(Problem 1) Compute the following integrals.

- $\displaystyle\oint_\gamma (z+\bar{z})^2\,dz$, where γ is the straight line segment from 1 to i.
- $\displaystyle\oint_\gamma (z+\bar{z})^2\,dz$, where $\gamma = \partial D(0,1)$ traversed once and oriented counterclockwise.
- $\displaystyle\oint_\gamma (z-\bar{z})^2\,dz$, where $\gamma = \partial D(0,1)$ traversed once and oriented clockwise.
- $\displaystyle\oint_\gamma z^{17}\,dz$, where γ is the straight line segment from 1 to i.
- $\displaystyle\oint_\gamma z^{17}\,dz$, where γ is the arc the circle of radius 1 from the point i to the point -1.
- $\displaystyle\oint_\gamma e^z\,dz$, where $\gamma = \partial D(0,2)$, traversed once and oriented counterclockwise.

(Problem 2) Let the complex derivative be given by $f'(z) = \displaystyle\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$. If f is C^1, write $f'(z)$ in terms of the partial derivatives $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$.

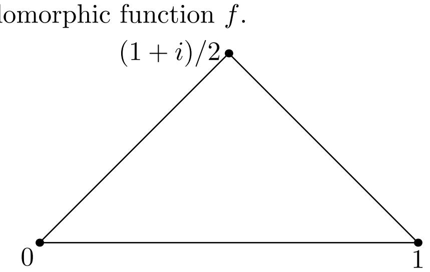
(Problem 3) If f is C^1, show that if $f'(z)$ exists then $\dfrac{\partial}{\partial\bar{z}}f(z) = 0$.

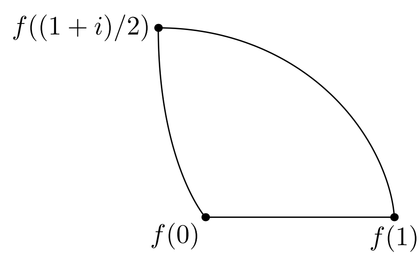
(Problem 4) Consider the second-order difference quotient $\displaystyle\lim_{h\to 0}\frac{f(z+h)+f(z-h)-2f(z)}{h^2}$.

(a) Show that if f is holomorphic, then $\dfrac{\partial^2 f(z)}{\partial z^2} = \displaystyle\lim_{h\to 0}\frac{f(z+h)+f(z-h)-2f(z)}{h^2}$.

(b) Show that if $\displaystyle\lim_{h\to 0}\frac{f(z+h)+f(z-h)-2f(z)}{h^2}$ exists for all z in a connected open set, then $\dfrac{\partial f}{\partial\bar{z}}$ is constant in that open set. What can you say about f?

(Problem 5) On the left are illustrated several curves γ_k. On the right are illustrated $f\circ\gamma_k$ for some holomorphic function f.





(a) Is there any number z such that you may be sure that $f'(z) = 0$?

(b) For each of the numbers z you found in part (a), what is the smallest number n such that $f^{(n)}(z) \neq 0$?

(Problem 6) Given Theorem 2.3.3, prove the Cauchy integral formula.

(Problem 7) Let $\displaystyle\sum_{k=0}^{\infty} a_k\,z^k$ be a power series. Show that there is some r with $0 \le r \le \infty$ such that the power series converges if $|z| < r$ and diverges if $|z| > r$.

(Problem 8) Let $\displaystyle\sum_{k=-\infty}^{\infty} a_k\,z^k$ be a Laurent series. Show that there is some r_1, r_2 with $0 \le r_1 \le r_2 \le \infty$ such that the power series converges if $r_1 < |z| < r_2$, diverges if $|z| > r_2$, and diverges if $|z| < r_1$.

(Problem 9) Let $\sum_{k=0}^{\infty} a_k\, z^k$ be a power series and let r be its radius of convergence. Show that if $\rho < r$ then the power series converges uniformly on $D(0,\rho)$.

(Problem 10) Where does a Laurent series converge uniformly?

(Problem 11) Let $f(z) = \sum_{k=0}^{\infty} a_k\, z^k$ with radius of convergence $r > 0$. Show that $f(z)$ is holomorphic and find a power series for $f'(z)$.

