Graded and supplementary homework, Math 2584, Section 4, Fall 2017

(AB 1)

(a) Is $y = \cos(2x)$ a solution to the differential equation $\dfrac{d^2y}{dx^2} + 4y = 0$?

(b) Is $y = e^{2x}$ a solution to the differential equation $\dfrac{d^2y}{dx^2} + 4y = 0$?

(c) Is $y = \sin(2x)$ a solution to the differential equation $\dfrac{d^2y}{dx^2} + 4y = 0$?

(AB 2) For each of the following initial-value problems, tell me whether you expect there to be a unique solution to the initial-value problem. If not, tell me whether you expect (i) infinitely many solutions to the initial-value problem, or (ii) no solutions to the initial-value problem. (You don't have to solve any of these problems.)

(a) $\dfrac{d^3y}{dt^3} + \cos t \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + t^2 y = 0$, $y(0) = 1$, $y'(0) = 2$.

(b) $\dfrac{dy}{dt} + \sin(t)y = \cos(t)$, $y(0) = 1$, $y'(0) = 2$.

(c) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + \sin(t)y = e^t$, $y(0) = 1$, $y'(0) = 2$.

(AB 3) For each of the following initial-value problems, tell me whether you expect there to be a unique solution to the initial-value problem. If not, tell me whether you expect (i) infinitely many solutions to the initial-value problem, or (ii) no solutions to the initial-value problem. (You don't have to solve any of these problems.)

(a) $\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + t^2 y = t^3$, $y(0) = 3$.

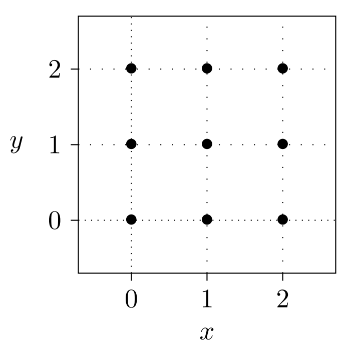
(b) $\dfrac{d^2y}{dt^2} + e^t \dfrac{dy}{dt} - e^{3t} y = \cos(t)$, $y(0) = 4$, $y'(0) = 3$.

(c) $\dfrac{d^2y}{dt^2} + t^2 \dfrac{dy}{dt} + \sin(t)y = \dfrac{1}{1+t^2}$, $y(0) = 5$, $y'(0) = 4$, $y''(0) = 3$.

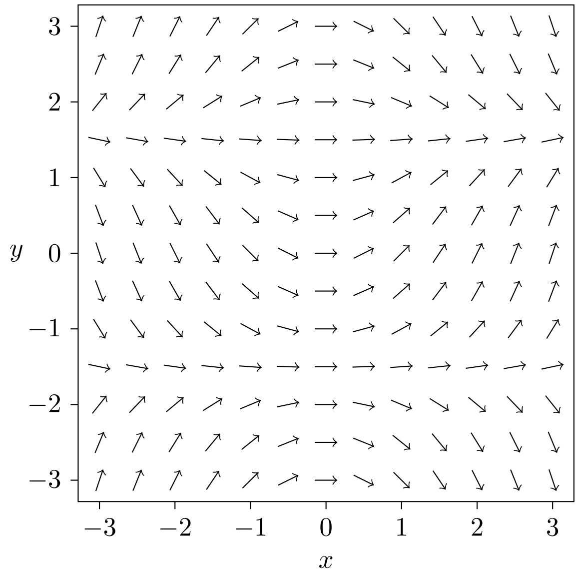
(AB 4) I borrow \$18,000 to buy a car. The dealer charges an interest rate of 4% per year. I plan to pay off my car at a constant rate to be determined.

Set up the differential equation and initial conditions that describe how much money I owe. Be sure to define your independent and dependent variables, as well as any unknown parameters, and to include units.

(AB 5) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\dfrac{dy}{dx} = 1 - x$.



(AB 6) Consider the differential equation $\dfrac{dy}{dx} = x\cos(y)$. Here is the direction field for this differential equation.



