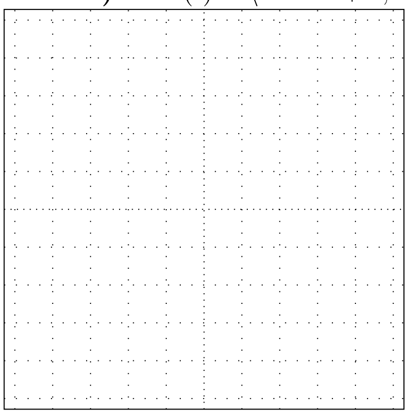
Math 2574, Fall 2017

You will be allowed a single-sided 8.5"×11" sheet of notes. You are responsible for writing down all formulas you might need.

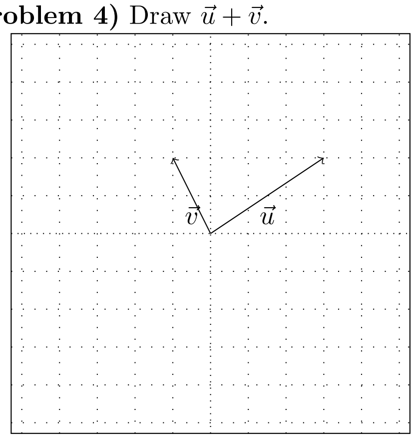
(Problem 1) Give a parametric equation for the ellipse $\{(x, y) : (x-3)^2/9 + (y+1)^2/25 = 1\}$ oriented clockwise.

(Problem 2) Let $\vec{r}(t) = \langle t^2 - 2t + 1, 2t^2 - 5\rangle$. Plot $\vec{r}(-1)$, $\vec{r}(0)$, $\vec{r}(1)$, and $\vec{r}(2)$.



(Problem 3) Give parametric equations for the curve $x = \sin(y)$, $-\pi/2 \le y \le \pi/2$.

(Problem 4) Draw $\vec{u} + \vec{v}$.



(Problem 5) Let $\vec{u} = \langle 2, 3, 0\rangle$ and let $\vec{v} = \langle 1, -2, 0\rangle$. Find a unit vector parallel to $\vec{u} \times \vec{v}$ without computing $\vec{u} \times \vec{v}$.

(Problem 6)

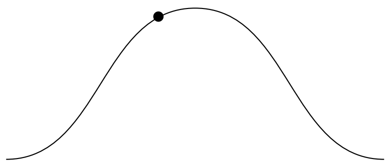
(a) Find all vectors $\vec{u}$ that satisfy the equation $\langle 1, -1, 1\rangle \times \vec{u} = \langle 3, 1, 2\rangle$.

(b) Find all vectors $\vec{u}$ that satisfy the equation $\langle 1, 3, 2\rangle \times \vec{u} = \langle 1, -1, 1\rangle$.

(Problem 7) Find the arc length of the polar curve $r = \theta^2$ from $\theta = 0$ to $\theta = \pi$.

(Problem 8) What is the curvature of the curve $\vec{r}(t) = \langle 3\cos t, 3\sin t\rangle$, $0 \le t < 2\pi$?

(Problem 9) Here is a curve. Sketch a circle that approximates the curve at the indicated point. Then sketch the unit normal vector.



(Problem 10) A circle has curvature $\frac{3}{5}$. What is its radius?

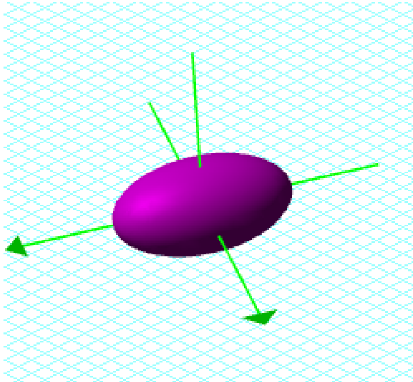
(Problem 11) Let $\vec{r}(t) = \langle \tan t - t, \ln|\sec t| \rangle$, $0 \leq t < \pi/2$. Find an arc length parameterization. Then find the unit tangent vector, the curvature, and the unit normal vector.

(Problem 12) Find a parametric equation for the intersection of the planes $x - y + z = 5$ and $3x - 2y + 5z = 7$.

(Problem 13) Find two non-parametric equations that describe the intersection of the planes $2x + y + 5z = 2$ and $3x - 2y + z = 2$. Simplify your answer as much as possible.

(Problem 14) Classify the following surfaces.

(a)



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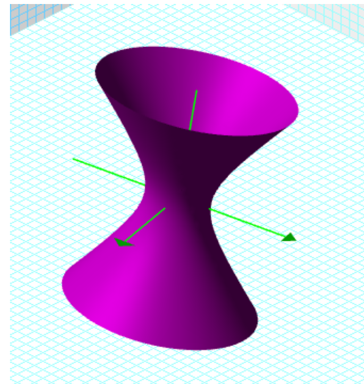

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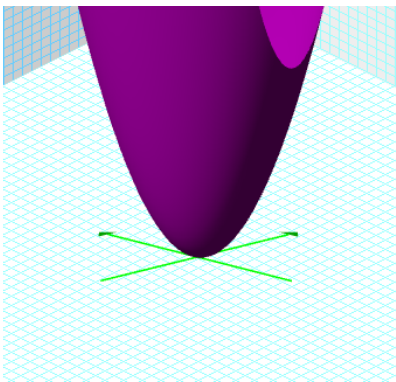

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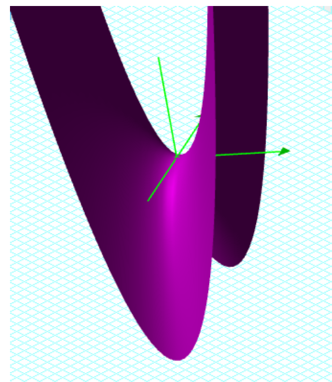

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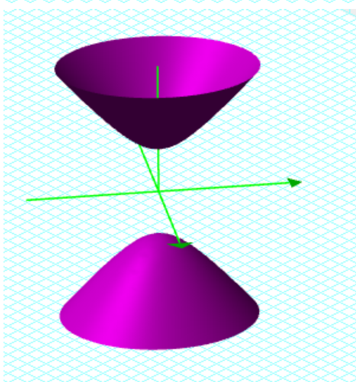

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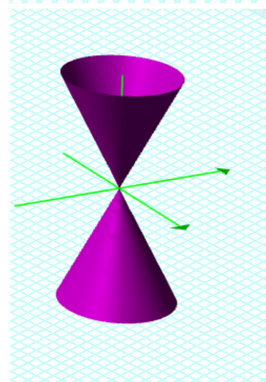


(Problem 15) Find the domain of the function $f(x, y) = \ln(x^2 + y^2)$. Simplify your answer as much as possible. If you have two conditions on the domain, clearly indicate whether they must both be true or whether at least one must be true.

(Problem 16) Find the domain of the function $f(x, y) = \ln(x^2 y^2)$. Simplify your answer as much as possible. If you have two conditions on the domain, clearly indicate whether they must both be true or whether at least one must be true.

(Problem 17) What is the range of the function $f(x, y) = \cos x + y$?

(Problem 18) What is the range of the function $f(x, y) = e^{x^2+y^2}$?

(Problem 19) Sketch a graph of $f(x, y) = e^{-x^2-y^2}$.

(Problem 20) Sketch a graph of $f(x, y) = \sin(xy)$.

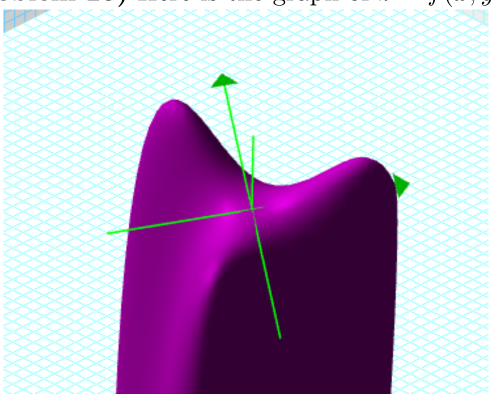
(Problem 21) Sketch a graph of $f(x, y) = \frac{1}{x-y}$.

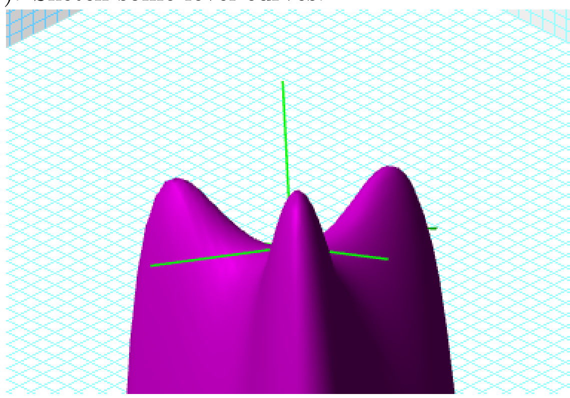
(Problem 22) Sketch some representative level curves of the function $z = x - 2y + 1$.

