Written homework, Math 2574, Fall 2017

(AB 1) Find parametric equations that describe the ellipse $\left\{(x,y) : \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1\right\}$.

(AB 2) Write the equation for the sphere centered at the point $(3, 2, 4)$ and passing through the point $(2, 4, 2)$.

(AB 3) Find the angle between the vectors $\langle 3, 2, 4\rangle$ and $\langle 5, 1, -3\rangle$.

(AB 4) Compute the cross product $\langle 1, 2, 0\rangle \times \langle 5, -1, 1\rangle$.

(AB 5) Find the area of the triangle with vertices at $(1, 0, -1)$, $(3, 2, 5)$, and $(7, -1, 3)$.

(AB 6) Let $\vec{r}(t) = \langle t\cos t, t\sin t\rangle$, $0 < t < 4\pi$. Sketch the curve parameterized by $\vec{r}(t)$. Then find the unit tangent vector to this curve.

(AB 7) Consider the two parameterizations $\vec{r}(t) = \langle \tan t, \sec t\rangle$, $0 < t < \pi/4$, and $\vec{R}(t) = \langle t, \sqrt{1+t^2}\rangle$, $c < t < d$.

(*a*) Find equations for the given curves in terms of x and y. Then find values of c and d such that $\vec{r}$ and $\vec{R}$ parameterize the same curve. Show all your work and carefully explain your reasoning.

(*b*) Find the arc length of the curve parameterized by $\vec{r}(t) = \langle \tan t, \sec t\rangle$, $0 < t < \pi/4$. (Your final answer should be in the form of a definite integral; you don't have to evaluate the integral.)

(*c*) Find the arc length of the curve parameterized by $\vec{R}(t) = \langle t, \sqrt{1+t^2}\rangle$, $c < t < d$. (Your final answer should be in the form of a definite integral.)

(*d*) (*Extra credit, 1 point*) Numerically evaluate the two integrals.

(*e*) Find a u-substitution or trigonometric substitution and show (using analytic methods, not numerical evaluation) that the two integrals are equal. Show all your work and carefully explain your reasoning.

(AB 8) Find the arc length of the cardioid with polar coordinates $r = 1 + \sin\theta$, $0 \le \theta < 2\pi$.

(AB 9) The parameterizations $\vec{r}(t) = \langle \sin t, \cos t\rangle$, $0 \le t \le \pi/2$, and $\vec{R}(u) = \langle \sqrt{u}, \sqrt{1-u}\rangle$, $0 \le t \le 1$, describe the same curve. Find a scalar-valued function $f(t)$ such that $\vec{r}(t) = \vec{R}(f(t))$.

(AB 10) Find an arc length parameterization for the curve $\vec{r}(t) = \langle t^3, 3t^2, t^3\rangle$, $1 \le t \le 5$. Be sure to include a range for your parameter.

(AB 11) Let $\vec{r}(t) = \langle 2\sqrt{t}\cos\sqrt{t} - 2\sin\sqrt{t}, 2\sqrt{t}\sin\sqrt{t} + 2\cos\sqrt{t}\rangle$, $0 \le t < \infty$.

(*a*) Show that $\vec{r}(t)$ is an arc length parameterization.

(*b*) Find the curvature $\kappa(t)$.

(*c*) Find the normal vector $\vec{N}(t)$.

(AB 12) Let $\vec{r}(t) = \langle t, t^2 \rangle$.

(a) Find $\vec{r}(0)$, $\kappa(0)$ and $\vec{N}(0)$.

(b) Using a computer, plot $\vec{r}(t)$ and the circle of radius $1/\kappa(0)$ centered at the point $\vec{r}(0) + \dfrac{1}{\kappa(0)}\vec{N}(0)$. A good way to graph functions is using Desmos, `https://www.desmos.com/calculator/selfsk9li6`

(c) Plot $\vec{r}(t)$ and the circle of radius R centered at the point $\vec{r}(0) + R\vec{N}(0)$ for $R = 1/4$, $1/3$, $2/3$, 1 and 2. Do these appear to be better or worse approximations to $\vec{r}(t)$ near $t = 0$?

(d) Find $\vec{r}(0.1)$.

(e) Find the points on the circle of radius R, centered at the point $\vec{r}(0) + R\vec{N}(0)$, with x-coordinate 0.1, for $R = 1/\kappa(0)$, $1/4$, $1/3$, $2/3$, 1 and 2. Express your answers to five decimal places. Which point is closest to $\vec{r}(0.1)$?

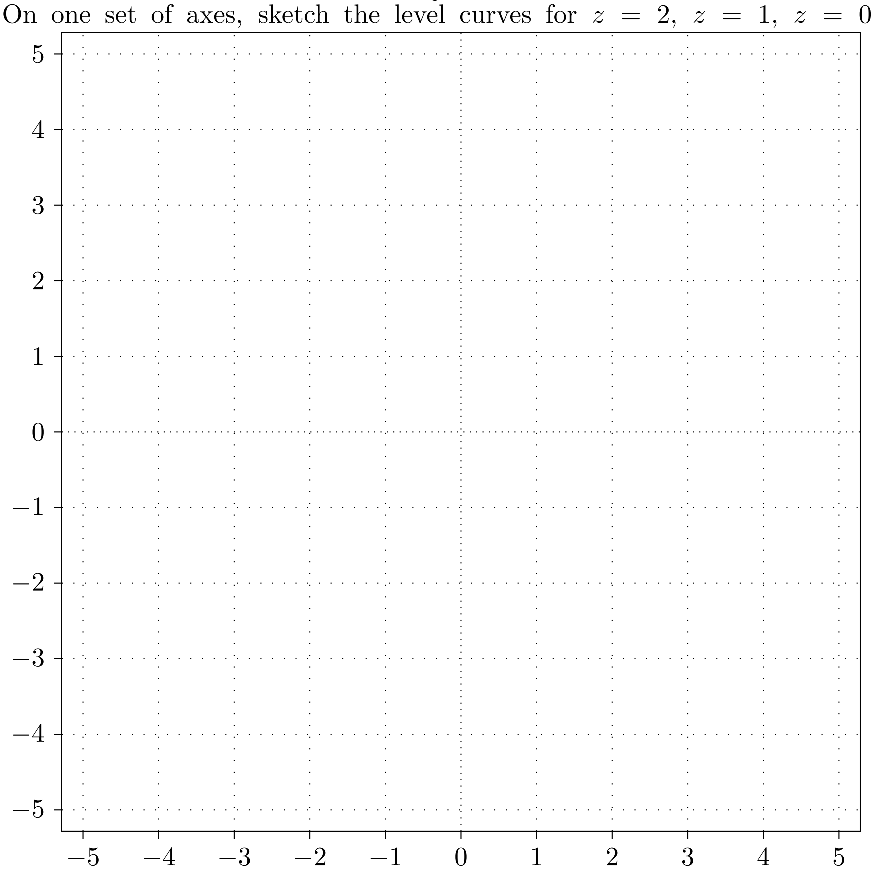
(AB 13) Let R be the plane $\{(x, y, z) : 5x - 4y + 3z = 7\}$, and let Q be the plane $\{(x, y, z) : 2x + 3y - z = 1\}$.

