

Math 2584, Fall 2017

You are allowed a double-sided, 3 inch by 5 inch card of notes. You are responsible for all of the formulas you will need, except for the following Laplace transforms, which will be written on the last page of the exam.

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0, \quad n \geq 0$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2 + k^2}, \quad s > 0$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2 + k^2}, \quad s > 0$$

If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$,

$$\mathcal{L}^{-1}\{F(s)\} = e^{at} \mathcal{L}^{-1}\{F(s+a)\},$$

$$\mathcal{L}\{\mathcal{U}(t-c)\} = \frac{e^{-cs}}{s}, \quad s > 0, \quad c \geq 0$$

$$\mathcal{L}\{\delta(t-c)\} = e^{-cs}, \quad s > 0, \quad c \geq 0$$

$$\mathcal{L}\{\mathcal{U}(t-c)f(t)\} = e^{-cs} \mathcal{L}\{f(t+c)\}, \quad c \geq 0$$

$$\mathcal{L}\{\mathcal{U}(t-c)g(t-c)\} = e^{-cs} \mathcal{L}\{g(t)\}, \quad c \geq 0$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s \mathcal{L}\{y(t)\} - y(0)$$

$$\mathcal{L}\{t f(t)\} = -\frac{d}{ds} \mathcal{L}\{f(t)\}$$

$$\mathcal{L}\left\{\int_0^t f(r)g(t-r) dr\right\} = \mathcal{L}\left\{\int_0^t f(t-r)g(r) dr\right\} = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\}$$

(AB 1) Is $y = e^t$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t} \frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 2) Is $y = e^{2t}$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t} \frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 3) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y, y(0) = 1$?

(AB 4) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y, y(0) = 2$?

(AB 5) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y, y(0) = 1$?

(AB 6) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y, y(0) = 2$?

(AB 7) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution on some interval. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + \arctan(t)y = e^t$, $y(3) = 7$.

(b) $\frac{dy}{dt} - t^5y = \cos(6t)$, $y(2) = -1$, $y'(2) = 3$.

(c) $\frac{d^2y}{dt^2} - 5\sin(t)y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.

(d) $\frac{d^2y}{dt^2} + 3(t-4)\frac{dy}{dt} + 4t^6y = 2$, $y(3) = 1$, $y'(3) = -1$.

(e) $\frac{d^2y}{dt^2} + \cos(t)\frac{dy}{dt} + 3\ln(1+t^2)y = 0$, $y(2) = 3$.

(AB 8) A lake initially has a population of 600 trout. The birth rate of trout is proportional to the number of trout living in the lake. Fishermen are allowed to harvest 30 trout/year from the lake. Write a differential equation for the fish population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 9) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write a differential equation for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 10) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C . Set up, but **do not solve**, the differential equation and initial conditions that describe the temperature of a cup of coffee. Be sure to define your independent and dependent variables, as well as all unknown parameters, and be sure to include units.

(AB 11) Suppose that an isolated town has 300 households. In 1920, two families install telephones in their homes. Write a differential equation for the number of telephones in the town if the rate at which families buy telephones is jointly proportional to the number of households with telephones and the number of households without telephones. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 12) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Set up, but **do not solve**, the differential equation and initial conditions that describe the amount of salt in the tank. Be sure to define your independent and dependent variables, as well as all unknown parameters, and be sure to include units.

(AB 13) I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write a differential equation for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 14) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write a differential equation for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 15) According to the Stefan-Boltzmann law, a black body in a dark vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write a differential equation for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

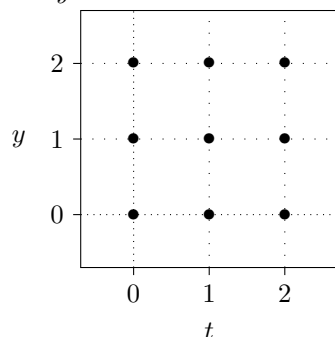
(AB 16) According to the Stefan-Boltzmann law, a black body in a vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). If there is some ambient light in the vacuum, then the black body absorbs energy at a rate proportional to the effective temperature of the light. The proportionality constant is the same in both cases; that is, if the object's temperature is equal to the effective temperature of the light, then its temperature will not change.

Suppose that a planet in space is currently at a temperature of 400 K. The effective temperature of its surroundings is 3 K. Write a differential equation for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 17) A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write a differential equation for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

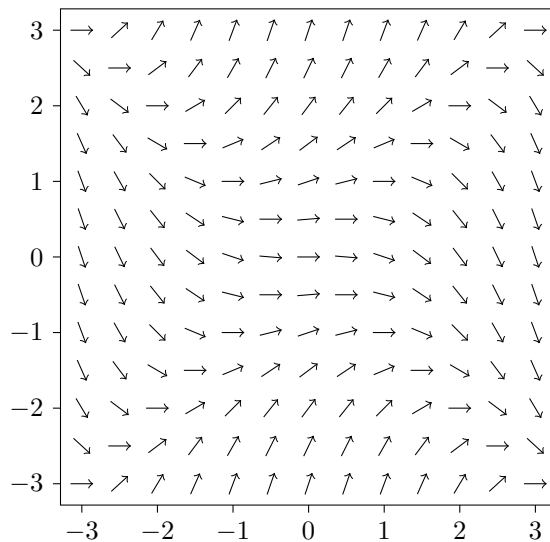
(AB 18) A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write a differential equation for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 19) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.

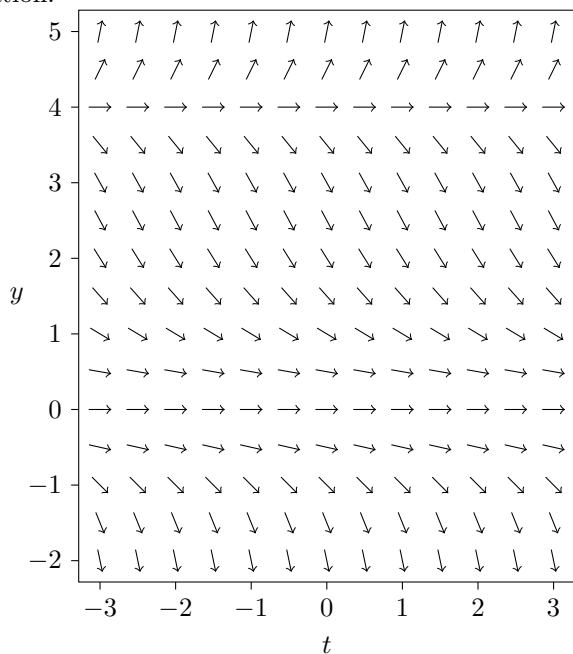


(AB 20) Consider the differential equation $\frac{dy}{dx} = (y^2 - x^2)/3$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-3) = 0.$$



(AB 21) Consider the differential equation $\frac{dy}{dt} = y^2(y - 4)/5$. Here is the direction field for this differential equation.



Based on the direction field, determine the behavior of y as $t \rightarrow \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency.

(AB 22) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y - 2)(y + 1)^2$. By hand, sketch some typical solutions.

(AB 23) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 24) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = y^2(y - 2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 25) I owe my bank a debt of B dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

- Formulate a differential equation for the amount of money I owe.
- Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

(AB 26) A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

- Write the differential equation for the amount of salt in the tank.
- Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

(AB 27) A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where v is her velocity in meters per second.

- Write a differential equation for her velocity. Assume her velocity is always negative.
- Find the (negative) critical points of this differential equation. Be sure to include units.
- What is the real-world meaning of these critical points?

(AB 28) Suppose that the population y of catfish in a certain lake, absent human intervention, satisfies the logistic equation $\frac{dy}{dt} = ry(1 - y/K)$, where r and K are constants and t denotes time.

- Assuming that $r > 0$ and $K > 0$, find the critical points of this equation and classify them as to stability.
- Suppose that humans intervene by harvesting fish at a rate proportional to the fish population. Write a new differential equation for the fish population. Find the critical points of your new equation and classify them as to stability.
- Suppose that humans intervene by harvesting fish at a constant rate. Write a new differential equation for the fish population. Find the critical points of your new equation and classify them as to stability.

(AB 29) For each of the following differential equations, determine whether it is linear, separable, homogeneous, a Bernoulli equation, or of the form $\frac{dy}{dt} = f(At + By + C)$. Then solve the differential equation.

- $\frac{dy}{dt} = \frac{t + \cos t}{\sin y - y}$
- $(t^2 + 1)\frac{dy}{dt} = ty - t^2 - 1$
- $t^2\frac{dy}{dt} = y^2 + t^2 - ty$
- $\frac{dy}{dt} = -y \tan 2t - y^3 \cos 2t$
- $(t + y)\frac{dy}{dt} = 5y - 3t$
- $\frac{dy}{dt} = \frac{1}{3t + 2y + 7}$
- $\frac{dy}{dt} = -y^3 \cos(2t)$
- $4ty\frac{dy}{dt} = 3y^2 - 2t^2$
- $t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$
- $\frac{dy}{dt} = \csc^2(y - t)$
- $\frac{dy}{dt} = 8y - y^8$
- $t\frac{dy}{dt} = -\cos t - 3y$
- $\frac{dy}{dt} = \cot(y/t) + y/t$

(AB 30) For each of the following differential equations, determine whether it is linear, separable, or homogeneous. Then solve the given initial-value problem.

(a) $2ty \frac{dy}{dt} = 4t^2 - y^2, y(1) = 3.$

(b) $t \frac{dy}{dt} = -1 - y^2, y(1) = 1$

(c) $\frac{dy}{dt} = (2y + 2t - 5)^2, y(0) = 3.$

(d) $\frac{dy}{dt} = 2y - \frac{6}{y^2}, y(0) = 7.$

(e) $\frac{dy}{dt} = -3y - \sin t e^{-3t}, y(0) = 2$

(AB 31) Solve the initial-value problem $\frac{dy}{dt} = \frac{t-5}{y}, y(0) = 3$ and determine the range of t -values in which the solution is valid.

(AB 32) Solve the initial-value problem $\frac{dy}{dt} = y^2, y(0) = 1/4$ and determine the range of t -values in which the solution is valid.

(AB 33) The function $y_1(t) = e^t$ is a solution to the differential equation $t \frac{d^2y}{dt^2} - (1 + 2t) \frac{dy}{dt} + (t + 1)y = 0.$ Find the general solution to this differential equation.

(AB 34) The function $y_1(t) = t$ is a solution to the differential equation $t^2 \frac{d^2y}{dt^2} - (t^2 + 2t) \frac{dy}{dt} + (t + 2)y = 0.$ Find the general solution to this differential equation.

