### 7.1. Basic properties of harmonic functions

[**Definition: Harmonic function**] We say that $u$ is harmonic in a domain $\Omega \subseteq \mathbb{C}$ if $u$ is $C^2$ in $\Omega$ and if $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ in $\Omega$.

**(Problem 1)** Write the definition of harmonic function using the operators $\dfrac{\partial}{\partial z}$ and $\dfrac{\partial}{\partial \bar{z}}$.

**(Problem 2)** Prove that if $F$ is holomorphic in an open set $\Omega$ and $u = \operatorname{Re} F$ then $u$ is harmonic.

**(Problem 3)** Prove that if $u$ is harmonic in a disc $\mathcal{D}$, then there is a holomorphic function $F$ such that $\operatorname{Re} F = u$. Do this by showing that there exists a function $v$ that satisfies the Cauchy-Riemann equations.

**(Problem 4)** Prove that if $u$ is harmonic in a disc $\mathcal{D}$, then there is a holomorphic function $F$ such that $\operatorname{Re} F = u$. Do this by considering the function $\dfrac{\partial u}{\partial z}$.

[**Definition: Harmonic conjugate**] Let $u$ and $v$ be two real-valued functions. If $F = u+iv$ is holomorphic, then we say that $v$ is a harmonic conjugate of $u$.

**(Problem 5)** Suppose that $v$ is a harmonic conjugate of $u$. Is $u$ also a harmonic conjugate of $v$?

**(Problem 6)** Show that harmonic functions are smooth.

**(Problem 7)** Provide an example of a domain $\Omega$ and a function $u$ that is harmonic on $\Omega$ but is not the real part of a holomorphic function on $\Omega$.

**(Problem 8)** Give a general class of domains $\Omega$ such that every function $u$ that is harmonic on $\Omega$ is the real part of a holomorphic function. Prove your assertion.

**(Problem 8a)** Prove that if every function $u$ that is harmonic on $\Omega$ is the real part of a holomorphic function, then $\operatorname{Ind}_\gamma(w) = 0$ for every closed piecewise-$C^1$ path $\gamma \subset \Omega$ and every point $w \notin \Omega$.

**(Problem 8b)** Prove that if $\Omega$ is holomorphically simply connected, then $\operatorname{Ind}_\gamma(w) = 0$ for every closed piecewise-$C^1$ path $\gamma \subset \Omega$ and every point $w \notin \Omega$.

**(Problem 8c)** Prove that if $\operatorname{Ind}_\gamma(w) = 0$ for every closed piecewise-$C^1$ path $\gamma \subset \Omega$ and every point $w \notin \Omega$, then $\Omega$ is simply connected.

**(Problem 9)** Suppose that $u$, $v$, and $w$ are real $C^2$ functions on a connected domain $\Omega$ and that $u+iv$ and $u+iw$ are both holomorphic. What can you say about $v$ and $w$?

**(Problem 10)** Suppose that $\varphi : \Omega \mapsto V$ is holomorphic and that $u$ is harmonic on $V$. Prove that $\tilde{u} = u \circ \varphi$ is harmonic on $\Omega$ by using the chain rule for complex differentiation.

**(Problem 10a)** Suppose that $\varphi : \Omega \mapsto V$ is holomorphic and that $u$ is harmonic on $V$. Prove that $\tilde{u} = u \circ \varphi$ is harmonic on $\Omega$ by using the multivariable chain rule for real-valued functions.

**(Problem 11)** Suppose that $\varphi : \Omega \mapsto V$ is holomorphic and that $u$ is harmonic on $V$. Prove that $\tilde{u} = u \circ \varphi$ is harmonic on $\Omega$ by using the fact that $u = \operatorname{Re} F$ (locally) for a holomorphic function $F$.

### 7.2. The maximum principle and the mean value property

**(Problem 12)** Prove that if $u$ is harmonic in a neighborhood of $\overline{D}(P,r)$, then $u(P) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} u(P+re^{i\theta})\,d\theta$.

[**Definition: Mean value property**] The formula given in Problem 12.

**(Problem 18)** Suppose that $u$ is harmonic on a neighborhood of $\overline{D}(0,1)$. If $z \in D(0,1)$, find a formula for $u(z)$ in terms of the values of $u$ on $\partial D(0,1)$. *Hint*: Start by recalling the set of holomorphic self-maps of $D(0,1)$.