(Problem 23) Sketch some representative level curves of the function $z = \frac{x^2}{9} + y^2$.

(Problem 24) Sketch some representative level curves of the function $z = \frac{x^2}{9} - y^2$.

(Problem 25) Here is the graph of $z = f(x, y)$. Sketch some level curves.





(Problem 26) Show that $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^3+y^3}$ does not exist.

(Problem 27) Show that $\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$ does not exist.

(Problem 28) Show that $\lim_{(x,y)\to(3,1)} \frac{x+2y-5}{x-2y-1}$ does not exist.

(Problem 29) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^6+y^3}$ does not exist.

(Problem 30) Let $f(x, y, z) = x^2 + yz$. Use the limit definition to find $D_{\langle 3/13,4/13,12/13\rangle} f(1, 2, 3)$.

(Problem 31) Let $f(x, y) = x^3 + 3y^2 - 2$.

(*a*) Find a unit vector in the direction of the steepest ascent of f at the point $(x, y) = (2, 1)$.

(*b*) Find a unit vector in the direction of the steepest descent of f at the point $(x, y) = (2, 1)$.

(*c*) Find a unit vector in the direction of no change in f at the point $(x, y) = (2, 1)$.

(*d*) Consider the level curve $x^3 + 3y^2 - 2 = 9$. Find a (parametric or nonparametric) equation for the tangent line to this curve at the point $(2, 1)$.

(Problem 32) Find the equation of the plane tangent to the surface $z = x^2 - y^2 + xy$ at the point $(2, 3, 1)$.

(Problem 33) Find the equation of the plane tangent to the surface $x^2 + 2y^2 + 3z^2 = 20$ at the point $(3, 2, 1)$.

(Problem 34) Find a linear approximation to the function $f(x, y) = \ln(x^2 + y^2)$ at the point $(x, y) = (0.8, -0.6)$. Then use it to estimate $f(0.81, -0.58)$.

(Problem 35) Find the maximum and minimum values of $f(x, y) = x^2 + y^2 - x$ on the region $\{(x, y) : (x - 3)^2 + (y - 6)^2 \leq 49\}$.

(Problem 36) Find $\iint_R x^2y\,dA$, where R is the triangle with vertices $(1,2)$, $(1,5)$ and $(3,2)$.

(Problem 37) Find $\iint_R xy\,dA$, where R is bounded by the curves $y=2x$ and $x=y^2$.

(Problem 38) Find $\int_0^3\int_{2x}^6 e^{-y^2}\,dy\,dx$.

(Problem 39) Find $\iint_R x^2+xy+y^2\,dx\,dy$, where $R=\{(x,y):0<x<y, 1<x^2+y^2<4\}$.

(Problem 40) Find $\iint_R x+y\,dA$, where $R=\{(x,y):x+y>0, 4<x^2+y^2<9\}$.

(Problem 41) Find $\iint_R x+3y\,dA$, where $R=\{(x,y):0<x<y, 1<x^2+y^2<4\}$.

(Problem 42) Find $\iint_R \frac{1}{\sqrt{x^2+y^2+1}}\,dA$, where R is the region in the first quadrant bounded by the semicircle of radius 2 centered at the origin.

(Problem 43) Describe the region $\{(\rho,\theta,\varphi):0<\rho<6\cos\varphi, 0\le\theta<2\pi, 0\le\varphi\le\pi\}$ using Cartesian coordinates.

