Math 2584, Spring 2016

If you need it, the following will be printed on the cover page of the exam:

Here are some Laplace transforms:

$f(t)$	$\mathcal{L}\{f(t)\}$	
t^n	$\frac{n!}{s^{n+1}}$,	$s > 0,\ n \geq 0$
e^{at}	$\frac{1}{s-a}$,	$s > a$
$\cos(bt)$	$\frac{s}{s^2+b^2}$,	$s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$,	$s > 0$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$,	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$,	$s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$,	$s > a$
$\mathcal{U}(t-c)$	$\frac{e^{-cs}}{s}$,	$s > 0,\ c \geq 0$
$\delta(t-c)$	e^{-cs},	$s > 0,\ c \geq 0$

$$\mathcal{L}\{\mathcal{U}(t-c)f(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}, \qquad c \geq 0$$

$$\mathcal{L}\{\mathcal{U}(t-c)g(t-c)\} = e^{-cs}\mathcal{L}\{g(t)\}, \qquad c \geq 0$$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\Big\{\int_0^t f(r)\,g(t-r)\,dr\Big\} = \mathcal{L}\Big\{\int_0^t f(t-r)\,g(r)\,dr\Big\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$$

$e^{i\theta} = \cos\theta + i\sin\theta, \qquad \sin(a+b) = \sin a\cos b + \sin b\cos a, \qquad \cos(a+b) = \cos a\cos b - \sin a\sin b.$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a vector, then $A\vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a\,v_1 + b\,v_2 \\ c\,v_1 + d\,v_2 \end{pmatrix}$.
If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix then $\det A = |A| = ad - bc$.
If A is a matrix, λ is a number (real or complex), and $\vec{v}$ is a vector, $\vec{v} \neq 0$, and if $A\vec{v} = \lambda\vec{v}$, then

$$\det(A - \lambda I) = 0 \qquad \text{and} \qquad (A - \lambda I)\vec{v} = 0$$

where I is the identity matrix of the same size as A. (If A is a 2×2 matrix then $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.)

Suggested review problems, Final Exam

(AB 1) Is $y = e^t$ a solution to the differential equation $y'' + \frac{4t}{1-2t}y' - \frac{4}{1-2t}y = 0$?

(AB 2) Is $y = e^{2t}$ a solution to the differential equation $y'' + \frac{4t}{1-2t}y' - \frac{4}{1-2t}y = 0$?

(AB 3) Are we guaranteed a unique solution to the following initial value problems?
(a) $x^2y' = y^2$, $y(0) = 1$.
(b) $x^2y' = y^2$, $y(1) = 0$.
(c) $y' = \sqrt[3]{y}$, $y(8) = 0$.
(d) $y' = \sqrt[3]{y}$, $y(8) = 1$.

(AB 4) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Set up, but **do not solve**, the differential equation and initial conditions that describe the amount of salt in the tank. Be sure to define your independent and dependent variables, as well as all unknown parameters, and be sure to include units.

(AB 5) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C. Set up, but **do not solve**, the differential equation and initial conditions that describe the temperature of a cup of coffee. Be sure to define your independent and dependent variables, as well as all unknown parameters, and be sure to include units.

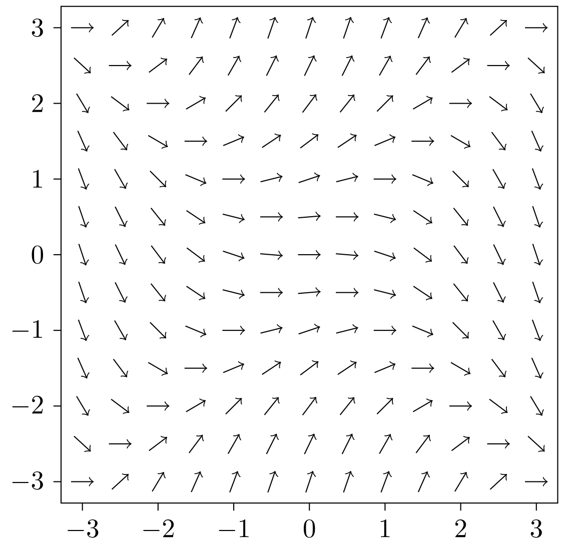
(AB 6) Suppose that an isolated town has 300 households. In 1920, two families install telephones in their homes. Write a differential equation for the number of telephones in the town if the rate at which families buy telephones is jointly proportional to the number of households with telephones and the number of households without telephones. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 7) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write a differential equation for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 8) A series circuit contains a resistor with resistance 100Ω, a capacitor with capacitance 3 μF, and an inductor with inductance 500 H. Initially there is no current in the circuit and a charge of 2 coulombs on the capacitor. Write a differential equation for the charge on the capacitor. Specify your independent and dependent variables, any unknown parameters, and your initial conditions.

(AB 9) Consider the differential equation $\frac{dy}{dx} = y^2 - x^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-3) = 0.$$



(AB 10) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y-2)(y+1)^2$. By hand, sketch some typical solutions.

(AB 11) Find the critical points and sketch the phase portrait of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 12) Find the critical points and sketch the phase portrait of the differential equation $\frac{dy}{dx} = y^2(y-2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 13) For each of the following differential equations, determine whether it is linear, separable, or exact. Then solve the differential equation.

(a) $1 + x^2 - xy + (x^2+1)\frac{dy}{dx} = 0$
(b) $\cos x + 3y + x\frac{dy}{dx} = 0$
(c) $\ln y + y + x + \left(\frac{x}{y} + x\right)\frac{dy}{dx} = 0$
(d) $y^3 \cos(2x) + \frac{dy}{dx} = 0$
(e) $\frac{x + \cos x}{y - \sin y} + \frac{dy}{dx} = 0$
(f) $4xy^3 + 6x^3 + (3 + 6x^2y^2 + y^5)\frac{dy}{dx} = 0$

(AB 14) For each of the following differential equations, determine whether it is linear, separable, or exact. Then solve the given initial-value problem.

