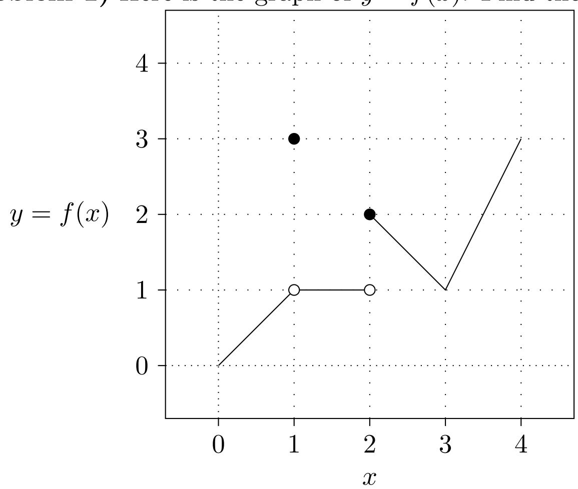
Math 2554H, Fall 2018

Throughout, if you are asked to find a limit, you are to either find a real number equal to the limit, state that the limit is ∞, state that the limit is $-\infty$, or state that the limit does not exist.

(Problem 1) Here is the graph of $y = f(x)$. Find the following values and limits.



(a) $f(1)$
(b) $\lim_{x\to 1} f(x)$
(c) $f(2)$
(d) $\lim_{x\to 2} f(x)$
(e) $\lim_{x\to 2^+} f(x)$
(f) $\lim_{x\to 2^-} f(x)$
(g) $f(3)$
(h) $\lim_{x\to 3} f(x)$

(Problem 2) Here is a table of values of x and of $f(x)$. Approximately how much is $\lim_{x\to 0} f(x)$?

x	$f(x)$
1	1
0.1	0.7177346254
0.01	0.6955550057
0.001	0.6933874626
0.0001	0.6931712038
0.00001	0.6931495828
-0.00001	0.6931447783
-0.0001	0.6931231585
-0.001	0.6929070095
-0.01	0.6907504563
-0.1	0.6696700846
-1	0.5

(Problem 3) Let $f(x) = \dfrac{|x|}{x}$. Complete the following table. What is $\lim_{x\to 0^+} f(x)$? What is $\lim_{x\to 0^-} f(x)$?

x	$f(x)$
1	
0.1	
0.01	
0.001	
-0.001	
-0.01	
-0.1	
-1	

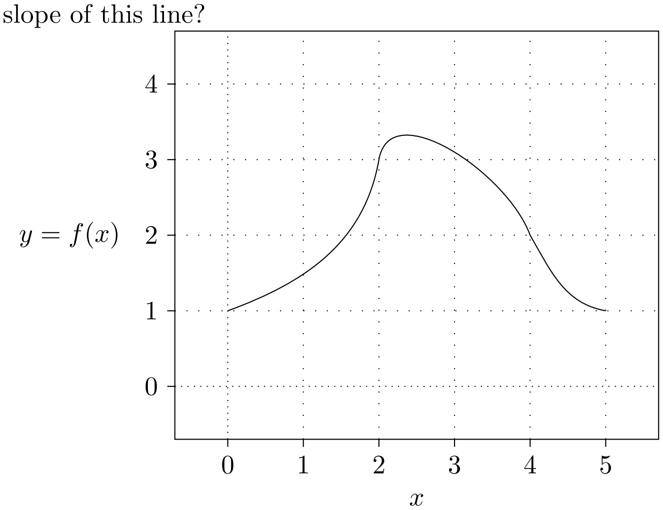
(Problem 4) What is $\lim_{x\to 3} \dfrac{x^3 - 9x^2 + 9x - 27}{x-3}$?

(Problem 5) Suppose that my position at time t is $x(t) = \sqrt{3t+10}$ meters from some fixed base point. Find my average velocity over the interval $[2, 5]$.