(a) Find a parametric equation for the intersection of these two planes.

(b) Find two non-parametric equations that describe the intersection of these two planes. Simplify your answer as much as possible.

(AB 14) Consider the equation $z = \dfrac{y^2}{4} - \dfrac{x^2}{9}$.

(a) On one set of axes, sketch the level curves for $z = 2$, $z = 1$, $z = 0$, $z = -1$ and $z = -2$.



(b) Classify the surface $z = x^2 - y^2$. Is it an ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, cone, elliptic paraboloid, or hyperbolic paraboloid?

(AB 15) What is the domain of the function $f(x, y) = \operatorname{arcsec}(x^2 + y^2) + \ln(x)$?

(AB 16) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^3}{x^2 + y^6}$ does not exist.

(AB 17) Let $V = \pi r^2 h$. Suppose that r and h are both functions of t. We have that $r(1) = 5$, $h(1) = 7$, $r'(1) = 2$ and $h'(1) = 3$. Find $\dfrac{dV}{dt}$ at $t = 1$.

(AB 18) Let $f(x,y) = x^2 + xy + y^2$.

(a) Let $g(h) = f(3 + \frac{3}{5}h, -2 + \frac{4}{5}h)$. Find a formula for $g(h)$. Then find $g'(0)$.

(b) Use the limit definition of the directional derivative to find $D_{\langle 3/5, 4/5\rangle} f(3,-2)$.

(c) Find $\nabla f(3,-2)$ and verify that $\langle 3/5, 4/5\rangle \cdot \nabla f(3,-2) = D_{\langle 3/5,4/5\rangle} f(3,-2)$.

(AB 19) Find and classify all the critical points of the function $f(x,y) = 6x^2 e^y - 3x^4 - e^{6y}$.

(AB 20) Consider the function $f(x,y) = 12xe^y - x^3 - 6e^{4y}$. This function has one local extremum.

(a) Where does the local extremum occur?

(b) Is the local extremum a local maximum or a local minimum?

(c) Show that $f(x,y)$ does not have a global extremum by finding points (c,d) and (e,g) such that $f(c,d) < f(a,b) < f(e,g)$, where (a,b) is the point you found in part (a).

(AB 21) Let $f(x,y) = 4x^2 + 9y^2 + 6x$. Find the absolute maximum and minimum values of $f(x,y)$ on the region $x^2 + y^2 \le 1$ or state that they do not exist.

(AB 22) Find the volume of the solid below the hyperbolic paraboloid $z = 3 + x^2 - y^2$ and above the rectangle $\{(x,y) : 3 \le x \le 4,\ 0 \le y \le 2\}$.

(AB 23) Evaluate the integral $\displaystyle\int_0^3 \int_{x^2}^9 \frac{x^7}{y^5+1}\,dy\,dx$. *Hint*: Change the order of integration.

(AB 24) Find $\displaystyle\iint_R 2xy\,dA$, where $R = \{(x,y) : y \ge x,\ x^2 + y^2 < 9\}$.

(AB 25) Let D be the region $\{(x,y,z) : x^2 + y^2 + z^2 < 25,\ x > 0,\ z^2 > x^2 + y^2\}$.

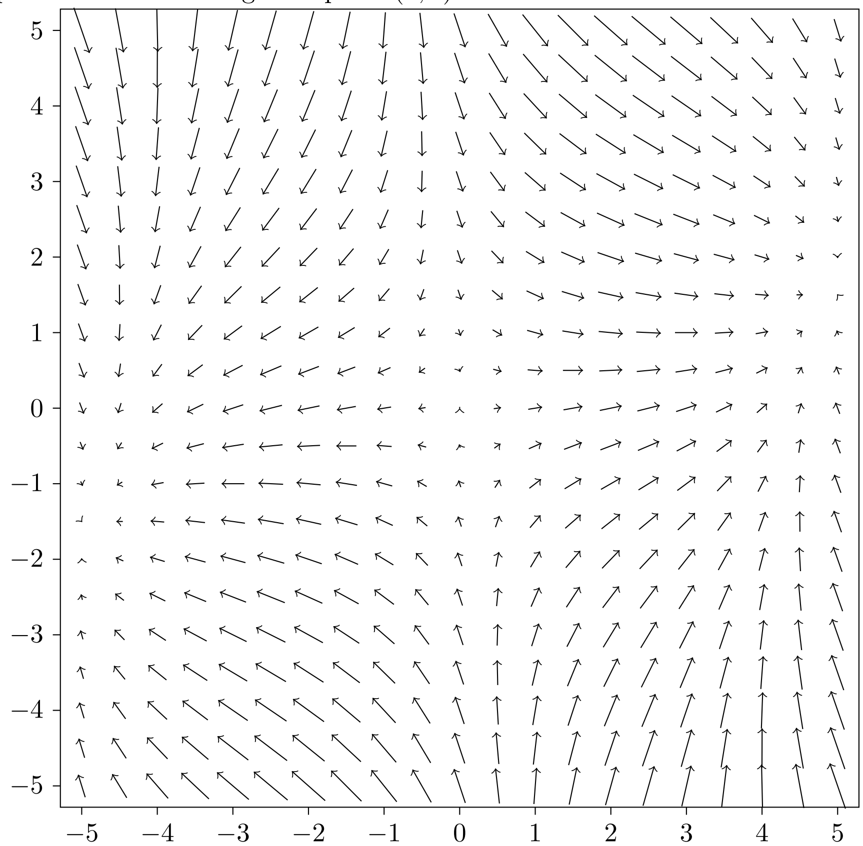
(a) Describe D using spherical coordinates.