(AB 35) The function $y_1(x) = x^3$ is a solution to the differential equation $x^2 \frac{d^2y}{dx^2} + (x^2 \tan x - 6x) \frac{dy}{dx} + (12 - 3x \tan x)y = 0.$ Find the general solution to this differential equation on the interval $0 < x < \pi/2.$

(AB 36) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $16 \frac{d^2y}{dt^2} - y = 0, y(0) = 1, y'(0) = 0.$

(b) $\frac{d^2y}{dt^2} + 49y = 0, y(\pi) = 3, y'(\pi) = 4.$

(c) $\frac{d^2y}{dt^2} + 3 \frac{dy}{dt} - 10y = 0, y(2) = 0, y'(2) = 3.$

(d) $9 \frac{d^2y}{dt^2} + 6 \frac{dy}{dt} + 2y = 0, y(0) = 3, y'(0) = 2.$

(e) $\frac{d^2y}{dt^2} + 10 \frac{dy}{dt} + 25y = 0, y(0) = 1, y'(0) = 4.$

(AB 37) Find the general solution to the following differential equations.

- (a) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0.$
- (b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0.$
- (c) $\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 24y = 0.$
- (d) $\frac{d^2y}{dt^3} + y = 0.$
- (e) $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0.$
- (f) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 3e^{4t}.$
- (g) $16\frac{d^2y}{dt^2} - y = e^{t/4} \sin t.$
- (h) $\frac{d^2y}{dt^2} + 49y = 3t \sin 7t.$
- (i) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}.$
- (j) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = t \sin(3t).$
- (k) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}.$
- (l) $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 6t^2 + \cos(2t).$
- (m) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = t^2e^{3t}.$
- (n) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}.$
- (o) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t).$
- (p) $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t).$
- (q) $\frac{d^2y}{dt^2} + 9y = 5\sin(3t).$
- (r) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2.$
- (s) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t.$
- (t) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t).$
- (u) $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2.$
- (v) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t.$

(AB 38) Solve the initial-value problem $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$.

(AB 39) Find the general solution to the equation $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^t \ln t$ on the interval $0 < t < \infty$.

(AB 40) Find the general solution to the equation $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sin(e^t)$.

(AB 41) Solve the initial-value problem $\frac{d^2y}{dt^2} + 9y = 9\sec^2 3t$, $y(0) = 4$, $y'(0) = 6$, on the interval $-\pi/6 < t < \pi/6$.

(AB 42) The general solution to the differential equation $t^2\frac{d^2y}{dt^2} - 2y = 0$, $t > 0$, is $y(t) = C_1t^2 + C_2t^{-1}$. Solve the initial-value problem $t^2\frac{d^2y}{dt^2} - 2y = 7t^3$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(AB 43) The general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 0$, $t > 0$, is $y(t) = C_1t^3 + C_2t^{-1}$. Find the general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

(AB 44) An object weighing 5 lbs stretches a spring 4 inches. It is attached to a viscous damper with damping constant 16 lb · sec/ft. The object is pulled down an additional 2 inches and is released with initial velocity 3 ft/sec upwards.

Write the differential equation and initial conditions that describe the position of the object.

(AB 45) A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant β . The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20 \text{ N} \cdot \text{s/m}$, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30 \text{ N} \cdot \text{s/m}$, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(AB 46) A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant β . The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of β for which the system is critically damped. Be sure to include units for β .

(AB 47) A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 N·s/m and is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3 \cos(20t)$ N upwards.

Write the differential equation and initial conditions that describe the position of the object.

(AB 48) An object weighing 4 lbs stretches a spring 2 inches. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3 \cos(\omega t)$ pounds, directed upwards. There is no damping.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of ω for which resonance occurs. Be sure to include units for ω .

(AB 49) Suppose that $\frac{d^2x}{dt^2} = 18x^3$, $x(0) = 1$, $x'(0) = 3$. Let $v = \frac{dx}{dt}$. Find a formula for v in terms of x . Then find a formula for x in terms of t .

(AB 50) Suppose that a bob of mass 300g hangs from a pendulum of length 15cm. The pendulum is set in motion from its equilibrium point with an initial velocity of 3 meters/second.

Write the differential equation and initial conditions that describe the pendulum's motion. Then find a formula for the pendulum's angular velocity in terms of its position.

(AB 51) Suppose that a rocket of mass $m = 1000$ kg is launched straight up from the surface of the earth with initial velocity 10 km/sec. The radius of the earth is 6,371 km. When the rocket is r meters from the center of the earth, it experiences a force due to gravity of magnitude GMm/r^2 , where $GM = 3.98 \times 10^{14}$ meters³/second².

- Formulate the initial value problem for the particle's position.
- Find the velocity of the particle as a function of position.
- How far away from the string is the particle when it stops moving and starts to fall back?

(AB 52) A particle of mass $m = 3$ kg a distance r from an infinitely long string experiences a force due to gravity of magnitude Gm/r , where $G = 2000$ meters²/second², directed directly toward the string. Suppose that the particle is initially 1000 meters from the string and takes off with initial velocity 200 meters/second directly away from the string.

- Formulate the initial value problem for the particle's position.
- Find the velocity of the particle as a function of position.
- How far away from the string is the particle when it stops moving and starts to fall back?

(AB 53) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$ (**not** the table on the front of the exam), find the Laplace transforms of the following functions.

(a) $f(t) = t$

(b) $f(t) = e^{-11t}$

(c) $f(t) = \begin{cases} 3, & 0 < t < 4, \\ 0, & 4 \leq t \end{cases}.$

(AB 54) Find the Laplace transforms of the following functions. You may use the table on the front of the exam.

(a) $f(t) = t^4 + 5t^2 + 4$

(b) $f(t) = (t + 2)^3$

(c) $f(t) = 9e^{4t+7}$

(d) $f(t) = -e^{3(t-2)}$

(e) $f(t) = (e^t + 1)^2$

(f) $f(t) = 8 \sin(3t) - 4 \cos(3t)$

(g) $f(t) = t^2 e^{5t}$

(h) $f(t) = 7e^{3t} \cos 4t$

(i) $f(t) = 4e^{-t} \sin 5t$

(j) $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$

(k) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$

(l) $f(t) = \begin{cases} 0, & t < 1, \\ t - 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$

(m) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

(n) $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

(o) $f(t) = \begin{cases} 0, & t < \pi, \\ \sin t, & t \geq \pi \end{cases}$

(p) $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi \end{cases}$

(q) $f(t) = t^2 \sin 5t$

(AB 55) For each of the following problems, find y .

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$

(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$

(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$

(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$

(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$

(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$

(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$

(h) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$

(i) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$

(j) $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$

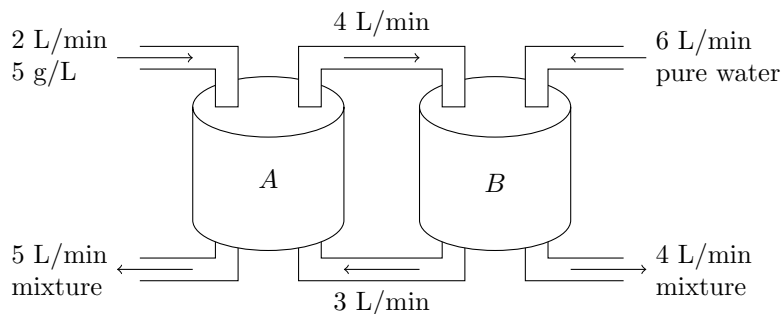
(k) $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$

(l) $\mathcal{L}\{y\} = \frac{s}{s^2-9} \mathcal{L}\{\sqrt{t}\}$

(AB 56) Solve the following initial-value problems using the Laplace transform. Do you expect the graphs of y , $\frac{dy}{dt}$, or $\frac{d^2y}{dt^2}$ to show any corners or jump discontinuities? If so, at what values of t ?

- (a) $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$
 (b) $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$
 (c) $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$
 (d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$
 (e) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$
 (f) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$
 (g) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$
 (h) $\frac{d^2y}{dt^2} - 4y = e^t \sin(3t)$, $y(0) = 0$, $y'(0) = 0$
 (i) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$
 (j) $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \leq t < 4, \\ 0, & 4 \leq t \end{cases}$, $y(0) = 2$, $y'(0) = 0$
 (k) $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \leq t < 2\pi, \\ 0, & 2\pi \leq t \end{cases}$, $y(0) = 0$, $y'(0) = 0$
 (l) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \leq t < 10, \\ 0, & 10 \leq t \end{cases}$, $y(0) = 0$, $y'(0) = 0$
 (m) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \leq t < 2, \\ 4, & 2 \leq t \end{cases}$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = 2$.
 (n) $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \leq t < \pi, \\ 0, & \pi \leq t \end{cases}$, $y(0) = 1$, $y'(0) = 0$
 (o) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \leq t < 2, \\ 3, & 2 \leq t \end{cases}$, $y(0) = 2$, $y'(0) = 1$
 (p) $\frac{d^2y}{dt^2} + 9y = t \sin(3t)$, $y(0) = 0$, $y'(0) = 0$
 (q) $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$
 (r) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$
 (s) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4u_2(t)$, $y(0) = 0$, $y'(0) = 1$.
 (t) $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$.
 (u) $\frac{d^2y}{dt^2} + 4y = -2\delta(t-4\pi)$, $y(0) = 1/2$, $y'(0) = 0$
 (v) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + u_2(t)$, $y(0) = 1$, $y'(0) = 0$
 (w) $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 5\delta(t-4)$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 2$.

(AB 57) Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



Write the differential equations and initial conditions that describe the amount of salt in each tank.