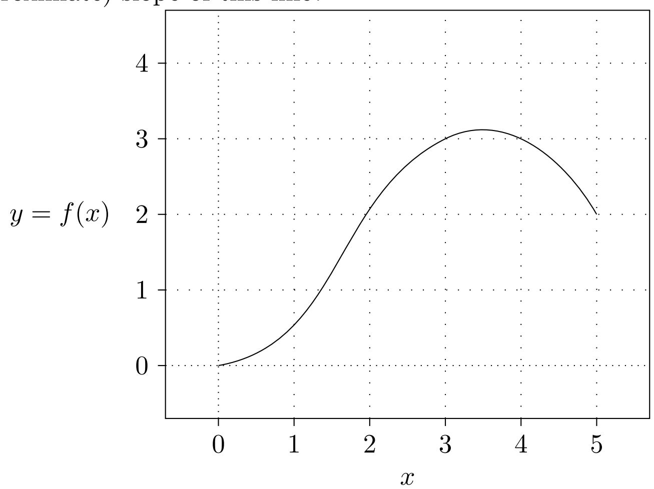
(Problem 6) Suppose that my position at time t is $x(t) = \sqrt{3t+10}$ meters from some fixed base point. Find my instantaneous velocity at time $t = 2$.

(Problem 7) Suppose that my position at time t is $x(t) = \sqrt[3]{t}$ meters from some fixed base point. Find my instantaneous velocity at time $t = 8$.

(Problem 8) Here is the graph of $y = f(x)$. Draw the secant line through $(2, f(2))$ and $(4, f(4))$. What is the slope of this line?



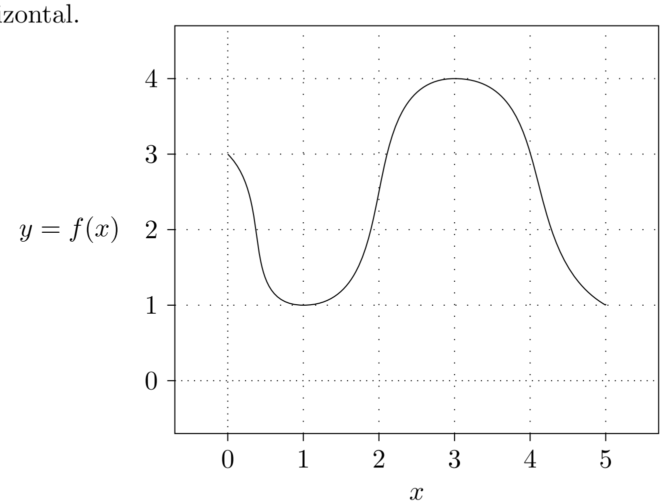
(Problem 9) Here is the graph of $y = f(x)$. Draw the tangent line at the point $(3, f(3))$. What is the (approximate) slope of this line?



(Problem 10) Find the equation for the tangent line to the graph of $y = x^2 - 3x$ at the point $(2, -2)$.

(Problem 11) Find the equation for the tangent line to the graph of $y = \dfrac{18}{x^2}$ at the point $(3, 2)$.

(Problem 12) Here is the graph of $y = f(x)$. Find all the points on this graph at which the tangent line is horizontal.



(Problem 13) Use the squeeze theorem to find $\lim_{x\to 3} x^2 + (x-3)^2 \sin(1/(x-3))$.

(Problem 14) Use the squeeze theorem to find $\displaystyle\lim_{x\to\pi/2} \frac{\cos x}{x - \pi/2}$.

(Problem 15) Use the squeeze theorem to find $\displaystyle\lim_{x\to\pi/2} \frac{1 - \sin x}{x - \pi/2}$.

(Problem 16) Use the squeeze theorem to find $\displaystyle\lim_{x\to\pi} \frac{\cos x + 1}{x - \pi}$.

(Problem 17) Use the squeeze theorem to find $\lim\limits_{x\to\pi} \dfrac{\sin x}{x-\pi}$.

(Problem 18) Use the squeeze theorem to find $\lim\limits_{x\to 3\pi/2} \dfrac{\sin x}{x-3\pi/2}$.

(Problem 19) Use the squeeze theorem to find $\lim\limits_{x\to 3\pi/2} \dfrac{1+\cos x}{x-3\pi/2}$.

(Problem 20) Let $f(x) = \dfrac{2}{(x-2)(x-3)^2(x-4)}$.

(a) Find all numbers a such that $\lim\limits_{x\to a^+} f(x) = \infty$.
(b) Find all numbers a such that $\lim\limits_{x\to a^+} f(x) = -\infty$.
(c) Find all numbers a such that $\lim\limits_{x\to a^-} f(x) = \infty$.
(d) Find all numbers a such that $\lim\limits_{x\to a^-} f(x) = -\infty$.
(e) Find all numbers a such that $\lim\limits_{x\to a} f(x) = \infty$.
(f) Find all numbers a such that $\lim\limits_{x\to a} f(x) = -\infty$.

