7.1. Basic properties of harmonic functions

[Definition: Harmonic function] We say that $u$ is harmonic in a domain $\Omega \subseteq \mathbb{C}$ if $u$ is $C^2$ in $\Omega$ and if
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ in } \Omega.$$

(Problem 1) Write the definition of harmonic function using the operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$.

(Problem 2) Prove that if $F$ is holomorphic in an open set $\Omega$ and $u = \text{Re } F$ then $u$ is harmonic.

(Problem 3) Prove that if $u$ is harmonic in a disc $\mathcal{D}$, then there is a holomorphic function $F$ such that $\text{Re } F = u$. Do this by showing that there exists a function $v$ that satisfies the Cauchy-Riemann equations.

(Problem 4) Prove that if $u$ is harmonic in a disc $\mathcal{D}$, then there is a holomorphic function $F$ such that $\text{Re } F = u$. Do this by considering the function $\frac{\partial u}{\partial z}$.

[Definition: Harmonic conjugate] Let $u$ and $v$ be two real-valued functions. If $F = u + iv$ is holomorphic, then we say that $v$ is a harmonic conjugate of $u$.

(Problem 5) Suppose that $v$ is a harmonic conjugate of $u$. Is $u$ also a harmonic conjugate of $v$?

(Problem 6) Show that harmonic functions are smooth.

(Problem 7) Provide an example of a domain $\Omega$ and a function $u$ that is harmonic on $\Omega$ but is not the real part of a holomorphic function on $\Omega$.

(Problem 8) Give a general class of domains $\Omega$ such that every function $u$ that is harmonic on $\Omega$ is the real part of a holomorphic function. Prove your assertion.

(Problem 8a) Prove that if every function $u$ that is harmonic on $\Omega$ is the real part of a holomorphic function, then $\text{Ind}_\gamma(w) = 0$ for every closed piecewise-$C^1$ path $\gamma \subset \Omega$ and every point $w \notin \Omega$.

(Problem 8b) Prove that if $\Omega$ is holomorphically simply connected, then $\text{Ind}_\gamma(w) = 0$ for every closed piecewise-$C^1$ path $\gamma \subset \Omega$ and every point $w \notin \Omega$, then $\Omega$ is simply connected.

(Problem 9) Suppose that $u$, $v$, and $w$ are real $C^2$ functions on a connected domain $\Omega$ and that $u + iv$ and $u + iw$ are both holomorphic. What can you say about $v$ and $w$?

(Problem 10) Suppose that $\varphi : \Omega \mapsto V$ is holomorphic and that $u$ is harmonic on $V$. Prove that $\tilde{u} = u \circ \varphi$ is harmonic on $\Omega$ by using the chain rule for complex differentiation.

(Problem 10a) Suppose that $\varphi : \Omega \mapsto V$ is holomorphic and that $u$ is harmonic on $V$. Prove that $\tilde{u} = u \circ \varphi$ is harmonic on $\Omega$ by using the multivariable chain rule for real-valued functions.

(Problem 11) Suppose that $\varphi : \Omega \mapsto V$ is holomorphic and that $u$ is harmonic on $V$. Prove that $\tilde{u} = u \circ \varphi$ is harmonic on $\Omega$ by using the fact that $u = \text{Re } F$ (locally) for a holomorphic function $F$.

7.2. The maximum principle and the mean value property

(Problem 12) Prove that if $u$ is harmonic in a neighborhood of $\overline{D}(P, r)$, then $u(P) = \frac{1}{2\pi} \int_0^{2\pi} u(P + re^{i\theta}) d\theta$.

[Definition: Mean value property] The formula given in Problem 12.

(Problem 18) Suppose that $u$ is harmonic on a neighborhood of $\overline{D}(0, 1)$. If $z \in D(0, 1)$, find a formula for $u(z)$ in terms of the values of $u$ on $\partial D(0, 1)$. Hint: Start by recalling the set of holomorphic self-maps of $D(0, 1)$. 
(Problem 13) Prove the maximum principle for harmonic functions by using the fact that harmonic functions are real parts of holomorphic functions. That is, prove that if $\Omega \subseteq \mathbb{C}$ is open and connected and if $u : \Omega \to \mathbb{R}$ is harmonic, and if there is some $P \in \Omega$ such that $u(P) \geq u(z)$ for all $z \in \Omega$, then $u$ is constant in $\Omega$.

(Problem 14) Prove the maximum principle for harmonic functions by using the mean value property. Hint: Show that $\{z \in \Omega : u(z) = u(P)\}$ and $\{z \in \Omega : u(z) < u(P)\}$ are both open and use the definition of connectedness in terms of open sets.

(Problem 15) Prove the minimum principle for harmonic functions.

(Problem 16) Suppose that $u$ is harmonic in $\Omega$ and continuous on $\partial \Omega$ for some bounded open set $\Omega$. What can you say about $\max_{\partial \Omega} u$ and $\max_{\overline{\Omega}} u$?

(Problem 17) Can you make the same statement if $\Omega$ is not bounded?

(Problem 17a) Prove that if $u$ and $v$ are both harmonic in $D(0, 1)$, continuous on $\partial D(0, 1)$, and $u(\zeta) = v(\zeta)$ for all $\zeta \in \partial D(0, 1)$, then $u(z) = v(z)$ for all $z \in D(0, 1)$.

7.3. The Poisson integral formula

[Definition: Poisson integral formula] We have that if $u$ is harmonic in a neighborhood of $\overline{D}(0, 1)$, then for all $|z| < 1$,

$$u(z) = \frac{1}{2\pi} \int_{0}^{2\pi} u(re^{i\theta}) \frac{1 - |z|^2}{|e^{i\theta} - z|^2} d\psi,$$

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{0}^{2\pi} u(e^{i\psi}) \frac{1 - r^2}{1 - 2r \cos(\theta - \psi) + r^2} d\psi.$$

Let $P_r(\theta - \psi) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \psi) + r^2}$, $P(z, \zeta) = \frac{|\zeta|^2 - |z|^2}{2\pi |\zeta - z|^2}$.

(Problem 19) Verify that the two formulas above are equivalent.

(Problem 20) Prove that if $\theta$ is real and $0 \leq r < 1$ then $0 < P_r(\theta) < \infty$ (in particular, the denominator is never zero).

(Bonus problem 20a) Show that $p(z) = P(z, \zeta)$ is harmonic on $\mathbb{C} \setminus \{\zeta\}$; in particular, if $\zeta = e^{i\theta}$ then $p(z)$ is harmonic in $D(0, 1)$.

(Bonus problem 20b) Find a holomorphic function $f(z) = F(z, \zeta)$ such that $p(z) = P(z, \zeta)$ is the real part of $f$.

(Problem 21) Suppose that $u$ is harmonic in a neighborhood of $\overline{D}(P, r)$. If $z \in D(P, r)$, find a formula for $u(z)$ in terms of the values of $u$ on $\partial D(P, r)$.

(Problem 22) Suppose that $u$ is continuous on $\overline{D}(0, 1)$ and harmonic in $D(0, 1)$. Show that the Poisson integral formula is still valid.

(Problem 23) Let $f$ be real-valued and continuous on $\partial D(0, 1)$. Let $u(z) = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{i\psi}) \frac{1 - |z|^2}{|e^{i\psi} - z|^2} d\psi$. Show that $u \in C^2(D(0, 1))$.

(Problem 24) Let $u$, $f$ be as in Problem 23. Show that $u$ is harmonic in $D(0, 1)$.

(Problem 25) Prove that if $0 \leq r < 1$ then $\int_{0}^{2\pi} P_r(\theta) d\theta = 1$.

(Problem 26) Prove that $\lim_{r \to 1^-} P_r(\theta) = 0$ for all $\theta \neq 2n\pi$.

(Problem 27) Let $0 < \delta < \pi$ be a small positive number. Prove that $\lim_{r \to 1^-} P_r(\theta) = 0$ uniformly for all $\delta < \theta < 2\pi - \delta$.

(Problem 28) Let $0 < \delta < \pi$. Prove that $\lim_{r \to 1^-} \int_{-\delta}^{\delta} P_r(\theta) d\theta = 1$.