(AB 58) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \quad \frac{dy}{dt} = -2x - 2y, \quad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 59) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y, \quad \frac{dy}{dt} = -x + 4y, \quad x(0) = -3, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 60) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \quad \frac{dy}{dt} = -18x - 17y, \quad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 61) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 13x - 39y, \quad \frac{dy}{dt} = 12x - 23y, \quad x(0) = 5, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 62) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y, \quad \frac{dy}{dt} = -5x + 6y, \quad x(0) = 1, \quad y(0) = -3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 63) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \quad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \quad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 64) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -2x + 2y, \quad \frac{dy}{dt} = 2x - 5y, \quad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 65) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \quad \frac{dy}{dt} = -2x - 2y, \quad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 66) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y + e^{2t} \cos t, \quad \frac{dy}{dt} = -x + 4y, \quad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

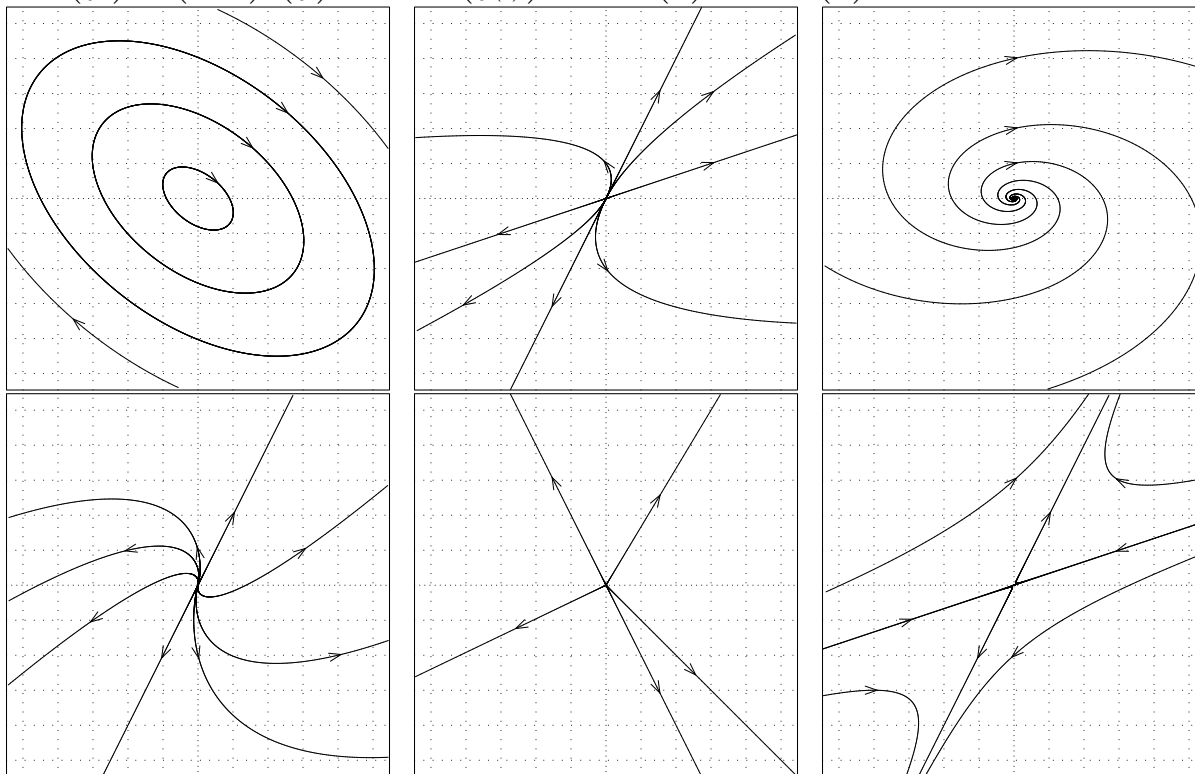
(AB 67) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \quad \frac{dy}{dt} = -4x - 5y, \quad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

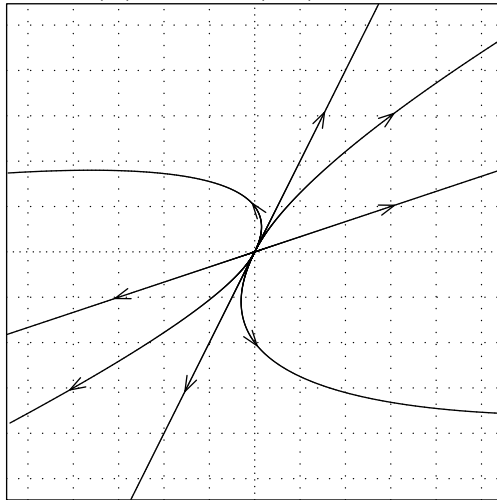
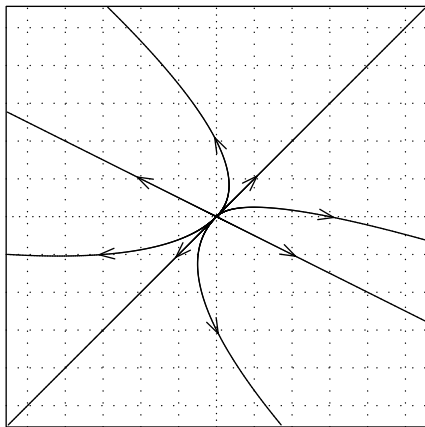
(AB 68) Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

- (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
- (b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2.6 & 1.8 \\ -1.2 & 1.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
- (c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4/3 & -1/6 \\ 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^t \begin{pmatrix} t+3 \\ 2t \end{pmatrix}$
- (d) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4.5 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 3 \sin 3t \\ 2 \cos 3t \end{pmatrix} + C_2 e^t \begin{pmatrix} -3 \cos 3t \\ 2 \sin 3t \end{pmatrix}$
- (e) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 5 \sin 4t \\ 4 \cos 4t - 2 \sin 4t \end{pmatrix} + C_2 \begin{pmatrix} 5 \cos 4t \\ -4 \sin 4t - 2 \cos 4t \end{pmatrix}$
- (f) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



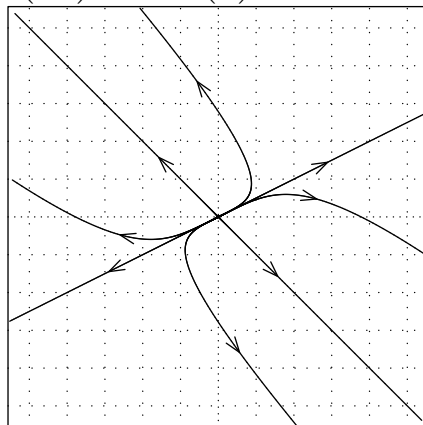
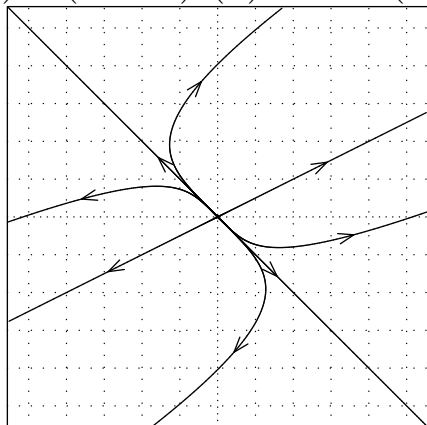
(AB 69) Here are some phase planes. To which of the following systems do these phase planes correspond?
How do you know?

- (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$
- (b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$



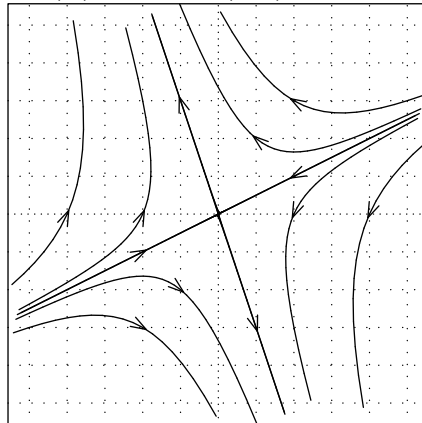
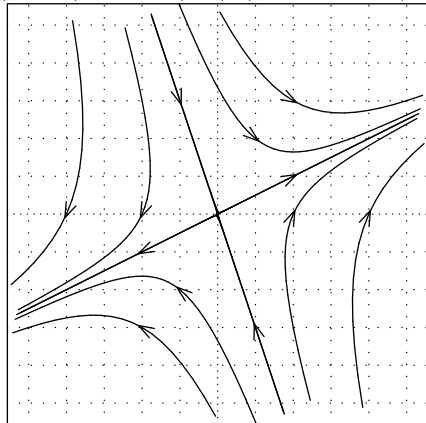
(AB 70) Here are some phase planes. To which of the following systems do these phase planes correspond?
How do you know?

- (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
- (b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



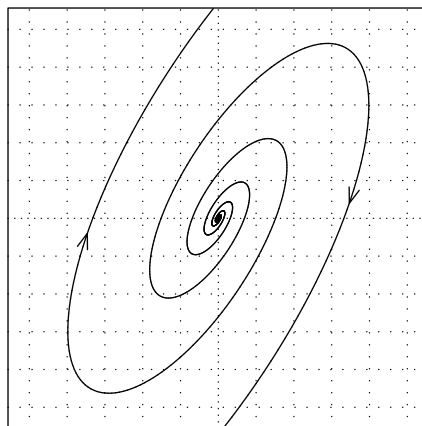
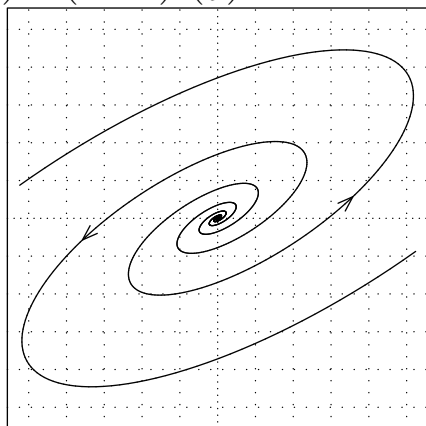
(AB 71) Here are some phase planes. To which of the following systems do these phase planes correspond?
How do you know?

- (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 9 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-14t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$
- (b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -4 & -6 \\ -9 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{14t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$



(AB 72) Here are some phase planes. To which of the following systems do these phase planes correspond?
How do you know?

- (a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -8 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, eigenvalues $r = 1 \pm 4i$
- (b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, eigenvalues $r = 1 \pm 4i$



(AB 73) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y - 15, \quad \frac{dy}{dt} = -5x + 6y - 8, \quad x(0) = 2, \quad y(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Answer key

(Answer 1) No, $y = e^t$ is not a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t} \frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(Answer 2) Yes, $y = e^{2t}$ is a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t} \frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(Answer 3) No.

(Answer 4) No.

(Answer 5) Yes.

(Answer 6) No.

(Answer 7)

- (a) We expect a unique solution.
- (b) We do not expect any solutions.
- (c) We do not expect any solutions.
- (d) We expect a unique solution.
- (e) We expect an infinite family of solutions.

(Answer 8) Independent variable: $t =$ time (in years).

Dependent variable: $P =$ Number of trout in the lake

Initial condition: $P(0) = 600$.

Parameters: $\alpha =$ birth rate (in 1/years).