(Problem 21) Find $\lim\limits_{x\to\pi^+} \dfrac{x^2}{\cos x+1}$.

(Problem 22) Find $\lim\limits_{x\to 4^+} 3\log_{10}(x-4)$.

(Problem 23) Find $\lim\limits_{x\to\infty} \dfrac{5x^2+4x+2}{3+2x+6x^2}$.

(Problem 24) Find $\lim\limits_{x\to\infty} \dfrac{3+2x+x^2}{2x^3+6x+5}$.

(Problem 25) Find $\lim\limits_{x\to\infty} \sqrt{3+\dfrac{2}{x^2}}$.

(Problem 26) Find $\lim\limits_{x\to\infty} \arctan x$.

(Problem 27) Find $\lim\limits_{x\to\infty} \operatorname{arccot} x$.

(Problem 28) Find $\lim\limits_{x\to\infty} \operatorname{arcsec} x$.

(Problem 29) Find $\lim\limits_{x\to-\infty} \arctan x$.

(Problem 30) Find $\lim\limits_{x\to-\infty} \operatorname{arccot} x$.

(Problem 31) Find $\lim\limits_{x\to-\infty} \operatorname{arcsec} x$.

(Problem 32) Find all the vertical asymptotes of $f(x) = \dfrac{x+3}{(x+3)(x-3)(x+4)^2}$.

(Problem 33) Find all the horizontal asymptotes of $f(x) = \dfrac{x^3+3x^2+5}{2-x+4x^3}$.

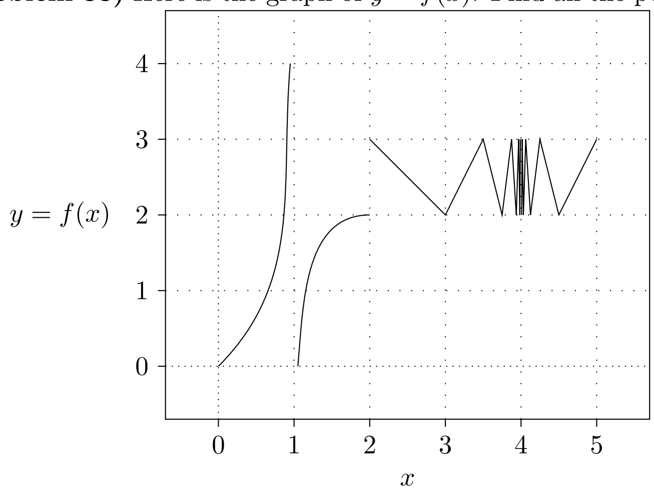
(Problem 34) Find all the horizontal asymptotes of $f(x) = 3\,\text{arcsec}\,x$.

(Problem 35) Find all the slant asymptotes of $f(x) = \dfrac{x^3 + 3x^2 + 2x + 1}{2x^2 + 3x + 5}$.

(Problem 36) Is $f(x) = x^2 \sin(1/x)$ continuous at $x = 0$? Why or why not?

(Problem 37) Is $f(x) = \arcsin x$ continuous at $x = 1$? Why or why not?

(Problem 38) Here is the graph of $y = f(x)$. Find all the points of discontinuity between $x = 0$ and $x = 5$.



(Problem 39) Sketch a function f that is left continuous at 1 but not continuous at 1.

(Problem 40) Give the intervals of continuity of $y = \arcsin x$, $y = \sqrt{5x - x^2 - 6}$, $y = \sqrt{x^2 - 5x + 4}$, $y = \dfrac{1}{3x^2 + 7x + 2}$, $y = \lfloor x \rfloor$.

(Problem 41) Let $f(x) = \dfrac{x^2 - 2x + 1}{x - 1}$. Let $g(x) = \begin{cases} 1 + x\sin(1/x), & x \neq 0, \\ 1, & x = 0. \end{cases}$

(a) Find $\lim_{x \to 0} g(x)$.
(b) Find $\lim_{x \to 1} f(x)$.
(c) Find $\lim_{x \to 0} f(g(x))$ or state that it does not exist.
(d) Find $\lim_{x \to 0} g(f(x))$ or state that it does not exist.

