Math 2584, Fall 2017

If you need it, the following will be printed on the back of the exam:

Euler's formula states that, if θ is any real number, then $e^{i\theta} = \cos\theta + i\sin\theta$.

The acceleration of gravity (on Earth) is 9.8 meters/second^2; alternatively, gravity exerts a force of 9.8 newtons per kilogram.

If an object of mass M kilograms is subjected to a force of F newtons, and its position is x meters and its velocity is v meters/second^2, then Newton's second law states that $F = M\frac{dv}{dt} = M\frac{d^2x}{dt^2}$, where t denotes time (in seconds).

$$\begin{aligned}
\sin x\cos y &= \frac{1}{2}\sin(x+y) + \frac{1}{2}\sin(x-y)\\
\cos x\cos y &= \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)\\
\sin x\sin y &= \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)
\end{aligned}$$

Here are some Laplace transforms:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\begin{aligned}
\mathcal{L}\{1\} &= \frac{1}{s}, && s>0\\
\mathcal{L}\{e^{at}\} &= \frac{1}{s-a}, && s>a\\
\mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}}, && s>0, \qquad n\geq 0\\
\mathcal{L}\{\cos(kt)\} &= \frac{s}{s^2+k^2}, && s>0\\
\mathcal{L}\{\sin(kt)\} &= \frac{k}{s^2+k^2}, && s>0\\
\text{If } \mathcal{L}\{f(t)\} &= F(s) \text{ then } \mathcal{L}\{e^{at}f(t)\} = F(s-a),\\
\mathcal{L}^{-1}\{F(s)\} &= e^{at}\,\mathcal{L}^{-1}\{F(s+a)\},\\
\mathcal{L}\{\mathcal{U}(t-c)\} &= \frac{e^{-cs}}{s}, && s>0, \qquad c\geq 0\\
\mathcal{L}\{\delta(t-c)\} &= e^{-cs}, && s>0, \qquad c\geq 0\\
\mathcal{L}\{\mathcal{U}(t-c)f(t)\} &= e^{-cs}\,\mathcal{L}\{f(t+c)\}, && c\geq 0\\
\mathcal{L}\{\mathcal{U}(t-c)g(t-c)\} &= e^{-cs}\,\mathcal{L}\{g(t)\}, && c\geq 0\\
\mathcal{L}\left\{\frac{dy}{dt}\right\} &= s\,\mathcal{L}\{y(t)\} - y(0)
\end{aligned}$$

(AB 1) An object weighing 5 lbs stretches a spring 4 inches. It is attached to a viscous damper with damping constant 16 lb$\cdot$sec/ft. The object is pulled down an additional 2 inches and is released with initial velocity 3 ft/sec upwards.

Write the differential equation and initial conditions that describe the position of the object.

(AB 2) A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant β. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of β for which the system is critically damped. Be sure to include units for β.

(AB 3) A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 $\text{N} \cdot \text{s/m}$ and is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(20t)$ N upwards.

Write the differential equation and initial conditions that describe the position of the object.

(AB 4) An object weighing 4 lbs stretches a spring 2 inches. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(\omega t)$ pounds, directed upwards. There is no damping.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of ω for which resonance occurs. Be sure to include units for ω.

(AB 5) Suppose that $\frac{d^2x}{dt^2} = 18x^3$, $x(0) = 1$, $x'(0) = 3$. Let $v = \frac{dx}{dt}$. Find a formula for v in terms of x. Then find a formula for x in terms of t.

(AB 6) Suppose that a bob of mass 300g hangs from a pendulum of length 15cm. The pendulum is set in motion from its equilibrium point with an initial velocity of 3 meters/second.

Write the differential equation and initial conditions that describe the pendulum's motion. Then find a formula for the pendulum's angular velocity in terms of its position.