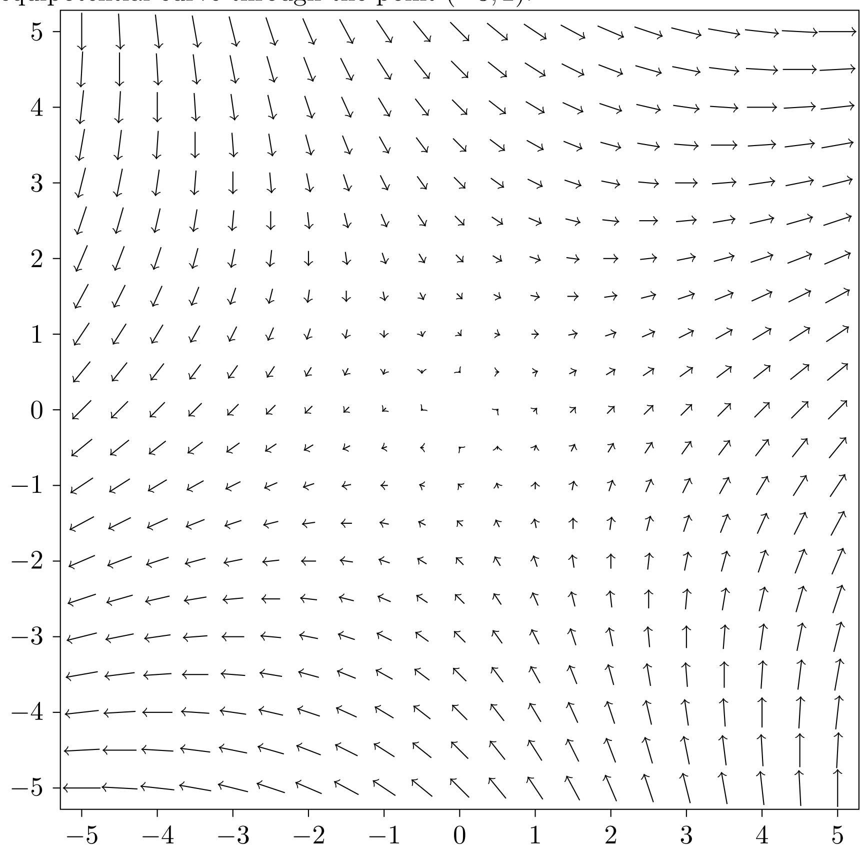
(Problem 44) Find $\iint_R xy\,dx\,dy$, where R is the parallelogram with vertices at $(3,2)$, $(5,4)$, $(4,-1)$, and $(6,1)$.

(Problem 45) Find $\iint_R \left(\frac{x-y}{x+y}\right)^6\,dx\,dy$, where R is the parallelogram with vertices at $(1,1)$, $(3,3)$, $(0,2)$, and $(2,4)$.

(Problem 46) Find the area of the region in the first quadrant bounded by the curves $y=x^3+1$, $y=x^3+2$, $y=3-x^3$ and $y=4-x^3$.

(Problem 47) Find the area of the region in the first quadrant bounded by the curves $y^2=3x^3$, $y^2=5x^3$, $y=2/x^2$ and $y=4/x^2$.

(Problem 48) Here is a graph of a vector field $\vec{F} = \nabla\varphi$ for some potential function φ. Sketch the flow line and equipotential curve through the point $(-3, 2)$.



(Problem 49) Let C be the curve parameterized by $\vec{r}(t) = \langle t^5 - 2t + 3, t^4 + 3t - 2\rangle$, $0 \le t \le 1$. Find $\int_C \vec{F} \cdot d\vec{r}$, where $\vec{F}(x, y) = \langle y \sin(\pi xy/2) + \sin(\pi x), x \sin(\pi xy/2) - 4y\rangle$.

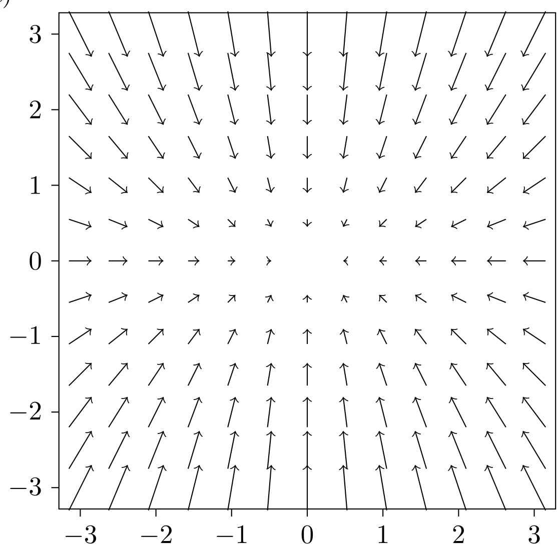
(Problem 50) For each of the following vector fields,

(i) Draw or visualize a simple closed curve oriented counterclockwise.

(ii) Is the outward flux across your curve positive or negative?

(iii) What can you say about the divergence of the vector field?