(Problem 42) Use the intermediate value theorem to show that the equation $x^3 + x = 3$ has a solution. Find an interval of length at most 0.125 containing a solution.

(Problem 43) Use the intermediate value theorem to show that the equation $\cos x - x = 0$ has a solution. Find an interval of length at most $\pi/6$ containing a solution.

(Problem 44) Using εs and δs, write the precise definition of $\lim_{x \to a^+} f(x) = L$.

(Problem 45) Using εs and δs, write the precise definition of $\lim_{x \to a^-} f(x) = L$.

(Problem 46) Using Ms and δs, write the precise definition of $\lim_{x \to a} f(x) = \infty$.

(Problem 47) Using Ms and δs, write the precise definition of $\lim_{x\to a^+} f(x) = \infty$.

(Problem 48) Using Ms and δs, write the precise definition of $\lim_{x\to a^-} f(x) = \infty$.

(Problem 49) Using Ms and δs, write the precise definition of $\lim_{x\to a} f(x) = -\infty$.

(Problem 50) Using Ms and δs, write the precise definition of $\lim_{x\to a^+} f(x) = -\infty$.

(Problem 51) Using Ms and δs, write the precise definition of $\lim_{x\to a^-} f(x) = -\infty$.

(Problem 52) Using εs and Ns, write the precise definition of $\lim_{x\to\infty} f(x) = L$.

(Problem 53) Using εs and Ns, write the precise definition of $\lim_{x\to-\infty} f(x) = L$.

(Problem 54) Using Ms and Ns, write the precise definition of $\lim_{x\to\infty} f(x) = \infty$.

(Problem 55) Using Ms and Ns, write the precise definition of $\lim_{x\to\infty} f(x) = -\infty$.

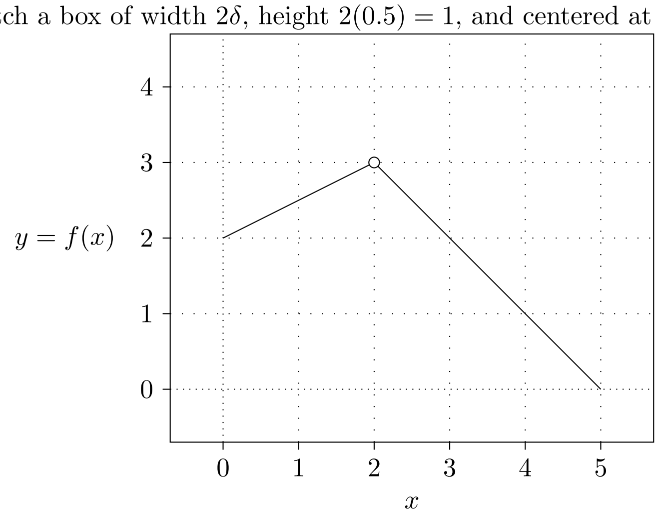
(Problem 56) Using Ms and Ns, write the precise definition of $\lim_{x\to-\infty} f(x) = \infty$.

(Problem 57) Using Ms and Ns, write the precise definition of $\lim_{x\to-\infty} f(x) = -\infty$.

(Problem 58) Find a δ such that, if $|x-3| < \delta$, then $|x^2 - 9| < 1$.

(Problem 59) Find a δ such that, if $|x-2| < \delta$, then $|x^2 + 4x - 8| < 5$.

(Problem 60) Here is the graph of $y = f(x)$. Find a δ such that, if $0 < |x-2| < \delta$, then $|f(x) - 3| < 0.5$. Sketch a box of width 2δ, height $2(0.5) = 1$, and centered at $(2, 3)$.



(Problem 61) Let $f(x) = \cos(1/x)$. Show that 0 is *not* $\lim_{x\to 0} f(x)$ by choosing an $\varepsilon > 0$ and showing that, for every $\delta > 0$, there is some x with $0 < |x - 0| < \delta$ and with $|f(x) - 0| \geq \varepsilon$.