(AB 7) Suppose that a rocket of mass $m = 1000$ kg is launched straight up from the surface of the earth with initial velocity 10 km/sec. The radius of the earth is 6,371 km. When the rocket is r meters from the center of the earth, it experiences a force due to gravity of magnitude GMm/r^2, where $GM = 3.98 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the particle's position.
(b) Find the velocity of the particle as a function of position.
(c) How far away from the string is the particle when it stops moving and starts to fall back?

(AB 8) A particle of mass $m = 3$ kg a distance r from an infinitely long string experiences a force due to gravity of magnitude Gm/r, where $G = 2000$ meters^2/second^2, directed directly toward the string. Suppose that the particle is initially 1000 meters from the string and takes off with initial velocity 200 meters/second directly away from the string.

(a) Formulate the initial value problem for the particle's position.
(b) Find the velocity of the particle as a function of position.
(c) How far away from the string is the particle when it stops moving and starts to fall back?

(AB 9) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ (**not** the table on the front of the exam), find the Laplace transforms of the following functions.

(a) $f(t) = t$
(b) $f(t) = e^{-11t}$
(c) $f(t) = \begin{cases} 3, & 0 < t < 4, \\ 0, & 4 \leq t \end{cases}$.

(AB 10) Find the Laplace transforms of the following functions. You may use the table on the front of the exam.

(a) $f(t) = t^4 + 5t^2 + 4$
(b) $f(t) = (t+2)^3$
(c) $f(t) = 9e^{4t+7}$
(d) $f(t) = -e^{3(t-2)}$
(e) $f(t) = (e^t + 1)^2$
(f) $f(t) = 8\sin(3t) - 4\cos(3t)$
(g) $f(t) = t^2 e^{5t}$
(h) $f(t) = 7e^{3t}\cos 4t$
(i) $f(t) = 4e^{-t}\sin 5t$
(j) $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$
(k) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$
(l) $f(t) = \begin{cases} 0, & t < 1, \\ t - 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$
(m) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$
(n) $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$
(o) $f(t) = \begin{cases} 0, & t < \pi, \\ \sin t, & t \geq \pi \end{cases}$
(p) $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi \end{cases}$

(AB 11) For each of the following problems, find y.

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$
(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$
(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$
(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$
(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$
(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$
(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$
(h) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$
(i) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$

(AB 12) Solve the following initial-value problems using the Laplace transform.

(a) $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$

(b) $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$

(c) $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

(e) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

(f) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$

(g) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$

(h) $\frac{d^2y}{dt^2} - 4y = e^t \sin(3t)$, $y(0) = 0$, $y'(0) = 0$

(i) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$

(j) $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

(k) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

(l) $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t \end{cases}$, $y(0) = 1$, $y'(0) = 0$

(m) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$

(AB 13) Solve the following initial-value problems using the Laplace transform.

(a) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4u_2(t)$, $y(0) = 0$, $y'(0) = 1$.

(b) $\frac{dy}{dt} + 9y = 7\delta(t - 2)$, $y(0) = 3$.

(c) $\frac{d^2y}{dt^2} + 4y = -2\delta(t - 4\pi)$, $y(0) = 1/2$, $y'(0) = 0$

(d) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t - 1) + u_2(t)$, $y(0) = 1$, $y'(0) = 0$

Answer key

(Answer 1) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then

$$\frac{5}{32}\frac{d^2x}{dt^2}+16\frac{dx}{dt}+15x=0, \qquad x(0)=-\frac{1}{6}, \quad x'(0)=3.$$

(Answer 2) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$2\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+80x=0, \qquad x(0)=-0.1, \quad x'(0)=0.$$

Critical damping occurs when $\beta = 8\sqrt{10}$ N$\cdot$s/m.

(Answer 3) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters).

Then

$$3\frac{d^2x}{dt^2}+27\frac{dx}{dt}+588x=3\cos(20t), \qquad u(0)=0, \quad u'(0)=0.$$

(Answer 4) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then

$$\frac{4}{32}\frac{d^2x}{dt^2}+24x=3\cos(\omega t), \qquad x(0)=0, \quad x'(0)=0.$$

Resonance occurs when $\omega = 8\sqrt{3}$ s^{-1}.