Differential equation: $\frac{dP}{dt} = \alpha P - 30$.

(Answer 9) Independent variable: $t =$ time (in years).

Dependent variable: $P =$ Number of birds on the island.

Parameters: $\alpha =$ birth rate parameter (in 1/years); $\beta =$ death rate parameter (in 1/(bird·years)).

Initial condition: $P(0) = 5$.

Differential equation: $\frac{dP}{dt} = \alpha P - \beta P^2$.

(Answer 10) Independent variable: $t =$ time (in seconds).

Dependent variable: $T =$ Temperature of the cup (in degrees Celsius)

Initial condition: $T(0) = 95$.

Differential equation: $\frac{dT}{dt} = -\alpha(T-25)$, where α is a parameter (constant of proportionality) with units of 1/seconds.

(Answer 11) Independent variable: $t =$ time (in years).

Dependent variable: $T =$ Number of telephones installed in the town.

Initial condition: $T(1920) = 2$.

Differential equation: $\frac{dT}{dt} = -\alpha T(300 - T)$, where α is a parameter (constant of proportionality) with units of 1/year · telephone.

(Answer 12) Independent variable: $t =$ time (in minutes).

Dependent variable: $Q =$ amount of dissolved salt (in kilograms).

Initial condition: $Q(0) = 3$.

Differential equation: $\frac{dQ}{dt} = 0.01 - \frac{3Q}{600-t}$.

(Answer 13) Independent variable: $t =$ time (in years).

Dependent variable: $B =$ balance of my loan (in dollars).

Initial condition: $B(0) = 300,000$.

Differential equation: $\frac{dB}{dt} = 0.04B - 12 \cdot 1600$

(Answer 14) Independent variable: $t =$ time (in minutes).

Dependent variables:

$h =$ depth of water in the hole (in centimeters)

$V =$ volume of water in the hole (in cubic centimeters); notice that $V = \frac{1}{3}\pi(5h)^2h$

Initial condition: $V(0) = 0$.

Differential equation: $\frac{dV}{dt} = 1 - \alpha\pi(5h)^2 = 1 - 25\alpha\pi(3V/25\pi)^{2/3}$, where α is a proportionality constant with units of cm/s.

(Answer 15) Independent variable: $t =$ time (in seconds).

Dependent variable: $T =$ object's temperature (in kelvins)

Parameter: $\sigma =$ proportionality constant (in $1/(\text{seconds}\cdot\text{kelvin}^3)$)

Initial condition: $T(0) = 400$.

Differential equation: $\frac{dT}{dt} = -\sigma T^4$.

(Answer 16) Independent variable: $t =$ time (in seconds).

Dependent variable: $T =$ object's temperature (in kelvins)

Parameter: $\sigma =$ proportionality constant (in $1/(\text{seconds}\cdot\text{kelvin}^3)$)

Initial condition: $T(0) = 400$.

Differential equation: $\frac{dT}{dt} = -\sigma T^4 + 81\sigma$.

(Answer 17) Independent variable: $t =$ time (in seconds).

Dependent variable: $v =$ velocity of the ball (in meters/second, where a positive velocity denotes upward motion).

Parameters: $\alpha =$ proportionality constant of the drag force (in newton-seconds/meter)

$m =$ mass of the ball (in kilograms)

Initial condition: $v(0) = 10$.

Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/m)v$.

(Answer 18) Independent variable: $t =$ time (in seconds).

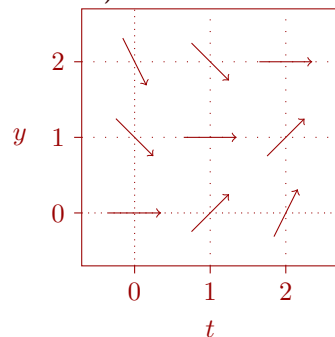
Dependent variable: $v =$ velocity of the ball (in meters/second, where a positive velocity denotes upward motion).

Parameter: $\alpha =$ proportionality constant of the drag force (in newton-seconds/meter).

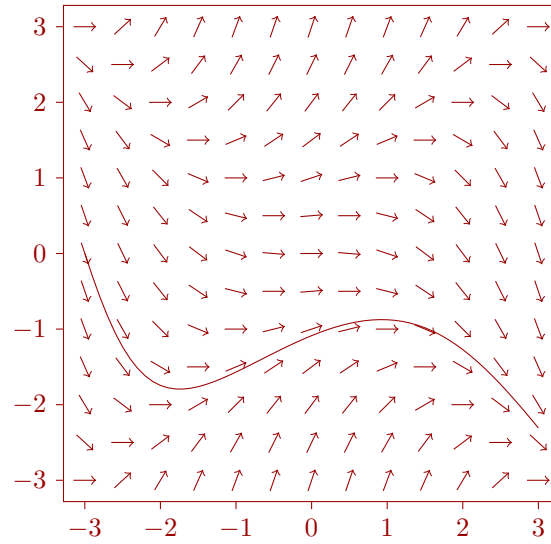
Initial condition: $v(0) = 20$.

Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/0.3)v|v|$, or $\frac{dv}{dt} = \begin{cases} -9.8 - (\alpha/0.3)v^2, & v \geq 0, \\ -9.8 + (\alpha/0.3)v^2, & v < 0. \end{cases}$

(Answer 19) Here is the direction field for the differential equation $\frac{dy}{dt} = t - y$.



(Answer 20)



(Answer 21)

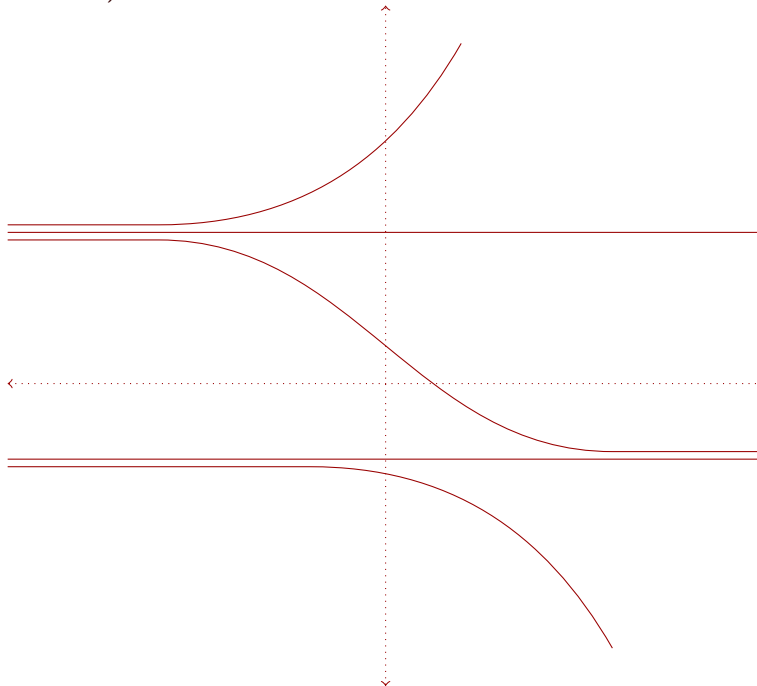
If $y(0) > 4$, then $y \rightarrow \infty$ as $t \rightarrow \infty$.

If $y(0) = 4$, then $y(t) = 4$ for all t .

If $0 \leq y(0) < 4$, then $y \rightarrow 0$ as $t \rightarrow \infty$. In particular, if $y(0) = 0$, then $y(t) = 0$ for all t .

If $y(0) < 0$, then $y \rightarrow -\infty$ as $t \rightarrow \infty$.

(Answer 22)



(Answer 23)

Critical points: $y = k\pi$ for any integer k .



If k is even then $y = k\pi$ is unstable.

If k is odd then $y = k\pi$ is stable.

(Answer 24)

Critical points: $y = 0$ and $y = 2$.



$y = 0$ is semistable. $y = 2$ is unstable.

(Answer 25)

(a) $\frac{dB}{dt} = 0.05B - 19200$, where t denotes time in years.

(b) The critical point is $B = \$384,000$. It is unstable. If my initial debt is less than $\$384,000$, then I will eventually pay it off (have a debt of zero dollars), but if my initial debt is greater than $\$384,000$, my debt will grow exponentially. The critical point $B = 384,000$ corresponds to the balance that will allow me to make interest-only payments on my debt.

(Answer 26)

(a) $\frac{dQ}{dt} = 10 - Q/300$, where Q denotes the amount of salt in grams and t denotes time in minutes.

(b) The critical point is $Q = 3000$. It is stable. No matter the initial amount of salt in the tank, as time passes the amount of salt will approach 3000 g or 3 kg.

(Answer 27)

(a) If $v \leq 0$ then $70\frac{dv}{dt} = -70 * 9.8 + 2v^2$. (If $v > 0$ then $70\frac{dv}{dt} = -70 * 9.8 - 2v^2$.)

(b) $v = -\sqrt{343}$ meters/second.

(c) As $t \rightarrow \infty$, her velocity will approach the stable critical point of $v = -\sqrt{343}$ meters/second.

(Answer 28)

(a) The critical points are $y = 0$ (unstable) and $y = K$ (stable).

(b) $\frac{dy}{dt} = ry(1 - y/K) - Ey$, where E is a proportionality constant. The critical points are $y = 0$ (unstable) and $y = K - KE/r$ (stable).

(c) $\frac{dy}{dt} = ry(1 - y/K) - h$, where h is the harvesting rate. If $h < Kr/4$, then the critical points are $y = K/2 - \sqrt{(K/2)^2 - Kh/r}$ (unstable) and $y = K/2 + \sqrt{(K/2)^2 - Kh/r}$ (stable). Notice that both critical points are positive (i.e., correspond to the physically meaningful case of at least zero fish in the lake.) If $h = Kr/4$, then there is one critical point at $y = K/2$; it is semistable. Finally, if $h > Kr/4$, then there are no critical points and the fish population will decrease until it goes extinct.