**(Problem 13)** Prove the maximum principle for harmonic functions by using the fact that harmonic functions are real parts of holomorphic functions. That is, prove that if $\Omega \subseteq \mathbb{C}$ is open and connected and if $u : \Omega \mapsto \mathbb{R}$ is harmonic, and if there is some $P \in \Omega$ such that $u(P) \geq u(z)$ for all $z \in \Omega$, then $u$ is constant in $\Omega$.

**(Problem 14)** Prove the maximum principle for harmonic functions by using the mean value property. *Hint*: Show that $\{z \in \Omega : u(z) = u(P)\}$ and $\{z \in \Omega : u(z) < u(P)\}$ are both open and use the definition of connectedness in terms of open sets.

**(Problem 15)** Prove the minimum principle for harmonic functions.

**(Problem 16)** Suppose that $u$ is harmonic in $\Omega$ and continuous on $\overline{\Omega}$ for some bounded open set $\Omega$. What can you say about $\max_{\overline{\Omega}} u$ and $\max_{\partial\Omega} u$?

**(Problem 17)** Can you make the same statement if $\Omega$ is not bounded?

**(Problem 17a)** Prove that if $u$ and $v$ are both harmonic in $D(0,1)$, continuous on $\overline{D}(0,1)$, and $u(\zeta) = v(\zeta)$ for all $\zeta \in \partial D(0,1)$, then $u(z) = v(z)$ for all $z \in D(0,1)$.

### 7.3. The Poisson integral formula

**[Definition: Poisson integral formula]** We have that if $u$ is harmonic in a neighborhood of $\overline{D}(0,1)$, then for all $|z| = r < 1$,

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\psi})\,\frac{1-|z|^2}{|e^{i\psi}-z|^2}\,d\psi, \qquad u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\psi})\,\frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2}\,d\psi.$$

Let $P_r(\theta-\psi) = \dfrac{1}{2\pi}\dfrac{1-r^2}{1-2r\cos(\theta-\psi)+r^2}$, $P(z,\zeta) = \dfrac{|\zeta|^2-|z|^2}{2\pi|\zeta-z|^2}$

**(Problem 19)** Verify that the two formulas above are equivalent.

**(Problem 20)** Prove that if $\theta$ is real and $0 \leq r < 1$ then $0 < P_r(\theta) < \infty$ (in particular, the denominator is never zero).

**(Bonus problem 20a)** Show that $p(z) = P(z,\zeta)$ is harmonic on $\mathbb{C} \setminus \{\zeta\}$; in particular, if $\zeta = e^{i\theta}$ then $p(z)$ is harmonic in $D(0,1)$.

**(Bonus problem 20b)** Find a holomorphic function $f(z) = F(z,\zeta)$ such that $p(z) = P(z,\zeta)$ is the real part of $f$.

**(Problem 21)** Suppose that $u$ is harmonic in a neighborhood of $\overline{D}(P,r)$. If $z \in D(P,r)$, find a formula for $u(z)$ in terms of the values of $u$ on $\partial D(P,r)$.

**(Problem 22)** Suppose that $u$ is continuous on $\overline{D}(0,1)$ and harmonic in $D(0,1)$. Show that the Poisson integral formula is still valid.

**(Problem 23)** Let $f$ be real-valued and continuous on $\partial D(0,1)$. Let $u(z) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(e^{i\psi})\,\frac{1-|z|^2}{|e^{i\psi}-z|^2}\,d\psi$. Show that $u \in C^2(D(0,1))$.

**(Problem 24)** Let $u$, $f$ be as in Problem 23. Show that $u$ is harmonic in $D(0,1)$.

**(Problem 25)** Prove that if $0 \leq r < 1$ then $\displaystyle\int_0^{2\pi} P_r(\theta)\,d\theta = 1$.

**(Problem 26)** Prove that $\lim_{r\to 1^-} P_r(\theta) = 0$ for all $\theta \neq 2n\pi$.

**(Problem 27)** Let $0 < \delta < \pi$ be a small positive number. Prove that $\lim_{r\to 1^-} P_r(\theta) = 0$ uniformly for all $\delta < \theta < 2\pi - \delta$.

**(Problem 28)** Let $0 < \delta < \pi$. Prove that $\displaystyle\lim_{r\to 1^-}\int_{-\delta}^{\delta} P_r(\theta)\,d\theta = 1$.