(Problem 62) Let $f(x) = \arctan(1/x)$. Show that $\pi/2$ is *not* $\lim_{x\to 0} f(x)$ by choosing an $\varepsilon > 0$ and showing that, for every $\delta > 0$, there is some x with $0 < |x - 0| < \delta$ and with $|f(x) - \pi/2| \geq \varepsilon$.

(Problem 63) Let $f(x) = \dfrac{1}{1 + 2^{1/x}}$. Show that 1 is *not* $\lim_{x\to 0} f(x)$ by choosing an $\varepsilon > 0$ and showing that, for every $\delta > 0$, there is some x with $0 < |x - 0| < \delta$ and with $|f(x) - 1| \geq \varepsilon$.

Answer key

(Answer 1)
(a) $f(1) = 3$.
(b) $\lim_{x\to 1} f(x) = 1$.
(c) $f(2) = 2$.
(d) $\lim_{x\to 2} f(x)$ does not exist.
(e) $\lim_{x\to 2^+} f(x) = 2$.
(f) $\lim_{x\to 2^-} f(x) = 1$.
(g) $f(3) = 1$.
(h) $\lim_{x\to 3} f(x) = 1$.

(Answer 2) $\lim_{x\to 0} f(x) \approx 0.69315$.

(Answer 3) $\lim_{x\to 0^+} f(x) = 1$; $\lim_{x\to 0^-} f(x) = -1$.

(Answer 4) $\lim_{x\to 3} \dfrac{x^3 - 9x^2 + 9x - 27}{x-3} = -18$.

(Answer 13) Because $-1 \leq \sin(1/(x-3)) \leq 1$ for all $x \neq 0$, and $(x-3)^2 \geq 0$ for we have that

$$x^2 - (x-3)^2 \leq x^2 + (x-3)^2 \sin(1/(x-3)) \leq x^2 + (x-3)^2$$

for all $x \neq 0$. Because $\lim_{x\to 3} x^2 - (x-3)^2 = \lim_{x\to 3} x^2 + (x-3)^2 = 9$, we have by the squeeze theorem that $\lim_{x\to 3} x^2 + (x-3)^2 \sin(1/(x-3)) = 9$.

(Answer 20)
(a) $\lim_{x\to a^+} f(x) = \infty$ for $a = 4$.
(b) $\lim_{x\to a^+} f(x) = -\infty$ for $a = 2$ and $a = 3$.
(c) $\lim_{x\to a^-} f(x) = \infty$ for $a = 2$.
(d) $\lim_{x\to a^-} f(x) = -\infty$ for $a = 3$ and $a = 4$.
(e) There are no numbers a such that $\lim_{x\to a} f(x) = \infty$.
(f) $\lim_{x\to a^-} f(x) = -\infty$ for $a = 3$.

(Answer 41)
(a) $\lim_{x\to 0} g(x) = 1$.
(b) $\lim_{x\to 1} f(x) = 0$.
(c) The limit does not exist.
(d) $\lim_{x\to 0} g(f(x)) = 1$.

(Answer 44) Suppose that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, if $a < x < a + \delta$, then $f(x)$ exists and $|f(x) - L| < \varepsilon$. Then we say that $\lim_{x\to a^+} f(x) = L$.

(Answer 45) Suppose that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that, if $a - \delta < x < a$, then $f(x)$ exists and $|f(x) - L| < \varepsilon$. Then we say that $\lim_{x\to a^-} f(x) = L$.

(Answer 46) Suppose that for every $M > 0$ there exists a $\delta > 0$ such that, if $0 < |x - a| < \delta$, then $f(x)$ exists and $f(x) > M$. Then we say that $\lim\limits_{x\to a} f(x) = \infty$.

(Answer 47) Suppose that for every $M > 0$ there exists a $\delta > 0$ such that, if $a < x < a + \delta$, then $f(x)$ exists and $f(x) > M$. Then we say that $\lim\limits_{x\to a^+} f(x) = \infty$.

(Answer 48) Suppose that for every $M > 0$ there exists a $\delta > 0$ such that, if $a - \delta < x < a$, then $f(x)$ exists and $f(x) > M$. Then we say that $\lim\limits_{x\to a^-} f(x) = \infty$.

(Answer 49) Suppose that for every $M > 0$ there exists a $\delta > 0$ such that, if $0 < |x - a| < \delta$, then $f(x)$ exists and $f(x) < -M$. Then we say that $\lim\limits_{x\to a} f(x) = -\infty$.