(a)



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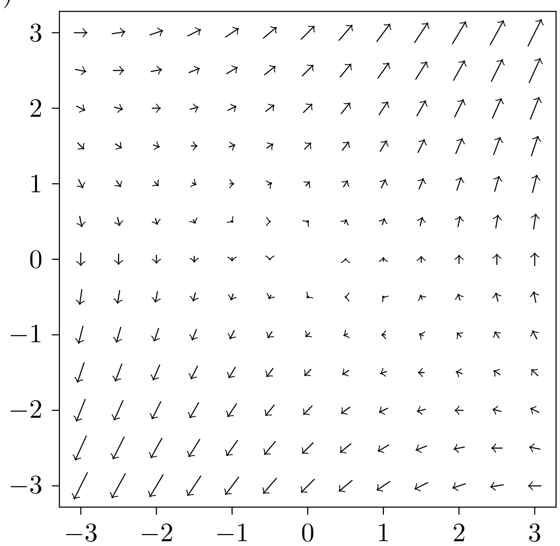

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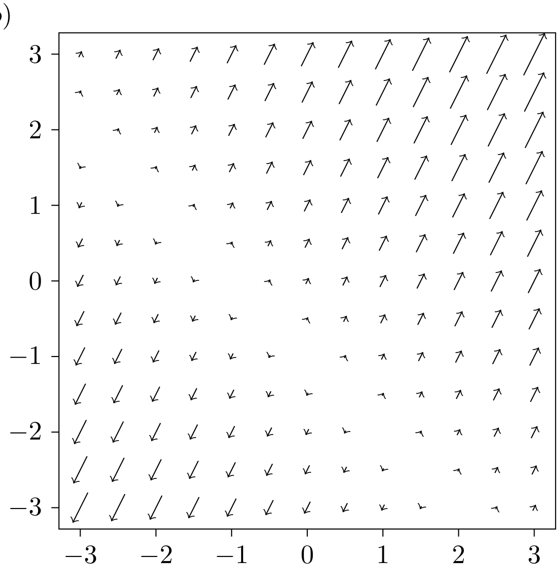


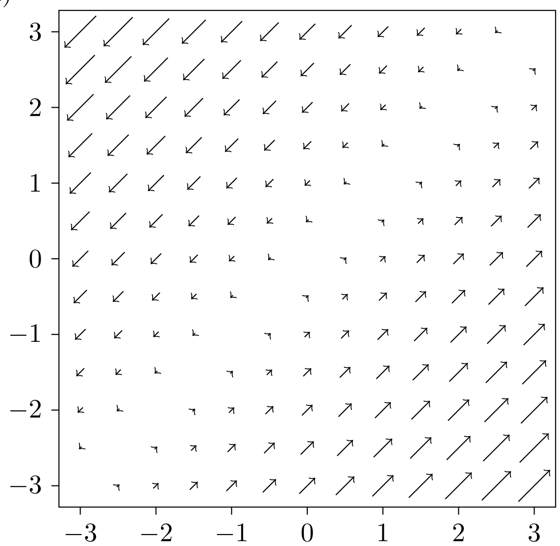
(Problem 51) For each of the following vector fields,

(i) Draw or visualize a simple closed curve oriented counterclockwise.

(ii) Is the circulation along your curve positive or negative?

(iii) What can you say about the curl of the vector field?

(a)



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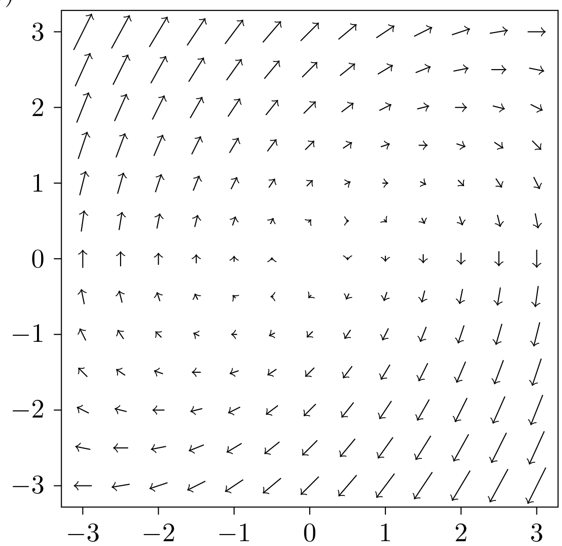