(b) Find the volume of D.

(AB 26) Evaluate the integral $\displaystyle\int\int_R \sqrt{4y^2 - x^2}\,dy\,dx$, where R is the triangle with vertices $(1,1)$, $(-1,2)$ and $(5,3)$.

(AB 27) Find the area of the region $\{(x,y) : (3x-y)^2 + (x+2y)^2 < 1\rangle$.

(AB 28) Here is a graph of a vector field $\vec{F} = \nabla\varphi$ for some potential function φ. Sketch the flow line and equipotential curve through the point $(2, 2)$.



(AB 29) Find the flux of $\vec{F}(x, y) = \langle x, y\rangle$ across the curve C, where C is the triangle with vertices at $(0, 0)$, $(0, 1)$ and $(1, 0)$ oriented counterclockwise.

(AB 30) Let C be the curve parameterized by $\vec{r}(t) = \langle t^3 + 3t + 1, 4t^2 - 4t + 7\rangle$, $0 \le t \le 7$. Find $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = \langle 3x^2y^2 + x, 2x^3y\rangle$.

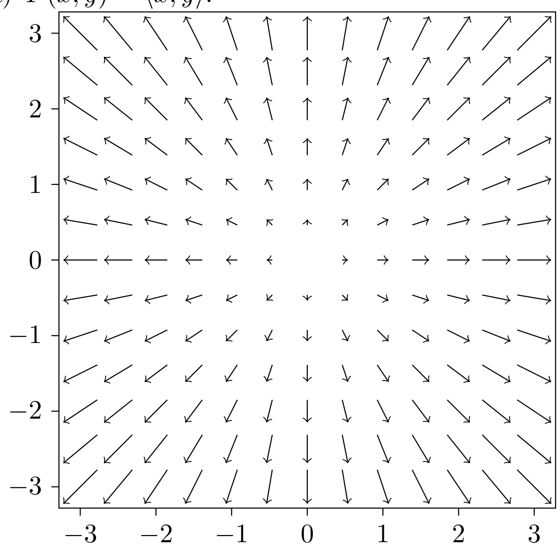
(AB 31) For each of the following vector fields,

(i) Draw or visualize a simple closed curve oriented counterclockwise. (You don't need to print out your curve; nothing need be turned in for this step.)

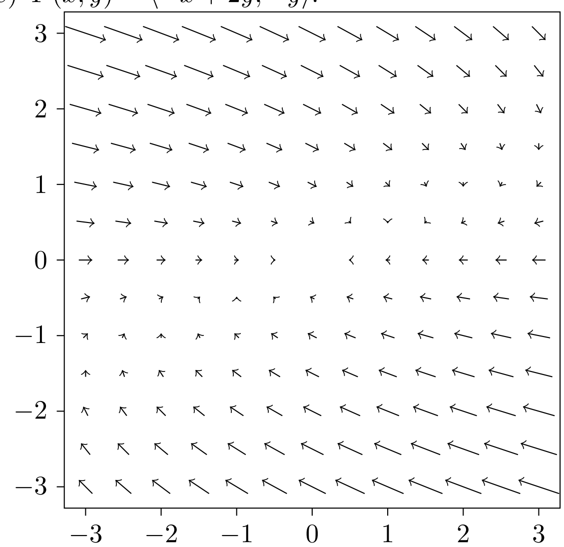
(ii) Is the outward flux across your curve positive or negative?

(iii) Compute the divergence of the vector field.

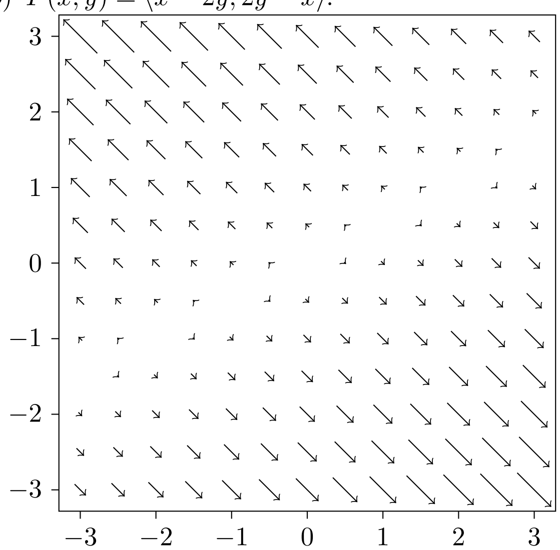
(a) $\vec{F}(x, y) = \langle x, y \rangle$.



(c) $\vec{F}(x, y) = \langle -x + 2y, -y \rangle$.



(b) $\vec{F}(x, y) = \langle x - 2y, 2y - x \rangle$.



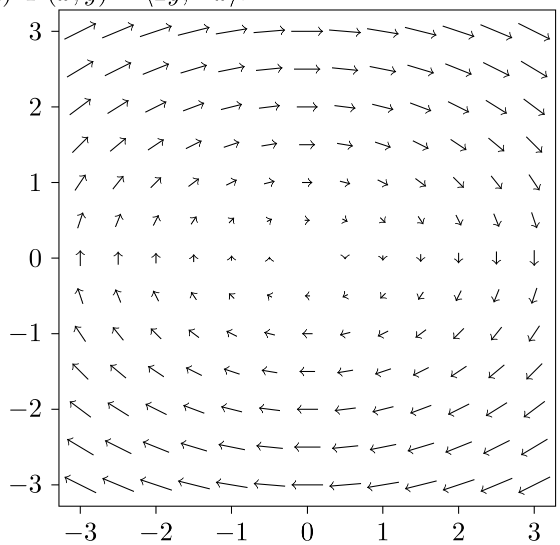
(AB 32) For each of the following vector fields,

(i) Draw or visualize a simple closed curve oriented counterclockwise. (You don't need to print out your curve; nothing need be turned in for this step.)

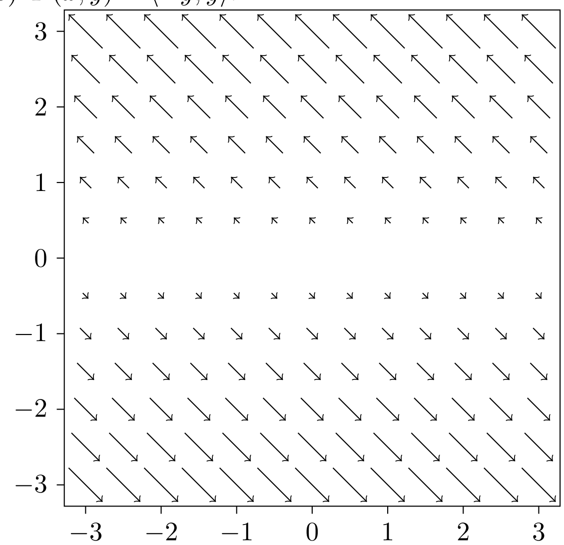
(ii) Is the circulation along your curve positive or negative?

(iii) Compute the curl of the vector field. (The two-dimensional curl of the vector field $\vec{F}(x,y) = \langle f(x,y), g(x,y)\rangle$ is the scalar $\dfrac{\partial f}{\partial y} - \dfrac{\partial g}{\partial x}$.)

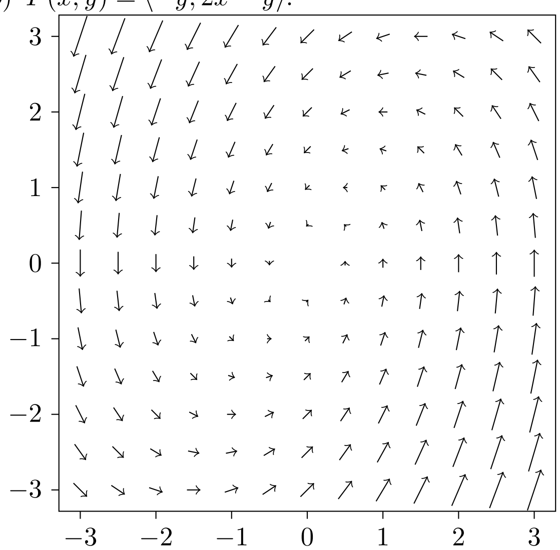
(a) $\vec{F}(x,y) = \langle 2y, -x\rangle$.



(c) $\vec{F}(x,y) = \langle -y, y\rangle$.



(b) $\vec{F}(x,y) = \langle -y, 2x - y\rangle$.



(AB 33) Write (but do not evaluate) a double integral giving the area of the ellipsoidal cap $\{(x,y,z) : (x/3)^2 + (y/4)^2 + (z/12)^2 = 1,\ z > 4\}$.

(AB 34) Consider the surface $z = x^2 + y^2$. Find a formula for the upward pointing unit normal vector to this surface.

Answer key

(AB 1) Find parametric equations that describe the ellipse $\left\{(x,y) : \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1\right\}$.

(Answer 1) The ellipse $\left\{(x,y) : \left(\frac{x}{3}\right)^2 + \left(\frac{y}{4}\right)^2 = 1\right\}$ is described by the parametric equations

$$x = 3\cos t, \quad y = 4\sin t, \quad 0 \le t < 2\pi.$$

(AB 2) Write the equation for the sphere centered at the point $(3,2,4)$ and passing through the point $(2,4,2)$.

(Answer 2) $(x-3)^2 + (y-2)^2 + (z-4)^2 = 9$.

(AB 3) Find the angle between the vectors $\langle 3,2,4\rangle$ and $\langle 5,1,-3\rangle$.

(Answer 3) We compute $\langle 3,2,4\rangle \cdot \langle 5,1,-3\rangle = 5$, $|\langle 3,2,4\rangle| = \sqrt{29}$, and $|\langle 5,1,-3\rangle| = \sqrt{35}$. Let θ be the angle between the two vectors. Then $\sqrt{29}\sqrt{35}\cos\theta = 5$, so $\boxed{\theta = \arccos \frac{5}{\sqrt{29 * 35}} = 1.4132 \text{ radians}}$.

(AB 4) Compute the cross product $\langle 1,2,0\rangle \times \langle 5,-1,1\rangle$.

(Answer 4) $\langle 1,2,0\rangle \times \langle 5,-1,1\rangle = \langle 2,-1,-11\rangle$.