**(Problem 29)** Let $u$, $f$ be as in Problem 23. Show that $\lim_{r\to 1^-} u(re^{i\theta}) = f(e^{i\theta})$ for all $0 \leq \theta \leq 2\pi$.

**(Problem 29a)** Let $u$, $f$ be as in Problem 23. Show that $u(re^{i\theta})$ converges to $f(e^{i\theta})$ as $r \to 1^-$ uniformly in $\theta$.

**(Problem 30)** Let $u$, $f$ be as in Problem 23. Show that $u$ is continuous on $\overline{D}(0,1)$.

**(Problem 31)** We can find a harmonic function in $D(0,1)$ with arbitrary boundary data using the Poisson integral formula. Why can't we find a holomorphic function in $D(0,1)$ with arbitrary boundary data by using the Cauchy integral formula?

### 7.4. Regularity of harmonic functions

[**Definition: The "small circle" mean value property**] Let $\Omega \subset \mathbb{C}$ be open and let $h : \Omega \mapsto \mathbb{R}$ be continuous. We say that $h$ has the SCMV property if, for every $P \in \Omega$, there is some number $\varepsilon_P > 0$ such that $\overline{D}(P, \varepsilon_P) \subset \Omega$ and such that $h(P) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} h(P + \varepsilon e^{i\theta})\, d\theta$ for all $0 < \varepsilon < \varepsilon_P$.

**(Problem 32)** Let $\Omega \subset \mathbb{C}$ be open and connected. Let $g$ be continuous on $\Omega$ and satisfy the "small circle" mean value property. Show that $g$ satisfies the maximum principle, that is, if there is some $P \in \Omega$ such that $g(P) \geq g(z)$ for all $z \in \Omega$ then $g$ is constant.

**(Problem 33)** Suppose that $g$ is continuous on $\overline{D}(P,r)$ and has the "small circle" mean value property in $D(P,r)$. Suppose further that $g = 0$ on $\partial D(P,r)$. Show that $g = 0$ in $D(P,r)$.

**(Problem 34)** Suppose that $g$ and $h$ are continuous on $\overline{D}(P,r)$ and that $u = h$ on $\partial D(P,r)$. Suppose that $h$ is harmonic in $D(P,r)$ and that $g$ has the "small circle" mean value property in $D(P,r)$. Show that $g = h$ in $D(P,r)$ as well.

**(Problem 35)** Let $\Omega \subset \mathbb{C}$ be open. Suppose that $g$ is continuous and has the "small circle" mean value property in $\Omega$. Show that $g$ is harmonic in $\Omega$.

**(Problem 36)** Let $\Omega \subset \mathbb{C}$ be open. Suppose that $\{h_j\}_{j=1}^{\infty}$ is a sequence of functions, each harmonic on $\Omega$, and that $h_j \to h$ uniformly on $\Omega$. Show that $h$ is also harmonic by showing that it has the "small circle" mean value property.

### 7.5. The Schwarz reflection principle

**(Problem 37)** Suppose $\Omega \subset \mathbb{C}$ is open and that $u$ is harmonic on $\Omega$. Let $v(z) = u(\overline{z})$. Show that $v$ is harmonic on $\widehat{\Omega} = \{z \in \mathbb{C} : \overline{z} \in \Omega\}$.

**(Problem 38)** Suppose $\Omega \subset \mathbb{C}$ is open and that $f$ is holomorphic on $\Omega$. Let $g(z) = \overline{f(\overline{z})}$. Show that $g$ is holomorphic on $\widehat{\Omega} = \{z \in \mathbb{C} : \overline{z} \in \Omega\}$.

**(Problem 39)** Let $\Psi \subset \mathbb{C}$ be open and connected. Suppose that $\Psi$ is symmetric about the real axis; that is, $z \in \Psi$ if and only if $\overline{z} \in \Psi$. Let $\Omega = \{z \in \Psi : \operatorname{Im} z > 0\}$.

Suppose that $v$ is harmonic in $\Omega$, continuous on $\overline{\Omega} \cap \Psi$, and that $v(x) = 0$ for any $x \in \Psi \cap \mathbb{R}$.

Sketch $\Psi$. Label $\Omega$, $\widehat{\Omega}$, the set where $v$ is harmonic, and the set where $v$ is equal to zero.