(Answer 29)

- (a) $\frac{dy}{dt} = \frac{t+\cos t}{\sin y-y}$ is separable and has solution $\frac{1}{2}y^2 + \cos y = -\frac{1}{2}t^2 - \sin t + C$.
- (b) $(t^2 + 1)\frac{dy}{dt} = ty - t^2 - 1$ is linear and has solution $y = -\sqrt{t^2 + 1} \ln(t + \sqrt{t^2 + 1}) + C\sqrt{t^2 + 1}$.
- (c) $t^2\frac{dy}{dt} = y^2 + t^2 - ty$ is homogeneous and has solution $y = \frac{t}{C-\ln|t|} + t$.
- (d) Let $v = y^{-2}$. Then $\frac{dv}{dt} = 2v \tan 2t + 2 \cos 2t$, so $v = \frac{1}{2} \sin(2t) + t \sec(2t) + C \sec(2t)$ and $y = \frac{1}{\sqrt{\frac{1}{2} \sin(2t) + t \sec(2t) + C \sec(2t)}}$.
- (e) $(t + y)\frac{dy}{dt} = 5y - 3t$ is homogeneous and has solution $(y - 3t)^2 = C(y - t)$.
- (f) If $\frac{dy}{dt} = \frac{1}{3t+2y+7}$, then $\frac{3t+2y+7}{3} - \frac{2}{9} \ln|3t + 2y + 7 + 2/3| = \ln|t| + C$.
- (g) $\frac{dy}{dt} = -y^3 \cos(2t)$ is separable and has solution $y = \pm \frac{1}{\sqrt{\sin(2t)+C}}$.
- (h) $4ty\frac{dy}{dt} = 3y^2 - 2t^2$ is homogeneous and has solution $2 \ln(y^2/t^2 + 2) = -\ln|t| + C$.
- (i) Let $v = y^6$. Then $t\frac{dv}{dt} = 18v - 6t^2$, so $v = Ct^{18} + \frac{3}{8}t^2$ and $y = \sqrt[6]{Ct^{18} + \frac{3}{8}t^2}$.
- (j) If $\frac{dy}{dt} = \csc^2(y - t)$, then $\tan(y - t) - y = C$.
- (k) Make the substitution $v = y^{-7}$. Then $\frac{dv}{dt} = -56v + 7$, so $v = Ce^{-56t} + \frac{1}{8}$ and $y = \frac{1}{\sqrt[7]{Ce^{-56t} + 1/8}}$.
- (l) $t\frac{dy}{dt} = -\cos t - 3y$ is linear and has solution $y = \frac{1}{t} \sin t + \frac{2}{t^2} \cos t - \frac{2}{t^3} \sin t + \frac{C}{t^3}$.
- (m) $\frac{dy}{dt} = \cot(y/t) + y/t$ is homogeneous and has solution $\sec(y/t) = Ct$.

(Answer 30)

- (a) If $2ty\frac{dy}{dt} = 4t^2 - y^2$, $y(1) = 3$, then $y = \sqrt{\frac{4}{3}t^2 + \frac{23}{3t}}$.
- (b) If $t\frac{dy}{dt} = -1 - y^2$, $y(1) = 1$, then $y = \tan(\pi/4 - \ln t)$.
- (c) If $\frac{dy}{dt} = (2y + 2t - 5)^2$, $y(0) = 3$, then $y = \frac{1}{2} \tan(2t + \pi/4) + \frac{5}{2} - t$.
- (d) If $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$, then $y = \sqrt[3]{340e^{6t} + 3}$.
- (e) If $\frac{dy}{dt} = -3y - \sin t e^{-3t}$, $y(0) = 2$, then $y = e^{-3t} \cos t + e^{-3t}$.

(Answer 31) $y = \sqrt{(t-5)^2 - 16} = \sqrt{(t-1)(t-9)} = \sqrt{t^2 - 10t + 9}$. The solution is valid for all $t < 1$.

(Answer 32) $y = \frac{1}{4-t}$. The solution is valid for all $t < 4$.

(Answer 33) $y(t) = C_1 e^t + C_2 t^2 e^t$.

(Answer 34) $y(t) = C_1 t + C_2 t e^t$.

(Answer 35) $y(x) = C_1 x^3 + C_2 x^3 \sin x$.

(Answer 36)

- (a) If $16\frac{d^2y}{dt^2} - y = 0$, $y(0) = 1$, $y'(0) = 0$, then $y(t) = \frac{1}{2}e^{t/4} + \frac{1}{2}e^{-t/4}$.
- (b) If $\frac{d^2y}{dt^2} + 49y = 0$, $y(\pi) = 3$, $y'(\pi) = 4$, then $y(t) = 3 \cos(7t) + \frac{4}{7} \sin(7t)$.
- (c) If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 0$, $y(2) = 0$, $y'(2) = 3$, then $y(t) = \frac{3}{7}e^{2t-4} - \frac{3}{7}e^{-5t+10}$.
- (d) If $9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 2y = 0$, $y(0) = 3$, $y'(0) = 2$, then $y = 3e^{-t/3} \cos(t/3) + 9e^{-t/3} \sin(t/3)$.
- (e) If $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$, $y(0) = 1$, $y'(0) = 4$, then $y = e^{-5t} + 9te^{-5t}$.

(Answer 37)

- (a) If $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$, then $y = C_1e^{-6t} \cos(7t) + C_2e^{-6t} \sin(7t)$.
- (b) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$, then $y = C_1e^{(-2+\sqrt{2})t} + C_2e^{(-2-\sqrt{2})t}$.
- (c) If $\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 24y = 0$, then $y = C_1e^{6t} + C_2e^{2t} + C_3e^{-2t}$.
- (d) If $\frac{d^3y}{dt^3} + y = 0$, then $y = C_1e^{-t} + C_2e^{t/2} \cos \frac{\sqrt{3}}{2}t + C_3e^{t/2} \sin \frac{\sqrt{3}}{2}t$.
- (e) If $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0$, then $y = C_1e^{2t} + C_2te^{2t} + C_3e^{-2t} + C_4te^{-2t}$.
- (f) The general solution to $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 0$ is $y_g = C_1e^{-t/2} + C_2e^{-t/3}$. To solve $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 2e^{4t}$ we make the guess $y_p = Ae^{4t}$. The solution is $y = C_1e^{-t/2} + C_2e^{-t/3} + \frac{1}{39}e^{4t}$.
- (g) The general solution to $16\frac{d^2y}{dt^2} - y = 0$ is $y_g = C_1e^{t/4} + C_2e^{-t/4}$. To solve $16\frac{d^2y}{dt^2} - y = e^{t/4} \sin t$ we make the guess $y_p = Ae^{t/4} \sin t + Be^{t/4} \cos t$. The solution is $y = C_1e^{t/4} + C_2e^{-t/4} - (1/20)e^{t/4} \sin t - (1/40)e^{t/4} \cos t$.
- (h) The general solution to $\frac{d^2y}{dt^2} + 49y = 0$ is $y_g = C_1 \sin 7t + C_2 \cos 7t$. To solve $\frac{d^2y}{dt^2} + 49y = 3t \sin 7t$ we make the guess $y_p = At^2 \sin 7t + Bt \sin 7t + Ct^2 \cos 7t + Dt \cos 7t$. The solution is $y = C_1 \sin 7t + C_2 \cos 7t - (3/28)t^2 \cos 7t + (3/4)t \sin 7t$.
- (i) The general solution to $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 0$ is $y_g = C_1 + C_2e^{4t}$. To solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$ we make the guess $y_p = Ae^{3t}$. The solution is $y = C_1 + C_2e^{4t} - (4/3)e^{3t}$.
- (j) The general solution to $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$ is $y_g = C_1e^{-6t} \cos(7t) + C_2e^{-6t} \sin(7t)$. To solve $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = t \sin(3t)$ we make the guess $y_p = At \sin(3t) + Bt \cos(3t) + C \sin(3t) + D \cos(3t)$. The solution is

$$y = C_1e^{-6t} \cos(7t) + C_2e^{-6t} \sin(7t) + \frac{19}{1768}t \sin(3t) - \frac{9}{1768}t \cos(3t) - \frac{1353}{781456} \sin(3t) + \frac{606}{781456} \cos(3t).$$

- (k) The general solution to $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 0$ is $y_g = C_1e^{2t} + C_2e^{-5t}$. To solve $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$ we make the guess $y_p = Ate^{-5t}$. The solution is $y = C_1e^{2t} + C_2e^{-5t} - (1/7)te^{-5t}$.
- (l) The general solution to $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 0$ is $y_g = C_1e^{(3/4)t} + C_2te^{(3/4)t}$. To solve $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 6t^2 + \cos(2t)$ we make the guess $y_p = At^2 + Bt + C + D \cos(2t) + E \sin(2t)$. The solution is $y = C_1e^{(3/4)t} + C_2te^{(3/4)t} + (2/3)t^2 + (32/9)t + 64/9 - (48/5329) \cos(2t) - (55/5329) \sin(2t)$.
- (m) The general solution to $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = 0$ is $y_g = C_1e^{3t} \cos(4t) + C_2e^{3t} \sin(4t)$. To solve $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = t^2e^{3t}$ we make the guess $y_p = At^2e^{3t} + Bte^{3t} + Ce^{3t}$. The solution is $y = C_1e^{3t} \cos(4t) + C_2e^{3t} \sin(4t) + (1/16)t^2e^{3t} - (1/128)e^{3t}$.
- (n) The general solution to $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$ is $y_g = C_1e^{-5t} + C_2te^{-5t}$. To solve $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}$ we make the guess $y_p = At^2e^{-5t}$. The solution is $y = C_1e^{-5t} + C_2te^{-5t} + (3/2)t^2e^{-5t}$.
- (o) The general solution to $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$ is $y_g = C_1e^{-2t} + C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3 \cos(2t)$, we make the guess $y_p = A \cos(2t) + B \sin(2t)$. The solution is $y = C_1e^{-2t} + C_2e^{-3t} + \frac{15}{52} \sin(2t) + \frac{3}{52} \cos(2t)$.
- (p) The general solution to $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$ is $y_g = C_1e^{-3t} + C_2te^{-3t}$. To solve $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5 \sin(4t)$, we make the guess $y_p = A \cos(4t) + B \sin(4t)$.
- (q) The general solution to $\frac{d^2y}{dt^2} + 9y = 0$ is $y_g = C_1 \cos(3t) + C_2 \sin(3t)$. To solve $\frac{d^2y}{dt^2} + 9y = 5 \sin(3t)$, we make the guess $y_p = C_1t \cos(3t) + C_2t \sin(3t)$. The solution is $y = C_1 \cos(3t) + C_2 \sin(3t) - \frac{5}{6}t \cos(3t)$.
- (r) The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ is $y_g = C_1e^{-t} + C_2te^{-t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$, we make the guess $y_p = At^2 + Bt + C$. The solution is $y = C_1e^{-t} + C_2te^{-t} + 2t^2 - 8t + 12$.
- (s) The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 0$ is $y_g = C_1 + C_2e^{-2t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$, we make the guess $y_p = At^2 + Bt$. The solution is $y = C_1 + C_2e^{-2t} + \frac{3}{4}t^2 - \frac{3}{4}t$.
- (t) The general solution to $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 0$ is $y_g = C_1e^{3t} + C_2e^{4t}$. To solve $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$, we make the guess $y_p = At + B + C \cos(2t) + D \sin(2t)$.
- (u) The general solution to $\frac{d^2y}{dt^2} - 9y = 0$ is $y_g = C_1e^{3t} + C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$, we make the guess $y_p = Ae^t + Be^{-t} + Ct + D$.