(Problem 29) Let $u$, $f$ be as in Problem 23. Show that $\lim_{r \to 1^-} u(re^{i\theta}) = f(e^{i\theta})$ for all $0 \leq \theta \leq 2\pi$. 
(Problem 29a) Let $u, f$ be as in Problem 23. Show that $u(re^{i\theta})$ converges to $f(e^{i\theta})$ as $r \to 1^-$ uniformly in $\theta$.

(Problem 30) Let $u, f$ be as in Problem 23. Show that $u$ is continuous on $\bar{D}(0, 1)$.

(Problem 31) We can find a harmonic function in $D(0, 1)$ with arbitrary boundary data using the Poisson integral formula. Why can’t we find a holomorphic function in $D(0, 1)$ with arbitrary boundary data by using the Cauchy integral formula?

7.4. Regularity of harmonic functions

[Definition: The “small circle” mean value property] Let $\Omega \subset \mathbb{C}$ be open and let $h : \Omega \to \mathbb{R}$ be continuous. We say that $h$ has the SCMV property if, for every $P \in \Omega$, there is some number $\varepsilon_P > 0$ such that $\bar{D}(P, \varepsilon_P) \subset \Omega$ and such that $h(P) = \frac{1}{2\pi} \int_0^{2\pi} h(P + \varepsilon e^{i\theta}) d\theta$ for all $0 < \varepsilon < \varepsilon_P$.

(Problem 32) Let $\Omega \subset \mathbb{C}$ be open and connected. Let $g$ be continuous on $\Omega$ and satisfy the “small circle” mean value property. Show that $g$ satisfies the maximum principle, that is, if there is some $P \in \Omega$ such that $g(P) \geq g(z)$ for all $z \in \Omega$ then $g$ is constant.

(Problem 33) Suppose that $g$ is continuous on $\bar{D}(P, r)$ and has the “small circle” mean value property in $D(P, r)$. Suppose further that $g = 0$ on $\partial D(P, r)$. Show that $g = 0$ in $D(P, r)$.

(Problem 34) Suppose that $g$ and $h$ are continuous on $\bar{D}(P, r)$ and that $u = h$ on $\partial D(P, r)$. Suppose that $h$ is harmonic in $D(P, r)$ and that $g$ has the “small circle” mean value property in $D(P, r)$. Show that $g = h$ in $D(P, r)$ as well.

(Problem 35) Let $\Omega \subset \mathbb{C}$ be open. Suppose that $g$ is continuous and has the “small circle” mean value property in $\Omega$. Show that $g$ is harmonic in $\Omega$.

(Problem 36) Let $\Omega \subset \mathbb{C}$ be open. Suppose that $\{h_j\}_{j=1}^\infty$ is a sequence of functions, each harmonic on $\Omega$, and that $h_j \to h$ uniformly on $\Omega$. Show that $h$ is also harmonic by showing that it has the “small circle” mean value property.

7.5. The Schwarz reflection principle

(Problem 37) Suppose $\Omega \subset \mathbb{C}$ is open and that $u$ is harmonic on $\Omega$. Let $v(z) = u(\overline{z})$. Show that $v$ is harmonic on $\Omega = \{z \in \mathbb{C} : \overline{z} \in \Omega\}$.

(Problem 38) Suppose $\Omega \subset \mathbb{C}$ is open and that $f$ is holomorphic on $\Omega$. Let $g(z) = \overline{f(\overline{z})}$. Show that $g$ is holomorphic on $\bar{\Omega} = \{z \in \mathbb{C} : \overline{z} \in \Omega\}$.

(Problem 39) Let $\Psi \subset \mathbb{C}$ be open and connected. Suppose that $\Psi$ is symmetric about the real axis; that is, $z \in \Psi$ if and only if $\overline{z} \in \Psi$. Let $\Omega = \{z \in \Psi : \text{Im } z > 0\}$.

Suppose that $v$ is harmonic in $\Omega$, continuous on $\overline{\Omega} \cap \Psi$, and that $v(x) = 0$ for any $x \in \Psi \cap \mathbb{R}$.

Sketch $\Psi$. Label $\Omega$, $\hat{\Omega}$, the set where $v$ is harmonic, and the set where $v$ is equal to zero.

(Problem 39a) Let $\Psi, \Omega, \hat{\Omega}$, and $v$ be as in Problem 39. Show that $\hat{v}$ is continuous in $\Psi$, where $\hat{v}(z) = \begin{cases} v(z), & z \in \Omega, \\ 0, & z \in \overline{\Omega} \cap \Psi, \\ -v(\overline{z}), & z \in \hat{\Omega} = \{w \in \mathbb{C} : \overline{w} \in \Omega\}. \end{cases}$

(Problem 40) Suppose that $v, \hat{v}$, and $\Psi$ are as in Problem 39. Show that $\hat{v}$ is harmonic in $\Psi$. Hint: Use the small circle mean value property.

(Problem 41) Suppose that $f$ is holomorphic in $D(x_0, r)$ for some $x_0 \in \mathbb{R}$ and some $r > 0$. Suppose further that $f(x)$ is real for all $x \in (x_0 - r, x_0 + r) = D(x_0, r) \cap \mathbb{R}$. Show that $f(z) = \overline{f(z)}$ for all $z \in D(x_0, r)$.
In Problems 49–50, let

\[ D \] continuous on \( \{ z \in D(x_0, r) : \text{Im} \, z \geq 0 \} \). Further suppose that \( f(x) \) is real for all \( x \in (x_0 - r, x_0 + r) \). Show that there is some function \( \hat{f} \) that is holomorphic in \( D(x_0, r) \) and equals \( f \) on \( \Omega = D(x_0, r) \). Find a formula for \( \hat{f}(z) \).

Let \( f \) be holomorphic in \( \Omega \) and continuous on \( \overline{\Omega} \cap \Psi = \Omega \cup (\Psi \cap \mathbb{R}) \). Suppose that \( \text{Im} \, f = 0 \) on \( \Psi \cap \mathbb{R} \). Show that \( f \) is holomorphic on \( \Psi \), where

\[
\hat{f}(z) = \begin{cases} 
  f(z), & z \in \overline{\Psi} \\
  \overline{f(z)}, & z \in \Omega = \{ w \in \mathbb{C} : \overline{w} \in \Omega \}.
\end{cases}
\]

Suppose that \( \lim_{u \to 0} u \) for \( u \in \overline{\Omega} \). Suppose further that \( f \) is real-valued on \( D(z_0, \varepsilon) \) and that \( f \) is holomorphic in \( D(z_0, \delta) \); that is, show that there exists a function \( \hat{f} \) that is holomorphic in \( D(z_0, \delta) \) and such that \( f = \hat{f} \) in \( D(z_0, \delta) \cap D(0, 1) \).

Let \( X \) be an open set and let \( \Omega = X \cap D(0, 1) \). Suppose that \( f \) is holomorphic on \( \Omega \) and continuous on \( \overline{\Omega} \cap X \), and that \( f \) is real on \( \partial D(0, 1) \cap X \). Find an open set \( \Psi \supset \Omega \) such that \( f \) may be extended to a holomorphic function \( \hat{f} \) on \( \Psi \). Give a formula for \( \hat{f}(z) \) whenever \( z \in \Psi \setminus \overline{\Omega} \).

Suppose that \( f \) is holomorphic on \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im} \, z > 0 \} \) and continuous on \( \mathbb{H} \) and that \( f(x) = 0 \) for all \( 0 < x < 1 \). Show that \( f(z) = 0 \) for all \( z \in \mathbb{H} \).

7.6. Harnack’s principle

Recall that if \( u \) is harmonic in \( D(P, R) \) and continuous on \( \overline{D}(P, R) \), then for any \( 0 \leq r < R \) and any \( 0 \leq \theta \leq 2\pi \),

\[
 u(P + re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(P + Re^{i\psi}) \frac{R^2 - r^2}{|Re^{i\psi} - re^{i\theta}|^2} \, d\psi.
\]

Find \( \min_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq 2\pi} |Re^{i\psi} - re^{i\theta}|^2 \) and \( \max_{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq 2\pi} |Re^{i\psi} - re^{i\theta}|^2 \).