**(Problem 39a)** Let $\Psi$, $\Omega$, $\widehat{\Omega}$, and $v$ be as in Problem 39. Show that $\widehat{v}$ is continuous in $\Psi$, where

$$\widehat{v}(z) = \begin{cases} v(z), & z \in \Omega \\ 0, & z \in \overline{\Omega} \cap \Psi, \\ -v(\overline{z}), & z \in \widehat{\Omega} = \{w \in \mathbb{C} : \overline{w} \in \Omega\}. \end{cases}$$

**(Problem 40)** Suppose that $v$, $\widehat{v}$, and $\Psi$ are as in Problem 39. Show that $\widehat{v}$ is harmonic in $\Psi$. *Hint:* Use the small circle mean value property.

**(Problem 41)** Suppose that $f$ is holomorphic in $D(x_0, r)$ for some $x_0 \in \mathbb{R}$ and some $r > 0$. Suppose further that $f(x)$ is real for all $x \in (x_0 - r, x_0 + r) = D(x_0, r) \cap \mathbb{R}$. Show that $f(z) = \overline{f(\overline{z})}$ for all $z \in D(x_0, r)$.

**(Problem 42)** Let $f$ be holomorphic on the half-circle $\Omega = \{z \in D(x_0, r) : \operatorname{Im} z > 0\}$ for some $x_0 \in \mathbb{R}$ and some $r > 0$. Suppose that $f$ is continuous on $\{z \in D(x_0, r) : \operatorname{Im} z \geq 0\}$. Further suppose that $f(x)$ is real for all $x \in (x_0 - r, x_0 + r)$. Show that there is some function $\widehat{f}$ that is holomorphic in $D(x_0, r)$ and equals $f$ on $\Omega = D(x_0, r)$. Find a formula for $\widehat{f}(z)$.

**(Problem 43)** Let $\Psi$ and $\Omega$ be as in Problem 39. Let $f$ be holomorphic in $\Omega$ and continuous on $\overline{\Omega} \cap \Psi = \Omega \cup (\Psi \cap \mathbb{R})$. Suppose that $\operatorname{Im} f = 0$ on $\Psi \cap \mathbb{R}$.

Sketch $\Psi$. Label $\Omega$, $\widehat{\Omega}$, the set where $f$ is holomorphic, and the set where $f$ is real-valued.

**(Problem 43a) The Schwarz reflection principle.** Let $\Psi$ and $\Omega$ be as in Problem 39. Let $f$ be holomorphic in $\Omega$ and continuous on $\overline{\Omega} \cap \Psi = \Omega \cup (\Psi \cap \mathbb{R})$. Suppose that $\operatorname{Im} f = 0$ on $\Psi \cap \mathbb{R}$. Show that $\widehat{f}$ is holomorphic on $\Psi$, where

$$\widehat{f}(z) = \begin{cases} f(z), & z \in \overline{\Omega} \cap \Psi \\ \overline{f(\overline{z})}, & z \in \widehat{\Omega} = \{w \in \mathbb{C} : \overline{w} \in \Omega\}. \end{cases}$$

**(Problem 44)** Suppose that $f$ is holomorphic on $D(0,1)$ and continuous on $\overline{D}(0,1) \setminus \{-1\}$, and that $f$ is real on $\partial D(0,1) \setminus \{-1\}$. Show that that $f$ may be extended to a holomorphic function $\widehat{f}$ on $\mathbb{C} \setminus \{-1\}$. Give a formula for $\widehat{f}(z)$ whenever $z \in \mathbb{C} \setminus \overline{D}(0,1)$.

**(Problem 44a)** Let $z_0 \in \partial D(0,1)$. Suppose that for some $\varepsilon > 0$, we have that $f$ is holomorphic in $D(z_0, \varepsilon) \cap D(0,1)$ and that $f$ is continuous on $D(z_0, \varepsilon) \cap \overline{D}(0,1)$. Suppose further that $f$ is real-valued on $D(z_0, \varepsilon) \cap \partial D(0,1)$. Show that there is some $\delta$ with $0 < \delta < \varepsilon$ such that $f$ may be extended to a holomorphic function $\widehat{f}$ in $D(z_0, \delta)$; that is, show that there exists a function $\widehat{f}$ that his holomorphic in $D(z_0, \delta)$ and such that $f = \widehat{f}$ in $D(z_0, \delta) \cap \overline{D}(0,1)$.

**(Problem 45)** Let $X$ be an open set and let $\Omega = X \cap D(0,1)$. