Math 2554H, Fall 2018

Exam 3 review

If you need it, the following information will be printed on the cover of the exam:
The volume of a cylinder of radius r and height h is $\pi r^2 h$.
The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.
The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.
The surface area of a sphere of radius r is $4\pi r^2$.

(Problem 1) A spherical snowball melts (loses volume) at a rate proportional to its surface area. Show that the rate of change of the radius of the snowball is constant.

(Problem 2) A small planet moves along the ellipse $\left(\frac{x-25}{65}\right)^2 + \left(\frac{y}{60}\right)^2 = 1$. When the planet's x-coordinate is 77, the x-component of its velocity is 3 units/second. What is the y-component of its velocity?

(Problem 3) A (simplified) hourglass is in the shape of a cone with height 10 cm and radius 6 cm, with the vertex pointed downwards, so that the (remaining) sand is also in the shape of a cone. Sand flows out of the cone at a rate of 3 cm^3/sec. What is the rate of change of the height of the sand left behind at the moment when the height is 4 cm?

(Problem 4) I fly a kite. My kite is ascending at a rate of 20 ft/sec. My kite is also caught in a strong wind that blows it westward at a rate of 30 ft/sec. When my kite is 300 feet above me and 400 feet west of me, how fast is the string running through my fingers? (Assume that the string is always a straight line between me and the kite.)

(Problem 5) A 50-ft ladder rests on the (horizontal) ground and leans on a (vertical) wall. The bottom of the ladder is pulled away from the wall at a rate of 3 ft/sec. How fast is the top of the ladder moving at the moment when the bottom of the ladder is 14 ft from the wall?

(Problem 6) A lighthouse is 300 m from the closest point to a long, straight wall built on shore. The focused beam of its light revolves three times per minute, and when it is aimed at land, it illuminates the wall. How fast is the illuminated point traveling when it passes by a window that is 340 m from the lighthouse?

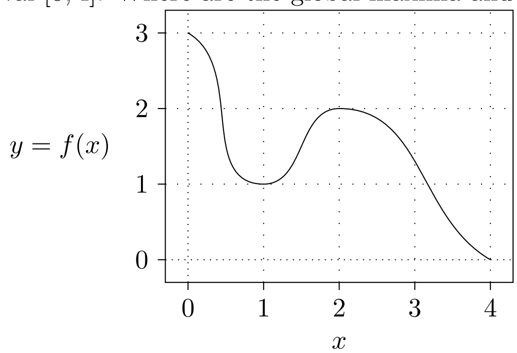
(Problem 7) I watch a spaceship being launched from a distance of 5 km. The ship launches straight up. I measure the line-of-sight angle between the rocket and the ground. At the moment when the angle between the rocket and the ground is $\pi/6$ radians, the angle is changing at a rate of 2 radians/second. How fast is the rocket moving?

(Problem 8) Sketch the graph of a continuous function that has an absolute maximum and a local minimum but no absolute minimum on the interval $(0, 3)$.

(Problem 9) Sketch the graph of a continuous function that has a local maximum value at $x = 2$ where $f'(2) = 0$.

(Problem 10) Sketch the graph of a continuous function that has a local minimum value at $x = 2$ where $f'(2)$ is undefined.

(Problem 11) Here is the graph of $y = f(x)$. Find all the local minima and local maxima of f on the interval $[0, 4]$. Where are the global maxima and minima?



(Problem 12) Find the critical point or points of the function $f(x) = 3x^2 - 6x + 9$.

(Problem 13) Find the critical point or points of the function $f(x) = x^3 - 3x^2 + 9x + 1$.

(Problem 14) Find the critical point or points of the function $f(x) = \sqrt[5]{3x - 2}$.

(Problem 15) Find the absolute maximum and minimum values of the function $f(x) = x^2 - 3x + 5$ on the interval $[1, 4]$ or state that they do not exist.

(Problem 16) Find the absolute maximum and minimum values of the function $f(x) = 32\sqrt{x} - x^2$ on the interval $[0, 9]$ or state that they do not exist.

(Problem 17) Find the absolute maximum and minimum values of the function $f(x) = \dfrac{1}{x^2} + 16x$ on the interval $(0, 3)$ or state that they do not exist.

(Problem 18) Does the Mean Value Theorem apply to the function $f(x) = \sqrt{x}$ on the interval $[1, 9]$? If so, find the point guaranteed to exist by the Mean Value Theorem.

(Problem 19) Does the Mean Value Theorem apply to the function $f(x) = \sqrt{x}$ on the interval $[0, 4]$? If so, find the point guaranteed to exist by the Mean Value Theorem.

(Problem 20) Does the Mean Value Theorem apply to the function $f(x) = \dfrac{1}{x}$ on the interval $[0, 4]$? If so, find the point guaranteed to exist by the Mean Value Theorem.

(Problem 21) Does the Mean Value Theorem apply to the function $f(x) = |x|$ on the interval $[-2, 3]$? If so, find the point guaranteed to exist by the Mean Value Theorem.