The Harnack inequality. Suppose that \( u \) is nonnegative and harmonic in \( D(P, R) \) and continuous on \( \overline{D}(P, R) \). Let \( z = P + re^{i\theta} \in D(P, R) \). Show that

\[
\frac{R - r}{R + r} u(P) \leq u(z) \leq \frac{R + r}{R - r} u(P).
\]

Did we need the assumption that \( u \) was continuous on \( \overline{D}(P, R) \)?

Suppose that \( u \) is harmonic in \( D(P, R) \) and continuous on \( \overline{D}(P, R) \). Find formulas for \( \partial_\theta u \) and \( \partial_\phi u \) in terms of \( u(P + Re^{i\phi}) \), \( 0 \leq \theta \leq 2\pi \).

Suppose that \( u \) is harmonic in \( D(P, R) \) and that \( |u| \leq M \) in \( D(P, R) \). Find an upper bound on \( |\nabla u(z)| \) for any \( z \in D(P, R) \) in terms of \( M \), \( R \) and \( |z - P| \).

In Problems 49–50, let \( \{ u_j \}_{j=1}^\infty \) be a sequence of real-valued functions harmonic in \( D(P, R) \) such that \( u_1(z) \leq u_2(z) \leq u_3(z) \leq \cdots \) for each \( z \in D(P, R) \).

Suppose that \( \lim_{j \to \infty} u_j(P) = \infty \). Show that \( u_j \to \infty \) uniformly on \( D(P, r) \) for any \( 0 < r < R \).

Suppose that \( \lim_{j \to \infty} u_j(P) < \infty \). Show that \( u_j \) converges to some (finite) harmonic function, uniformly on \( D(P, r) \) for any \( 0 < r < R \).
In Problems 51–52, let $\Omega \subseteq \mathbb{C}$ be a connected open set and let $\{u_j\}_{j=1}^\infty$ be a sequence of real-valued functions harmonic in $\Omega$ such that $u_1(z) \leq u_2(z) \leq \cdots$ for each $z \in \Omega$.

(Problem 51) Show that either $\lim_{j \to \infty} u_j(z) = \infty$ for all $z \in \Omega$ or $\lim_{j \to \infty} u_j(z) < \infty$ for all $z \in \Omega$. Hint: Show that $\{z : \lim_{j \to \infty} u_j(z) = \infty\}$ and $\{z : \lim_{j \to \infty} u_j(z) < \infty\}$ are both open.

(Problem 52) Harnack’s principle. Show that either $\lim_{j \to \infty} u_j(z) = \infty$ for all $z \in \Omega$, uniformly on compact sets, or that there is some function $u_0$ harmonic in $\Omega$ such that $u_j \to u_0$ uniformly on compact sets.

7.7. Subharmonic functions

[Definition: Subharmonic functions] Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \mapsto \mathbb{R}$ be continuous. Suppose that for every $D(P, r) \subset \Omega$, we have that

$$f(P) \leq \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta}) \, d\theta.$$

Then we say that $f$ is subharmonic in $\Omega$.

[Definition: Superharmonic functions] Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \mapsto \mathbb{R}$ be continuous. Suppose that for every $D(P, r) \subset \Omega$, we have that

$$f(P) \geq \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta}) \, d\theta.$$

Then we say that $f$ is superharmonic in $\Omega$.

(Problem 53) Show that $f$ is subharmonic if and only if $-f$ is superharmonic.

(Problem 54) Suppose that $f$ is subharmonic in an open set $\Omega$ and that $\alpha \geq 0$ is a nonnegative real number. Show that $\alpha f$ is subharmonic in $\Omega$. Did we need the assumption $\alpha \geq 0$?

(Problem 55) Suppose that $f$ and $g$ are both subharmonic in an open set $\Omega$. Show that $f + g$ is subharmonic in $\Omega$. Is $f - g$ subharmonic in $\Omega$?

(Problem 56) Suppose that $f$ is subharmonic and $g$ is superharmonic in an open set $\Omega \subseteq \mathbb{C}$. Show that $f - g$ is subharmonic in $\Omega$.

(Problem 57) Suppose that $f$ is a continuous, real-valued function in an open set $\Omega \subseteq \mathbb{C}$. Show that $f$ is harmonic if and only if $f$ is both subharmonic and superharmonic.

(Problem 58) Suppose that $u$ and $v$ are both subharmonic in an open set $\Omega$. Let $f(z) = \max(u(z), v(z))$. Show that $f$ is subharmonic in $\Omega$. (In particular, if $u$ and $v$ are real and harmonic then $f$ is subharmonic.)

(Problem 58a) Let $\Omega \subset \mathbb{C}$ be open and let $f : \Omega \mapsto \mathbb{C}$ be holomorphic. Show that $u(z) = |f(z)|$ is subharmonic in $\Omega$.

(Bonus problem 58b) Let $\Omega \subset \mathbb{C}$ be open and let $u : \Omega \mapsto \mathbb{C}$ be subharmonic. Let $\varphi : \mathbb{R} \mapsto \mathbb{R}$ be nondecreasing and convex, so that if $0 < t < 1$ and $a, b \in \mathbb{R}$ then $\varphi(ta + (1-t)b) \leq t\varphi(a) + (1-t)\varphi(b)$. Show that $v(z) = \varphi(u(z))$ is subharmonic in $\Omega$.

(Problem 59) Give eight examples of functions that are subharmonic in a domain $\Omega$ but are not harmonic in that domain.

(Problem 60) Prove the maximum principle for subharmonic functions.

(Problem 61) Is there a minimum principle for subharmonic functions?

(Problem 69) Let $\Omega \subseteq \mathbb{C}$ be open and connected. Suppose that $f : \Omega \mapsto \mathbb{R}$ is continuous and satisfies the small circle sub-mean-value property: for every $P \in \Omega$, there is some $\varepsilon_P > 0$ such that $D(P, \varepsilon_P) \subset \Omega$ and such that

$$f(P) \leq \frac{1}{2\pi} \int_0^{2\pi} f(P + \varepsilon e^{i\theta}) \, d\theta \quad \text{for all } 0 < \varepsilon < \varepsilon_P.$$

Show that $f$ satisfies the maximum principle in $\Omega$. 
(Problem 62) Suppose that \( f \) is continuous on \( \overline{D}(P,r) \) and subharmonic in \( D(P,r) \). Suppose that \( h \) is continuous on \( \overline{D}(P,r) \) and harmonic in \( D(P,r) \). Suppose that \( f \leq h \) on \( \partial D(P,r) \). Show that \( f \leq h \) in \( D(P,r) \).

(Problem 63) Suppose that \( \Omega \subseteq \mathbb{C} \) is open and that \( f : \Omega \mapsto \mathbb{R} \) is continuous. Suppose further that whenever \( \overline{D}(P,r) \subset \Omega \), \( h \) is harmonic in \( D(P,r) \) and continuous on \( \overline{D}(P,r) \), and \( h \geq f \) on \( \partial D(P,r) \), we have that \( h \geq f \) in \( D(P,r) \). Prove that \( f \) is subharmonic.

(Problem 70) Let \( \Omega \subseteq \mathbb{C} \) be open. Suppose that \( f : \Omega \mapsto \mathbb{R} \) is continuous and satisfies the small circle sub-mean-value property in \( \Omega \) (as in Problem 69). Use Problem 63 to show that \( f \) is subharmonic in \( \Omega \).

7.7. The Dirichlet problem

[Definition: The Dirichlet problem] Let \( \Omega \subseteq \mathbb{C} \) be a bounded open connected set. We say that the Dirichlet problem is well posed on \( \Omega \) if, for every function \( f \) defined and continuous on \( \partial \Omega \), there is exactly one function \( u \) that is harmonic in \( \Omega \), continuous on \( \overline{\Omega} \), and such that \( u = f \) on \( \partial \Omega \).