(Answer 50) Suppose that for every $M > 0$ there exists a $\delta > 0$ such that, if $a < x < a + \delta$, then $f(x)$ exists and $f(x) < -M$. Then we say that $\lim\limits_{x\to a^+} f(x) = -\infty$.

(Answer 51) Suppose that for every $M > 0$ there exists a $\delta > 0$ such that, if $a - \delta < x < a$, then $f(x)$ exists and $f(x) < -M$. Then we say that $\lim\limits_{x\to a^-} f(x) = -\infty$.

(Answer 52) Suppose that for every $\varepsilon > 0$ there exists a $N > 0$ such that, if $x > N$, then $f(x)$ exists and $|f(x) - L| < \varepsilon$. Then we say that $\lim\limits_{x\to\infty} f(x) = L$.

(Answer 53) Suppose that for every $\varepsilon > 0$ there exists a $N > 0$ such that, if $x < -N$, then $f(x)$ exists and $|f(x) - L| < \varepsilon$. Then we say that $\lim\limits_{x\to-\infty} f(x) = L$.

(Answer 54) Suppose that for every $M > 0$ there exists a $N > 0$ such that, if $x > N$, then $f(x)$ exists and $f(x) > M$. Then we say that $\lim\limits_{x\to\infty} f(x) = \infty$.

(Answer 55) Suppose that for every $M > 0$ there exists a $N > 0$ such that, if $x > N$, then $f(x)$ exists and $f(x) < -M$. Then we say that $\lim\limits_{x\to\infty} f(x) = -\infty$.

(Answer 56) Suppose that for every $M > 0$ there exists a $N > 0$ such that, if $x < -N$, then $f(x)$ exists and $f(x) > M$. Then we say that $\lim\limits_{x\to-\infty} f(x) = \infty$.

(Answer 57) Suppose that for every $M > 0$ there exists a $N > 0$ such that, if $x < -N$, then $f(x)$ exists and $f(x) < -M$. Then we say that $\lim\limits_{x\to-\infty} f(x) = -\infty$.

(Answer 58) We want to find a δ such that, if $|x - 3| < \delta$, then $|x^2 - 9| < 1$.

We know that if $-1 < x^2 - 9 < 1$, then $|x^2 - 9| < 1$.

Adding 9 to all sides, we see that if $8 < x^2 < 10$, then $|x^2 - 9| < 1$.

The function $f(x) = x^2$ is increasing when $x > 0$. Thus, if $\sqrt{8} < x < \sqrt{10}$, then $8 < x^2 < 10$.

Subtracting 3, we see that if $\sqrt{8} - 3 < x - 3 < \sqrt{10} - 3$, then $|x^2 - 9| < 1$.

If $|x - 3| < \min(\sqrt{10} - 3, 3 - \sqrt{8})$, then $\sqrt{8} - 3 < x - 3 < \sqrt{10} - 3$ and so $|x^2 - 9| < 1$.

Thus, $\boxed{\delta = \min(\sqrt{10} - 3, 3 - \sqrt{8})}$ is a solution.

(If you have a calculator, you can compute $\sqrt{10} - 3 = 0.16228$ and $3 - \sqrt{8} = 0.17157$, so $\delta = \sqrt{10} - 3 = 0.16228$.)

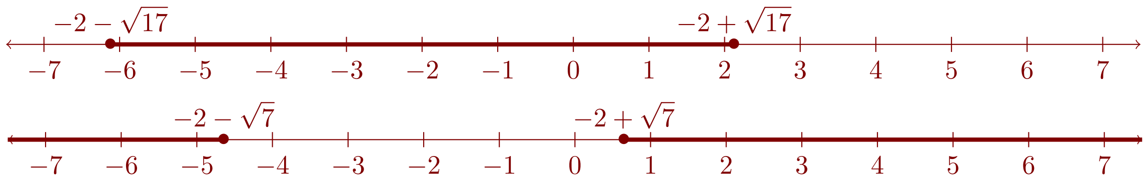
(Answer 59) We want to find a δ such that, if $|x-2|<\delta$, then $|x^2+4x-8|<5$.

We know that if $-5<x^2+4x-8<5$, then $|x^2+4x-8|<5$.

Thus, we need both that $x^2+4x-13<0$ and $x^2+4x-3>0$.

