{-r}$. For what values of r is $u \in L^1_{loc}(\mathbb{R}^n)$?

(Problem 2000) Let $r > 0$ and let $u(x) = |x|^{-r}$. Find the weak first derivatives of u or state that they do not exist.

(Problem 2010) Let $r > 0$ and let $u(x) = |x|^{-r}$. For what values of p is $u \in W^{1,p}(B(0,1))$?

5.2.3. Elementary properties

(Problem 2020) Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in L^1_{loc}(\Omega)$. Suppose that α, β are multiindices, and that $D^\alpha u$, $D^\beta u$, and $D^{\alpha+\beta} u$ exist in the weak sense. Show that $D^\alpha(D^\beta u)$ also exists. What can you say about $D^\alpha(D^\beta u)$ and $D^{\alpha+\beta} u$?

(Problem 2021) Give an example of a function $u \in L^1_{loc}(\mathbb{R}^2)$ such that $D_{xy} u$ exists in the weak sense but such that $D_x u$ does not exist.

(Problem 2030) Let $\Omega \subset \mathbb{R}^n$ be an open set and let u, $v \in L^1_{loc}(\Omega)$. Let μ, $\lambda \in \mathbb{R}$. Suppose that $D^\alpha u$ and $D^\alpha v$ exist. Show that $D^\alpha(\mu u + \lambda v)$ exists.

(Problem 2031*) Show that $D^\alpha(\zeta\varphi) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta}\varphi$ for any multiindex α and any φ, $\zeta \in C^\infty_c(\mathbb{R}^n)$.

(Problem 2032*) Show that $\zeta D^\alpha \varphi = \sum_{\beta\leq\alpha} (-1)^{|\beta|} \binom{\alpha}{\beta} D^{\alpha-\beta}(\varphi D^\beta \zeta)$ for any α, ζ, φ as in Problem 2031.

(Problem 2040) Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $\zeta \in C^\infty_c(\Omega)$ and let $u \in W^{k,p}(\Omega)$. Show that $\zeta u \in W^{k,p}(\Omega)$. If $|\alpha| \leq k$, find a formula for $D^\alpha(\zeta u)$.

[Definition: Convergence] We say that the sequence $\{u_m\}_{m=1}^\infty$ converges to u in $W^{k,p}(\Omega)$ (or $u_m \to u$) if $\lim_{m\to\infty} \|u - u_m\|_{W^{k,p}(\Omega)} = 0$.

[Definition: Local convergence] We say that the sequence $\{u_m\}_{m=1}^\infty$ converges to u in $W^{k,p}_{loc}(\Omega)$ if $\lim_{m\to\infty} \|u - u_m\|_{W^{k,p}(V)} = 0$ for every bounded open set V with $\overline{V} \subset \Omega$.

(Problem 2041) Give an example of a sequence that converges in $W^{k,p}_{loc}(\Omega)$ but not in $W^{k,p}(\Omega)$.

[Definition: Cauchy sequence] We say that the sequence $\{u_m\}_{m=1}^\infty$ is Cauchy if, for every $\varepsilon > 0$, there is a $N > 0$ such that if m, $n > N$, then $\|u_m - u_n\| < \varepsilon$.

(Problem 2050) Show that if $\Omega \subset \mathbb{R}^n$ is an open set, $k \geq 1$ is an integer, and $1 \leq p \leq \infty$, then $W^{k,p}(\Omega)$ is a Banach space. That is, show that

- If u, $v \in W^{k,p}(\Omega)$ and μ, $\lambda \in \mathbb{R}$ then $\mu u + \lambda v \in W^{k,p}(\Omega)$.
- If u, $v \in W^{k,p}(\Omega)$ then $\|u + v\|_{W^{k,p}(\Omega)} \leq \|u\|_{W^{k,p}(\Omega)} + \|v\|_{W^{k,p}(\Omega)}$.
- If $u \in W^{k,p}(\Omega)$ and $\lambda \in \mathbb{R}$ then $\|\lambda u\|_{W^{k,p}(\Omega)} = |\lambda| \|u\|_{W^{k,p}(\Omega)}$,
- $\|u\|_{W^{k,p}(\Omega)} = 0$ if and only if $u = 0$.
- Every Cauchy sequence in $W^{k,p}(\Omega)$ converges to some limit in $W^{k,p}(\Omega)$.