(v) The general solution to $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$ is $y_g = C_1e^{2t} + C_2te^{2t}$. To solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$, we make the guess $y_p = At^2e^{2t} + B\cos t + C\sin t$.

(Answer 38) If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$, then $y(t) = -\frac{4}{3}e^{3t} + \frac{7}{4}e^{4t} + \frac{7}{12}$.

(Answer 39) If $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^t \ln t$, then $y(t) = c_1 e^t + c_2 t e^t + \frac{1}{2}t^2 e^t \ln t - \frac{3}{4}t^2 e^t$ for all $t > 0$.

(Answer 40) If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sin(e^t)$, then $y = c_1 e^{-t} + c_2 e^{-2t} - e^{-2t} \sin(e^t)$.

(Answer 41) If $\frac{d^2y}{dt^2} + 9y = 9\sec^2 3t$, $y(0) = 4$, $y'(0) = 6$, then

$$y(t) = 5\cos(3t) + 2\sin(3t) + (\sin(3t))\ln(\tan(3t) + \sec(3t)) - 1$$

for all $-\pi/6 < t < \pi/6$.

(Answer 42) If $t^2\frac{d^2y}{dt^2} - 2y = 7t^3$, $y(1) = 1$, $y'(1) = 2$, then $y(t) = \frac{7}{4}t^3 - \frac{4}{3}t^2 + \frac{7}{12}t^{-1}$ for all $0 < t < \infty$.

(Answer 43) If $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$, then $y(t) = -\frac{3}{2}t^{-1}\ln t + C_1t^3 + C_2t^{-1}$ for all $t > 0$.

(Answer 44) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then

$$\frac{5}{32}\frac{d^2x}{dt^2} + 16\frac{dx}{dt} + 15x = 0, \quad x(0) = -\frac{1}{6}, \quad x'(0) = 3.$$

(Answer 45) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$2\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 80x = 0, \quad x(0) = -0.1, \quad x'(0) = 0.$$

If $\beta = 20 \text{ N}\cdot\text{s}/\text{m}$, then the system is underdamped, and we do expect to see decaying oscillations.

If $\beta = 30 \text{ N}\cdot\text{s}/\text{m}$, then the system overdamped, and we do not expect to see decaying oscillations.

(Answer 46) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$4\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 70x = 0, \quad x(0) = 0.05, \quad x'(0) = -3.$$

Critical damping occurs when $\beta = 4\sqrt{70} \text{ N}\cdot\text{s}/\text{m}$.

(Answer 47) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters).

Then

$$3\frac{d^2x}{dt^2} + 27\frac{dx}{dt} + 588x = 3\cos(20t), \quad u(0) = 0, \quad u'(0) = 0.$$

(Answer 48) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then

$$\frac{4}{32}\frac{d^2x}{dt^2} + 24x = 3\cos(\omega t), \quad x(0) = 0, \quad x'(0) = 0.$$

Resonance occurs when $\omega = 8\sqrt{3} \text{ s}^{-1}$.

(Answer 49) We have that $\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}$ and also $\frac{dv}{dt} = \frac{d^2x}{dt^2}$. Thus $v \frac{dv}{dx} = 18x^3$, $v(1) = 3$. Solving, we see that $v = 3x^2$.

But then $\frac{dx}{dt} = 3x^2$, $x(0) = 1$, and so $x = \frac{1}{1-3t}$.

(Answer 50) Let θ be the angle between the pendulum and a vertical line (in radians), and let t denote time in seconds. Then

$$0.3 \frac{d^2\theta}{dt^2} = -\frac{9.8}{0.5} \sin \theta, \quad \theta(0) = 0, \quad \theta'(0) = 20.$$

Let $\omega = \frac{d\theta}{dt}$ be the pendulum's angular velocity. We have that $\frac{d\omega}{dt} = \frac{d\omega}{d\theta} \frac{d\theta}{dt} = \omega \frac{d\omega}{d\theta}$ and also $\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$. Thus $\omega \frac{d\omega}{d\theta} = -\frac{196}{3} \sin \theta$ and $\omega(0) = 20$. Solving, we see that $\frac{1}{2}\omega^2 = \frac{196}{3} \cos \theta + \frac{404}{3}$.

(Answer 51)

(a) The initial value problem is

$$1000 \frac{d^2r}{dt^2} = -\frac{3.98 \times 10^{17}}{r^2}, \quad r(0) = 6371000, \quad r'(0) = 10000$$

where r denotes the distance to the center of the earth in meters and t denotes time in seconds.

(b) Let v be the particle's velocity in meters/second. We have that

$$1000v \frac{dv}{dr} = -\frac{3.98 \times 10^{17}}{r^2}, \quad v(6371000) = 10000$$

and so

$$v^2 = \frac{7.96 \times 10^{14}}{r} - 2.494 \times 10^7$$

(c) $v = 0$ when $r = 3.191 \times 10^7$ meters.

(Answer 52)

(a) The initial value problem is

$$3 \frac{d^2r}{dt^2} = -\frac{6000}{r}, \quad r(0) = 1000, \quad r'(0) = 200$$

where r denotes the distance to the string in meters and t denotes time in seconds.

(b) Let v be the particle's velocity in meters/second. We have that

$$3v \frac{dv}{dr} = -\frac{6000}{r}, \quad v(1000) = 200$$

and so

$$v^2 = -4000 \ln r + 4000 \ln 1000 + 40000.$$

(c) $v = 0$ when $r = 1000e^{10}$ meters.

(Answer 53)

(a) $\mathcal{L}\{t\} = \frac{1}{s^2}$.

(b) $\mathcal{L}\{e^{-11t}\} = \frac{1}{s+11}$.

(c) $\mathcal{L}\{f(t)\} = \frac{3-3e^{-4s}}{s}$.

(Answer 54)

(a) $\mathcal{L}\{t^4 + 5t^2 + 4\} = \frac{96}{s^5} + \frac{10}{s^3} + \frac{4}{s}$.

(b) $\mathcal{L}\{(t+2)^3\} = \mathcal{L}\{t^3 + 6t^2 + 12t + 8\} = \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{8}{s}$.

(c) $\mathcal{L}\{9e^{4t+7}\} = \mathcal{L}\{9e^7 e^{4t}\} = \frac{9e^7}{s-4}$.

(d) $\mathcal{L}\{-e^{3(t-2)}\} = \mathcal{L}\{-e^{-6} e^{3t}\} = -\frac{e^{-6}}{s-3}$.

(e) $\mathcal{L}\{(e^t + 1)^2\} = \mathcal{L}\{e^{2t} + 2e^t + 1\} = \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s}$.

(f) $\mathcal{L}\{8 \sin(3t) - 4 \cos(3t)\} = \frac{24}{s^2+9} - \frac{4s}{s^2+9}$.

(g) $\mathcal{L}\{t^2 e^{5t}\} = \frac{2}{(s-5)^3}$.

(h) $\mathcal{L}\{7e^{3t} \cos 4t\} = \frac{7s-21}{(s-3)^2+16}$.

(i) $\mathcal{L}\{4e^{-t} \sin 5t\} = \frac{20}{(s+1)^2+25}$.

(j) If $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-3s}e^3}{s-1}$.

(k) If $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$ then $\mathcal{L}\{f(t)\} = e^{-s} \left(\frac{1}{s} + \frac{2}{s^3} \right)$.

(l) If $f(t) = \begin{cases} 0, & t < 1, \\ t-2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2}$.

(m) If $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{5}{s-2} - \frac{5e^6 e^{-3s}}{s-2}$.

(n) If $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{14}{(s+1)^3} - \frac{14e^{-3s-3}}{(s+1)^3} - \frac{42e^{-3s-3}}{(s+1)^2} - \frac{63e^{-3s-3}}{s+1}$.

(o) If $f(t) = \begin{cases} 0, & t < \pi, \\ \sin t, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s} \frac{1}{s^2+1}$.

(p) If $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s/2} \frac{1}{s^2+1} + e^{-\pi s} \frac{s}{s^2+1}$.

(q) $\mathcal{L}\{t^2 \sin 5t\} = \frac{40s^2 - 10(s^2+25)}{(s^2+25)^3}$.

(Answer 55)

(a) If $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$, then $y = 2e^{-t} \cos(2t) + 3e^{-t} \sin(2t)$

(b) If $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$, then $y = \frac{5}{2}t^2 - \frac{7}{6}t^3$

(c) If $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$, then $y = \frac{1}{2}t^2 e^{-t} + \frac{1}{6}t^3 e^{-t}$

(d) If $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$, then $y = (1/4)e^{2t} + (7/4)e^{-2t}$

(e) If $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$, then $y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{1}{9}te^{3t}$

(f) If $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$, then $y = \frac{1}{2} - \frac{1}{2} \cos(2t) + \frac{1}{2} \sin(2t)$

(g) If $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$, then $y = \frac{1}{8} \cos t - \frac{1}{8} \cos 3t$

(h) If $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$, then $y = 2u_2(t) e^{t-2} \cos(t-2) + u_2(t) e^{t-2} \sin(t-2)$

(i) If $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$, then $y = \frac{1}{2}u_1(t) e^{3(t-1)} + \frac{1}{2}u_1(t) e^{t-1}$

(j) If $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$, then $y = \int_0^t \frac{1}{3} \sin 3r \cos(3t-3r) dr = \frac{1}{6}t \sin(3t)$

(k) If $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$, then $y = e^{-2t} \int_0^t \sin(2r) \sin(2t-2r) dr = \frac{1}{4}e^{-2t} \sin(2t) - \frac{1}{2}e^{-2t}t \cos(2t)$

(l) If $\mathcal{L}\{y\} = \frac{s}{s^2-9} \mathcal{L}\{\sqrt{t}\}$, then $y = \int_0^t \frac{e^{3r} + e^{-3r}}{2} \sqrt{t-r} dr$.