(Problem 64) Give an example of an unbounded domain \( \Omega \) and two functions \( u \) and \( v \) that are harmonic in \( \Omega \), continuous on \( \overline{\Omega} \) and equal zero on \( \partial \Omega \).

(Problem 65) Prove that we have uniqueness for the Dirichlet problem in any bounded domain; that is, show that if \( \Omega \subseteq \mathbb{C} \) is bounded, if \( u \) and \( v \) are both harmonic in \( \Omega \) and continuous on \( \overline{\Omega} \), and if \( u = v \) on \( \partial \Omega \), then \( u = v \) in \( \Omega \). Clearly explain how you used the fact that \( \Omega \) is bounded.

(Problem 66) Let \( 0 < r < 1 \). Find a function \( u \) that is harmonic in the annulus \( \Omega = D(0,1) \setminus D(0,r) \), continuous on \( \overline{\Omega} \) and such that \( u(e^{i\theta}) = 0 \), \( u(re^{i\theta}) = 1 \) for any \( 0 \leq \theta \leq 2\pi \).

(Problem 67) Let \( \Omega = D(0,1) \setminus \{0\} \). Let \( u \) be harmonic in \( \Omega \) and continuous on \( \overline{\Omega} \). Suppose that \( u(e^{i\theta}) \) is constant for \( 0 \leq \theta \leq 2\pi \). Show that \( u \) is radial; that is, for any fixed \( r \) with \( 0 < r < 1 \), \( u(re^{i\theta}) = u(re^{i\psi}) \) for any \( 0 \leq \theta \leq 2\pi \), \( 0 \leq \psi \leq 2\pi \).

(Problem 68) Let \( \Omega = D(0,1) \setminus \{0\} \). Suppose that \( u \) is harmonic in \( \Omega \), continuous on \( \overline{\Omega} \), and that \( u = 0 \) on \( \partial D(0,1) \). Prove that \( u(0) = 0 \). Is the Dirichlet problem well posed in \( \Omega \)?

7.8. The Perron method and the solution to the Dirichlet problem

Our goal is to use subharmonic functions to construct solutions to the Dirichlet problem.

(Problem 71) Let \( \Omega \subset \mathbb{C} \) be a bounded open set. Let \( f : \partial \Omega \mapsto \mathbb{R} \) be continuous. Let
\[
S = \{ \psi : \psi \text{ is subharmonic in } \Omega, \text{ continuous on } \overline{\Omega} \text{ and } \psi(w) \leq f(w) \text{ for all } w \in \partial \Omega \}.
\]
Show that \( S \) is nonempty.

(Problem 72) Let \( S \) be as in Problem 71. For each \( z \in \Omega \), let \( u(z) = \sup \{ \psi(z) : \psi \in S \} \). Show that \( u \) is finite for all \( z \in \Omega \) and, in fact, is bounded above and below.

(Problem 73) Suppose that \( \Omega = D(0,1) \setminus \{0\} \) and that \( f(e^{\theta}) = 1 \), \( f(0) = 0 \). Let \( u \) be as in Problem 72. Show that \( u(z) = 1 \) for all \( z \in \Omega \).

(Problem 74) Let \( \Omega \subseteq \mathbb{C} \) be open. Let \( u \) be as in Problem 72. Show that \( u \) is lower semicontinuous on \( \overline{\Omega} \); that is, for each \( P \in \overline{\Omega} \) and each \( \varepsilon > 0 \), show that there is some \( \delta > 0 \) such that if \( z \in D(P,\delta) \cap \overline{\Omega} \), then \( u(z) > u(P) - \varepsilon \).

(Problem 75) Can we show that \( u \) is continuous on \( \overline{\Omega} \)?

(Problem 76) Let \( \Omega \subseteq \mathbb{C} \) be open and let \( f \) be subharmonic in \( \Omega \). Suppose that \( \overline{D}(P,r) \subset \Omega \). Let \( h \) be harmonic in \( D(P,r) \) with \( h = f \) on \( \partial D(P,r) \); we may construct \( h \) using the Poisson integral. Let
\[
\psi(z) = \begin{cases} 
  h(z), & z \in D(P,r) \\
  f(z), & z \in \Omega \setminus D(P,r).
\end{cases}
\]
Show that \( \psi \) is subharmonic in \( \Omega \).

(Problem 77) Let \( u, \Omega \) and \( S \) be as in Problems 71–72. Let \( w \in \Omega \). Show that there is a sequence of functions \( \{ \psi_j \}_{j=1}^{\infty} \subset S \) such that \( u(w) = \lim_{j \to \infty} \psi_j(w) \).
(Problem 78) Show that there is a sequence of functions \( \{\varphi_j^w\}_{j=1}^\infty \subset S \) such that \( u(w) = \lim_{j \to \infty} \varphi_j^w(w) \) and such that \( \varphi_j^w(z) \leq \varphi_j^w(z) \leq \varphi_j^\Omega(z) \) for all \( z \in \Omega \).

(Problem 79) Let \( w \in D(P, r) \) for some \( D(P, r) \subset \Omega \). Show that there is a sequence of functions \( \{\eta_j^w\}_{j=1}^\infty \subset S \) such that \( u(w) = \lim_{j \to \infty} \eta_j^w(w) \), such that \( \eta_j^w(z) \leq \eta_j^w(z) \leq \eta_j^\Omega(z) \) for all \( z \in \Omega \), and such that \( \eta_j^w \) is harmonic in \( D(P, r) \).

(Problem 80) Suppose that \( w \in D(P, r) \) and \( \overline{D}(P, r) \subset \Omega \). Let \( \eta_j^w = \lim_{j \to \infty} \eta_j^w \). Prove that \( \eta_j^w \) is harmonic in \( D(P, r) \).

(Problem 80a) Suppose that \( w_1, w_2 \in D(P, r) \) and \( \overline{D}(P, r) \subset \Omega \). Let \( \eta_j^{w_1} = \lim_{j \to \infty} \eta_j^{w_1} \) and let \( \eta_j^{w_2} = \lim_{j \to \infty} \eta_j^{w_2} \). Prove that \( \eta_j^{w_1}(z) = \eta_j^{w_2}(z) \) for all \( z \in D(P, r) \). Hint: Let \( \varphi_j^{w_1,w_2}(z) = \max(\varphi_j^{w_1}(z), \varphi_j^{w_2}(z)) \) and construct \( \eta_j^{w_1,w_2} \) from \( \varphi_j^{w_1,w_2} \) as before. What can you say about \( \eta_j^{w_1,w_2}(z) \) and \( \eta_j^{w_1,w_2}(z) = \lim_{j \to \infty} \eta_j^{w_1,w_2}(z) \) for arbitrary \( z \in D(P, r) \), and for \( z = w_1 \) and \( z = w_2 \) in particular?

(Problem 81) Let \( u \) be as in Problem 72. Prove that \( u \) is harmonic in \( \Omega \).

[Definition: Barriers] Let \( \Omega \subset \subset \mathbb{C} \) be open and let \( P \in \partial \Omega \). We say that \( b : \overline{\Omega} \to \mathbb{R} \) is a barrier for \( \Omega \) at \( P \) if:

(i) \( b \) is continuous on \( \overline{\Omega} \),
(ii) \( b \) is subharmonic in \( \Omega \),
(iii) \( b(P) > b(z) \) for all \( z \in \overline{\Omega} \setminus \{P\} \). (Often we take \( b(P) = 0 \).)

(Problem 84) Let \( \Omega, f \text{ and } S \) be as in Problem 71. Let \( P \in \partial \Omega \).

Suppose that a barrier \( b \) at \( P \) exists.
Let \( \varepsilon > 0 \). Use \( b \) to construct a function \( w_\varepsilon \) such that \( w_\varepsilon(P) = f(P) - \varepsilon \) and such that \( w_\varepsilon \in S \).

(Problem 85) Let \( u \) be as in Problem 72. Use the functions \( w_\varepsilon \) to show that \( u(P) = f(P) \).

(Problem 82) Let \( \Omega, f \text{ and } S \) be as in Problem 71. Let \( P \in \partial \Omega \).