5.3.1. Interior approximation by smooth functions

(Problem 2060) Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in L^1_{loc}(\Omega)$. Let $\Omega_\varepsilon = \{x \in \Omega : \overline{B(x,\varepsilon)} \subset \Omega\}$. Let η_ε be a smooth mollifier, as in Problem 270. Show that $u * \eta_\varepsilon$ is smooth in Ω_ε.

(Problem 2070) Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in W^{k,p}(\Omega)$ for some integer $k \geq 1$ and some $1 \leq p \leq \infty$. Let η_ε be as in Problem 270. Show that $D^\alpha(u * \eta_\varepsilon) = (D^\alpha u) * \eta_\varepsilon$ in Ω_ε for all $|\alpha| \leq k$.

(Problem 2071) Let $\Omega \subset \mathbb{R}^n$ be an open set and let η_ε be as in Problem 270. Let g be uniformly continuous in a neighborhood of $\overline{\Omega}$. Show that $g * \eta_\varepsilon \to g$ as $\varepsilon \to 0^+$, uniformly in $\overline{\Omega}$.

(Problem 2080) Let $\Omega \subset \mathbb{R}^n$ be an open set and let η_ε be as in Problem 270. Show that if $f \in L^1_{loc}(\Omega)$ then $\lim_{\varepsilon \to 0^+} f * \eta_\varepsilon(x) = f(x)$ for almost every $x \in \Omega$.

(Problem 2082) Let $\Omega \subset \mathbb{R}^n$ be an open set and let η_ε be as in Problem 270. Let $1 \leq p \leq \infty$. Show that if $f \in L^p(\Omega)$ then $\|f * \eta_\varepsilon\|_{L^p(\Omega_\varepsilon)} \leq \|f\|_{L^p(\Omega)}$.

(Problem 2083) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let η_ε be as in Problem 270. Let $1 \leq p < \infty$ and let $f \in L^p(\Omega)$. Show that $\lim_{\varepsilon \to 0^+} \|f * \eta_\varepsilon - f\|_{L^p(\Omega_\varepsilon)} = 0$.

(Problem 2090) Let $\Omega \subset \mathbb{R}^n$ be an open set and let η_ε be as in Problem 270. Let $u \in W^{k,p}(\Omega)$ for some integer $k \geq 1$ and some $1 \leq p \leq \infty$. Show that $u * \eta_\varepsilon$ converges to u in $W^{k,p}_{loc}(\Omega)$.

5.3.2. Approximation by smooth functions

(Problem 2100) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $\Omega_j = \{x \in \Omega : \overline{B(x, 2^{-j})} \subset \Omega\}$. Show that there exist smooth functions ξ_j such that $\xi_j = 1$ in Ω_{j-1} and $\xi_j = 0$ outside of Ω_j. *Hint*: Convolve a characteristic function with a mollifier.

(Problem 2110) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $u \in W^{k,p}(\Omega)$. Let ξ_j be as in Problem 2100 and let $\zeta_1 = \xi_1$, $\zeta_j = \xi_j - \xi_{j-1}$ for $j \geq 2$. Show that $(u\zeta_j) \in W^{k,p}(\Omega)$. What is $\sum_{j=1}^{\infty} u\zeta_j$?

(Problem 2120) Let η_ε be as in Problem 270 and let ζ_j and Ω_j be as in Problems 2100–2110. Suppose that $\varepsilon < 2^{-j-1}$. Show that $\eta_\varepsilon * (u\zeta_j)$ is supported in $\overline{\Omega_{j+1}} \setminus \Omega_{j-2}$.

(Problem 2130) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $u \in W^{k,p}(\Omega)$. Show that there are smooth functions v_m such that $v_m \to u$ in $W^{k,p}(\Omega)$.

5.3.3. Global approximation by smooth functions

(Problem 2140*) Let $1 \leq p < \infty$ and let $f \in L^p(\mathbb{R}^n)$. Let $\vec{e} \in \mathbb{R}^n$ be a unit vector and let $f_r(x) = f(x + r\vec{e})$. Show that $f_r \to f$ in $L^p(\mathbb{R}^n)$ as $r \to 0$.