(Answer 5) We have that $\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=v\frac{dv}{dx}$ and also $\frac{dv}{dt}=\frac{d^2x}{dt^2}$. Thus $v\frac{dv}{dx}=18x^3$, $v(1)=3$. Solving, we see that $v=3x^2$.

But then $\frac{dx}{dt}=3x^2$, $x(0)=1$, and so $x=\frac{1}{1-3t}$.

(Answer 6) Let θ be the angle between the pendulum and a vertical line (in radians), and let t denote time in seconds. Then

$$0.3\frac{d^2\theta}{dt^2}=-\frac{9.8}{0.5}\sin\theta, \quad \theta(0)=0, \quad \theta'(0)=20.$$

Let $\omega=\frac{d\theta}{dt}$ be the pendulum's angular velocity. We have that $\frac{d\omega}{dt}=\frac{d\omega}{d\theta}\frac{d\theta}{dt}=\omega\frac{d\omega}{d\theta}$ and also $\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2}$. Thus $\omega\frac{d\omega}{d\theta}=-\frac{196}{3}\sin\theta$ and $\omega(0)=20$. Solving, we see that $\frac{1}{2}\omega^2=\frac{196}{3}\cos\theta+\frac{404}{3}$.

(Answer 7)

(a) The initial value problem is

$$1000\frac{d^2r}{dt^2} = -\frac{3.98\times10^{17}}{r^2}, \quad r(0) = 6371000, \quad r'(0) = 10000$$

where r denotes the distance to the center of the earth in meters and t denotes time in seconds.

(b) Let v be the particle's velocity in meters/second. We have that

$$1000v\frac{dv}{dr} = -\frac{3.98\times10^{17}}{r^2}, \quad v(6371000) = 10000$$

and so

$$v^2 = \frac{7.96\times10^{14}}{r} - 2.494\times10^7$$

(c) $v = 0$ when $r = 3.191\times10^7$ meters.

(Answer 8)

(a) The initial value problem is

$$3\frac{d^2r}{dt^2} = -\frac{6000}{r}, \quad r(0) = 1000, \quad r'(0) = 200$$

where r denotes the distance to the string in meters and t denotes time in seconds.

(b) Let v be the particle's velocity in meters/second. We have that

$$3v\frac{dv}{dr} = -\frac{6000}{r}, \quad v(1000) = 200$$

and so

$$v^2 = -4000\ln r + 4000\ln 1000 + 40000.$$

(c) $v = 0$ when $r = 1000e^{10}$ meters.

(Answer 9)

(a) $\mathcal{L}\{t\} = \frac{1}{s^2}$.

(b) $\mathcal{L}\{e^{-11t}\} = \frac{1}{s+11}$.

(c) $\mathcal{L}\{f(t)\} = \frac{3-3e^{-4s}}{s}$.

(Answer 10)

(a) $\mathcal{L}\{t^4+5t^2+4\} = \frac{96}{s^5}+\frac{10}{s^3}+\frac{4}{s}$.

(b) $\mathcal{L}\{(t+2)^3\} = \mathcal{L}\{t^3+6t^2+12t+8\} = \frac{6}{s^4}+\frac{12}{s^3}+\frac{12}{s^2}+\frac{8}{s}$.

(c) $\mathcal{L}\{9e^{4t+7}\} = \mathcal{L}\{9e^7e^{4t}\} = \frac{9e^7}{s-4}$.

(d) $\mathcal{L}\{-e^{3(t-2)}\} = \mathcal{L}\{-e^{-6}e^{3t}\} = -\frac{e^{-6}}{s-3}$.

(e) $\mathcal{L}\{(e^t+1)^2\} = \mathcal{L}\{e^{2t}+2e^t+1\} = \frac{1}{s-2}+\frac{2}{s-1}+\frac{1}{s}$.