Suppose that a barrier \( b \) at \( P \) exists.
Let \( \varepsilon > 0 \). Use \( b \) to construct a function \( g_\varepsilon \) that is continuous on \( \overline{\Omega} \), superharmonic in \( \Omega \), and satisfies \( g_\varepsilon \geq f \) on \( \partial \Omega \), and such that \( g_\varepsilon(P) = f(P) + \varepsilon \).

(Problem 83) Let \( u \) be as in Problem 72. Use the functions \( g_\varepsilon \) to show that \( u \) is upper semicontinuous at \( P \).

(Problem 86) Let \( \Omega \subset \subset \mathbb{C} \) be open and bounded. Give a condition on \( \Omega \) that ensures that the Dirichlet problem is well-posed in \( \Omega \).

(Problem 86a) Let \( \Omega \subset \subset \mathbb{C} \) be open and bounded. Suppose that the Dirichlet problem is well-posed in \( \Omega \). Show that for any \( P \in \partial \Omega \), there exists a function \( b \) that is a barrier at \( P \).

(Problem 87) Let \( \Omega = D(0, 1) \) and let \( P = e^{i\theta} \in \partial \Omega \). Give an example of a function \( u \) that is a barrier at \( P \).

(Problem 93) Suppose that \( \Omega \subset \subset \mathbb{C} \) and \( \Psi \subset \subset \mathbb{C} \) are two open connected sets. Suppose that \( \varphi : \overline{\Omega} \to \mathbb{W} \) is continuous and that \( \varphi : \Omega \to \mathbb{W} \) is holomorphic. Suppose that there is some \( P \in \partial \Psi \) and some function \( b : \mathbb{W} \to \mathbb{R} \) that is a barrier for \( \Psi \) at \( P \). Show that if there is exactly one \( Q \in \Omega \) with \( \varphi(Q) = P \), then \( \hat{b} = b \circ \varphi \) is a barrier for \( \Omega \) at \( Q \).

(Problem 88) Let \( \Omega = \mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\} \) and let \( P = x \in \partial \Omega \). Give an example of a function \( b \) that is a barrier at \( P \).

(Problem 89) Let \( \Omega = \mathbb{C} \setminus \{x + 0i : 0 \leq x \leq \infty\} \) denote the complex plane with an infinite slit removed. Let \( P = 0 \in \partial \Omega \). Give an example of a function \( b \) that is a barrier at \( P \).

(Problem 90) Let \( \Omega = \mathbb{C} \setminus \{x + 0i : 0 \leq x \leq 1\} \) denote the complex plane with a finite slit removed. Let \( P = 0 \in \partial \Omega \). Give an example of a function \( b \) that is a barrier at \( P \).
(Problem 91) Let Ω ⊆ C and Ψ ⊆ C be two open sets, and suppose Ω ⊂ Ψ. Suppose that P ∈ ∂Ω ∩ ∂Ψ. Show that there is a barrier for Ψ at P. Suppose that there is a barrier for Ψ at P.

(Problem 91a) Let Ω ⊆ C be open. Suppose that for every point P ∈ ∂Ω, there is some point Q_P such that the line segment from P to Q_P is contained in C \ Ω. Show that the Dirichlet problem is well-posed in Ω.

(Problem 92) Show that having a barrier is a local property. That is, let Ω ⊆ C and Ψ ⊆ C be two open sets, and suppose that P ∈ ∂Ω ∪ ∂Ψ and that for some ε > 0, it is the case that Ω ∩ D(P, ε) = Ψ ∩ D(P, ε). Suppose that b is a barrier for Ω at P. Construct a barrier for Ψ at P.

(Problem 94) Give an example of a domain Ω ⊆ C and a point P ∈ ∂Ω such that there is no function b that is a barrier at P.

7.9. Conformal mappings of annuli

(Problem 95) Let 0 < r_1 < R_1 < ∞ and 0 < r_2 < R_2 < ∞. Let P_1 ∈ C and P_2 ∈ C. Show that, if R_1/r_1 = R_2/r_2, then A_1 = {z ∈ C : r_1 < |z - P_1| < R_1} and A_2 = {z ∈ C : r_2 < |z - P_2| < R_2} are conformally equivalent; that is, there is a holomorphic bijection ϕ : A_1 → A_2.

(Problem 96) Let Ω ⊆ C and Ψ ⊆ C be two open sets, and suppose that ϕ : Ω → Ψ is a conformal mapping (holomorphic bijection). Suppose that \{z_n\}_{n=1}^∞ ⊂ Ω and that z_n → z_∞ for some z_∞ ∈ ∂Ω. Suppose that ϕ(z_n) converges to some w_∞ ∈ Ψ. Show that w_∞ ∈ ∂Ψ.

(Problem 97) Let Ω ⊆ C and Ψ ⊆ C be two bounded open sets, and suppose that ϕ : Ω → Ψ. Suppose that \{z_n\}_{n=1}^∞ ⊂ Ω. Show that there is a subsequence \{z_{n_k}\}_{k=1}^∞ such that ϕ(z_{n_k}) converges as k → ∞.

(Problem 98) Suppose that 1 < R_1 < ∞ and that 1 < R_2 < ∞. Let A_1 = \{z ∈ C : 1/R_1 < |z| < R_1\} and A_2 = \{z ∈ C : 1/R_2 < |z| < R_2\}. Let ϕ : A_1 → A_2 be a holomorphic bijection. Let 1/R_2 < ρ < R_2. Let \rho_n = \inf\{|ϕ^{-1}(ρe^{iθ})| : 0 ≤ θ ≤ 2π\} and let s_n = \sup\{|ϕ^{-1}(ρe^{iθ})| : 0 ≤ θ ≤ 2π\}. Show that 1/R_1 < \rho_n < s_n < R_1.

(Problem 99) Let z_∞ ∈ ∂A_1 with |z_∞| = R_1. Suppose there is some \{w_n\}_{n=1}^∞ ⊂ A_1 such that w_n → z_∞ and |ϕ(w_n)| → R_2 as n → ∞.

Suppose further that \{z_n\}_{n=1}^∞ ⊂ A_1, that z_n → z_∞ as n → ∞, and that ϕ(z_n) converges. Show that |ϕ(z_n)| → R_2 as n → ∞. Hint: Use the fact that \{z ∈ A_1 : |ϕ(z)| < 1\} and \{z ∈ A_1 : |ϕ(z)| > 1\} are disjoint open sets and \{z ∈ A_1 : |z| > s_1\} is connected.

(Problem 100) If z_n, w_n ∈ A_1, |z_∞| = R_1, z_n → z_∞, w_n → z_∞, |ϕ(w_n)| → 1/R_2, and ϕ(z_n) converges, what can you say about \lim_{n→∞} |ϕ(z_n)|?

(Problem 100a) If z_n, w_n ∈ A_1, |z_∞| = 1/R_1, z_n → z_∞, w_n → z_∞, |ϕ(w_n)| → R_2, and ϕ(z_n) converges, what can you say about \lim_{n→∞} |ϕ(z_n)|?

(Problem 100b) If z_n, w_n ∈ A_1, |z_∞| = 1/R_1, z_n → z_∞, w_n → z_∞, |ϕ(w_n)| → 1/R_2, and ϕ(z_n) converges, what can you say about \lim_{n→∞} |ϕ(z_n)|?

(Problem 101) Let z_∞ ∈ ∂A_1. Show that there is a sequence of points \{w_n\}_{n=1}^∞ ⊂ A_1 such that w_n → z_∞ and that either |ϕ(w_n)| → R_2 or |ϕ(w_n)| → 1/R_2 as n → ∞.

(Problem 101a) Let z_∞ ∈ ∂A_1. Show that one of the following is true:

- For every sequence \{z_n\}_{n=1}^∞ ⊂ A_1 that satisfies z_n → z_∞, we have that |ϕ(z_n)| → R_2 as n → ∞.
- For every sequence \{z_n\}_{n=1}^∞ ⊂ A_1 that satisfies z_n → z_∞, we have that |ϕ(z_n)| → 1/R_2 as n → ∞.