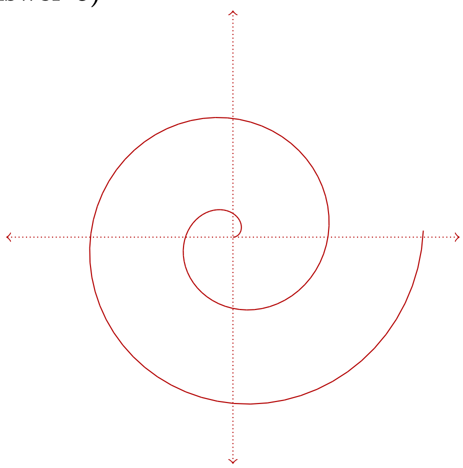
(AB 5) Find the area of the triangle with vertices at $(1,0,-1)$, $(3,2,5)$, and $(7,-1,3)$.

(Answer 5) The triangle has edges $\langle 2,2,6\rangle$, $\langle 4,-3,-2\rangle$ and $\langle 6,-1,4\rangle$. It therefore has area

$$A = \frac{1}{2}|\langle 2,2,6\rangle \times \langle 4,-3,-2\rangle| = \frac{1}{2}|\langle 14,28,-14\rangle| = \boxed{7\sqrt{6}.}$$

(AB 6) Let $\vec{r}(t) = \langle t\cos t, t\sin t\rangle$, $0 < t < 4\pi$. Sketch the curve parameterized by $\vec{r}(t)$. Then find the unit tangent vector to this curve.

(Answer 6)



$\vec{r}\,'(t) = \langle \cos t - t\sin t, \sin t + t\cos t\rangle$, so

$$\vec{T}(t) = \frac{\langle \cos t - t\sin t, \sin t + t\cos t\rangle}{|\langle \cos t - t\sin t, \sin t + t\cos t\rangle|} = \left\langle \frac{\cos t - t\sin t}{\sqrt{1+t^2}}, \frac{\sin t + t\cos t}{\sqrt{1+t^2}} \right\rangle.$$

(AB 7) Consider the two parameterizations $\vec{r}(t) = \langle \tan t, \sec t\rangle$, $0 < t < \pi/4$, and $\vec{R}(t) = \langle t, \sqrt{1+t^2}\rangle$, $c < t < d$.

(a) Find equations for the given curves in terms of x and y. Then find values of c and d such that $\vec{r}$ and $\vec{R}$ parameterize the same curve. Show all your work and carefully explain your reasoning.

(a) Recall that $\sec^2 t = 1 + \tan^2 t$ for all real numbers t. Thus, $\vec{r}(t)$ parameterizes the curve $y^2 = x^2 + 1$, $0 < x < 1$ and $1 < y < \sqrt{2}$.

$\vec{R}(t)$ parameterizes the curve $y = \sqrt{1+x^2}$, $c < x < d$.

If $c = 0$ and $d = 1$, then $\vec{r}(t)$ and $\vec{R}(t)$ parameterize the same curve.

(b) Find the arc length of the curve parameterized by $\vec{r}(t) = \langle \tan t, \sec t\rangle$, $0 < t < \pi/4$. (Your final answer should be in the form of a definite integral; you don't have to evaluate the integral.)

(b) $\displaystyle\int_0^{\pi/4} \sqrt{sec^4 t + \sec^2 t\tan^2 t}\,dt$

(c) Find the arc length of the curve parameterized by $\vec{R}(t) = \langle t, \sqrt{1+t^2}\rangle$, $c < t < d$. (Your final answer should be in the form of a definite integral.)

(c) $\displaystyle\int_0^1 \sqrt{1 + \frac{t^2}{1+t^2}}\,dt$

(d) (*Extra credit, 1 point*) Numerically evaluate the two integrals.

(d) $\displaystyle\int_0^{\pi/4} \sqrt{sec^4 t + \sec^2 t\tan^2 t}\,dt = \int_0^1 \sqrt{1 + \frac{t^2}{1+t^2}}\,dt = 1.09969.$

(e) Find a u-substitution or trigonometric substitution and show (using analytic methods, not numerical evaluation) that the two integrals are equal. Show all your work and carefully explain your reasoning.

(e) Let $u = \tan t$.

Then $du = \sec^2 t\,dt$ and $\sec^2 t = 1 + \tan^2 t = 1 + u^2$, and so

$$\int_{t=0}^{t=\pi/4} \sqrt{sec^4 t + \sec^2 t\tan^2 t}\,dt = \int_{u=0}^{u=1} \sqrt{(1+u^2)^2 + (1+u^2)u^2}\,\frac{du}{1+u^2} = \int_{u=0}^{u=1} \sqrt{1 + \frac{u^2}{1+u^2}}\,du.$$

(AB 8) Find the arc length of the cardioid with polar coordinates $r = 1 + \sin\theta$, $0 \le \theta < 2\pi$.

(Answer 8)

$$\int_0^{2\pi} \sqrt{(1+\sin\theta)^2+\cos^2\theta}\,d\theta = \int_0^{2\pi} \sqrt{2+2\sin\theta}\,d\theta = 8.$$

(AB 9) The parameterizations $\vec{r}(t) = \langle \sin t, \cos t\rangle$, $0 \le t \le \pi/2$, and $\vec{R}(u) = \langle \sqrt{u}, \sqrt{1-u}\rangle$, $0 \le t \le 1$, describe the same curve. Find a scalar-valued function $f(t)$ such that $\vec{r}(t) = \vec{R}(f(t))$.

(Answer 9) $f(t) = \sin^2 t$

(AB 10) Find an arc length parameterization for the curve $\vec{r}(t) = \langle t^3, 3t^2, t^3\rangle$, $1 \le t \le 5$. Be sure to include a range for your parameter.