(f) $\mathcal{L}\{8\sin(3t)-4\cos(3t)\} = \frac{24}{s^2+9}-\frac{4s}{s^2+9}$.

(g) $\mathcal{L}\{t^2e^{5t}\} = \frac{2}{(s-5)^3}$.

(h) $\mathcal{L}\{7e^{3t}\cos 4t\} = \frac{7s-21}{(s-3)^2+16}$.

(i) $\mathcal{L}\{4e^{-t}\sin 5t\} = \frac{20}{(s+1)^2+25}$.

(j) If $f(t) = \begin{cases} 0, & t<3, \\ e^t, & t\geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-3s}e^3}{s-1}$.

(k) If $f(t) = \begin{cases} 0, & t<1, \\ t^2-2t+2, & t\geq 1, \end{cases}$ then $\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{1}{s}+\frac{2}{s^3}\right)$.

(l) If $f(t) = \begin{cases} 0, & t<1, \\ t-2, & 1\leq t<2, \\ 0, & t\geq 2 \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2}-\frac{e^{-s}}{s}-\frac{e^{-2s}}{s^2}$.

(m) If $f(t) = \begin{cases} 5e^{2t}, & t<3, \\ 0, & t\geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{5}{s-2}-\frac{5e^6e^{-3s}}{s-2}$.

(n) If $f(t) = \begin{cases} 7t^2e^{-t}, & t<3, \\ 0, & t\geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{14}{(s+1)^3}-\frac{14e^{-3s-3}}{(s+1)^3}-\frac{42e^{-3s-3}}{(s+1)^2}-\frac{63e^{-3s-3}}{s+1}$.

(o) If $f(t) = \begin{cases} 0, & t<\pi, \\ \sin t, & t\geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s}\frac{1}{s^2+1}$.

(p) If $f(t) = \begin{cases} 0, & t<\pi/2, \\ \cos t, & \pi/2\leq t<\pi, \\ 0, & t\geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s/2}\frac{1}{s^2+1}+e^{-\pi s}\frac{s}{s^2+1}$.

(Answer 11)

(a) If $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$, then $y = 2e^{-t}\cos(2t)+3e^{-t}\sin(2t)$

(b) If $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$, then $y = \frac{5}{2}t^2-\frac{7}{6}t^3$

(c) If $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$, then $y = \frac{1}{2}t^2e^{-t}+\frac{1}{6}t^3e^{-t}$

(d) If $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$, then $y = (1/4)e^{2t}+(7/4)e^{-2t}$

(e) If $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$, then $y = \frac{2}{27}+\frac{1}{9}t-\frac{2}{27}e^{3t}+\frac{1}{9}t\,e^{3t}$

(f) If $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$, then $y = \frac{1}{2}-\frac{1}{2}\cos(2t)+\frac{1}{2}\sin(2t)$

(g) If $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$, then $y = \frac{1}{8}\cos t-\frac{1}{8}\cos 3t$

(h) If $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$, then $y = 2\,u_2(t)\,e^{t-2}\,\cos(t-2)+u_2(t)\,e^{t-2}\,\sin(t-2)$

(i) If $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$, then $y = \frac{1}{2}u_1(t)\,e^{3(t-1)}+\frac{1}{2}u_1(t)\,e^{t-1}$

(Answer 12)

(a) If $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$, then $y(t) = -\frac{1}{30}\sin 3t - \frac{1}{90}\cos 3t + \frac{31}{30}e^{9t}$.

(b) If $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$, then $y = 3te^{2t} + 2e^{2t}$.

(c) If $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$, then $y = \frac{1}{5}t^3 - \frac{3}{25}t^2 + \frac{6}{125}t - \frac{6}{625} + \frac{1881}{625}e^{-5t}$.

(d) If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$, then $y(t) = e^{2t} - te^{2t}$.

(e) If $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$, then $y(t) = \frac{1}{5}(e^{-t} - e^t\cos t + 7e^t\sin t)$.