(Problem 102) Let h(z) = |ϕ(z)|. Show that there is a function ũ that is continuous on \overline{A_1} and satisfies ũ = h on A_1.

(Problem 102a) What values can ũ take on ∂A_1?

(Problem 103) Show that ũ(z) is constant on each of the two boundary components of A_1.
(Problem 104) Show that \( g(z) = \log h(z) = \log|\varphi(z)| \) is harmonic in \( A_1 \).

(Problem 104a) Can \( \hat{h} = |\varphi(z)| \) be equal on the two boundary components of \( A_1 \)?

(Problem 105) For fixed \( R_1 \) and \( R_2 \), there are two possible values of \( g(z) = \log|\varphi(z)| \). Find them.

(Problem 106) Suppose \( f : A_1 \to A_2 \) is holomorphic and \( \log|f(z)| = \beta \log|z| \) for some real number \( \beta \). Find \( f(z) \). Are there any restrictions on \( \beta \)? If we require that \( f \) be one-to-one, are there any additional restrictions on \( \beta \)?

(Problem 107) Let \( 0 < r_1 < R_1 < \infty \) and \( 0 < r_2 < R_2 < \infty \). Let \( P_1 \subset \mathbb{C} \) and \( P_2 \subset \mathbb{C} \). Suppose that \( \{z \in \mathbb{C} : r_1 < |z - P_1| < R_1 \} \) and \( \{z \in \mathbb{C} : r_2 < |z - P_2| < R_2 \} \) are conformally equivalent. Show that \( R_1/r_1 = R_2/r_2 \).

(Problem 108) Let \( A = \{z \in \mathbb{C} : 1/R < |z| < R\} \) be an annulus for some \( R > 1 \). Find all conformal self-maps of \( A \).

8.1. Basic concepts for infinite products

(Problem 109) Let \( \{A_j\}_{j=1}^{\infty} \) be a sequence of complex numbers. Suppose \( A_j \neq 0 \) for all \( j \) and that

\[
\lim_{N \to \infty} \prod_{j=1}^{N} A_j \text{ exists and is nonzero. Show that } \lim_{j \to \infty} A_j = 1.
\]

(Problem 110) Let \( \{A_j\}_{j=1}^{\infty} \) be a sequence of complex numbers. Suppose \( A_j \neq 0 \) for all \( j \) and that

\[
\lim_{N \to \infty} \prod_{j=1}^{N} A_j = 0. \text{ Can we conclude that } \lim_{j \to \infty} A_j = 1? \]

(Problem 111) Let \( \{A_j\}_{j=1}^{\infty} \) be a sequence of complex numbers. Suppose \( A_k = 0 \) for some \( k \). What is

\[
\lim_{N \to \infty} \prod_{j=1}^{N} A_j? \text{ If the sequence of partial products converges, can we conclude that } \lim_{j \to \infty} A_j = 1? \]

(Problem 112) Show that if \( 0 \leq x \leq 1 \), then \( 1 + x \leq e^x \leq 1 + 2x \).

(Problem 113) Show that if \( a_j \in \mathbb{C} \) with \( |a_j| < 1 \), then

\[
\exp \left( \frac{1}{2} \sum_{j=1}^{n} |a_j| \right) \leq \prod_{j=1}^{n} |1 + a_j| \leq \exp \left( \sum_{j=1}^{n} |a_j| \right).
\]

(Problem 114) Show that if \( a_j \in \mathbb{C} \) and \( \sum_{j=1}^{\infty} |a_j| \) converges, then

\[
\lim_{N \to \infty} \prod_{j=1}^{N} 1 + |a_j| \text{ exists. Can the limit be zero?}
\]

(Problem 115) Show that if \( a_j \in \mathbb{C} \) and \( \lim_{N \to \infty} \prod_{j=1}^{N} 1 + |a_j| \) exists, then \( \sum_{j=1}^{\infty} |a_j| \) converges.

(Problem 116) Suppose that \( \lim_{N \to \infty} \prod_{j=1}^{N} 1 + |a_j| \) exists. Show that there is some \( N_0 > 0 \) such that \( 1 + a_j \neq 0 \) for any \( j \geq N_0 \).

(Problem 117) Suppose that \( a_j \in \mathbb{C} \) and \( N \geq M \). Show that

\[
\left| \prod_{j=M}^{N} 1 + a_j \right| \leq \left| \prod_{j=M}^{N} 1 + |a_j| \right| - 1.
\]

Hint: Use induction.

(Problem 118) Suppose that \( a_j \in \mathbb{C} \) and \( a_j \neq -1 \). Suppose that \( \lim_{N \to \infty} \prod_{j=1}^{N} 1 + |a_j| \) exists. Show that

\[
\lim_{N \to \infty} \prod_{j=1}^{N} 1 + a_j \text{ exists and is nonzero.}
\]

(Problem 119) Suppose that \( a_j \in \mathbb{C} \) and \( \sum_{j=1}^{\infty} |a_j| \) converges. Show that \( \lim_{N \to \infty} \prod_{j=1}^{N} 1 + a_j \) exists; if \( a_j \neq -1 \) for all \( j \), the limit is nonzero.
(Problem 120) Let $K \subset \mathbb{C}$ and let $\{f_j\}_{j=1}^\infty$ be a sequence of functions $f_j : K \to \mathbb{C}$. Suppose that $|f_j(z)| < 1$ for all $j \geq 1$ and all $z \in K$ and that $\sum_{j=1}^\infty |f_j(z)|$ converges uniformly for all $z \in K$. Show that $\prod_{j=1}^N 1 + f_j(z)$ converges as $N \to \infty$ to a function $F(z)$, uniformly for all $z \in K$.

(Problem 120a) Let $K \subset \mathbb{C}$ and let $\{f_j\}_{j=1}^\infty$ be a sequence of functions $f_j : K \to \mathbb{C}$. Suppose that the $f_j$s are uniformly bounded (although not necessarily by 1) and that $\sum_{j=1}^\infty |f_j(z)|$ converges uniformly for all $z \in K$. Show that $\prod_{j=1}^N 1 + f_j(z)$ converges as $N \to \infty$ to a function $F(z)$, uniformly for all $z \in K$.

(Problem 121) Let $f_j$, $K$ and $F$ be as in Problem 120a. Let $z_0 \in K$. Show that $1 + f_j(z_0) = 0$ for at most finitely many numbers $j$.

(Problem 122) Let $\Omega \subset \mathbb{C}$ be open and let $f_j : \Omega \to \mathbb{C}$ be holomorphic. Suppose that $\sum_{j=1}^\infty |f_j(z)|$ converges normally (that is, uniformly on compact sets). Show that $\prod_{j=1}^N 1 + f_j(z)$ converges normally as $N \to \infty$ to a holomorphic function $F$.

(Problem 123) Let $\Omega$, $f_j$ and $F$ be as in Problem 122. Let $z_0 \in \Omega$. Show that $F(z_0) = 0$ if and only if $f_j(z_0) = -1$ for some $j \geq 1$.

(Problem 124) Let $\Omega$, $f_j$ and $F$ be as in Problem 122. Suppose that $F$ is not identically equal to zero. Let $z_0 \in \Omega$ and suppose that $F(z_0) = 0$. Show that the multiplicity of the zero of $F$ at $z_0$ is equal to the sum of the multiplicities of the zeros of $1 + f_j$ at $z_0$.

(Problem 124a) Let $\Omega$, $f_j$ and $F$ be as in Problem 122. Show that if $F$ is identically equal to zero then $f_j$ is identically equal to $-1$ for some $j$.

8.2. The Weierstrass factorization theorem

(Problem 125) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that $f$ has finitely many zeros. Show that there is an entire function $g$, an integer $N$, and complex numbers $a_n$ such that

$$f(z) = e^{g(z)} \prod_{n=1}^N (z - a_n).$$

(Problem 126) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function that is not identically zero. Show that $f$ can have at most countably many zeros (counted with multiplicity).

[Definition: Elementary factors] If $p \geq 0$ is an integer, we let $E_p(z) = (1 - z) \exp \left( z + \frac{z^2}{2} + \ldots + \frac{z^p}{p} \right)$.