Sketch the solution to the initial-value problem

$$\frac{dy}{dx} = x\cos(y), \qquad y(-2) = 0.$$

(AB 7) Recall that Newton's law of cooling states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and that of its surroundings. Suppose that a thermometer, initially at temperature 0° C, is taken into a warm room. After 1 minute, the thermometer's temperature is 10° C. After 2 minutes, the thermometer reads 16°. What is the ambient temperature of the room?

(AB 8) Solve the initial value problem

$$\sin x \frac{dy}{dx} = y + \sin^2 x, \quad y(\pi/2) = 1$$

Give the largest interval I over which the solution is defined.

(AB 9) Solve the initial value problem

$$\frac{dy}{dx} = \frac{y}{x(1 + \ln x - \ln y)}, \quad y(e^2) = e.$$

(AB 10) Solve the initial value problem

$$(1 + t^2)\frac{dy}{dt} = y(y^4 + t), \quad y(0) = \frac{1}{2}.$$

Give the largest interval over which the solution is defined. Express the endpoints of the interval as a decimal with four significant figures.

(AB 11) A spherical snowball is taken into a warm room. The snowball melts at a rate proportional to its surface area. Initially the snowball has a mass of 27 grams. After 10 minutes, the snowball has a mass of 8 grams. When will the snowball be completely melted?

(AB 12) The function $y_1(t) = t^3$ is a particular solution to the differential equation $t^2\dfrac{d^2y}{dt^2} - 6t\dfrac{dy}{dt} + 12y = 0$. Solve the initial value problem

$$t^2\frac{d^2y}{dt^2} - 6t\frac{dy}{dt} + 12y = 6t^2, \quad y(1) = 6, \quad y'(1) = 13.$$

(AB 13) The function $y_1(t) = e^{3t}$ is a particular solution to the differential equation $t\dfrac{d^2y}{dt^2} - (1+3t)\dfrac{dy}{dt} + 3y = 0$. Find the general solution.

(AB 14) Solve the initial value problem

$$3\frac{d^2y}{dt^2} - 14\frac{dy}{dt} + 8y = 0, \quad y(0) = 1, \quad y'(0) = -16.$$

(AB 15) Solve the initial value problem

$$\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 8y = 0, \quad y(0) = 1, \quad y'(0) = 15, \quad y''(0) = 24.$$

(AB 16) Solve the initial-value problem $5\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 5y = 0$, $y(0) = 2$, $y'(1) = 0$.

(AB 17) Solve the initial-value problem $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 3y = 13\sin 2t + 6e^{3t}$, $y(0) = 0$, $y'(0) = 0$.

(AB 18) Solve the initial-value problem $\dfrac{d^2y}{dt^2} + 8\dfrac{dy}{dt} + 16y = 5t^2e^t + 6e^{-4t}$, $y(0) = 0$, $y'(0) = 1$.

(AB 19) Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-2t}\arctan t.$$

(AB 20) A spring is suspended vertically. When a 30-g object is hung from the spring, its weight stretches the spring to an equilibrium position 8mm lower. The object is also attached to a viscous damper with damping constant 7.5 newton·seconds/meter. The mass is pulled down to 3cm below equilibrium and released. Formulate the initial value problem for the object's position. Then solve the initial value problem and find the object's position at any time.

(AB 21) A spring is suspended vertically. When a 2-lb object is hung from the spring, its weight stretches the spring to an equilibrium position 1.5in lower. The object is also attached to a viscous damper with damping constant β. The object is set in motion from its equilibrium position with initial velocity 3 feet/second upwards.

(a) Formulate the initial value problem for the object's position.

(b) Find the value of β for which the system is critically damped. Be sure to include units for β.

(AB 22) A spring with constant $k = 12$ N/m is suspended vertically. A 30-g object is hung from the spring. At time t seconds, there is an external force of $7\cos(\omega t)$ newtons, directed upward, acting on the object. There is no damping and the object is initially at rest at equilibrium.