(Answer 56)

- (a) If $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$, then $y(t) = -\frac{1}{30} \sin 3t - \frac{1}{90} \cos 3t + \frac{31}{30} e^{9t}$.
- (b) If $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$, then $y = 3te^{2t} + 2e^{2t}$.
- (c) If $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$, then $y = \frac{1}{5}t^3 - \frac{3}{25}t^2 + \frac{6}{125}t - \frac{6}{625} + \frac{1881}{625}e^{-5t}$.
- (d) If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$, then $y(t) = e^{2t} - te^{2t}$.
- (e) If $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$, then $y(t) = \frac{1}{5}(e^{-t} - e^t \cos t + 7e^t \sin t)$.
- (f) If $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$, then $y(t) = 4e^{3t} - 2e^{4t} - te^{3t}$.
- (g) If $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$, then $y(t) = -\frac{15}{8}e^{4t} + \frac{39}{8}e^{2t} - \frac{1}{4}te^{2t} - \frac{1}{2}t^2e^{2t} - t^3e^{2t}$.
- (h) If $\frac{d^2y}{dt^2} - 4y = e^t \sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{40}e^{2t} - \frac{1}{72}e^{-2t} - \frac{1}{90}e^t \cos(3t) - \frac{1}{45}e^t \sin(3t)$.
- (i) If $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$, then $y(t) = \frac{1}{5} \cos 2t + \cos 3t + \frac{5}{3} \sin 3t$.
- (j) If $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \leq t < 4, \\ 0, & 4 \leq t \end{cases}$, $y(0) = 2$, then

$$y(t) = \frac{2}{3} - \frac{2}{3}e^{-3t} - \frac{2}{3}\mathcal{U}(t-4) + \frac{2}{3}\mathcal{U}(t-4)e^{12-3t}.$$

The graph of y has a corner at $t = 4$. The graph of $\frac{dy}{dt}$ has a jump at $t = 4$.

- (k) If $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \leq t < 2\pi, \\ 0, & 2\pi \leq t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = (1/6)(1 - u_{2\pi}(t))(2 \sin t - \sin 2t).$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2\pi$.

- (l) If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \leq t < 10, \\ 0, & 10 \leq t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - u_{10}(t) \left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)} \right].$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 10$. The graph of $\frac{d^2y}{dt^2}$ has a jump discontinuity at $t = 10$.

- (m) If $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \leq t < 2, \\ 4, & 2 \leq t \end{cases}$, $y(0) = 3$, $y'(0) = 2$, $y''(0) = 1$, then

$$y(t) = 3e^{-t} + 5te^{-t} + 4t^2e^{-t} + \mathcal{U}(t-2)(4 - 4e^{2-t} - 4te^{2-t} - 2(t-2)^2e^{2-t}).$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2$. The graph of $\frac{d^3y}{dt^3}$ has a jump discontinuity at $t = 2$.

- (n) If $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \leq t < \pi, \\ 0, & \pi \leq t \end{cases}$, $y(0) = 1$, $y'(0) = 0$, then

$$\begin{aligned} y(t) &= e^{-t/2} \cos t + \frac{1}{2}e^{-t/2} \sin t \\ &+ u_{\pi}(t) \left(-\frac{16}{17} \cos(t-\pi) + \frac{4}{17} \sin(t-\pi) \right) \\ &+ u_{\pi}(t) \left(\frac{4}{17} e^{-(t-\pi)/2} \sin(t-\pi) + \frac{16}{17} e^{-(t-\pi)/2} \cos(t-\pi) \right). \end{aligned}$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = \pi$.

- (o) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \leq t < 2, \\ 3, & 2 \leq t \end{cases}$, $y(0) = 2$, $y'(0) = 1$, then

$$y(t) = 2e^{-2t} + 5te^{-2t} + \frac{3}{4}u_2(t) - \frac{3}{4}e^{-2t+4}u_2(t) - \frac{3}{2}(t-2)e^{-2t+4}u_2(t).$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 2$. The graph of $\frac{d^2y}{dt^2}$ has a jump at $t = 2$.

- (p) If $\frac{d^2y}{dt^2} + 9y = t \sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3} \int_0^t r \sin 3r \sin(3t - 3r) dr$.
- (q) If $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3} \int_0^t \sin 3r \sin(3t - 3r) dr = \frac{1}{6} \sin 3t - \frac{1}{2} t \cos 3t$.
- (r) If $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$, then $y(t) = 7e^{-2t} - 5e^{-3t} + \int_0^t (e^{-2r} - e^{-3r})\sqrt{t-r+1} dr$.
- (s) If $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4u_2(t)$, $y(0) = 0$, $y'(0) = 1$, then

$$y = 6e^{-t/3} - 6e^{-t/2} + 4u_2(t) - 12u_2(t)e^{-(t-2)/3} + 8u_2(t)e^{-(t-2)/2}.$$

The graph of $y'(t)$ has a corner at $t = 2$, and the graph of $y''(t)$ has a jump at $t = 2$. The graph of $y'(t)$ has a corner at $(2, 0.0768)$, and at $t = 2$, the graph of $y''(t)$ jumps from -0.2095 to 0.45712 .

- (t) If $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$, then $y(t) = 3e^{-9t} + 7\mathcal{U}(t-2)e^{-9t+18}$. The graph of $y(t)$ has a jump at $t = 2$.
- (u) If $\frac{d^2y}{dt^2} + 4y = \delta(t-4\pi)$, $y(0) = 1/2$, $y'(0) = 0$, then

$$y = \frac{1}{2} \cos(2t) - u_{4\pi}(t) \sin(2t).$$

The graph of $y(t)$ has a corner at $t = 4\pi$, and graph of $y'(t)$ has a jump at $t = 4\pi$. The graph of $y(t)$ has a corner at $(4\pi, 1/2)$, and at $t = 4\pi$, the graph of $y'(t)$ jumps from 0 to -2 . ($y''(t)$ is hard to graph, because the impulse function $\delta(t-4\pi)$ is part of $y''(t)$.)

- (v) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + u_2(t)$, $y(0) = 1$, $y'(0) = 0$, then

$$y = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} + \frac{1}{2}u_1(t)e^{-t+1} - \frac{1}{2}u_1(t)e^{-3t+3} + \frac{1}{3}u_2(t) - \frac{1}{2}e^{-t+2}u_2(t) + \frac{1}{6}u_2(t)e^{-3t+6}.$$

The graph of $y(t)$ has a corner at $t = 1$. The graph of $y'(t)$ has a corner at $t = 2$, and a jump at $t = 1$. $y''(t)$ has an impulse at $t = 1$, and a jump at $t = 2$. The graph of $y(t)$ has a corner at $(1, 0.5269)$. The graph of $y'(t)$ has a corner at $(2, -0.3085)$, and at $t = 1$ the graph of $y'(t)$ jumps from -0.4771 to 0.5229 . $y''(t)$ has an impulse at $t = 1$, and at $t = 2$, $y''(t)$ jumps from 0.1517 to 1.1517 .

- (w) If $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - \frac{dy}{dt} + 2y = 5\delta(t-4)$, $y(0) = 3$, $y'(0) = 0$, $y''(0) = 0$, then

$$y = \frac{3}{5}e^{2t} + \frac{4}{5} \cos t - \frac{2}{5} \sin t + \mathcal{U}(t-4)(e^{2t} - \cos t - 2 \sin t)$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 4$, and the graph of $\frac{d^2y}{dt^2}$ has a jump at $t = 4$. The function $\frac{d^3y}{dt^3}$ has an impulse at $t = 4$.

(Answer 57) Let t denote time (in minutes).

Let x denote the amount of salt (in grams) in tank A.

Let y denote the amount of salt (in grams) in tank B.

Then $x(0) = 3000$ and $y(0) = 2000$.

If $t < 50$, then

$$\frac{dx}{dt} = -\frac{9x}{200-4t} + \frac{3y}{300+3t}, \quad \frac{dy}{dt} = \frac{4x}{200-4t} - \frac{7y}{300+3t}.$$

(Answer 58)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 4t-5 \\ -2t+3 \end{pmatrix}.$$

(Answer 59)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} -3 \cos t + 16 \sin t \\ 2 \cos t + 7 \sin t \end{pmatrix}.$$

(Answer 60)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{3}{20} e^{-20t} \begin{pmatrix} 1 \\ 6 \end{pmatrix} + \frac{1}{20} e^{10t} \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

(Answer 61)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 5 \cos 12t + \sin 12t \\ 2 \cos 12t + 2 \sin 12t \end{pmatrix}.$$

(Answer 62)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos 3t - 11 \sin 3t \\ -3 \cos 3t - (23/3) \sin 3t \end{pmatrix}.$$

(Answer 63)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 15t + 1 \\ (15/4)t + 4 \end{pmatrix}.$$

(Answer 64)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} + e^{-6t} \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix}.$$

(Answer 65) If

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \quad \frac{dy}{dt} = -2x - 2y, \quad x(0) = 0, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} (38/45)t^{10} + (1/9)t^9 \\ -(29/45)t^{10} \end{pmatrix}.$$

(Answer 66) If

$$\frac{dx}{dt} = 5y + e^{2t} \cos t, \quad \frac{dy}{dt} = -x + 4y, \quad x(0) = 0, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} -t \sin t + (1/2)t \cos t + (1/2) \sin t \\ -(1/2)t \sin t \end{pmatrix}.$$

(Answer 67) If

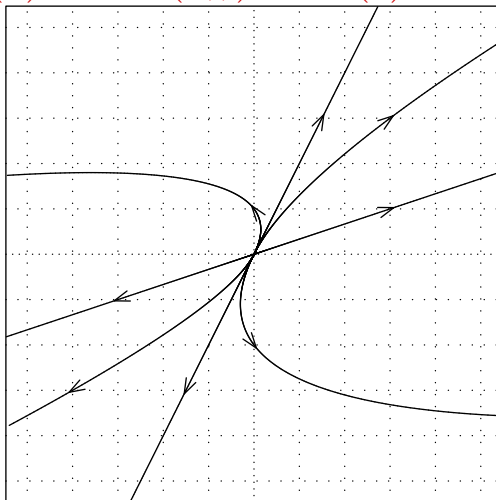
$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \quad \frac{dy}{dt} = -4x - 5y, \quad x(0) = 0, \quad y(0) = 0$$

then

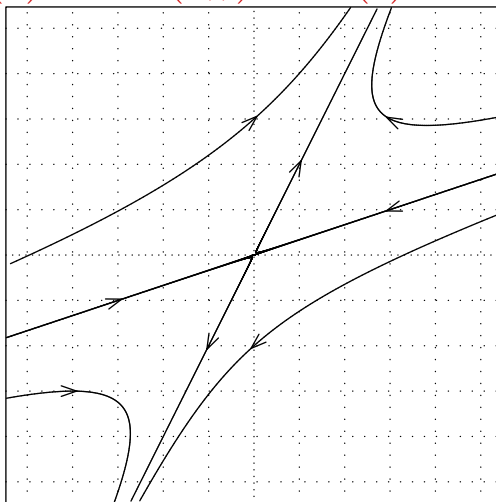
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 4 - 4 \cos(e^t) \\ 4 \cos(e^t) - 4 \end{pmatrix} + e^{-2t} \begin{pmatrix} 3 \sin(e^t) - 3e^t \cos(e^t) - 3 \sin 1 + 3 \cos 1 \\ 4e^t \cos(e^t) - 4 \sin(e^t) - 4 \cos 1 + 4 \sin 1 \end{pmatrix}.$$