(Answer 10) Let $s(t)$ denote the arc length of the curve from $\vec{r}(1)$ to $\vec{r}(t)$. Then

$$s(t) = \int_1^t \sqrt{(3u^2)^2+(6u)^2+(3u^2)^2}\,du.$$

Integrating, we see that

$$s = \frac{(2t^2+4)^{3/2}}{2} - \frac{6^{3/2}}{2}.$$

Solving for t, we see that

$$t = \sqrt{\frac{(2s+6\sqrt{6})^{2/3}-4}{2}}$$

and so the arc length parameterization is

$$\vec{r}(s) = \left\langle \left(\frac{(2s+6\sqrt{6})^{2/3}-4}{2}\right)^{3/2}, 3\frac{(2s+6\sqrt{6})^{2/3}-4}{2}, \left(\frac{(2s+6\sqrt{6})^{2/3}-4}{2}\right)^{3/2}\right\rangle, \quad 0 < s < 78\sqrt{6}.$$

(AB 11) Let $\vec{r}(t) = \langle 2\sqrt{t}\cos\sqrt{t} - 2\sin\sqrt{t}, 2\sqrt{t}\sin\sqrt{t} + 2\cos\sqrt{t}\rangle$, $0 \le t < \infty$.

(a) Show that $\vec{r}(t)$ is an arc length parameterization.

(a) We compute $\vec{r}\,'(t) = \langle -\sin\sqrt{t}, \cos\sqrt{t}\rangle$, so $|\vec{r}\,'(t)| = 1$.

(b) Find the curvature $\kappa(t)$.

(b) $\vec{r}\,''(t) = \langle -\frac{1}{2\sqrt{t}}\cos\sqrt{t}, -\frac{1}{2\sqrt{t}}\sin\sqrt{t}\rangle$, so $\kappa(t) = |\vec{r}\,''(t)| = \frac{1}{2\sqrt{t}}$.

(c) Find the normal vector $\vec{N}(t)$.

(c) $\vec{r}\,''(t) = \langle -\frac{1}{2\sqrt{t}}\cos\sqrt{t}, -\frac{1}{2\sqrt{t}}\sin\sqrt{t}\rangle$, so $\vec{N}(t) = \frac{1}{|\vec{r}\,''(t)|}\vec{r}\,''(t) = \langle -\cos\sqrt{t}, -\sin\sqrt{t}\rangle$.

(AB 12) Let $\vec{r}(t) = \langle t, t^2 \rangle$.

(a) Find $\vec{r}(0)$, $\kappa(0)$ and $\vec{N}(0)$.

(a) $\vec{r}(0) = \langle 0, 0 \rangle$. We compute $\vec{r}\,'(0) = \langle 1, 0 \rangle$ and $\vec{r}\,''(0) = \langle 0, 2 \rangle$, so $\kappa(0) = \dfrac{|\langle 1,0,0\rangle \times \langle 0,2,0\rangle|}{|\langle 1,0\rangle|^3} = 2$ and $\vec{N}(0) = \dfrac{\langle 0,2\rangle - \mathrm{proj}_{<1,0>}\langle 0,2\rangle}{|\langle 0,2\rangle - \mathrm{proj}_{<1,0>}\langle 0,2\rangle|} = \langle 0, 1 \rangle$.

(d) Find $\vec{r}(0.1)$.

(d) $\vec{r}(0.1) = \langle 0.1, 0.01 \rangle$.

(e) Find the points on the circle of radius R, centered at the point $\vec{r}(0) + R\vec{N}(0)$, with x-coordinate 0.1, for $R = 1/\kappa(0)$, 1/4, 1/3, 2/3, 1 and 2. Express your answers to five decimal places. Which point is closest to $\vec{r}(0.1)$?

(e) If $R = 1/2$, then the points on the circle with x-coordinate 0.1 are $(0.1, 0.98990)$ and $(0.1, 0.01010)$.
If $R = 1/4$, then the points on the circle with x-coordinate 0.1 are $(0.1, 0.47913)$ and $(0.1, 0.02087)$.
If $R = 1/3$, then the points on the circle with x-coordinate 0.1 are $(0.1, 0.65131)$ and $(0.1, 0.01535)$.
If $R = 2/3$, then the points on the circle with x-coordinate 0.1 are $(0.1, 1.32579)$ and $(0.1, 0.00754)$.
If $R = 1$, then the points on the circle with x-coordinate 0.1 are $(0.1, 1.99499)$ and $(0.1, 0.00501)$.
If $R = 2$, then the points on the circle with x-coordinate 0.1 are $(0.1, 3.99750)$ and $(0.1, 0.00250)$.

(AB 13) Let R be the plane $\{(x,y,z) : 5x - 4y + 3z = 7\}$, and let Q be the plane $\{(x,y,z) : 2x + 3y - z = 1\}$.

(a) Find a parametric equation for the intersection of these two planes.

(a) $\vec{r}(t) = t\langle -5, 11, 23 \rangle + \langle 0, 2, 5 \rangle$, $-\infty < t < \infty$.

(b) Find two non-parametric equations that describe the intersection of these two planes. Simplify your answer as much as possible.

(b) $y = 2 - \dfrac{11}{5}x$; $z = 5 - \dfrac{23}{5}x$
or
$x = \dfrac{25}{23} - \dfrac{5}{23}z$, $y = -\dfrac{9}{23} + \dfrac{11}{23}z$
or
$x = \dfrac{10}{11} - \dfrac{5}{11}y$, $z = \dfrac{9}{11} + \dfrac{23}{11}y$.

(AB 14) Consider the equation $z = \dfrac{y^2}{4} - \dfrac{x^2}{9}$.

(b) Classify the surface $z = x^2 - y^2$. Is it an ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, cone, elliptic paraboloid, or hyperbolic paraboloid?

(b) A hyperbolic paraboloid.

(AB 15) What is the domain of the function $f(x,y) = \mathrm{arcsec}(x^2 + y^2) + \ln(x)$?

(Answer 15) The domain is $\{(x,y) : x^2 + y^2 \geq 1 \text{ and } x > 0\}$.

(AB 16) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^3}{x^2 + y^6}$ does not exist.