(Problem 127) Show that $E_p$ is an entire function and that $E_p(z) = 0$ if and only if $z = 1$. What is the multiplicity of the zero of $E_p(z)$ at 1?

(Problem 128) Let $b_n$ be the components of the power series for $E_p$ centered at zero, so $E_p(z) = \sum_{n=0}^\infty b_n z^n$. Find $b_0$.

(Problem 129) Compute $E_p'(z)$ using the definition given above and also using the power series expansion.

(Bonus problem 129a) Write a recurrence relation for the coefficients $b_n$. That is, write the power series for $(1 - z)E_p'(z) - z^p E_p(z)$ in terms of $b_n$, and then use your formula for $(1 - z)E_p'(z) - z^p E_p(z)$ to find a formula for $b_n$ in terms of $b_0, b_1, \ldots, b_{n-1}$.

(Problem 130) What can you say about $b_n$ for $1 \leq n \leq p$?

(Problem 131) Show that $b_n$ is real and that $b_n \leq 0$ for any $n > p$. 

(Problem 132) Compute $\sum_{n=p+1}^{\infty} |b_n|$. Hint: Start by computing $E_p(1)$.

(Problem 133) Show that if $|z| \leq 1$ then $|E_p(z) - 1| \leq |z|^{p+1}$.

(Problem 134) Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence of nonzero complex numbers. Suppose that the $a_n$s have no accumulation point. Show that the zero set of $\prod_{n=1}^{\infty} (z - a_n)$ converges uniformly for all $|z| < 1$.

(Problem 134a) Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence of nonzero complex numbers. Suppose that the $a_n$s have no accumulation point. Fix an $r > 0$. Show that $\sum_{n=1}^{\infty} |1 - E_n(z/a_n)|$ converges uniformly for all $|z| < r$.

(Problem 135) Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence of complex numbers. Suppose that the $a_n$s have no accumulation point. Show that there exists a sequence of points $\{\hat{a}_n\}_{n=1}^{\infty} \subset C \backslash \Omega$ such that $|a_n - \hat{a}_n| \to 0$ as $n \to \infty$.

(Problem 136) The Weierstrass factorization theorem. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function that is not identically equal to zero. Show that there is an entire function $g(z)$, an integer $m \geq 0$, and complex numbers $a_n \in \mathbb{C}$ such that

$$f(z) = e^{g(z)} z^m \prod_{n=1}^{\infty} E_n \left( \frac{z}{a_n} \right)$$

where the infinite product converges uniformly on compact sets.

8.3. The Theorems of Weierstrass and Mittag-Leffler: interpolation problems

(Problem 137) Let $\Omega \subseteq \mathbb{C}$ be an open set, let $R > 0$, and let $\{a_n\}_{n=1}^{\infty} \subset D(0, R) \cap \Omega$ be a sequence with no accumulation points in $\Omega$. Show that $\lim_{n \to \infty} \text{dist}(a_n, \partial \Omega) = 0$.

(Problem 137a) Show that there exists a sequence of points $\{\hat{a}_n\}_{n=1}^{\infty} \subset \mathbb{C} \backslash \Omega$ and such that $|a_n - \hat{a}_n| \to 0$ as $n \to \infty$.

(Problem 138) Let $\Omega$, $\{a_n\}_{n=1}^{\infty}$ and $\{\hat{a}_n\}_{n=1}^{\infty}$ be as in Problem 137a. Let $N \geq 1$ be an integer. Show that $\prod_{n=1}^{N} E_n \left( \frac{a_j - \hat{a}_j}{z - \hat{a}_j} \right)$ is holomorphic in $\Omega$.

(Problem 139) Let $\Omega$, $\{a_n\}_{n=1}^{\infty}$ and $\{\hat{a}_n\}_{n=1}^{\infty}$ be as in Problem 137a. Show that $\prod_{n=1}^{N} E_n \left( \frac{a_j - \hat{a}_j}{z - \hat{a}_j} \right)$ converges as $N \to \infty$ for all $z \in \Omega$, uniformly on compact sets.

(Problem 140) Let $\Omega$, $\{a_n\}_{n=1}^{\infty}$ and $\{\hat{a}_n\}_{n=1}^{\infty}$ be as in Problem 137a. What can you say about $f(z) = \lim_{N \to \infty} \prod_{n=1}^{N} E_n \left( \frac{a_j - \hat{a}_j}{z - \hat{a}_j} \right)$?

(Problem 141) Weierstrass’s theorem. Let $\Omega \subseteq \mathbb{C}$ be open and let $\{a_n\}_{n=1}^{\infty} \subset \Omega$ have no accumulation points in $\Omega$. We do not require that $\{a_n\}$ be bounded. Show that there is a function $f$ that is holomorphic in $\Omega$ and such that the zero set of $f$ (with multiplicity) is precisely equal to $\{a_n\}_{n=1}^{\infty}$.

(Problem 142) Let $\Omega \subseteq \mathbb{C}$ be open. Let $m$ be meromorphic on $\Omega$. Show that there are functions $f$ and $g$ that are holomorphic in $\Omega$ and such that $m(z) = f(z)/g(z)$ for all $z \in \Omega \setminus A$, where $A$ is the set of poles of $m$. 
(Problem 143) Let \( f(e^{i\theta}) = \sum_{n=0}^{\infty} \frac{5^n}{6^n} \cos(7^n \theta) \). (This is a special case of the Weierstrass function.) Show that \( f \) is well-defined (the sum converges) for all \( 0 \leq \theta \leq 2\pi \) and that \( f \) is continuous on \( \partial D(0,1) \).

(Problem 144) Plot the first few partial sums for the Weierstrass function.

(Problem 145) Let \( u \) be the function that is harmonic in \( D(0,1) \), continuous on \( \overline{D}(0,1) \) and with \( u(e^{i\theta}) = f(e^{i\theta}) \) for \( 0 \leq \theta \leq 2\pi \). Let \( H \) be the function that is holomorphic in \( D(0,1) \) with real part \( u \).

Show that \( D(0,1) \) is the domain of existence of \( H \); that is, if \( \tilde{H} = H \) in \( D(0,1) \) and \( \tilde{H} \) is holomorphic on some open set \( \Omega \supseteq D(0,1) \), then \( \Omega = D(0,1) \). Hint: Use the fact (proven by Weierstrass in 1872) that \( f(\theta) \) is nowhere differentiable.

(Problem 146) Let \( T = \{\left[k2^j, (k+1)2^j\right) \times \left[\ell2^j, (\ell+1)2^j\right) : k, \ell \text{ are integers}\} \) be the grid of squares in \( \mathbb{C} \) with side-length \( 2^j \) aligned with the axes. Sketch \( T \).

(Problem 147) Suppose that \( S \subseteq T \). Let \( P(S) \) be the “dyadic parent” of \( S \), so \( S \subseteq P(S) \subseteq T \). Let \( 2S \) be the square concentric to \( S \) of side-length \( 2^{j+1} \).

Sketch \( S, 2S \) and the four possibilities for \( P(S) \).

(Problem 151) If \( S \subseteq T \), let \( \ell(S) = 2^j \) be the side-length of \( S \). Show that if \( S \subseteq T \) and \( z \subseteq S \), then \( D(z, \ell(S)/2) \subseteq 2S \) and \( 2P(S) \subseteq D(z, 3\sqrt{2}\ell(S)) \).

(Problem 148) Let \( \mathcal{Q} = \bigcup_{n=-\infty}^{\infty} Q_n \). Let \( \Omega \subseteq \mathbb{C} \) be open. Let \( \mathcal{G} = \{S \in \mathcal{Q} : 2S \subseteq \Omega, 2P(S) \not\subseteq \Omega\} \). We call \( \mathcal{G} \) a dyadic Whitney decomposition of \( \Omega \). Show that \( \bigcup_{S \in \mathcal{G}} S \subseteq \Omega \).

(Problem 148a) Show that if \( z \in \Omega \), then there is some \( S \in \mathcal{G} \) with \( z \in S \).