(a) $\cos(x+y^3) + 2x + 3y^2\cos(x+y^3)\frac{dy}{dx} = 0$, $y(\pi/2) = 0$
(b) $1 + y^2 + x\frac{dy}{dx} = 0$, $y(1) = 1$
(c) $\sin x\, e^{-3x} + 3y + \frac{dy}{dx} = 0$, $y(0) = 2$

(AB 15) Solve the initial-value problem $\frac{dy}{dx} = \frac{x-5}{y}$, $y(0) = 3$ and determine the range of x-values in which the solution is valid.

(AB 16) Solve the initial-value problem $\frac{dy}{dt} = y^2$, $y(0) = 1/4$ and determine the range of x-values in which the solution is valid.

(AB 17) Solve the differential equation $\frac{dy}{dx} = \cot(y/x) + y/x$.

(AB 18) Solve the differential equation $x^2 \frac{dy}{dx} = x^2 - xy + y^2$.

(AB 19) Solve the differential equation $\frac{dy}{dx} = \frac{1}{3x+2y+7}$.

(AB 20) Solve the differential equation $\frac{dy}{dx} = \csc^2(y - x)$.

(AB 21) For each of the following initial-value problems, tell me whether I am guaranteed an infinite family of solutions, am not guaranteed any solutions, or am guaranteed a unique solution on some interval. If I am guaranteed a unique solution on some interval, tell me the interval on which I am guaranteed a solution. Do not find the solution to the differential equation.
(a) $ty'' - 5y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.
(b) $x(x-4)y'' + 3xy' + 4y = 2$, $y(3) = 1$, $y'(3) = -1$.
(c) $y'' + (\cos t)y' + 3(\ln|t|)y = 0$, $y(2) = 3$.

(AB 22) Consider the initial-value problem

$$t^2y'' + ty' + 5y = 0, \qquad y(-3) = 0, \quad y(-3) = 2.$$

What is the largest interval in which we are guaranteed that a unique solution exists?

(AB 23) Consider the initial-value problem

$$y'' + 5y' + 6y = \frac{1}{t^3 - 6t^2 + 8t}, \qquad y(1) = 0, \quad y(1) = 2.$$

What is the largest interval in which we are guaranteed that a unique solution exists?

(AB 24) The function $y_1(t) = t$ is a solution to the differential equation $(1-t)y'' + ty' - y = 0$. Find the general solution to this differential equation.

(AB 25) The function $y_1(t) = e^t$ is a solution to the differential equation $ty'' - (1+2t)y' + (t+1)y = 0$. Find the general solution to this differential equation.

(AB 26) The function $y_1(x) = x^3$ is a solution to the differential equation $x^2y'' + (x^2 \tan x - 6x)y' + (12 - 3\,x \tan x)y = 0$. Find the general solution to this differential equation on the interval $0 < x < \pi/2$.

(AB 27) Solve the following initial-value problems. Express your answers in terms of real functions.
(a) $16y'' - y = 0$, $y(0) = 1$, $y'(0) = 0$.
(b) $y'' + 49y = 0$, $y(\pi) = 3$, $y'(\pi) = 4$.
(c) $y'' + 3y' - 10y = 0$, $y(2) = 0$, $y'(2) = 3$.

(AB 28) Solve the following differential equations. Express your answers in terms of real functions.
(a) $y'' + 12y' + 85y = 0$.
(b) $y'' + 4y' + 2y = 0$.

(AB 29) Find a differential equation whose general solution is $y(t) = C_1e^{7t} + C_2e^{5t}$.

(AB 30) Solve the following differential equations. Express your answers in terms of real functions.

(a) $\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 24y = 0$.

(b) $\frac{d^3y}{dt^3} + y = 0$.

(c) $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0$.

(AB 31) Solve the initial-value problem $y'' - 4y' = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$.

(AB 32) Find the general solution to the equation $y'' - 2y' + y = \frac{e^t}{1+t^2}$.

(AB 33) Solve the initial-value problem $y'' + 9y = 9\sec^2 3t$, $y(0) = 4$, $y'(0) = 6$, on the interval $-\pi/6 < t < \pi/6$.

(AB 34) The general solution to the differential equation $t^2y'' - 2y = 0$, $t > 0$, is $y(t) = C_1t^2 + C_2t^{-1}$. Solve the initial-value problem $t^2y'' - 2y = 7t^3$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(AB 35) The general solution to the differential equation $t^2y'' - ty' - 3y = 0$, $t > 0$, is $y(t) = C_1t^3 + C_2t^{-1}$. Find the general solution to the differential equation $t^2y'' - ty' - 3y = 6t^{-1}$.

(AB 36) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ (**not** the table on the front of the exam), find the Laplace transforms of the following functions.

(a) $f(t) = t$

(b) $f(t) = e^{-11t}$

(c) $f(t) = \begin{cases} 3, & 0 < t < 4, \\ 0, & 4 \leq t \end{cases}$.

(AB 37) Find the Laplace transforms of the following functions. You may use the table on the front of the exam.

(a) $f(t) = t^4 + 5t^2 + 4$

(b) $f(t) = (t+2)^3$

(c) $f(t) = 9e^{4t+7}$

(d) $f(t) = -e^{3(t-2)}$

(e) $f(t) = (e^t + 1)^2$

(f) $f(t) = 8\sin(3t) - 4\cos(3t)$

(g) $f(t) = t^2e^{5t}$

(h) $f(t) = 7e^{3t}\cos 4t$

(i) $f(t) = 4e^{-t}\sin 5t$

(j) $f(t) = t^2 \sin 5t$

(k) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$

(l) $f(t) = \begin{cases} 0, & t < 1, \\ t - 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$

(m) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

(n) $f(t) = \begin{cases} 7t^2e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

(o) $f(t) = \begin{cases} 0, & t < \pi, \\ \sin t, & t \geq \pi \end{cases}$

(p) $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi \end{cases}$

(AB 38) Find the inverse Laplace transforms of the following functions:

(a) $F(s) = \frac{2s+8}{s^2+2s+5}$
(b) $F(s) = \frac{5s-7}{s^4}$
(c) $F(s) = \frac{s+2}{(s+1)^4}$
(d) $F(s) = \frac{2s-3}{s^2-4}$
(e) $F(s) = \frac{1}{s^2(s-3)^2}$
(f) $F(s) = \frac{s+2}{s(s^2+4)}$
(g) $F(s) = \frac{s}{(s^2+1)(s^2+9)}$
(h) $F(s) = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$
(i) $F(s) = \frac{(s-2)e^{-s}}{s^2-4s+3}$
(j) $F(s) = \frac{s}{(s^2+9)^2}$

(AB 39) Solve the following initial-value problems using the Laplace transform.