(Answer 16) Let $f(x,y) = \dfrac{2xy^3}{x^2+y^6}$. Suppose we approach $(0,0)$ along the x-axis, so $y = 0$. Then $\displaystyle\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{2x0^3}{x^2 + 0^6} = \lim_{x\to 0} 0 = 0$.

Suppose we approach $(0,0)$ along the path $x = y^3$. If $x = y^3$, then $\dfrac{2xy^3}{x^2+y^6} = \dfrac{2y^6}{y^6+y^6} = 1$, and so $\displaystyle\lim_{y\to 0} f(y^3, y) = \lim_{y\to 0} \frac{2y^6}{y^6+y^6} = 1$.

By the two-path test, the limit does not exist.

(AB 17) Let $V = \pi r^2 h$. Suppose that r and h are both functions of t. We have that $r(1) = 5$, $h(1) = 7$, $r'(1) = 2$ and $h'(1) = 3$. Find $\dfrac{dV}{dt}$ at $t = 1$.

(Answer 17) $\left.\dfrac{dV}{dt}\right|_{t=1} = 215\pi$.

(AB 18) Let $f(x, y) = x^2 + xy + y^2$.

(a) Let $g(h) = f(3 + \frac{3}{5}h, -2 + \frac{4}{5}h)$. Find a formula for $g(h)$. Then find $g'(0)$.

(a) $g(h) = 7 + \frac{8}{5}h + \frac{37}{25}h^2$, so $g'(0) = \frac{8}{5}$.

(b) Use the limit definition of the directional derivative to find $D_{\langle 3/5,4/5\rangle} f(3, -2)$.

(b)

$$D_{\langle 3/5,4/5\rangle} f(3, -2) = \lim_{h\to 0} \frac{f(3 + \frac{3}{5}h, -2 + \frac{4}{5}h) - f(3, -2)}{h} = \lim_{h\to 0} \frac{\frac{8}{5}h + \frac{37}{25}h^2}{h} = \frac{8}{5}.$$

(c) Find $\nabla f(3, -2)$ and verify that $\langle 3/5, 4/5\rangle \cdot \nabla f(3, -2) = D_{\langle 3/5,4/5\rangle} f(3, -2)$.

(c) $\nabla f(x, y) = \langle 2x + y, x + 2y\rangle$, so $\nabla f(3, -2) = \langle 4, -1\rangle$ and $\langle 3/5, 4/5\rangle \cdot \nabla f(3, -2) = \langle 3/5, 4/5\rangle \cdot \langle 4, -1\rangle = \frac{8}{5} = D_{\langle 3/5,4/5\rangle} f(3, -2)$.

(AB 19) Find and classify all the critical points of the function $f(x, y) = 6x^2 e^y - 3x^4 - e^{6y}$.

(Answer 19) The critical points are $(x, y) = (1, 0)$ and $(x, y) = (-1, 0)$. They are both local maxima. $(D = (-24)(-30) - 12^2 = 576)$

(AB 20) Consider the function $f(x, y) = 12xe^y - x^3 - 6e^{4y}$. This function has one local extremum.

(a) Where does the local extremum occur?

(a) At $(x, y) = (2, 0)$.

(b) Is the local extremum a local maximum or a local minimum?

(b) A local maximum.

(c) Show that $f(x, y)$ does not have a global extremum by finding points (c, d) and (e, g) such that $f(c, d) < f(a, b) < f(e, g)$, where (a, b) is the point you found in part (a).

(c) We have $f(2, 0) = 10$. There are many possible answers. For example, $f(0, 0) = -6$ and $f(-3, -\ln 10) = -36/10 + 27 - 6/10000$.

(AB 21) Let $f(x, y) = 4x^2 + 9y^2 + 6x$. Find the absolute maximum and minimum values of $f(x, y)$ on the region $x^2 + y^2 \le 1$ or state that they do not exist.

(Answer 21) The minimum value $-\dfrac{9}{4}$ occurs at $(x, y) = (-3/4, 0)$. The maximum value $\dfrac{54}{5}$ occurs at $(x, y) = (\frac{3}{5}, \pm\frac{4}{5})$.

(AB 22) Find the volume of the solid below the hyperbolic paraboloid $z = 3 + x^2 - y^2$ and above the rectangle $\{(x, y) : 3 \le x \le 4,\ 0 \le y \le 2\}$.

(Answer 22) $\displaystyle\int_3^4 \int_0^2 3 + x^2 - y^2 \, dy\, dx = 28.$

(AB 23) Evaluate the integral $\displaystyle\int_0^3 \int_{x^2}^9 \frac{x^7}{y^5 + 1}\, dy\, dx$. *Hint*: Change the order of integration.

(Answer 23) $\displaystyle\int_0^3 \int_{x^2}^9 \frac{x^7}{y^5+1}\,dy\,dx = \int_0^9 \int_0^{\sqrt{y}} \frac{x^7}{y^5+1}\,dx\,dy = \frac{1}{8}\int_0^9 \frac{y^4}{y^5+1}\,dy = \frac{1}{40}\ln(59050)$.

(AB 24) Find $\displaystyle\iint_R 2xy\,dA$, where $R = \{(x,y) : y \geq x,\ x^2+y^2 < 9\}$.

(Answer 24) $\displaystyle\iint_R 2xy\,dA = \int_0^3 \int_{\pi/4}^{5\pi/4} 2r^3 \sin\theta\cos\theta\,d\theta\,dr = 0$

(AB 25) Let D be the region $\{(x,y,z) : x^2+y^2+z^2 < 25,\ x > 0,\ z^2 > x^2+y^2\}$.

(a) Describe D using spherical coordinates.

(a) $D = \{(\rho,\varphi,\theta) : 0 \leq \rho < 5,\ -\pi/2 < x < \pi/2,\ 0 \leq \varphi < \pi/4 \text{ or } 3\pi/4 < \varphi \leq \pi\}$.