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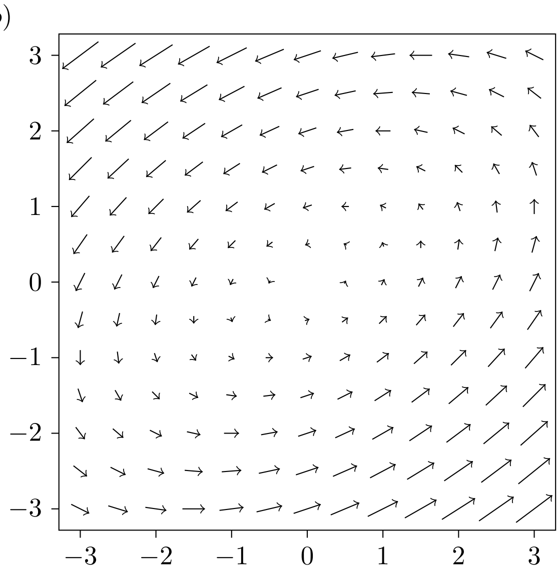
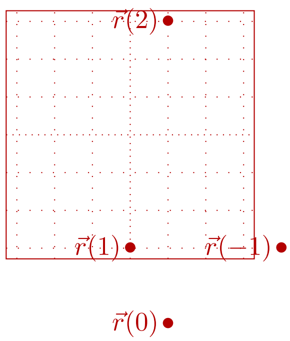


Answer key

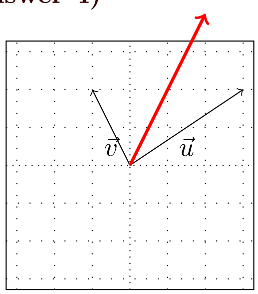
(Answer 1) There are many possible answers, including $\vec{r}(t) = \langle 3 + 3\cos t, -1 - 5\sin t\rangle$, $0 \le t < 2\pi$.

(Answer 2)



(Answer 3) $\vec{r}(t) = \langle \sin t, t\rangle$, $-\pi/2 \le t \le \pi/2$.

(Answer 4)



(Answer 5) $\vec{u}$ and $\vec{v}$ are both perpendicular to $\langle 0, 0, 1\rangle$, and so this vector must be parallel to $\vec{u} \times \vec{v}$.

(Answer 6)

(a) $\langle 1, -1, 1\rangle \cdot \langle 3, 1, 2\rangle = 4 \ne 0$. The vectors are not orthogonal and so there is no $\vec{u}$ that satisfies the given equation.

(b) $\vec{u} = \alpha\langle 1, 3, 2\rangle + \langle 0, 1, 1\rangle$ for some scalar α.

(Answer 11) An arc length parameterization is $\vec{r}(s) = \langle \sqrt{(s+1)^2 - 1} - \operatorname{arcsec}(s+1), \ln(s+1)\rangle$, $0 \le s < \infty$. We compute $\vec{r}'(s) = \left\langle \frac{\sqrt{s^2+2s}}{(s+1)}, \frac{1}{s+1}\right\rangle$.

(Answer 14)

(a) An ellipsoid.
(b) A hyperboloid of one sheet.
(c) An elliptic paraboloid.
(d) A hyperbolic paraboloid.
(e) A hyperboloid of two sheets.
(f) An elliptic cone.

(Answer 15) The domain of the function $f(x, y) = \ln(x^2 + y^2)$ is $\{(x, y) : x \ne 0 \text{ or } y \ne 0\}$.

(Answer 16) The domain of the function $f(x, y) = \ln(x^2y^2)$ is $\{(x, y) : x \ne 0 \text{ and } y \ne 0\}$.

(Answer 17) The range of the function $f(x, y) = \cos x + y$ is the interval $[-1, 1]$ (or $\{z : -1 \leq z \leq 1\}$).

(Answer 18) The range of the function $f(x, y) = e^{x^2+y^2}$ is the interval $[0, \infty)$ (or $\{z : z \geq 0\}$).

(Answer 35) The minimum value is $f(1/2, 0) = -1/4$. The maximum value is $f(74/13, 162/13) = 182$.

(Answer 46) Make the change of variables $u = y + x^3$ and $v = y - x^3$. Then the area of the region is $\frac{3}{2}[(1/2)^{4/3} + (3/2)^{4/3}] - 3$.

(Answer 47) Make the change of variables $u = x^2/y^3$ and $v = x^2 y$.