(Problem 2150) Let $\Omega = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > \psi(x')\}$ for some C^1 function $\psi : \mathbb{R}^{n-1} \mapsto \mathbb{R}$. Suppose that $u \in W^{k,p}(\Omega)$. Let $u_\varepsilon(x_1, \dots, x_n) = u(x_1, \dots, x_{n-1}, x_n + \varepsilon)$. Show that $u_\varepsilon \to u$ as $\varepsilon \to 0^+$ in $W^{k,p}(\Omega)$.

(Problem 2160) Let Ω be as in Problem 2150, and suppose in addition that $|D\psi(x')| \leq M < \infty$ for all $x' \in \mathbb{R}^{n-1}$. Fix some $R > 0$ and let $\Omega^\varepsilon = \{(x', x_n - \varepsilon) : (x', x_n) \in \Omega\} \cap B(0, R + \varepsilon)$. Let Ω^ε_j be as in Problem 2100, that is, $\Omega^\varepsilon_j = \{x : B(x, 2^{-j}) \subset \Omega_\varepsilon\}$. Show that if j is large enough then $\Omega \cap B(0, R) \subset \Omega^\varepsilon_j$.

(Problem 2170) Let u and Ω be as in Problem 2160. Suppose in addition that there is some $R < \infty$ such that $u = 0$ in $\Omega \setminus B(0, R)$. Show that there are functions $v_m \in C^\infty_0(\mathbb{R}^n)$ such that $v_m \to u$ in $W^{k,p}(\Omega)$.

(Problem 2180*) Can you bound $\|v_m\|_{W^{k,p}(\Omega)}$? Can you bound $\|v_m\|_{W^{k,p}(\mathbb{R}^n)}$?

(Problem 2190) Let Ω be a bounded open set. We say that Ω is C^1 if, for every $x \in \partial\Omega$, there is a number $r_x > 0$ such that $B(x, r_x) \cap \Omega = B(x, r_x) \cap V_x$, where $V_x = \{T_x(y', y_n) : y' \in \mathbb{R}^{n-1}, y_n > \psi_x(y')\}$ for some C^1 function $\psi_x : \mathbb{R}^{n-1} \mapsto \mathbb{R}$ and some rotation T_x.

Show that if Ω is a bounded C^1 open set, then there are finitely many smooth functions ξ_j such that $\displaystyle\sum_{j=0}^{N} \xi_j = 1$ in Ω, such that there is some $r > 0$ such that $\xi_0(x) = 0$ whenever $B(x, r) \not\subset \Omega$, and such that there exist points $\{x_j\}_{j=1}^{N} \subset \partial\Omega$ such that if $j \geq 1$, then ξ_j is supported in $B(x_j, r_{x_j}/2)$.

(Problem 2200) Show that if Ω is a bounded C^1 open set and if $u \in W^{k,p}(\Omega)$, then there are functions $v_m \in C_0^\infty(\mathbb{R}^n)$ such that $v_m \to u$ in $W^{k,p}(\Omega)$.

5.4. Extensions

(Problem 2210) Let v_m be as in Problem 2200. Recall that $v_m \in C_0^\infty(\mathbb{R}^n)$. Can you say anything about $\|v_m\|_{W^{k,p}(\mathbb{R}^n)}$?

(Problem 2220) Let $1 \leq p < \infty$. We let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$. Let $u \in W^{1,p}(\mathbb{R}^n_+)$ and suppose that $u = 0$ outside of $B(0,R)$ for some large number R. Let v_m be as in Problem 2170 or 2200. Let

$$w_m(x) = \begin{cases} v_m(x), & x \in \overline{\mathbb{R}^n_+}, \\ 4v_m(x_1, \dots, x_{n-1}, -x_n/2) - 3v_m(x_1, \dots, x_{n-1}, -x_n), & x \notin \mathbb{R}^n_+. \end{cases}$$

Show that $w_m \in C^1(\mathbb{R}^n)$.

(Problem 2230) Show that $\|w_m\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|v_m\|_{W^{1,p}(\mathbb{R}^n_+)}$ for some constant C.

(Problem 2240*) Show that $\|w_m\|_{W^{2,p}(\mathbb{R}^n)} \leq C\|v_m\|_{W^{2,p}(\mathbb{R}^n_+)}$ for some constant C. Is $w_m \in C^2(\mathbb{R}^n)$?

(Problem 2250) Show that $\{w_m\}_{m=1}^\infty$ is a Cauchy sequence in $W^{1,p}(\mathbb{R}^n)$.