(*a*) Formulate the initial value problem for the object's position.

(*b*) Find the value of ω for which resonance occurs. Be sure to include units for ω.

(AB 23) A particle of mass $m = 2$ kg a distance r from an infinitely long string experiences a force due to gravity of magnitude Gm/r, where $G = 6000$ meters^2/second^2, directed directly toward the string. Suppose that the particle is initially 2000 meters from the string and takes off with initial velocity 90 meters/second directly away from the string.

(*a*) Formulate the initial value problem for the particle's position.

(*b*) Find the velocity of the particle as a function of position.

(*c*) How far away from the string is the particle when it stops moving and starts to fall back?

(AB 24) Use the Laplace transform to solve the initial value problem $\dfrac{dy}{dt} + 3y = e^{2t}$, $y(0) = 1$.

(AB 25) Find $\mathcal{L}\{e^{2t}t^6 + e^{-3t}\sin(4t)\}$.

(AB 26) Use the integral definition (Definition 7.1.1 in your book) of the Laplace transform (not Theorem 7.3.2 in your book) to find $\mathcal{L}\{f(t)\}$, where

$$f(t) = \begin{cases} 0, & 0 \le t < 2, \\ e^{3t}, & 2 \le t < 4, \\ 0, & 4 \le t. \end{cases}$$

(AB 27) Let $f(t) = \begin{cases} 3, & t < 2, \\ 2, & 2 \le t < 4, \\ 0, & 4 \le t. \end{cases}$ Express $f(t)$ in terms of the unit step function $\mathcal{U}$.

(AB 28) Use the Laplace transform to solve the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = f(t), \quad y(0) = 1, \quad y'(0) = 0$$

where $f(t)$ is as in Problem **(AB 27)**.

(AB 29) Use the Laplace transform to solve the initial value problem

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = g(t), \quad y(0) = 0, \quad y'(0) = 3, \quad \text{where} \quad g(t) = \begin{cases} 0, & t < \pi/2, \\ \cos 3t, & \pi/2 \le t. \end{cases}$$

(AB 30) Use the Laplace transform to solve the initial value problem

$$\frac{dy}{dt} + 3y = t\cos 3t, \quad y(0) = 2.$$

Simplify your answer as much as possible.

(AB 31)

(a) Solve the initial value problem $4\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + y = 4e^{-t}\mathcal{U}(t-1) + 4\delta(t-3)$, $y(0) = 1$, $y'(0) = 3$.

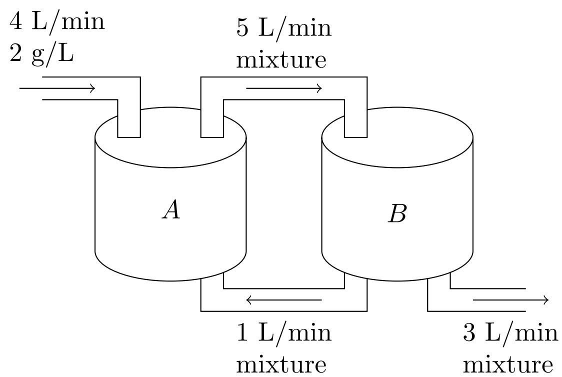
(b) Without plotting $y(t)$, predict the special features you expect the graph of y to display at $t = 1$ and $t = 3$.

(c) What special features do you expect the graph of $\dfrac{dy}{dt}$ to display at $t = 1$ and $t = 3$?

(d) What special features do you expect the graph of $\dfrac{d^2y}{dt^2}$ to display at $t = 1$? What occurs at $t = 3$?

(e) Plot $y(t)$ and $y'(t)$. (It is best to plot them using a computer; print out your plots or sketch the computer plots by hand.) Do you see the features you expected?

(AB 32) Consider the following two tanks.



Initially, tank A contains 200 liters of pure water, and tank B contains 300 liters in which 4 kg of salt has been dissolved. Every minute, 4 liters of water containing 2 grams of salt per liter flows into tank A, and the well-stirred solution flows from tank A to tank B and back and out as shown in the figure.