(a) $y' - 9y = \sin 3t$, $y(0) = 1$
(b) $y' - 2y = 3e^{2t}$, $y(0) = 2$
(c) $y' + 5y = t^3$, $y(0) = 3$
(d) $y'' - 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$
(e) $y'' - 2y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$
(f) $y'' - 7y' + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$
(g) $y'' + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$
(h) $y'' + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$
(i) $y'' + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$
(j) $y'' + 3y' + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$
(k) $y'' + y' + \frac{5}{4}y = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t \end{cases}$, $y(0) = 1$, $y'(0) = 0$

(AB 40) Solve the following initial-value problems using the Laplace transform. Do the graphs of y, y', or y'' have any corners or jump discontinuities? If so, where?

(a) $6y'' + 5y' + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$.
(b) $y' + 9y = 7\delta(t-2)$, $y(0) = 3$.
(c) $y'' + 4y = -2\delta(t-4\pi)$, $y(0) = 1/2$, $y'(0) = 0$
(d) $y'' + 4y' + 3y = 2\delta(t-1) + \mathcal{U}(t-2)$, $y(0) = 1$, $y'(0) = 0$

(AB 41) Find the solution to the initial-value problem

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\frac{8}{3}x + \frac{14}{3}y, \qquad x(0) = 4, \quad y(0) = 6.$$

(AB 42) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 43) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 44) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 45) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 46) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 47) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 48) Find the solution to the initial-value problem

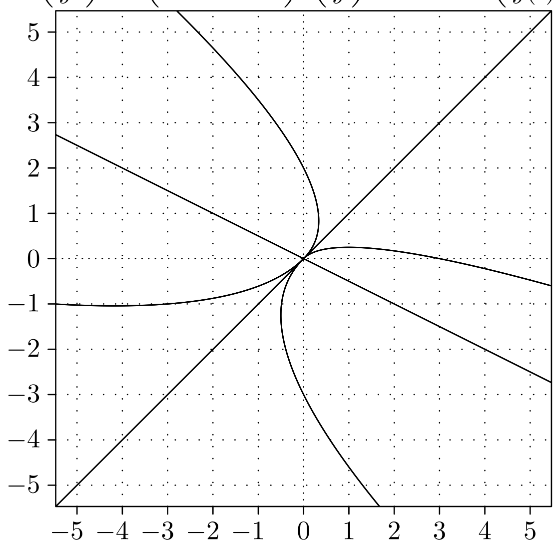
$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0.$$

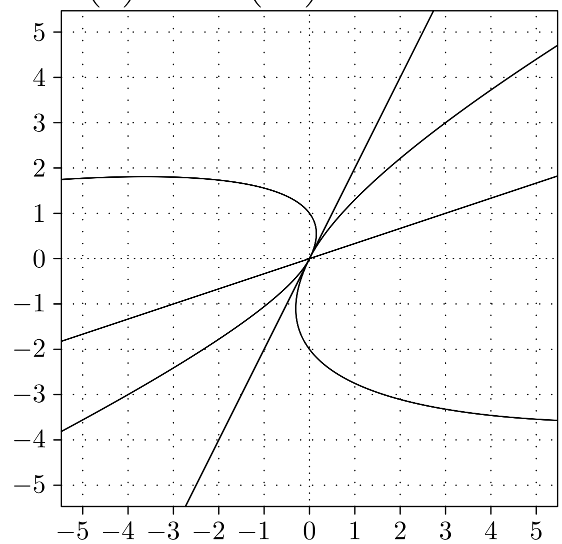
Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 49) Here are some phase portraits. To which of the following systems do these phase portraits correspond? How do you know?