(Problem 2251) Show that there is a bounded linear operator $E : W^{1,p}(\mathbb{R}^n_+) \mapsto W^{1,p}(\mathbb{R}^n)$ with $Eu = u$ in $\mathbb{R}^n_+$.

(Problem 2260) Let Ω be a C^1 graph domain, as in Problem 2160. Let $u \in W^{1,p}(\Omega)$. Let $\tilde{u}(x', x_n) = u(x', x_n + \psi(x'))$. Show that there is a constant $C < \infty$ such that $\frac{1}{C}\|u\|_{W^{1,p}(\Omega)} \leq \|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n_+)} \leq C\|u\|_{W^{1,p}(\Omega)}$.

(Problem 2270) Let Ω be a C^1 graph domain, as in Problem 2160. Show that there is a bounded linear operator $E : W^{1,p}(\Omega) \mapsto W^{1,p}(\mathbb{R}^n)$ with $Eu = u$ in Ω.

(Problem 2280) Let Ω be a C^1 bounded open set, as in Problem 2190. Let $u \in W^{1,p}(\Omega)$. Show that there is a bounded linear operator $E : W^{1,p}(\Omega) \mapsto W^{1,p}(\mathbb{R}^n)$ with $Eu = u$ in Ω.

(Problem 2290*) Show that if Ω is C^2 then $\|Eu\|_{W^{2,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{2,p}(\Omega)}$.

5.5. Traces

(Problem 2300) Let $1 \leq p \leq \infty$. Show that if $u \in C^1(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ and $v_t(x') = u(x', t) - u(x', 0)$ for all $x' \in \mathbb{R}^{n-1}$, then $\|v_t\|_{L^p(\mathbb{R}^{n-1})} \leq t^{1-1/p}\|u\|_{W^{1,p}(\mathbb{R}^n_+)}$.

(Problem 2310) Let $1 \leq p \leq \infty$. Show that if $u \in C^1(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ and $v(x') = u(x', 0)$ then $\|v\|_{L^p(\mathbb{R}^{n-1})} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n_+)}$ for some constant C.

(Problem 2320) Let $1 \leq p \leq \infty$. Show that there is a bounded operator $T : W^{1,p}(\mathbb{R}^n_+) \mapsto L^p(\mathbb{R}^{n-1})$ such that if $u \in C(\overline{\mathbb{R}^n_+}) \cap W^{1,p}(\mathbb{R}^n_+)$ then $Tu(x') = u(x', 0)$ for all $x' \in \mathbb{R}^{n-1}$.

(Problem 2330) Let Ω be a C^1 graph domain, as in Problem 2160. Let $1 \leq p \leq \infty$. Show that there is a bounded operator $T : W^{1,p}(\Omega) \mapsto L^p(\partial\Omega)$ such that if $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ then $Tu(x) = u(x)$ for all $x \in \partial\Omega$.

(Problem 2340) Let Ω be a C^1 bounded open set, as in Problem 2190. Let $1 \leq p \leq \infty$. Show that there is a bounded operator $T : W^{1,p}(\Omega) \mapsto L^p(\partial\Omega)$ such that if $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$ then $Tu(x) = u(x)$ for all $x \in \partial\Omega$.

(Problem 2350) Let Ω be a C^1 bounded open set, as in Problem 2190. Let $1 \leq p < \infty$ and let $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ under the $W^{1,p}(\Omega)$-norm. Show that if $u \in W_0^{1,p}(\Omega)$ then $Tu = 0$ in $L^p(\partial\Omega)$.

(Problem 2360) Let $1 \leq p < \infty$. Let $u \in W^{1,p}(\mathbb{R}^n_+)$ with $Tu = 0$. Show that $\int_{\mathbb{R}^n} \int_0^t |u(x', s)|^p \, ds \, dx \leq Ct^p\|u\|_{W^{1,p}(\mathbb{R}^{n-1} \times \{0,t\})}$, where C does not depend on t or u.

(Problem 2370) Let $1 \leq p < \infty$ and let $W_0^{1,p}(\mathbb{R}^n_+)$ be the closure of $C_0^\infty(\mathbb{R}^n_+)$ under the $W^{1,p}(\mathbb{R}^n_+)$-norm. Show that if $u \in W^{1,p}(\mathbb{R}^n_+)$ and $Tu = 0$ in $L^p(\mathbb{R}^{n-1})$, then $u \in W_0^{1,p}(\mathbb{R}^n_+)$.