(Problem 12) Show that $\displaystyle\sum_{k=-\infty}^{\infty} a_k\, z^k$ is holomorphic inside its annulus of convergence.

(Problem 13) Show that if $\rho > 0$ and $\displaystyle\sum_{k=-\infty}^{\infty} a_k z^k = \sum_{k=-\infty}^{\infty} b_k z^k$ for all $|z| = \rho$, then $a_k = b_k$ for all k.

(Problem 14) Show that any power series is its own Taylor series.

(Problem 15) Suppose that f is holomorphic in $D(0,r)$. Use the Cauchy integral formula and the geometric series summation formula to find a power series expansion for f that is valid in $D(0, r-\varepsilon)$ for some $\varepsilon > 0$.

(Problem 16) Show that the coefficients of the power series that you found in 15 do not depend on ε and so there is a unique power series for f around 0. What is the radius of convergence of the power series for f?

(Problem 17) Let $f(z) = e^z/\cos z$. Let $\sum_{k=0}^{\infty} a_k z^k$ be the power series for $f(z)$ about the point $z = 0$. What is the radius of convergence of this series?

(Problem 18) Let $f(z) = 1/(1-z)$. Find two Laurent series for f. What region does each converge in?

(Problem 19) Let $f(z) = 1/(1-z)$. Using a partial fraction decomposition, compute $\displaystyle\frac{1}{2\pi i}\oint_{\partial D(0,3)} \frac{f(z)}{z}\, dz$. Is this equal to $f(0)$? Why does this not contradict Cauchy's theorem?

(Problem 20) Let $f(z) = \sin z/(z^2 - 6z + 8)$. Let $a_k = \displaystyle\frac{1}{2\pi i}\oint_{\partial D(0,3)} \frac{f(z)}{z^{k+1}}\, dz$. What is the largest open set in which $\displaystyle\sum_{k=-\infty}^{\infty} a_k z^k$ converges?

(Problem 21) Use Problems 14 and 15 to find a formula for $f^{(k)}(0)$ in terms of f.

(Problem 22) Prove the Cauchy estimates; that is, show that if f is holomorphic in $D(0,r)$ and continuous on $\overline{D}(0,r)$ then

$$\left|\frac{\partial^k f}{\partial z^k}(0)\right| \leq \frac{M k!}{r^k}$$

where $M = \max_{\overline{D}(0,r)} |f|$.

(Problem 23) By differentiating under the integral sign, find a formula for $f^{(k)}(z)$ for all $z \in D(0,r)$; that is, prove Theorem 3.1.1. Given the power series representation for $f^{(k)}(z)$ above, can this theorem be proven without differentiating under the integral sign?

(Problem 24) Prove Liouville's theorem.

(Problem 25) Prove that if f is entire and $|f(z)| < 8z^2$ for all $|z| > 100$ then f is a quadratic.

(Problem 26) Suppose that f_n is a sequence of functions holomorphic on an open set U that converge uniformly. Prove that the limit function is holomorphic.

(Problem 27) Suppose that f is holomorphic in $D(0,1)$ and $f(z_n) = 0$ for some points z_n with $z_n \neq 0$ and $z_n \to 0$, then $f(z) = 0$ for all $z \in D(0,1)$.

(Problem 28) Show that the conclusion of Problem 27 is still true if $D(0,1)$ is replaced by any connected open set and if the points z_n converge to any point z_0 in that set.

(Problem 29) Suppose that $\{f_k\}_{k=0}^{\infty}$ is a sequence of uniformly bounded holomorphic functions on $D(0,1)$, and that for some points z_n with $z_n \neq 0$ and $z_n \to 0$, we have that $\lim\limits_{k\to\infty} f_k(z_n) = 0$ for each n. Show that $f_k(z) \to 0$ for each $z \in D(0,1)$.

(Problem 30) Prove the Riemann removable singularities theorem.

(Problem 31) The Riemann removable singularities theorem states that if f is holomorphic in $D(P,r) \setminus \{P\}$ and is bounded in $D(P,r) \setminus \{P\}$, then f may be extended to a function holomorphic in $D(P,r)$. Consider the set $U = D(0,1) \setminus \{x + 0y : 0 \leq x < 1\}$.