(Answer 68)

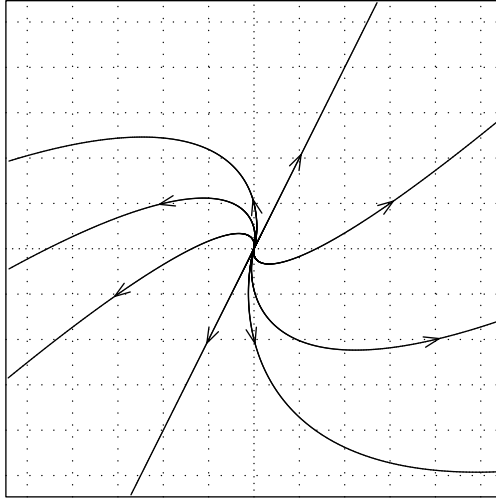
(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



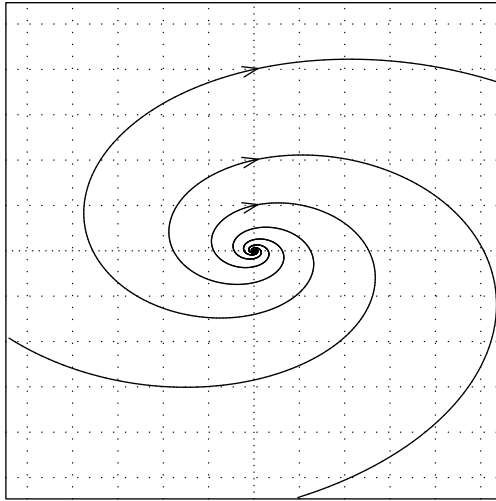
(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2.6 & 1.8 \\ -1.2 & 1.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



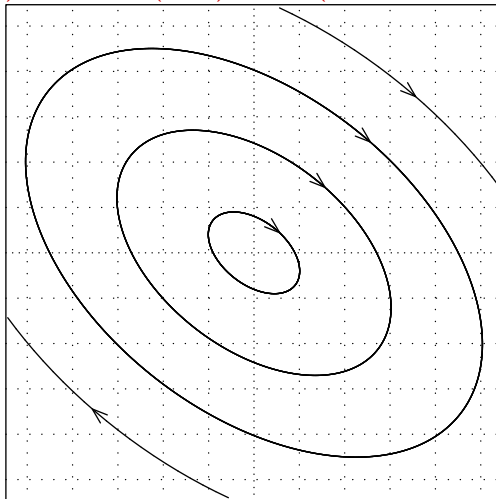
(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4/3 & -1/6 \\ 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^t \begin{pmatrix} t+3 \\ 2t \end{pmatrix}$



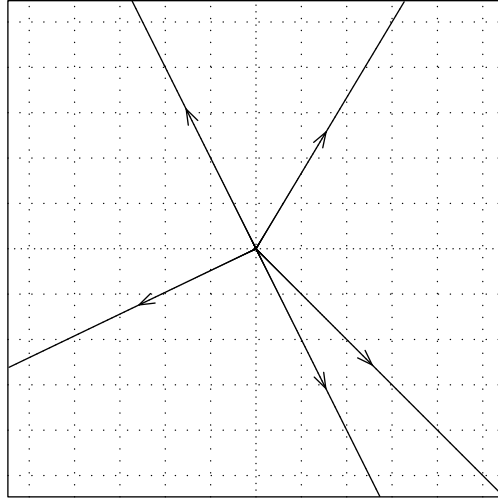
(d) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4.5 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 3 \sin 3t \\ 2 \cos 3t \end{pmatrix} + C_2 e^t \begin{pmatrix} -3 \cos 3t \\ 2 \sin 3t \end{pmatrix}$



(e) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ -4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 5 \sin 4t \\ 4 \cos 4t - 2 \sin 4t \end{pmatrix} + C_2 \begin{pmatrix} 5 \cos 4t \\ -4 \sin 4t - 2 \cos 4t \end{pmatrix}$

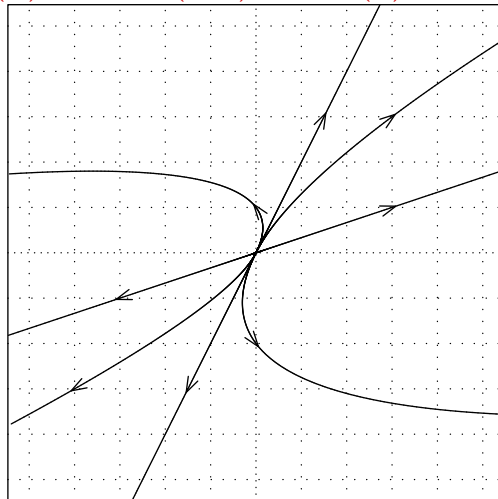


(f) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

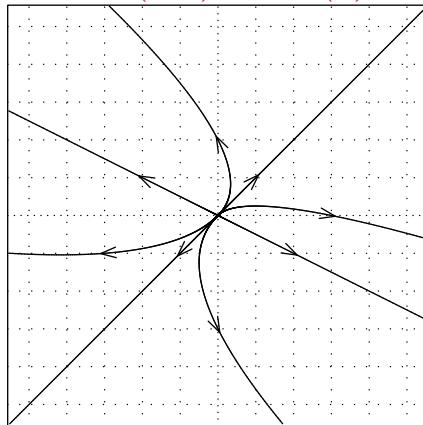


(Answer 69)

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

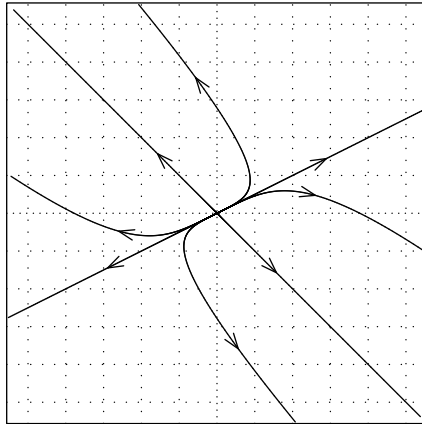


(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

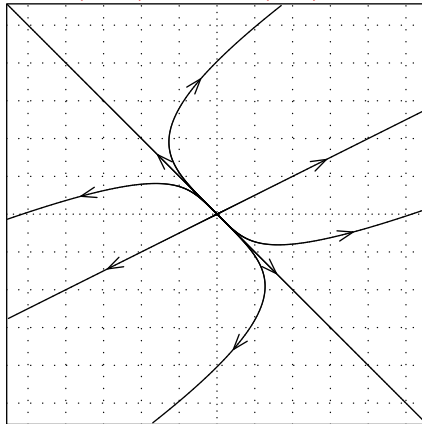


(Answer 70)

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

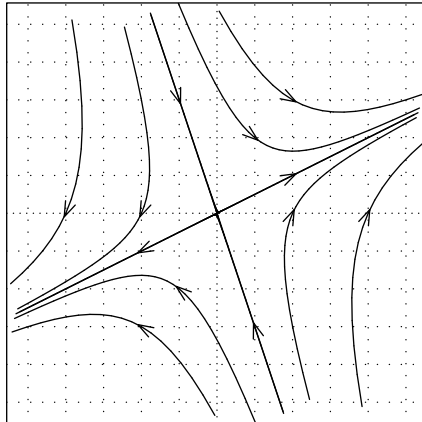


(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

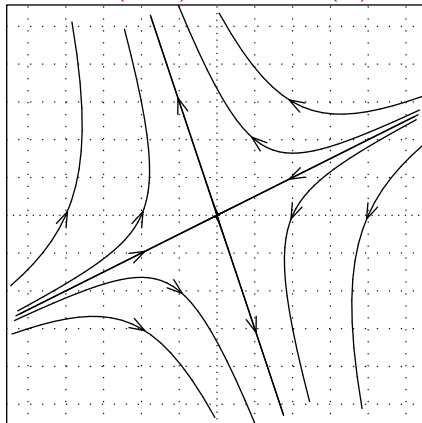


(Answer 71)

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 9 & -11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-14t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

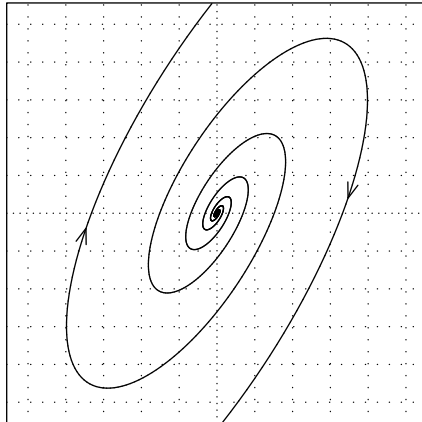


(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -4 & -6 \\ -9 & 11 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{-7t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{14t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

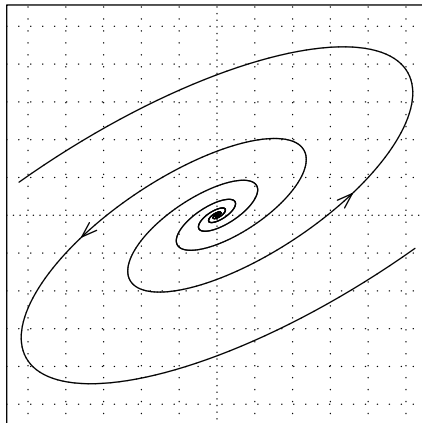


(Answer 72)

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -8 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, eigenvalues $r = 1 \pm 4i$



(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 5 & -8 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, eigenvalues $r = 1 \pm 4i$



(Answer 73)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 6 \sin 3t \\ 4 \sin 3t + 2 \cos 3t \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$