(a) Define the independent and dependent variables you would like to use to describe this situation. Be sure to include units.

(b) What is the volume of liquid in each tank? (Your answer should be a function of time.)

(c) What are the initial conditions for this situation? (Your answer should be in the form of one or more equations.)

(d) Write the differential equation or equations for the amount of salt in each tank at any time before the tanks are emptied.

(AB 33) Find all eigenvalues and eigenvectors for the matrix $\begin{pmatrix} 0 & 4 & -2 \\ 3 & 2 & -3 \\ 4 & 4 & -6 \end{pmatrix}$.

(AB 34) Solve the initial value problem

$$\frac{dx}{dt} = x - 2y + 2z, \quad \frac{dy}{dt} = 2x + 3y - 4z, \quad \frac{dz}{dt} = 2x + y - 2z, \quad x(0) = 0, \quad y(0) = 1, \quad z(0) = 0.$$

Express your answer in terms of real functions. Simplify your answer as much as possible.

(AB 35) Solve the initial value problem

$$\frac{dx}{dt} = 11x + 18y + 9z, \quad \frac{dy}{dt} = x + 8y + 3z, \quad \frac{dz}{dt} = -6x - 18y - 7z, \quad x(0) = 1, \quad y(0) = 0, \quad z(0) = 0.$$

Express your answer in terms of real functions. Simplify your answer as much as possible.

(AB 36) Solve the initial value problem

$$\frac{dx}{dt} = 2x - y + z, \quad \frac{dy}{dt} = -x + 2y + z, \quad \frac{dz}{dt} = -x - y + 4z, \quad x(0) = 0, \quad y(0) = 0, \quad z(0) = 1.$$

Express your answer in terms of real functions. Simplify your answer as much as possible.

(AB 37) Solve the initial value problem

$$\frac{dx}{dt} = 5y - 6x, \quad \frac{dy}{dt} = -16x + 10y, \quad x(0) = 1, \quad y(0) = 0.$$

Express your answer in terms of real functions. Simplify your answer as much as possible.

Answer key

(AB 1)

(a) Is $y = \cos(2x)$ a solution to the differential equation $\dfrac{d^2y}{dx^2} + 4y = 0$?
Yes.

(b) Is $y = e^{2x}$ a solution to the differential equation $\dfrac{d^2y}{dx^2} + 4y = 0$?
No.

(c) Is $y = \sin(2x)$ a solution to the differential equation $\dfrac{d^2y}{dx^2} + 4y = 0$?
Yes.

(AB 2) For each of the following initial-value problems, tell me whether you expect there to be a unique solution to the initial-value problem. If not, tell me whether you expect (i) infinitely many solutions to the initial-value problem, or (ii) no solutions to the initial-value problem. (You don't have to solve any of these problems.)

(a) $\dfrac{d^3y}{dt^3} + \cos t\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + t^2y = 0$, $y(0) = 1$, $y'(0) = 2$.
There are infinitely many solutions to the initial value problem.

(b) $\dfrac{dy}{dt} + \sin(t)y = \cos(t)$, $y(0) = 1$, $y'(0) = 2$.
There are no solutions to the initial value problem; if $y(0) = 1$ and $y'(0) = 2$, then $y'(0)+\sin(0)y(0) = 2 \neq 1 = \cos(0)$ and so the differential equation cannot be satisfied at $t = 0$.

(c) $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + \sin(t)y = e^t$, $y(0) = 1$, $y'(0) = 2$.
There is a unique solution to the initial value problem.

(AB 3) For each of the following initial-value problems, tell me whether you expect there to be a unique solution to the initial-value problem. If not, tell me whether you expect (i) infinitely many solutions to the initial-value problem, or (ii) no solutions to the initial-value problem. (You don't have to solve any of these problems.)

(a) $\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + t^2y = t^3$, $y(0) = 3$.
There are infinitely many solutions to the initial value problem.