- (a) Why can't we adapt the proof of the Riemann removable singularities theorem to apply to the domain U?
- (b) Provide an example of a function f that is holomorphic in U and bounded in U but that cannot be extended to a function holomorphic in $D(0,1)$.

(Problem 32) Prove that if f has an essential singularity at 0 then $f(D(0,\varepsilon) \setminus \{0\})$ is dense. That is, show that if f is holomorphic in $D(0,\varepsilon) \setminus \{0\}$ and $f(D(0,\varepsilon) \setminus \{0\})$ is not dense in $\mathbb{C}$, then either f is bounded near 0 or $\lim\limits_{z\to 0} |f(z)| = \infty$.

(Problem 33) Suppose that f is holomorphic in $D(0,r) \setminus \{0\}$. Prove that f has a pole at 0 if and only if the Laurent series for f about 0 has only finitely many terms with negative exponents.

(Problem 34) Let f be an entire function. Prove that if f is not constant then $f(\mathbb{C})$ is dense in $\mathbb{C}$.

(Problem 35) Show that if γ is a closed curve, $P \in \mathbb{C} \setminus \gamma$, and k is a (possibly negative) integer with $k \neq 1$, then

$$\oint_\gamma (z-P)^k \, dz = 0.$$

(Problem 36) Show that if γ is a closed curve, $P \in \mathbb{C} \setminus \gamma$, then $\dfrac{1}{2\pi i} \displaystyle\oint_\gamma \frac{1}{z-P} \, dz$ is an integer.

(Problem 37) Find the following residues.

- $\operatorname{Res}_{\csc}(0)$
- $\operatorname{Res}_{\csc}(\pi)$
- $\operatorname{Res}_{\cot}(0)$
- $\operatorname{Res}_{\cot}(\pi)$
- $\operatorname{Res}_{\cot}(2\pi)$
- $\operatorname{Res}_f(0)$, where $f(z) = \sin z/z^2$
- $\operatorname{Res}_f(i)$, where $f(z) = \log z/(z^2+1)$. Specify the branch of log that you use.
- $\operatorname{Res}_f(1)$, where $f(z) = \sqrt{z}/(z-1)^3$. Specify the branch of the square root that you use.

(Problem 38) Evaluate the following integrals and infinite sums.

- $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx$
- $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+2x^2+1}\,dx$
- $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+9}\,dx$
- $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x - 1}{x^2}\,dx$
- $\displaystyle\int_{0}^{\infty} \frac{1}{x^3+1}\,dx$
- $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}}{x^2+1}\,dx$
- $\displaystyle\sum_{k=0}^{\infty} \frac{1}{1+k^2}$
- $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$

(Problem 39) Prove the argument principle. That is, suppose that f is a meromorphic function on an open set $U \subseteq \mathbb{C}$, that $\overline{D}(P,r) \subset U$ and that f has neither poles nor zeros on $\partial D(P,r)$. Use the residue theorem to prove that

$$\frac{1}{2\pi i}\oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\,d\zeta = \sum_{j=1}^{p} n_j - \sum_{k=1}^{q} m_k,$$

where $n_1, n_2, \dots, n_p$ are the multiplicities of the zeros $z_1, z_2, \dots, z_p$ of f in $D(P,r)$ and $m_1, m_2, \dots, m_q$ are the orders of the poles $w_1, w_2, \dots, w_q$ of f in $D(P,r)$.

(Problem 40) Prove the open mapping theorem, that is, that if f is holomorphic on a connected open set U and f is not constant then $f(U)$ is open

(Problem 41) Suppose that U is a connected open set, that $P \in U$, and that f is holomorphic in $U \setminus \{P\}$ with a pole at P. Prove that there is some $R > 0$ such that $\{z \in \mathbb{C} : |z| > R\} \subset f(U)$. Do this (a) by considering $g(z) = 1/f(z)$, and (b) by using the argument principle.

(Problem 42) Suppose that f is holomorphic in some open set $U \subseteq \mathbb{C}$ and that $f'(P) \neq 0$ for some $P \in U$. Prove that f is one-to-one in a neighborhood of P.

(Problem 43) Suppose that f is holomorphic in some open set $U \subseteq \mathbb{C}$ and that $f(P) = 0$ for some $P \in U$. Let k be the multiplicity of the zero at P. Prove that there is some open set $V \subset U$ with $P \in V$ such that f is k-to-one on $V \setminus \{P\}$.