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$





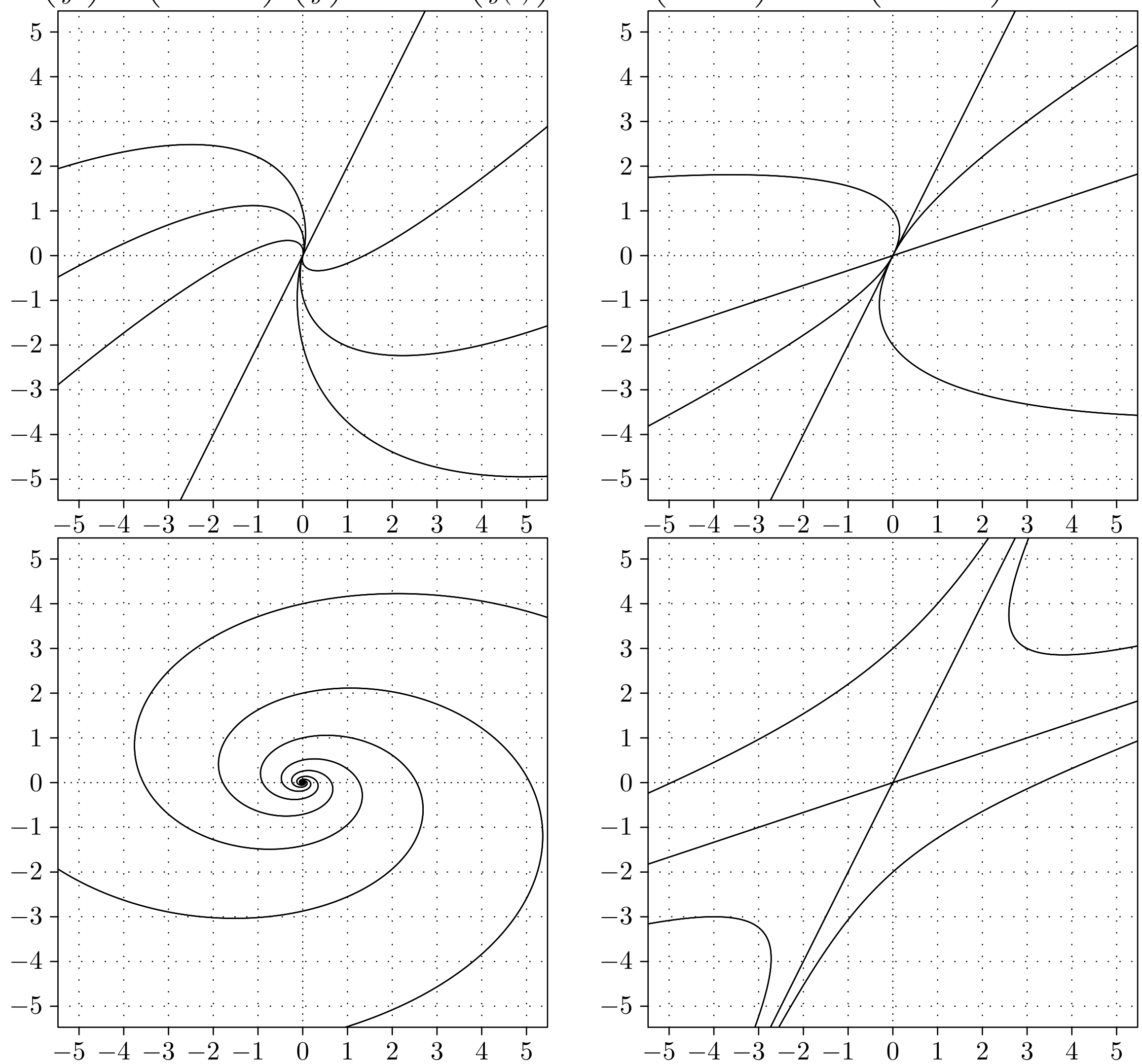
(AB 50) Here are some phase portraits. To which of the following systems do these phase portraits correspond? How do you know?

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2.6 & 1.8 \\ -1.2 & 1.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4/3 & -1/6 \\ 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} t+3 \\ 2t \end{pmatrix}$

(d) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4.5 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 3\sin 3t \\ 2\cos 3t \end{pmatrix} + C_2 e^{t} \begin{pmatrix} -3\cos 3t \\ 2\sin 3t \end{pmatrix}$



Answer key

(Answer 1) No, $y = e^t$ is not a solution to the differential equation $y'' + \frac{4t}{1-2t}y' - \frac{4}{1-2t}y = 0$.

(Answer 2) Yes, $y = e^{2t}$ is a solution to the differential equation $y'' + \frac{4t}{1-2t}y' - \frac{4}{1-2t}y = 0$.

(Answer 3)
(a) We are not guaranteed a unique solution to the problem $x^2y' = y^2$, $y(0) = 1$.
(b) We are guaranteed a unique solution to the problem $x^2y' = y^2$, $y(1) = 0$.
(c) We are not guaranteed a unique solution to the problem $y' = \sqrt[3]{y}$, $y(8) = 0$.
(d) We are guaranteed a unique solution to the problem $y' = \sqrt[3]{y}$, $y(8) = 1$.

(Answer 4) Independent variable: $t =$ time (in minutes).
Dependent variable: $Q =$ amount of dissolved salt (in kilograms).
Initial condition: $Q(0) = 3$.
Differential equation: $\frac{dQ}{dt} = 0.01 - \frac{3Q}{600-t}$.

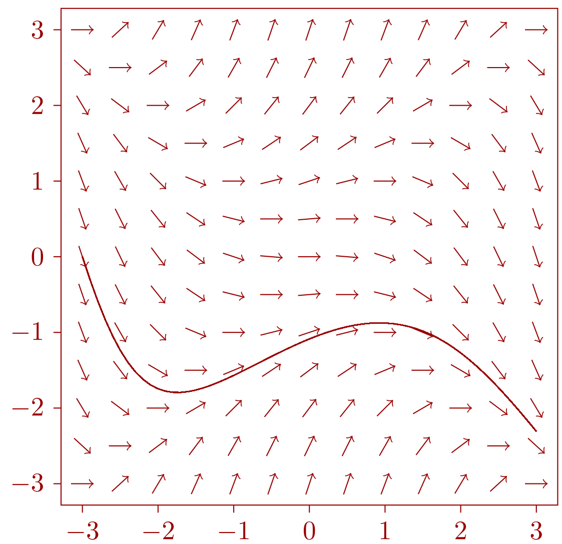
(Answer 5) Independent variable: $t =$ time (in seconds).
Dependent variable: $T =$ Temperature of the cup (in degrees Celsius)
Initial condition: $T(0) = 95$.
Differential equation: $\frac{dT}{dt} = -\alpha(T-25)$, where α is a parameter (constant of proportionality) with units of 1/seconds.

(Answer 6) Independent variable: $t =$ time (in years).
Dependent variable: $T =$ Number of telephones installed in the town.
Initial condition: $T(1920) = 2$.
Differential equation: $\frac{dT}{dt} = -\alpha T(300 - T)$, where α is a parameter (constant of proportionality) with units of $1/\text{year} \cdot \text{telephone}$.