(b) $\dfrac{d^2y}{dt^2} + e^t\dfrac{dy}{dt} - e^{3t}y = \cos(t)$, $y(0) = 4$, $y'(0) = 3$.
There is a unique solution to the initial value problem.

(c) $\dfrac{d^2y}{dt^2} + t^2\dfrac{dy}{dt} + \sin(t)y = \dfrac{1}{1+t^2}$, $y(0) = 5$, $y'(0) = 4$, $y''(0) = 3$.
There are no solutions to the initial value problem; if $y(0) = 5$, $y'(0) = 4$ and $y''(0) = 3$, then $y''(0) + 0^2y'(0) + \sin(0)y(0) = 3 \neq 1 = \dfrac{1}{1+0^2}$ and so the differential equation cannot be satisfied at $t = 0$.

(AB 4) I borrow \$18,000 to buy a car. The dealer charges an interest rate of 4% per year. I plan to pay off my car at a constant rate to be determined.

Set up the differential equation and initial conditions that describe how much money I owe. Be sure to define your independent and dependent variables, as well as any unknown parameters, and to include units.

(Answer 4)

Independent variable: t = time (in years)

Dependent variable: B = amount of money owed (in dollars)

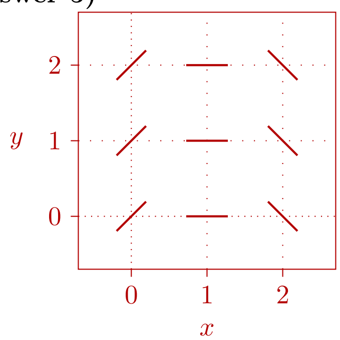
Parameter: k = payment rate (in dollars per month)

Initial condition: $B(0) = 18{,}000$

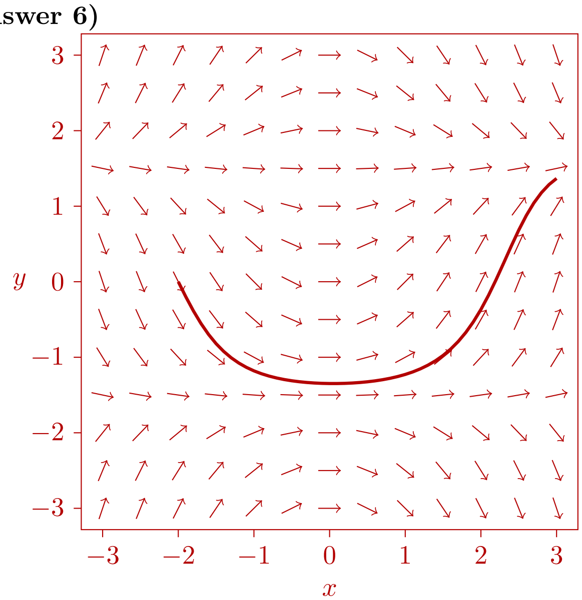
Differential equation: $\dfrac{dB}{dt} = 0.04B - 12k$

(AB 5) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\dfrac{dy}{dx} = 1 - x$.

(Answer 5)



(Answer 6)



(AB 7) Recall that Newton's law of cooling states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and that of its surroundings. Suppose that a thermometer, initially at temperature 0° C, is taken into a warm room. After 1 minute, the thermometer's temperature is 10° C. After 2 minutes, the thermometer reads 16°. What is the ambient temperature of the room?

(Answer 7) Independent variable: t = time (in minutes)
Dependent variable: T = temperature of the thermometer (in degrees C)
Parameters: α = proportionality constant (in $1/minutes$), R = temperature of the room (in degrees C)
Differential equation: $\frac{dT}{dt} = -\alpha(T - R)$
Conditions: $T(0) = 0$, $T(1) = 16$, $T(2) = 20$
Solution to differential equation: $T(t) = R + Ce^{-\alpha t}$;
Using the conditions $T(0) = 0$, $T(1) = 10$ and $T(2) = 16$, we find that $\alpha = \ln 5/3$, $C = -25$ and $R = 25$.
Thus the ambient temperature is $\boxed{25^\circ.}$

(AB 8) Solve the initial value problem

$$\sin x \frac{dy}{dx} = y + \sin^2 x, \quad y(\pi/2) = 1$$

Give the largest interval I over which the solution is defined.