(Problem 44) Suppose that U is a connected open set, that $P \in U$, and that f is holomorphic in $U \setminus \{P\}$ with a pole at P. Let k be the order of the pole at P. Prove that there is an open set $V \subset U$ containing P such that f is k-to-one in $V \setminus \{P\}$ for some $\varepsilon > 0$. Do this (a) by considering $g(z) = 1/f(z)$, and (b) by using the argument principle.

(Problem 45) Prove Rouché's theorem and Hurwitz's theorem.

(Problem 46) Use Hurwitz's theorem to show that if $f_n \to f$ normally on some connected open set U and if each f_n is holomorphic and one-to-one, then f is also one-to-one.

(Problem 47) Let f be holomorphic on a connected open set U. Suppose that there is some $P \in U$ such that $\Re f(P) \leq \Re f(z)$ for all $z \in U$. Prove that f is constant.

(Problem 48) Suppose that f is holomorphic on $D(0,1)$, that $|f(z)| \leq 1$ for all $z \in D(0,1)$, and that $f(0) = 0$. Prove that $|f(z)| \leq |z|$ for all $z \in D(0,1)$.

(Problem 49) Suppose that f is holomorphic on $D(0,1)$, that $|f(z)| \leq 1$ for all $z \in D(0,1)$, and that $f(0) = 0$ and $f'(0) = 0$. Prove that $|f(z)| \leq |z|^2$ for all $z \in D(0,1)$. If $|f(z)| = |z|^2$, what can you say about $f(z)$?

(Problem 50) Let $a \in D(0,1)$. Let $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$.

- Prove that $\varphi_a(D(0,1)) \subset D(0,1)$.
- Prove that φ_a is a bijection from $D(0,1)$ to itself.

(Problem 51) Let $a \in D(0,1)$. Let $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$. Extend φ_a to a map defined on the Riemann sphere by letting $\varphi_a(\infty) = -1/\bar{a}$ and $\varphi_a(1/\bar{a}) = \infty$. Let $U = \mathbb{C} \cup \{\infty\} \setminus \overline{D}(0,1) = \{z \in \mathbb{C} \cup \{\infty\} : |z| > 1\}$.

- Prove that $\varphi_a(U) \subset U$.
- Prove that φ_a is a bijection from U to itself.

(Problem 52) If $|a| > 1$, what does φ_a do to $D(0,1)$? To $\mathbb{C} \cup \{\infty\} \setminus \overline{D}(0,1)$?

(Problem 53) Let $f(z) = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$ be a fractional linear transformation. Show that if γ is a circle or a line, then $f \circ \gamma$ is a circle or a line.

(Problem 54) Find all biholomorphic self-maps of the following domains. Prove that you have found all self-maps (do not rely on remembering the maps given in the book).

- $\mathbb{C}$.
- $D(0,1)$.
- The Riemann sphere $\mathbb{C} \cup \{\infty\}$.
- $\mathbb{C} \setminus \{0\}$.
- $\mathbb{C} \setminus \{1, -1\}$.
- $D(0,1) \setminus \{0\}$.
- $\mathbb{C} \setminus \overline{D}(0,1)$.
- The upper half-plane $\{z \in \mathbb{C} : \Im z > 0\}$.

(Problem 55) Find a conformal mapping between each of the following pairs of domains.

- $D(0,1)$ and the upper half-plane.
- The upper half-plane and a strip $\{(x+iy) : x \in \mathbb{R}, 0 < y < \pi\}$
- $D(0,1)$ and a strip
- A quarter-circle $\{x+iy : x > 0, y > 0, x^2+y^2 < 1\}$ and the upper half-plane.
- $D(0,1) \setminus \{0\}$ and $\mathbb{C} \setminus \overline{D}(0,1)$
- $D(0,1) \setminus \{x+0i : 0 \leq x < 1\}$ and $D(0,1)$

(Problem 56) Prove that if U is holomorphically simply connected, f is holomorphic on U, and $f \neq 0$ on U, then there is some holomorphic function h on U such that $e^h = f$.