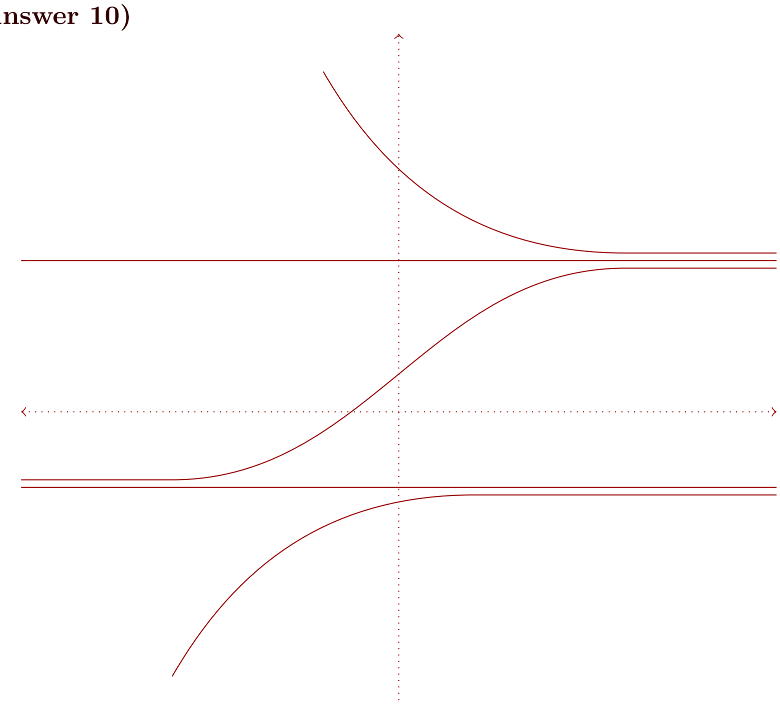
(Answer 7) Independent variable: $t =$ time (in years).
Dependent variable: $P =$ Number of birds on the island.
Initial condition: $P(0) = 5$.
Differential equation: $\frac{dP}{dt} = \alpha P - \beta P^2$, where α and β are parameters.

(Answer 8) Independent variable: $t =$ time (in seconds).
Dependent variable: $Q =$ charge on the capacitor (in coulombs).
Initial condition: $Q(0) = 2$, $Q'(0) = 0$.
Differential equation: $500\frac{d^2Q}{dt^2} + 100Q + \frac{1000}{3}Q = 0$.

(Answer 9)

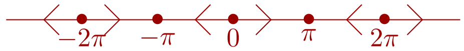


(Answer 10)



(Answer 11)

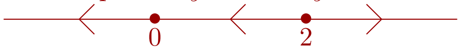
Critical points: $y = k\pi$ for any integer k.



If k is even then $y = k\pi$ is unstable.
If k is odd then $y = k\pi$ is stable.

(Answer 12)

Critical points: $y = 0$ and $y = 2$.



$y = 0$ is semistable. $y = 2$ is unstable.

(Answer 13)

(a) $1 + x^2 - xy + (x^2 + 1)\frac{dy}{dx} = 0$ is linear and has solution $y = -\sqrt{x^2+1}\ln(x + \sqrt{x^2+1}) + C\sqrt{x^2+1}$.

(b) $\cos x + 3y + x\frac{dy}{dx} = 0$ is linear and has solution $y = \frac{1}{x}\sin x + \frac{2}{x^2}\cos x - \frac{2}{x^3}\sin x + \frac{C}{x^3}$.

(c) $\ln y + y + x + \left(\frac{x}{y} + x\right)\frac{dy}{dx} = 0$ is exact and has solution $x\ln y + xy + \frac{1}{2}x^2 = C$.

(d) $y^3\cos(2x) + \frac{dy}{dx} = 0$ is separable and has solution $y = \pm\frac{1}{\sqrt{\sin(2x)+C}}$.

(e) $\frac{x+\cos x}{y-\sin y} + \frac{dy}{dx} = 0$ is separable and has solution $\frac{1}{2}y^2 + \cos y = -\frac{1}{2}x^2 - \sin x + C$.

(f) $4xy^3 + 6x^3 + (3 + 6x^2y^2 + y^5)\frac{dy}{dx} = 0$ is exact and has solution $2x^2y^3 + \frac{3}{2}x^4 + 3y + \frac{1}{6}y^6 = C$.

(Answer 14)

(a) If $\cos(x + y^3) + 2x + 3y^2\cos(x + y^3)\frac{dy}{dx} = 0$, $y(\pi/2) = 0$, then $\sin(x + y^3) + x^2 = 1 + \pi^2/4$.

(b) If $1 + y^2 + x\frac{dy}{dx} = 0$, $y(1) = 1$, then $y = \tan(\pi/4 - \ln x)$.

(c) If $\sin x\, e^{-3x} + 3y + \frac{dy}{dx} = 0$, $y(0) = 2$, then $y = e^{-3x}\cos x + e^{-3x}$.

(Answer 15) $y = \sqrt{(x-5)^2 - 16}$. The solution is valid for all $x < 1$.

(Answer 16) $y = \frac{1}{4-x}$. The solution is valid for all $x < 4$.

(Answer 17) $\sec(y/x) = Cx$.

(Answer 18) $y = \frac{x}{C - \ln|x|} + x$.

(Answer 19) $\frac{3x+2y+7}{3} - \frac{2}{9}\ln|3x + 2y + 7 + 2/3| = \ln|x| + C$.

(Answer 20) $\tan(y - x) - y = C$.

(Answer 21)

(a) We are not guaranteed any solutions.

(b) We are guaranteed a unique solution on the interval

(c) We are guaranteed an infinite family of solutions.

(Answer 22) The interval $t < 0$.

(Answer 23) The interval $0 < t < 2$.

(Answer 24) $y(t) = C_1\, t + C_2\, e^t$.

(Answer 25) $y(t) = C_1\, e^t + C_2\, t^2 e^t$.

(Answer 26) $y(x) = C_1\, x^3 + C_2\, x^3\sin x$.

(Answer 27)

(a) If $16y'' - y = 0$, $y(0) = 1$, $y'(0) = 0$, then $y(t) = \frac{1}{2}e^{t/4} + \frac{1}{2}e^{-t/4}$.

(b) If $y'' + 49y = 0$, $y(\pi) = 3$, $y'(\pi) = 4$, then $y(t) = 3\cos(7t) + \frac{4}{7}\sin(7t)$.

(c) If $y'' + 3y' - 10y = 0$, $y(2) = 0$, $y'(2) = 3$, then $y(t) = \frac{3}{7}e^{2t-4} - \frac{3}{7}e^{-5t+10}$.