(Answer 8) $y = \frac{x + \sin x - (\pi/2)}{\csc x + \cot x}$, defined for $0 < x < \pi$.

(AB 9) Solve the initial value problem

$$\frac{dy}{dx} = \frac{y}{x(1 + \ln x - \ln y)}, \quad y(e^2) = e.$$

(Answer 9) $\ln|\ln|y/x|| - \ln|y/x| = \ln|x| - 1$.

(AB 10) Solve the initial value problem

$$(1 + t^2)\frac{dy}{dt} = y(y^4 + t), \quad y(0) = \frac{1}{2}.$$

Give the largest interval over which the solution is defined. Express the endpoints of the interval as a decimal with four significant figures.

(Answer 10)

$$\frac{(1 + t^2)^2}{y^4} = -4t - \frac{4}{3}t^3 + 16, \quad -\infty < t < 1.859.$$

(AB 11) A spherical snowball is taken into a warm room. The snowball melts at a rate proportional to its surface area. Initially the snowball has a mass of 27 grams. After 10 minutes, the snowball has a mass of 8 grams. When will the snowball be completely melted?

(Answer 11) The snowball's mass after t minutes is $M(t) = (3 - t/10)^3$ grams. Thus, the mass will be zero when $\boxed{t = 30}$ minutes.

(AB 12) The function $y_1(t) = t^3$ is a particular solution to the differential equation $t^2\frac{d^2y}{dt^2} - 6t\frac{dy}{dt} + 12y = 0$. Solve the initial value problem

$$t^2\frac{d^2y}{dt^2} - 6t\frac{dy}{dt} + 12y = 6t^2, \quad y(1) = 6, \quad y'(1) = 13.$$

(Answer 12) $y(t) = 3t^2 + 5t^3 - 2t^4$.

(AB 13) The function $y_1(t) = e^{3t}$ is a particular solution to the differential equation $t\dfrac{d^2y}{dt^2} - (1+3t)\dfrac{dy}{dt} + 3y = 0$. Find the general solution.

(Answer 13) $y(t) = C_1(3t+1) + C_2 e^{3t}$.

(AB 14) Solve the initial value problem

$$3\frac{d^2y}{dt^2} - 14\frac{dy}{dt} + 8y = 0, \quad y(0) = 1, \quad y'(0) = -16.$$

(Answer 14) $y(t) = 6e^{(2/3)t} - 5e^{4t}$

(AB 15) Solve the initial value problem

$$\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 8y = 0, \quad y(0) = 1, \quad y'(0) = 15, \quad y''(0) = 24.$$

(Answer 15)

$$y(t) = 3e^{2t} + 5te^{2t} - 2e^{-2t}.$$

If you did not take into account the third initial condition, we would expect the answer

$$y(t) = C_1 e^{2t} + C_2 te^{2t} + C_3 e^{-2t}, \quad \text{where} \quad C_1 + C_3 = 1 \text{ and } 2C_1 + C_2 - 2C_3 = 15.$$

(AB 16) Solve the initial-value problem $5\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 5y = 0$, $y(0) = 2$, $y'(1) = 0$.

(Answer 16)

$$y(t) = 2e^{-(3/5)t}\cos\frac{4t}{5} + \frac{6\cos(4/5) + 8\sin(4/5)}{4\cos(4/5) - 3\sin(4/5)}e^{-(3/5)t}\sin\frac{4t}{5} = 2e^{-(3/5)t}\cos\frac{4t}{5} + 15.6266e^{-(3/5)t}\sin\frac{4t}{5}.$$

(AB 17) Solve the initial-value problem $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 3y = 13\sin 2t + 6e^{3t}$, $y(0) = 0$, $y'(0) = 0$.