(f) If $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$, then $y(t) = 4e^{3t} - 2e^{4t} - te^{3t}$.

(g) If $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$, then $y(t) = -\frac{15}{8}e^{4t} + \frac{39}{8}e^{2t} - \frac{1}{4}te^{2t} - \frac{1}{2}t^2e^{2t} - t^3e^{2t}$.

(h) If $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{40}e^{2t} - \frac{1}{72}e^{-2t} - \frac{1}{90}e^t\cos(3t) - \frac{1}{45}e^t\sin(3t)$.

(i) If $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$, then $y(t) = \frac{1}{5}\cos 2t + \cos 3t + \frac{5}{3}\sin 3t$.

(j) If $\frac{d^2y}{dt^2} + 4y = \begin{cases}\sin t, & 0 \le t < 2\pi,\\ 0, & 2\pi \le t\end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = (1/6)(1 - u_{2\pi}(t))(2\sin t - \sin 2t).$$

(k) If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases}1, & 0 \le t < 10,\\ 0, & 10 \le t\end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - u_{10}(t)\left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right].$$

(l) If $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases}\sin t, & 0 \le t < \pi,\\ 0, & \pi \le t\end{cases}$, $y(0) = 1$, $y'(0) = 0$, then

$$\begin{aligned} y(t) = {}& e^{-t/2}\cos t + \frac{1}{2}e^{-t/2}\sin t \\ & - u_\pi(t)\left(\frac{16}{17}\cos(t-\pi) - \frac{4}{17}\sin(t-\pi)\right) \\ & + u_\pi(t)\left(\frac{4}{17}e^{-(t-\pi)/2}\sin(t-\pi) + \frac{16}{17}e^{-(t-\pi)/2}\cos(t-\pi)\right). \end{aligned}$$

(m) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases}0, & 0 \le t < 2,\\ 3, & 2 \le t\end{cases}$, $y(0) = 2$, $y'(0) = 1$, then $y(t) = 2e^{-2t} + 5te^{-2t} + \frac{3}{4}u_2(t) - \frac{3}{4}e^{-2t+4}u_2(t) - \frac{3}{2}(t-2)e^{-2t+4}u_2(t)$.

(Answer 13)

(a) If $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4u_2(t)$, $y(0) = 0$, $y'(0) = 1$, then

$$y = 6e^{-t/3} - 6e^{-t/2} + 4u_2(t) - 12u_2(t)e^{-(t-2)/3} + 8u_2(t)e^{-(t-2)/2}.$$

The graph of $y'(t)$ has a corner at $t = 2$, and the graph of $y''(t)$ has a jump at $t = 2$.

(b) If $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$, then $y(t) = 3e^{-9t} + 7\mathcal{U}(t-2)e^{-9t+18}$.

(c) If $\frac{d^2y}{dt^2} + 4y = \delta(t - 4\pi)$, $y(0) = 1/2$, $y'(0) = 0$, then

$$y = \frac{1}{2}\cos(2t) - u_{4\pi}(t)\sin(2t).$$

The graph of $y(t)$ has a corner at $t = 4\pi$, and graph of $y'(t)$ has a jump at $t = 4\pi$. ($y''(t)$ is hard to graph, because the impulse function $\delta(t - 4\pi)$ is part of $y''(t)$.)

(d) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + u_2(t)$, $y(0) = 1$, $y'(0) = 0$, then

$$y = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} + \frac{1}{2}u_1(t)e^{-t+1} - \frac{1}{2}u_1(t)e^{-3t+3} + \frac{1}{3}u_2(t) - \frac{1}{2}e^{-t+2}u_2(t) + \frac{1}{6}u_2(t)e^{-3t+6}.$$

The graph of $y(t)$ has a corner at $t = 1$. The graph of $y'(t)$ has a corner at $t = 2$, and a jump at $t = 1$. $y''(t)$ has an impulse at $t = 1$, and a jump at $t = 2$.