(Answer 28)

(a) If $y'' + 12y' + 85y = 0$, then $y = C_1 e^{-6t}\cos(7t) + C_2 e^{-6t}\sin(7t)$.

(b) If $y'' + 4y' + 2y = 0$, then $y = C_1 e^{(-2+\sqrt{2})t} + C_2 e^{(-2-\sqrt{2})t}$.

(Answer 29) $y'' - 12y' + 35y = 0$.

(Answer 30)

(a) If $\frac{d^3y}{dt^3} - 6\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 24y = 0$, then $y = C_1 e^{6t} + C_2 e^{2t} + C_3 e^{-2t}$.

(b) If $\frac{d^3y}{dt^3} + y = 0$, then $y = C_1 e^{-t} + C_2 e^{t/2}\cos\frac{\sqrt{3}}{2}t + C_3 e^{t/2}\sin\frac{\sqrt{3}}{2}t$.

(c) If $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0$, then $y = C_1 e^{2t} + C_2 t e^{2t} + C_3 e^{-2t} + C_4 t e^{-2t}$.

(Answer 31) If $y'' - 4y' = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$, then $y(t) = -\frac{4}{3}e^{3t} + \frac{7}{4}e^{4t} + \frac{7}{12}$.

(Answer 32) $y(t) = e^t + c_2\, t\, e^t - \frac{1}{2}e^t\, \ln(1+t^2) + t\, e^t \arctan t$.

(Answer 33) $y(t) = 5\cos(3t) + 2\sin(3t) + (\sin(3t))\ln(\tan(3t) + \sec(3t)) - 1$.

(Answer 34) $y(t) = \frac{7}{4}t^3 - \frac{4}{3}t^2 + \frac{7}{12}t^{-1}$.

(Answer 35) $y(t) = -\frac{3}{2}t^{-1}\ln t + C_1 t^3 + C_2 t^{-1}$.

(Answer 36)

(a) $\mathcal{L}\{t\} = \frac{1}{s^2}$.

(b) $\mathcal{L}\{e^{-11t}\} = \frac{1}{s+11}$.

(c) $\mathcal{L}\{f(t)\} = \frac{3-3e^{-4s}}{s}$.

(Answer 37)

(a) $\mathcal{L}\{t^4+5t^2+4\} = \frac{96}{s^5}+\frac{10}{s^3}+\frac{4}{s}$.

(b) $\mathcal{L}\{(t+2)^3\} = \mathcal{L}\{t^3+6t^2+12t+8\} = \frac{6}{s^4}+\frac{12}{s^3}+\frac{12}{s^2}+\frac{8}{s}$.

(c) $\mathcal{L}\{9e^{4t+7}\} = \mathcal{L}\{9e^7e^{4t}\} = \frac{9e^7}{s-4}$.

(d) $\mathcal{L}\{-e^{3(t-2)}\} = \mathcal{L}\{-e^{-6}e^{3t}\} = -\frac{e^{-6}}{s-3}$.

(e) $\mathcal{L}\{(e^t+1)^2\} = \mathcal{L}\{e^{2t}+2e^t+1\} = \frac{1}{s-2}+\frac{2}{s-1}+\frac{1}{s}$.

(f) $\mathcal{L}\{8\sin(3t)-4\cos(3t)\} = \frac{24}{s^2+9}-\frac{4s}{s^2+9}$.

(g) $\mathcal{L}\{t^2e^{5t}\} = \frac{2}{(s-5)^3}$.

(h) $\mathcal{L}\{7e^{3t}\cos 4t\} = \frac{7s-21}{(s-3)^2+16}$.

(i) $\mathcal{L}\{4e^{-t}\sin 5t\} = \frac{20}{(s+1)^2+25}$.

(j) $\mathcal{L}\{t^2\sin 5t\} = \frac{40s^2-10(s^2+25)}{(s^2+25)^3}$.

(k) If $f(t) = \begin{cases} 0, & t<1, \\ t^2-2t+2, & t\geq 1, \end{cases}$ then $\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{1}{s}+\frac{2}{s^3}\right)$.

(l) If $f(t) = \begin{cases} 0, & t<1, \\ t-2, & 1\leq t<2, \\ 0, & t\geq 2 \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2}-\frac{e^{-s}}{s}-\frac{e^{-2s}}{s^2}$.

(m) If $f(t) = \begin{cases} 5e^{2t}, & t<3, \\ 0, & t\geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{5}{s-2}-\frac{5e^6e^{-3s}}{s-2}$.

(n) If $f(t) = \begin{cases} 7t^2e^{-t}, & t<3, \\ 0, & t\geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{14}{(s+1)^3}-\frac{14e^{-3s-3}}{(s+1)^3}-\frac{42e^{-3s-3}}{(s+1)^2}-\frac{63e^{-3s-3}}{s+1}$.

(o) If $f(t) = \begin{cases} 0, & t<\pi, \\ \sin t, & t\geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s}\frac{1}{s^2+1}$.

(p) If $f(t) = \begin{cases} 0, & t<\pi/2, \\ \cos t, & \pi/2\leq t<\pi, \\ 0, & t\geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s/2}\frac{1}{s^2+1}+e^{-\pi s}\frac{s}{s^2+1}$.

(Answer 38)

(a) $\mathcal{L}^{-1}\left\{\frac{2s+8}{s^2+2s+5}\right\} = 2e^{-t}\cos(2t)+3e^{-t}\sin(2t)$

(b) $\mathcal{L}^{-1}\left\{\frac{5s-7}{s^4}\right\} = \frac{5}{2}t^2-\frac{7}{6}t^3$

(c) $\mathcal{L}^{-1}\left\{\frac{s+2}{(s+1)^4}\right\} = \frac{1}{2}t^2e^{-t}+\frac{1}{6}t^3e^{-t}$

(d) $\mathcal{L}^{-1}\left\{\frac{2s-3}{s^2-4}\right\} = (1/4)e^{2t}+(7/4)e^{-2t}$

(e) $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-3)^2}\right\} = \frac{2}{27}+\frac{1}{9}t-\frac{2}{27}e^{3t}+\frac{1}{9}t\,e^{3t}$