(b) Find the volume of D.

(b) The volume is $\displaystyle 2\int_0^5 \int_0^\pi \int_0^{\pi/4} \rho^2 \sin\varphi\,d\varphi\,d\theta\,d\rho = \frac{125\pi(2-\sqrt{2})}{3}$.

(AB 26) Evaluate the integral $\displaystyle\int\int_R \sqrt{4y^2-x^2}\,dy\,dx$, where R is the triangle with vertices $(1,1)$, $(-1,2)$ and $(5,3)$.

(Answer 26) We let $u = 2y-x$ and $v = 2y+x$. Then $x = \frac{1}{2}v - \frac{1}{2}u$ and $y = \frac{1}{4}v + \frac{1}{4}u$, and so

$$\begin{aligned}\iint_R \sqrt{4y^2-x^2}\,dy\,dx &= \int_1^5 \int_3^{13-2u} \sqrt{uv}\,\frac{1}{4}\,dv\,du \\ &= \int_1^5 \sqrt{u}((13-2u)^{3/2} - 3^{3/2})\,\frac{1}{6}\,du \\ &\approx 13.824\end{aligned}$$

(AB 27) Find the area of the region $\{(x,y) : (3x-y)^2 + (x+2y)^2 < 1\rangle$.

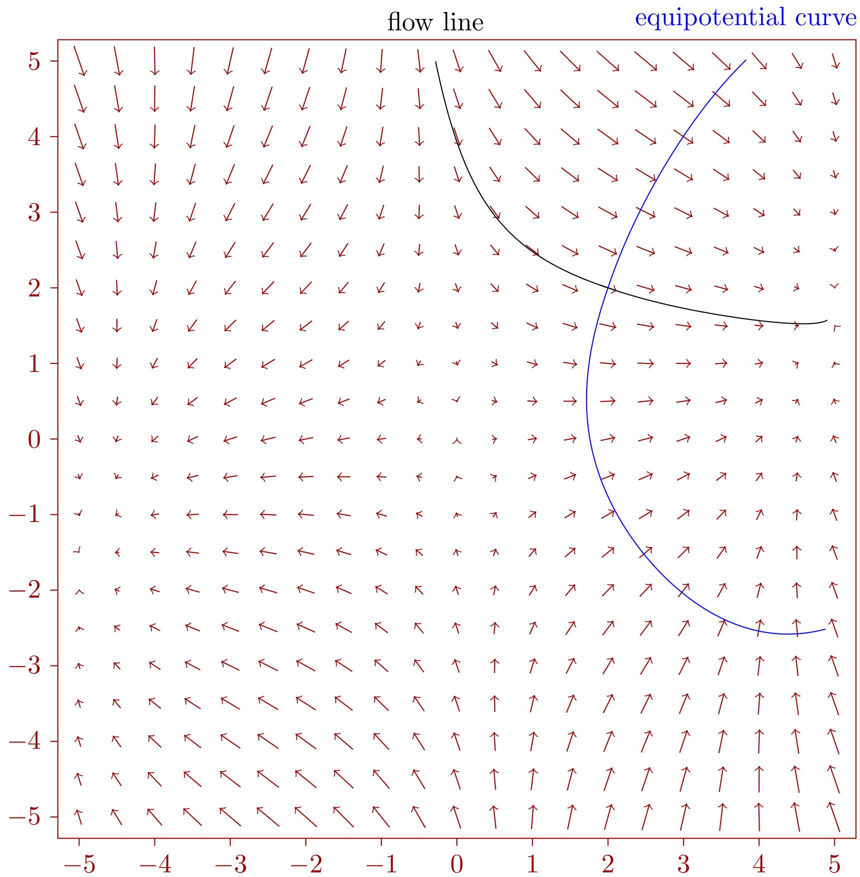
(Answer 27) Let $R = \{(x,y) : (3x-y)^2 + (x+2y)^2 < 1\rangle$. Make the change of variables $u = 3x-y$, $v = x+2y$, we see that $x = \frac{2}{7}u + \frac{1}{7}v$, $y = \frac{3}{7}v - \frac{1}{7}u$, and so

$$\iint_R 1\,dx\,dy = \iint_S \frac{1}{7}\,du\,dv$$

where $S = \{(u,v) : u^2+v^2 < 1\}$. Since S has area π, R has area $\pi/7$.

(AB 28) Here is a graph of a vector field $\vec{F} = \nabla\varphi$ for some potential function φ. Sketch the flow line and equipotential curve through the point $(2,2)$.

(Answer 28)



(AB 29) Find the flux of $\vec{F}(x, y) = \langle x, y \rangle$ across the curve C, where C is the triangle with vertices at $(0, 0)$, $(0, 1)$ and $(1, 0)$ oriented counterclockwise.

(Answer 29)

$$\int_C \vec{F} \cdot \vec{n}\, ds = \int_0^1 \langle t, 0\rangle \cdot \langle 0, -1\rangle\, dt + \int_0^1 \langle 1-t, t\rangle \cdot \langle 1, 1\rangle\, dt + \int_0^1 \langle 0, 1-t\rangle \cdot \langle -1, 0\rangle\, dt = 1.$$

(AB 30) Let C be the curve parameterized by $\vec{r}(t) = \langle t^3 + 3t + 1, 4t^2 - 4t + 7\rangle$, $0 \leq t \leq 7$. Find $\displaystyle\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = \langle 3x^2y^2 + x, 2x^3y\rangle$.

(Answer 30) $\vec{F}(x, y, z) = \nabla\varphi$, where $\varphi(x, y) = x^3y^2 + \dfrac{1}{2}x^2$. Therefore, $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \varphi(365, 175) - \varphi(1, 7) = 365^3 175^2 + \frac{365^2}{2} - 49 - \frac{1}{2} =$ 1,489,205,769,688.