Factoring (and using the quadratic formula), we see that we need $(x+2-\sqrt{13})(x+2+\sqrt{17})<0$ and $(x+2-\sqrt{11})(x+2+\sqrt{7})>0$.

Thus, we need $-2-\sqrt{17}<x<-2+\sqrt{17}$ and either $x>-2+\sqrt{7}$ or $x<-2-\sqrt{7}$.



Thus, we need either $-2-\sqrt{17}<x<-2-\sqrt{7}$ or $-2+\sqrt{7}<x<-2+\sqrt{17}$.

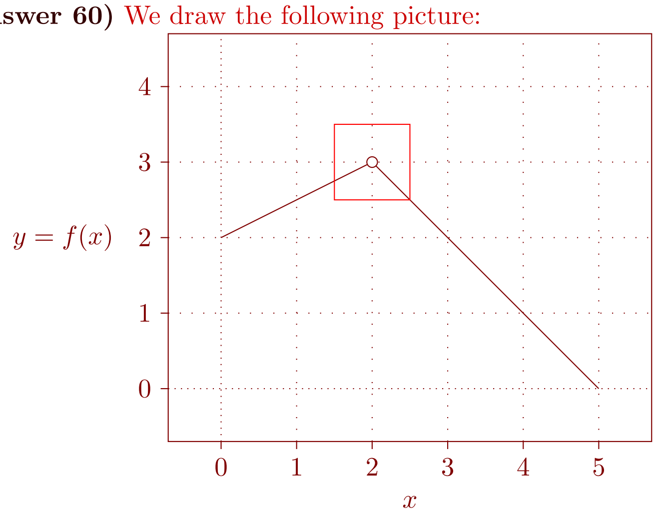
Notice that $2=-2+\sqrt{16}$, so $-2+\sqrt{7}<2<-2+\sqrt{17}$; thus, we care about the interval $-2+\sqrt{7}<x<-2+\sqrt{17}$.

Subtracting 2, we see that we need $-4+\sqrt{7}<x-2<-4+\sqrt{17}$.

Thus, if $|x-2|<\min(\sqrt{17}-4, 4-\sqrt{7})$, then $-4+\sqrt{7}<x-2<-4+\sqrt{17}$ and so $|x^2+4x-8|<5$.

Thus, $\boxed{\delta=\min(\sqrt{17}-4,4-\sqrt{7})}$ is a solution.

(Answer 60) We draw the following picture:



Observe that the box has width 1; thus, $\boxed{\delta=1/2}$ is a solution.

(Answer 61) I choose $\varepsilon=1/2$. Let $\delta>0$. Let N be an integer larger than $\dfrac{1}{\pi\delta}$, so

$$\frac{1}{\pi\delta}<N.$$

Multiplying both sides of the inequality by the positive number δ/N, we see that

$$\frac{1}{N\pi}<\delta.$$

Let $x=1/(N\pi)$. Then $0<x<\delta$ and so $0<|x-0|<\delta$. We have that $f(x)=\cos(1/x)=\cos(N\pi)=\pm1$. Then $|f(x)-0|=|\pm1|=1>0.5=\varepsilon$, and so 0 is not $\lim_{x\to0}f(x)$.

(Answer 62) I choose $\varepsilon = \pi/2$. Let $\delta > 0$. Let x satisfy $-\delta < x < 0$. Then $0 < |x - 0| < \delta$. We have that $f(x) = \arctan(1/x)$, and $\arctan(z) < 0$ whenever $z < 0$. So $|f(x) - \pi/2| = \pi/2 - \arctan(1/x) \geq \pi/2 = \varepsilon$, and so $\pi/2$ is not $\lim_{x\to 0} f(x)$.

(Answer 63) I choose $\varepsilon = 1/2$. Let $\delta > 0$. Let x satisfy $0 < x < \delta$. Then $0 < |x - 0| < \delta$. Furthermore, $1/x > 0$, so $2^{1/x} > 1$ and $1 + 2^{1/x} > 2$ and $f(x) = \dfrac{1}{1+2^{1/x}} < \dfrac{1}{2}$. Thus, $f(x) - 1 < -1/2$ and so $|f(x) - 1| > 1/2 = \varepsilon$, and so 1 is not $\lim_{x\to 0} f(x)$.