(f) $\mathcal{L}^{-1}\left\{\frac{s+2}{s(s^2+4)}\right\} = \frac{1}{2}-\frac{1}{2}\cos(2t)+\frac{1}{2}\sin(2t)$

(g) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+9)}\right\} = \frac{1}{8}\cos t-\frac{1}{8}\cos 3t$

(h) $\mathcal{L}^{-1}\left\{\frac{(2s-1)e^{-2s}}{s^2-2s+2}\right\} = 2\mathcal{U}(t-2)\,e^{t-2}\,\cos(t-2)+\mathcal{U}(t-2)\,e^{t-2}\,\sin(t-2)$

(i) $\mathcal{L}^{-1}\left\{\frac{(s-2)e^{-s}}{s^2-4s+3}\right\} = \frac{1}{2}\mathcal{U}(t-1)\,e^{3(t-1)}+\frac{1}{2}\mathcal{U}(t-1)\,e^{t-1}$

(j) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\} = \int_0^t \frac{1}{3}\sin 3r\cos(3t-3r)\,dr$

(Answer 39)

(a) If $y' - 9y = \sin 3t$, $y(0) = 1$, then $y(t) = -\frac{1}{30}\sin 3t - \frac{1}{90}\cos 3t + \frac{31}{30}e^{9t}$.
(b) If $y' - 2y = 3e^{2t}$, $y(0) = 2$, then $y = 3te^{2t} + 2e^{2t}$.
(c) If $y' + 5y = t^3$, $y(0) = 3$, then $y = \frac{1}{5}t^3 - \frac{3}{25}t^2 + \frac{6}{125}t - \frac{6}{625} + \frac{1881}{625}e^{-5t}$.
(d) If $y'' - 4y' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$, then $y(t) = e^{2t} - te^{2t}$.
(e) If $y'' - 2y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$, then $y(t) = \frac{1}{5}(e^{-t} - e^t\cos t + 7e^t\sin t)$.
(f) If $y'' - 7y' + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$, then $y(t) = 4e^{3t} - 2e^{4t} - te^{3t}$.
(g) If $y'' + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$, then $y(t) = \frac{1}{5}\cos 2t + \cos 3t + \frac{5}{3}\sin 3t$.
(h) If $y'' + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t r\sin 3r\sin(3t - 3r)\,dr$.
(i) If $y'' + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = (1/6)(1 - \mathcal{U}(t - 2\pi))(2\sin t - \sin 2t).$$

(j) If $y'' + 3y' + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - \mathcal{U}(t - 10)\left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right].$$

(k) If $y'' + y' + \frac{5}{4}y = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t \end{cases}$, $y(0) = 1$, $y'(0) = 0$, then

$$\begin{aligned} y(t) = e^{-t/2}\cos t &+ \frac{1}{2}e^{-t/2}\sin t \\ &- \mathcal{U}(t-\pi)\left(\frac{16}{17}\cos(t-\pi) - \frac{4}{17}\sin(t-\pi)\right) \\ &+ \mathcal{U}(t-\pi)\left(\frac{4}{17}e^{-(t-\pi)/2}\sin(t-\pi) + \frac{16}{17}e^{-(t-\pi)/2}\cos(t-\pi)\right). \end{aligned}$$

(Answer 40)

(a) If $6y'' + 5y' + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$, then

$$y = 6e^{-t/3} - 6e^{-t/2} + 4\mathcal{U}(t-2) - 12\mathcal{U}(t-2)e^{-(t-2)/3} + 8\mathcal{U}(t-2)e^{-(t-2)/2}.$$

The graph of $y'(t)$ has a corner at $(2, 0.0768)$, and at $t = 2$, the graph of $y''(t)$ jumps from $y'' = -0.2095$ to $y'' = 0.45712$.

(b) If $y' + 9y = 7\delta(t-2)$, $y(0) = 3$, then

(c) If $y'' + 4y = \delta(t - 4\pi)$, $y(0) = 1/2$, $y'(0) = 0$, then

$$y = \frac{1}{2}\cos(2t) - \mathcal{U}(t-4\pi)\sin(2t).$$

The graph of $y(t)$ has a corner at $(4\pi, 1/2)$, and at $t = 4\pi$, the graph of $y'(t)$ jumps from $y' = 0$ to $y' = -2$. ($y''(t)$ is hard to graph, because the impulse function $\delta(t-4\pi)$ is part of $y''(t)$.)

(d) If $y'' + 4y' + 3y = 2\delta(t-1) + \mathcal{U}(t-2)$, $y(0) = 1$, $y'(0) = 0$, then

$$y = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} + \frac{1}{2}\mathcal{U}(t-1)e^{-t+1} - \frac{1}{2}\mathcal{U}(t-1)e^{-3t+3} + \frac{1}{3}\mathcal{U}(t-2) - \frac{1}{2}e^{-t+2}\mathcal{U}(t-2) + \frac{1}{6}\mathcal{U}(t-2)e^{-3t+6}.$$

The graph of $y(t)$ has a corner at $(1, 0.5269)$. The graph of $y'(t)$ has a corner at $(2, -0.3085)$, and at $t = 1$ the graph of $y'(t)$ jumps from $y' = -0.4771$ to $y' = 0.5229$. $y''(t)$ has an impulse at $t = 1$, and at $t = 2$, $y''(t)$ jumps from $y'' = 0.1517$ to $y'' = 1.1517$.