(Problem 57) Let $U = \mathbb{C} \setminus \{x+0i : x \leq 0\}$ be the complex plane minus a slit. Let $f(z) = z^2$ be holomorphic on U.

- Find an explicit formula for a function $h(z)$ such that $e^h = f$ on U.
- Compute $f(i)$ and $f(-i)$. Are they equal?
- Compute $h(i)$ and $h(-i)$. Are they equal?
- Is there some branch of log such that $h(z) = \log f(z)$?

(Problem 58) Use the function h of Problem 56 to show that if U is holomorphically simply connected, f is holomorphic on U, and $f \neq 0$ on U, then there is some holomorphic function g on U such that $g^3 = f$.

(Problem 59) Prove that if U is holomorphically simply connected, f is holomorphic on U, and $f \neq \pm 1$ on U, then there is some holomorphic function h on U such that $\sin h = f$.

(Problem 60) Suppose that U is holomorphically simply connected and that $0 \notin U$. Find a one-to-one holomorphic function $f : U \mapsto D(0,1)$ using (*a*) the function h of Problem 56; (*b*) the function g of Problem 58.

(Problem 61) Suppose that U is holomorphically simply connected and that $\pm 1 \notin U$. Find a one-to-one holomorphic function $f : U \mapsto D(0,1)$ using the function h of Problem 59. *Hint:* First show that $\sin(z+2\pi) = \sin z$ for all $z \in \mathbb{C}$.

(Problem 62) Let U be a holomorphically simply connected open set with $0 \in U$. Let $f : U \mapsto D(0,1)$ be one-to-one holomorphic function function such that $f(0) = 0$, $f'(0)$ is a positive real and $\{x+0i : -1 < x \leq -r\} \cap f(U) = \emptyset$ for some $r > 0$. Prove that $g(z) = \varphi_{\sqrt{r}} \circ s \circ \varphi_{-r} \circ f(z)$ is also a one-to-one holomorphic function with $g(U) \subset D(0,1)$, $g(0) = 0$ and $g'(0) > f'(0)$. Here $s(re^{i\theta}) = \sqrt{r}e^{i\theta/2}$ provided $r \geq 0$ and $-\pi < \theta \leq \pi$; that is, $s(z) = z^{1/2}$ with branch cut along the negative reals.

(Problem 63) Modify the proof of Problem 62 so as to only use the hypothesis $r \notin f(U)$ for a single point r.

(Problem 64) Montel's theorem states that if $\{f_j\}_{j=1}^{\infty}$ is a sequence of uniformly bounded holomorphic functions on some open set U, then a subsequence converges normally. Suppose that $\{f_j\}_{j=1}^{\infty}$ is a sequence of holomorphic functions on some open set U, and for each compact set $K \subset U$, there is some constant M_K such that $|f_j(z)| \leq M_k$ for each $j \geq 1$ and each $z \in K$. Prove that a subsequence converges normally. *Hint:* Start by showing that $U = \cup_{k=1}^{\infty} U_k$, where U_k is an open set such that $\overline{U_k}$ is a compact subset of U.

(Problem 65) Suppose that $\{a_0^j\}_{j=1}^{\infty}$, $\{a_1^j\}_{j=1}^{\infty}$, $\{a_2^j\}_{j=1}^{\infty}\ldots$ are all convergent sequences, with $\lim_{j\to\infty} a_k^j = a_k$. Suppose further that $|a_k^j| \leq M/r^k$ for some constants $M > 0$ and $r > 0$.

Prove that the functions $g_j(z) = \displaystyle\sum_{k=0}^{\infty} a_k^j z^k$ converge uniformly on $D(0, r/2)$.

(Problem 66) Suppose that $\{a_0^j\}_{j=1}^\infty$, $\{a_{\pm 1}^j\}_{j=1}^\infty$, $\{a_{\pm 2}^j\}_{j=1}^\infty \dots$ are all convergent sequences. Suppose further that there are some positive constants M, r and R such that if $k \geq 0$ then $|a_k^j| \leq M/R^k$ and if $k \leq 0$ then $|a_k^j| \leq M/r^k$.

Prove that the functions $g_j(z) = \displaystyle\sum_{k=-\infty}^{\infty} a_k^j z^k$ converge for any z with $r < |z| < R$.