(Problem 22)
(a) Do the conditions of the Mean Value theorem hold for the function $f(x) = x^3$ on the interval $[-1, 1]$?
(b) Does the conclusion hold?

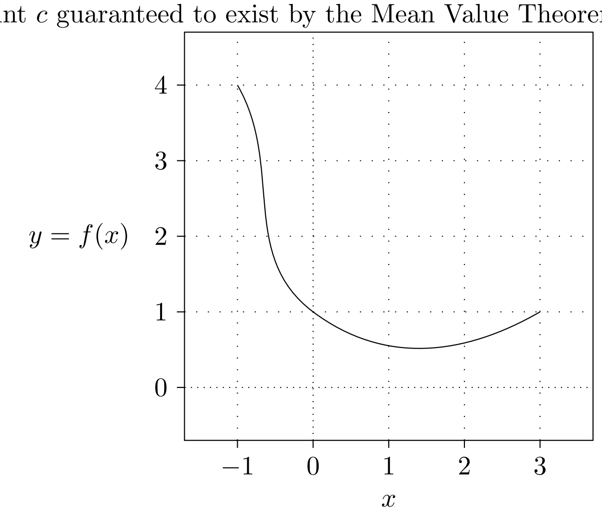
(Problem 23)
(a) Do the conditions of the Mean Value theorem hold for the function $f(x) = \sqrt[3]{x}$ on the interval $[-1, 1]$?
(b) Does the conclusion hold?

(Problem 24)

(a) Do the conditions of the Mean Value theorem hold for the function $f(x) = x^{2/3}$ on the interval $[-1, 1]$?

(b) Does the conclusion hold?

(Problem 25) Here is the graph of $y = f(x)$ on the interval $[a, b] = [-1, 3]$. By visual inspection, find the point c guaranteed to exist by the Mean Value Theorem. (There may be more than one such point).



(Problem 26) A police department has no speedometers. However, they have a number of traffic cameras and can record the license plate and transit time of every car that passes through a given point. Explain a way that the police department can catch (at least some of the) people who exceed the speed limit over a given stretch of road.

(Problem 27) A rocket ship is stationary just before it is launched. Four minutes after the ship is launched, it is moving at a rate of 12 km/sec. One hour after launch, the ship is in free-fall (and feels no acceleration).

(a) Show that at some point, the crew felt an acceleration of 50 m/sec.

(b) Show that there were at least two points when the crew felt an acceleration of 30 m/sec.

(Problem 28) Sketch a function that is continuous on $[0, 5]$, where $f' > 0$ on $(0, 2)$, $f' < 0$ on $(2, 4)$, and $f' > 0$ on $(4, 5)$.

(Problem 29) Sketch a function that is continuous on $[0, 5]$ and satisfies $f' > 0$ on $(0, 2)$, $f'(2) = 0$, and $f' > 0$ on $(2, 5)$.

(Problem 30) Find the intervals on which $f(x) = 2x^4 - x^2$ is increasing and the intervals on which it is decreasing.

(Problem 31) Find the intervals on which $f(x) = x^4 - 8x^3 + 18x^2 + 5$ is increasing and the intervals on which it is decreasing.

(Problem 32) Find all the critical points of $f(x) = 5x^3 - 3x^5$. Use the First Derivative Test to identify the local maxima and minima.

(Problem 33) Find all the critical points of $f(x) = x^5 - 5x^4 + 5x^3$. Use the First Derivative Test to identify the local maxima and minima.

(Problem 34) Sketch a function that is continuous on $[0, 5]$, where $f' > 0$ on $(0, 1)$, $f' < 0$ on $(1, 3)$, and $f' > 0$ on $(3, 5)$, and where $f'' < 0$ on $(0, 2)$ and $f'' > 0$ on $(2, 5)$.

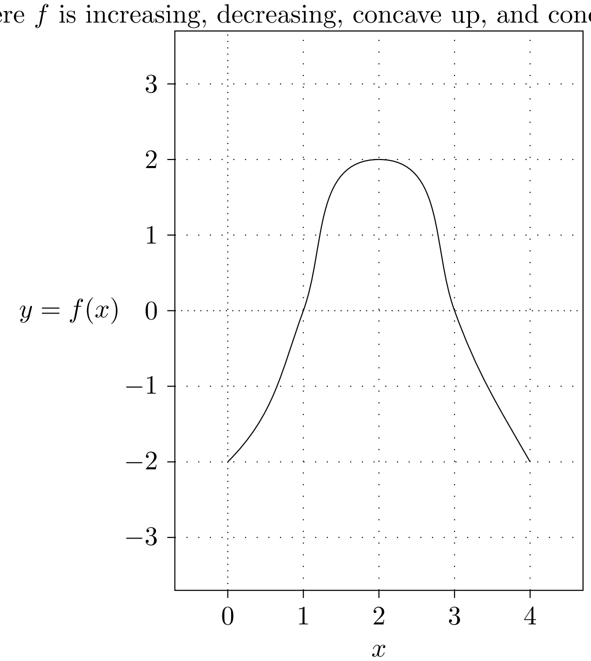
(Problem 35) Sketch a function that is continuous on $[0,4]$, where $f' > 0$ on $(0,2)$ and $f' < 0$ on $(2,4)$, and where $f'' > 0$ on $(0,2)$ and $f'' > 0$ on $(2,4)$. Is it possible to sketch such an f that is differentiable everywhere?