(Problem 149) Show that if \( S \in \mathcal{G} \) and \( T \in \mathcal{G} \), then either \( S = T \) or \( S \cap T = \emptyset \).

(Problem 149a) If \( z \in \Omega \), then how many cubes \( S \in \mathcal{G} \) can satisfy \( z \in S \)?

(Problem 150) Show that \( \mathcal{G} \) is a countable set.

(Problem 152) Suppose that \( S, T \in \mathcal{G} \) and that \( \text{dist}(S, T) = 0 \); that is, the closures of \( S \) and \( T \) intersect. Show that \( \ell(S) \leq 8\ell(T) \) and that \( \ell(T) \leq 8\ell(S) \).

(Problem 152a) If \( S \in \mathcal{G} \), let \( z_S \) be the midpoint of \( S \). Let \( A = \{z_S : S \in \mathcal{G}\} \).

Let \( z \in \Omega \). Show that \( z \) is not an accumulation point for \( A \). Hint: if \( z \in T \in \mathcal{G} \), then how many midpoints \( z_S \) can appear in \( D(z, \ell(T)/16) \)?

(Problem 153) Let \( z \in \partial \Omega \). Show that \( z \) is an accumulation point for \( A \).

(Problem 154) Show that there is a function \( f \) that is holomorphic in \( \Omega \) and such that \( f(z) = 0 \) if and only if \( z \in A \).

(Problem 155) Show that the domain of existence of \( f \) is \( \Omega \); that is, if \( \tilde{f} = f \) in \( \Omega \) and \( \tilde{f} \) is holomorphic on some open set \( \Psi \subseteq \Omega \), then \( \Psi = \Omega \).

In Problems 156–159, let \( \{a_n\}_{n=1}^{\infty} \subseteq \mathbb{C} \) be a sequence of distinct points with no accumulation points.

(Problem 156) Let \( \beta \in \mathbb{C} \) and let \( k > 0 \) be an integer. Find an entire function \( f \) such that \( f(a_k) = \beta \) and such that \( f(a_n) = 0 \) for all \( n \neq k \).

(Problem 157) Let \( \beta \in \mathbb{C} \) and let \( k > 0 \) be an integer. Find an entire function \( f \) such that \( f(a_k) = \beta \) and such that \( f(a_n) = 0 \) for all \( n \neq k \), and such that \( |f(z)| < 2^{-k} \) for all \( |z| < \frac{1}{2}|a_k| \).

(Problem 158) Let \( \{\beta_n\}_{n=1}^{\infty} \subseteq \mathbb{C} \) be a sequence of complex numbers. Find an entire function \( f \) such that \( f(a_n) = \beta_n \) for all \( n \geq 1 \).

(Problem 159) Let \( \{\beta_n\}_{n=1}^{\infty} \subseteq \mathbb{C} \) and \( \{\gamma_n\}_{n=1}^{\infty} \subseteq \mathbb{C} \) be two sequences of complex numbers. Find an entire function \( f \) such that \( f(a_n) = \beta_n \) and \( f'(a_n) = \gamma_n \) for all \( n \geq 1 \).
10.1. Definition of an analytic function element

(Problem 160) Let $\Omega \subseteq \mathbb{C}$ be open. Let $\Psi \subseteq \Omega$ be open and nonempty. Suppose that $f$ is holomorphic in $\Psi$. Show that there is at most one function $F$ that is holomorphic in $\Omega$ and such that $F = f$ in $\Psi$.

(Problem 161) Let $\Psi$ be the open sector $\{ re^{i\theta} : r > 0, 0 < \theta < \pi/2 \}$. Let $f(z)$ be the branch of the logarithm given by $f(re^{i\theta}) = \log r + i\theta$ whenever $0 < \theta < \pi/2$.

Find two functions $F$ and $\tilde{F}$ and domains $\Omega \supseteq \Psi$ and $\Omega \supseteq \Psi$ such that $F$ is holomorphic in $\Omega$, $\tilde{F}$ is holomorphic in $\tilde{\Omega}$, $\Omega \cap \tilde{\Omega}$ is nonempty, $F \neq \tilde{F}$ on $\Omega \cap \tilde{\Omega}$, and $F = f = \tilde{F}$ in $\Psi$.

(Problem 162) The gamma function $\Gamma(z)$ is defined by $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, dt$. Find $\Gamma(n)$, where $n$ is a positive integer.

(Problem 163) For what values of $z$ does the integral in the definition of the gamma function converge?

(Problem 164) Integrate by parts to find a new formula for $\Gamma(z)$ that converges for $z$ in a larger set.

[Definition: Function elements] A function element is an ordered pair $(f, D(P, r))$ where $P \in \mathbb{C}$, $r > 0$ and $f$ is a holomorphic function defined on $D(P, r)$.

[Definition: Direct analytic continuation] If $(f, D(P, r))$ and $(g, D(Q, s))$ are function elements, if $D(P, r) \cap D(Q, s) \neq \emptyset$, and if $f = g$ on $D(P, r) \cap D(Q, s)$, we say that $(g, D(Q, s))$ is a direct analytic continuation of $(f, D(P, r))$.

[Definition: Analytic continuation] Suppose that we have a finite sequence of function elements $\{(f_j, D(P_j, r_j))\}_{j=1}^k$ such that $(f_j, D(P_j, r_j))$ is a direct analytic continuation of $(f_{j-1}, D(P_{j-1}, r_{j-1}))$ for all $1 < j \leq k$. Then $(f_k, D(P_k, r_k))$ is an analytic continuation of $(f_1, D(P_1, r_1))$.

(Problem 165) Find a function element $(f, D(P, r))$ and two distinct function elements $(g, D(Q, s))$ and $(\tilde{g}, D(Q, s))$, with the same disc $D(Q, s)$, such that $(g, D(Q, s))$ and $(\tilde{g}, D(Q, s))$ are both analytic continuations of $(f, D(P, r))$.

(Problem 166) Can you do this for a direct analytic continuation?

10.2. Analytic continuation along a curve

[Definition: Analytic continuation along a curve] Let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be a continuous function (we will call $\gamma$ a curve). Let $(f, D(\gamma(0), r))$ be a function element. An analytic continuation of $(f, D(\gamma(0), r))$ along $\gamma$ is a collection of function elements $\{(f_t, D(\gamma(t), r_t))\}_{0 \leq t \leq 1}$ such that $(f_0, D(\gamma(0), r_0)) = (f, D(\gamma(0), r))$ and such that if $0 \leq t \leq 1$, then there is an $\varepsilon > 0$ such that, if $0 \leq s \leq 1$ and $|t - s| < \varepsilon$, then $(f_s, D(\gamma(s), r_s))$ is a direct analytic continuation of $(f_t, D(\gamma(t), r_t))$.

(Problem 167) Let $\gamma : [0, 1] \rightarrow \mathbb{C}$ be a curve and let $(f, D(\gamma(0), r))$ be a function element. Suppose that $\{(f_t, D(\gamma(t), r_t))\}_{0 \leq t \leq 1}$ and $\{(f_s, D(\gamma(t), r_s))\}_{0 \leq s \leq 1}$ are two analytic continuations of $(f, D(\gamma(0), r))$ along $\gamma$.

Let $S = \{ s : 0 \leq s \leq 1, f_s = f_s \text{ on } D(\gamma(s), \min(r_s, r_t)) \}$. Let $T = \{ t : 0 \leq t \leq 1, s \in S \text{ for all } 0 \leq s \leq t \}$.

Show that $T$ is not empty.

(Problem 168) Show that $T$ is closed.

(Problem 169) Show that $T$ is open in $[0, 1]$.

(Problem 170) Is there a sense in which an analytic continuation along a curve is unique?

(Problem 171) Suppose that $\gamma : [0, 1] \rightarrow \mathbb{C}$ is a closed curve (so $\gamma(1) = \gamma(0)$). Let $\{(f_t, D(\gamma(t), r_t))\}_{0 \leq t \leq 1}$ be an analytic continuation of $(f, D(\gamma(0), r))$ along $\gamma$. Is it necessarily true that $f_1 = f_0$ on $D(\gamma(0), \min(r_0, r_1))$?