(Answer 17) $y(t) = 3te^{3t} - \dfrac{1}{2}e^{3t} - \dfrac{11}{10}e^t + \dfrac{8}{5}\cos 2t - \dfrac{1}{5}\sin 2t$

(AB 18) Solve the initial-value problem $\dfrac{d^2y}{dt^2} + 8\dfrac{dy}{dt} + 16y = 5t^2e^t + 6e^{-4t}$, $y(0) = 0$, $y'(0) = 1$.

(Answer 18) $y = 3t^2e^{-4t} + \dfrac{1}{5}t^2e^t - \dfrac{1}{100}te^t - \dfrac{3}{250}e^t + \dfrac{3}{250}e^{-4t} + \dfrac{107}{100}te^{-4t}$

(AB 19) Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-2t}\arctan t.$$

(Answer 19) $y = \frac{1}{2}(t^2 \arctan t - t\ln(1+t^2) - t + \arctan t)e^{-2t} + C_1 e^{-2t} + C_2 t e^{-2t}$.

(AB 20) A spring is suspended vertically. When a 30-g object is hung from the spring, its weight stretches the spring to an equilibrium position 8mm lower. The object is also attached to a viscous damper with damping constant 7.5 newton·seconds/meter. The mass is pulled down to 3cm below equilibrium and released. Formulate the initial value problem for the object's position. Then solve the initial value problem and find the object's position at any time.

(Answer 20) The initial value problem is

$$0.03\frac{d^2x}{dt^2} = -7.5\frac{dx}{dt} - \frac{9.8\cdot 0.03}{0.008}x, \quad x(0) = -0.03, \quad x'(0) = 0$$

where x denotes the displacement above weighted equilibrium in meters and t denotes time in seconds.

The object's position at time t is

$$x(t) = -\frac{49}{1600}e^{-5t} + \frac{1}{1600}e^{-245t}.$$

(AB 21) A spring is suspended vertically. When a 2-lb object is hung from the spring, its weight stretches the spring to an equilibrium position 1.5in lower. The object is also attached to a viscous damper with damping constant β. The object is set in motion from its equilibrium position with initial velocity 3 feet/second upwards.

(a) Formulate the initial value problem for the object's position.
The initial value problem is

$$\frac{2}{32}\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - 16x, \quad x(0) = 0, \quad x'(0) = 3$$

where x denotes the displacement above weighted equilibrium in feet and t denotes time in seconds.

(b) Find the value of β for which the system is critically damped. Be sure to include units for β.
Critical damping occurs when $\beta = 2$ lb · sec/foot.

(AB 22) A spring with constant $k = 12$ N/m is suspended vertically. A 30-g object is hung from the spring. At time t seconds, there is an external force of $7\cos(\omega t)$ newtons, directed upward, acting on the object. There is no damping and the object is initially at rest at equilibrium.

(a) Formulate the initial value problem for the object's position.
The initial value problem is

$$0.03\frac{d^2x}{dt^2} = -12x + 7\cos(\omega t), \quad x(0) = 0, \quad x'(0) = 0$$

where x denotes the displacement above weighted equilibrium in meters and t denotes time in seconds.

(b) Find the value of ω for which resonance occurs. Be sure to include units for ω.
Resonance occurs when $\omega = 20\ \text{sec}^{-1}$.

(AB 23) A particle of mass $m = 2$ kg a distance r from an infinitely long string experiences a force due to gravity of magnitude Gm/r, where $G = 6000$ meters^2/second^2, directed directly toward the string. Suppose that the particle is initially 2000 meters from the string and takes off with initial velocity 90 meters/second directly away from the string.

(a) Formulate the initial value problem for the particle's position.
The initial value problem is

$$2\frac{d^2r}{dt^2} = -\frac{12000}{r}, \quad r(0) = 2000, \quad r'(0) = 90$$

where r denotes the distance to the string in meters and t denotes time in seconds.