(Problem 36) Where is $f(x) = 2x^4 - 2x^3 + 4$ concave up and concave down? What are the inflection points?

(Problem 37) Find the critical points of the function $f(x) = x^3 - 6x^2$. Use the second derivative test to identify which points are local maxima and which are local minima.

(Problem 38) Find the critical points of the function $f(x) = 2x^4 - 3x^6$. What does the second derivative test tell you about these critical points?

(Problem 39) Here is the graph of $y = f'(x)$. (This is **not** the graph of $y = f(x)$!) Find the intervals where f is increasing, decreasing, concave up, and concave down.



(Problem 40) Suppose that $f'(x) = (x-2)^2(x-3)^3$. Sketch a possible graph of f.

(Problem 41) Suppose that $f'(x) = \cos(\pi x^2)$. Sketch a possible graph of f on $[-2, 2]$.

(Problem 42) Sketch the graph of $y = \dfrac{x^2+12}{3x+1}$.

(Problem 43) Sketch the graph of $y = x^{2/3} - x$.

(Problem 44) Sketch the graph of $y = \sqrt[3]{x-2} - x$.

(Problem 45) Sketch the graph of $y = \dfrac{1}{\pi} \arccos x$.

(Problem 46) Sketch the graph of $y = \dfrac{1}{\pi} \operatorname{arcsec} x$.

(Problem 47) What two positive real numbers whose product is 70 have the smallest possible sum?

(Problem 48) A rancher plans to make three identical and adjacent rectangular pens against a barn, each with an area of 25 m^2. What are the dimensions of each pen that minimize the amount of fence that must be used?

(Problem 49)

(a) A rectangle is constructed with its base on the $x$axis and two of its vertices on the parabola $y = 64 - x^2$. What are the dimensions of the rectangle with the maximum area?

(b) Suppose that the rectangle is constrained to lie in the upper half-space. What are the dimensions of the rectangle with the maximum area?

(Problem 50) A cylindrical tank with a circular base and an open top is to be constructed out of 300π square meters of sheet steel. What are the dimensions of the largest possible tank?

(Problem 51) An island is 6 miles from the nearest point on a straight shoreline; that point is 10 miles from a rest area on shore.

(a) An athlete can bicycle at a rate of 7.8 miles/hour and swim at a rate of 3 miles/hour. She wants to get from the rest area to the island as quickly as possible. How far should she bicycle before she starts swimming?

(b) Another athlete can swim at a rate of 4.5 miles/hour and walk at a rate of 5.1 miles/hour. How far should he walk before he starts to swim to minimize the travel time?

(Problem 52) I want to make a rectangular box by cutting square corners out of an 8.5" by 11" piece of paper and gluing adjacent cut edges. What should the height of the box (that is, the side-length of the removed squares) be to maximize the volume of the box?

(Problem 53) I have \$800 available to fence in a rectangular garden. The fencing for the side of the garden facing the road costs \$15 per foot, and the fencing for the other three sides costs \$5 per foot. Find the dimensions of the garden with the largest possible area.

(Problem 54) Using a linear approximation, estimate $\ln(0.97)$.

(Problem 55) Using a linear approximation, estimate $\arctan(1.01) - \pi/4$.

(Problem 56) Using a linear approximation, estimate $(10.2)^4$.

(Problem 57) Using a linear approximation, estimate $\sqrt[3]{997}$.

Answer key

(Answer 18) Yes; $c = 4$.

(Answer 19) Yes; $c = 1$.

(Answer 20) No; f is not right continuous at 0 and therefore f is not continuous on $[0, 4]$.

(Answer 21) No; $f(x)$ is not differentiable at $x = 0$ and 0 is in $(-2, 3)$.

(Answer 22)
(a) Yes; $f(x)$ is continuous on $[-1, 1]$ and $f'(x) = 3x^2$ exists everywhere in $(-1, 1)$.
(b) By the mean value theorem, if the conditions holds, then the conclusion holds.

(Answer 23)
(a) No; $f(x)$ is not differentiable at $x = 0$.
(b) Yes; $f'(\frac{1}{\sqrt{27}}) = 1 = \dfrac{f(1) - f(-1)}{1 - (-1)}$.

(Answer 24)
(a) No; $f(x)$ is not differentiable at $x = 0$.
(b) No; $f'(x) = \frac{2}{3}x^{-1/3}$ is never zero, and $\dfrac{f(1) - f(-1)}{1 - (-1)} = 0$.

(Answer 31) $f(x)$ is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.