(Answer 41)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{(2/3)t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + e^{4t}\begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

(Answer 42)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t}\begin{pmatrix} 8t-5 \\ -4t+3 \end{pmatrix}.$$

(Answer 43)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t}\begin{pmatrix} -3\cos t + 16\sin t \\ 2\cos t + 7\sin t \end{pmatrix}.$$

(Answer 44)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{3}{20}e^{-20t}\begin{pmatrix} 1 \\ 6 \end{pmatrix} + \frac{1}{20}e^{10t}\begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

(Answer 45)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t}\begin{pmatrix} 5\cos 12t + \sin 12t \\ 2\cos 12t + 2\sin 12t \end{pmatrix}.$$

(Answer 46)

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos 3t - 11\sin 3t \\ -3\cos 3t - (23/3)\sin 3t \end{pmatrix}.$$

(Answer 47)

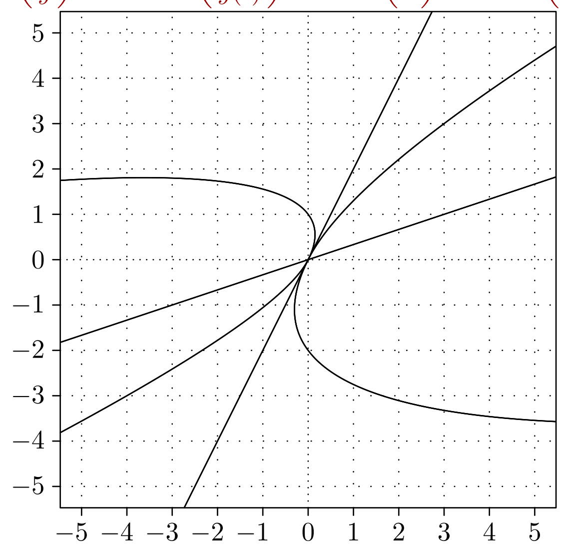
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 15t+1 \\ (15/4)t+4 \end{pmatrix}.$$

(Answer 48)

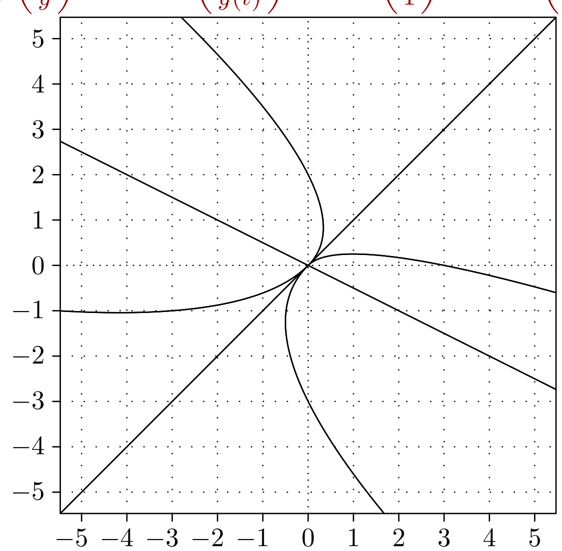
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} + e^{-6t} \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix}.$$

(Answer 49)

(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

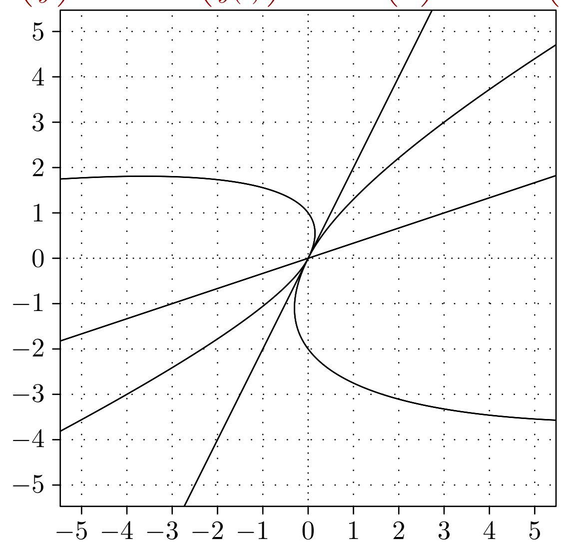


(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

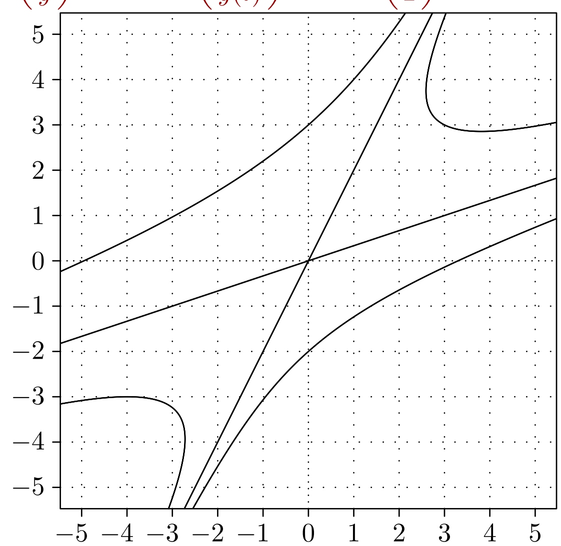


(Answer 50)

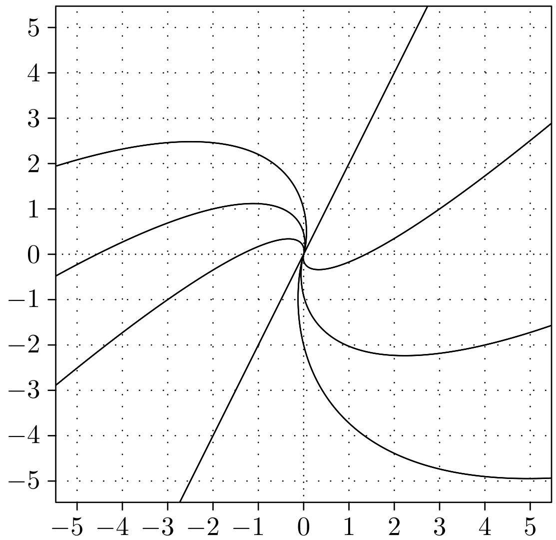
(a) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



(b) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -2.6 & 1.8 \\ -1.2 & 1.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$



(c) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 4/3 & -1/6 \\ 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^t \begin{pmatrix} t+3 \\ 2t \end{pmatrix}$



(d) $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 4.5 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 3\sin 3t \\ 2\cos 3t \end{pmatrix} + C_2 e^t \begin{pmatrix} -3\cos 3t \\ 2\sin 3t \end{pmatrix}$