(b) Find the velocity of the particle as a function of position.
Let v be the particle's velocity in meters/second. We have that

$$2v\frac{dv}{dr} = -\frac{12000}{r}, \quad v(2000) = 90$$

and so

$$v^2 = -12000\ln r + 8100 + 12000\ln(2000) = -12000\ln r + 99310.8.$$

(c) How far away from the string is the particle when it stops moving and starts to fall back?
$v = 0$ when $r = 2000e^{81/120} = 3928$ meters.

(AB 24) Use the Laplace transform to solve the initial value problem $\dfrac{dy}{dt} + 3y = e^{2t}$, $y(0) = 1$.

(Answer 24) $\mathcal{L}\{y\} = \dfrac{s-1}{(s-2)(s+3)}$, so $y(t) = \dfrac{1}{5}e^{2t} + \dfrac{4}{5}e^{-3t}$.

(AB 25) Find $\mathcal{L}\{e^{2t}t^6 + e^{-3t}\sin(4t)\}$.

(Answer 25) $\mathcal{L}\{e^{2t}t^6 + e^{-3t}\sin(4t)\} = \dfrac{720}{(s-2)^7} + \dfrac{4}{(s+3)^2+16}$.

(AB 26) Use the integral definition (Definition 7.1.1 in your book) of the Laplace transform (not Theorem 7.3.2 in your book) to find $\mathcal{L}\{f(t)\}$, where

$$f(t) = \begin{cases} 0, & 0 \le t < 2, \\ e^{3t}, & 2 \le t < 4, \\ 0, & 4 \le t. \end{cases}$$

(Answer 26)

$$\mathcal{L}\{f(t)\} = \int_2^4 e^{-st}e^{3t}\,dt = \frac{e^{-4s+12} - e^{-2s+6}}{3-s}.$$

(AB 27) Let $f(t) = \begin{cases} 3, & t < 2, \\ 2, & 2 \le t < 4, \\ 0, & 4 \le t. \end{cases}$ Express $f(t)$ in terms of the unit step function $\mathcal{U}$.

(Answer 27) $f(t) = 3 - \mathcal{U}(t-2) - 2\mathcal{U}(t-4)$.

(AB 28) Use the Laplace transform to solve the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = f(t), \quad y(0) = 1, \quad y'(0) = 0$$

where $f(t)$ is as in Problem **(AB 37)**.

(Answer 28) $\mathcal{L}\{y\} = \dfrac{s+2}{s^2+2s+5} + \dfrac{3 - e^{-2s} - 2e^{-4s}}{s(s^2+2s+5)}$, so

$$\begin{aligned} y(t) &= \frac{3}{5} + \frac{2}{5}e^{-t}\cos(2t) + \frac{1}{5}e^{-t}\sin(2t) \\ &\quad + \mathcal{U}(t-2)\left(-\frac{1}{5} + \frac{1}{5}e^{-t}\cos(2t) + \frac{1}{10}e^{-t}\sin(2t)\right) \\ &\quad + \mathcal{U}(t-4)\left(-\frac{2}{5} + \frac{2}{5}e^{-t}\cos(2t) + \frac{1}{5}e^{-t}\sin(2t)\right). \end{aligned}$$

(AB 29) Use the Laplace transform to solve the initial value problem

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = g(t), \quad y(0) = 0, \quad y'(0) = 3, \quad \text{where} \quad g(t) = \begin{cases} 0, & t < \pi/2, \\ \cos 3t, & \pi/2 \le t. \end{cases}$$

(Answer 29) $\mathcal{L}\{y\} = \dfrac{3}{s^2+5s+4} - \dfrac{3e^{-\pi s/2}}{(s^2+9)(s^2+5s+4)}$. Thus,

$$y(t) = e^{-t} - e^{-4t} + \mathcal{U}(t-\pi/2)\left(\frac{1}{10}e^{\pi/2-t} - \frac{1}{25}e^{2\pi-4t} - \frac{3}{50}\cos(3t-3\pi/2) - \frac{1}{50}\sin(3t-3\pi/2)\right)$$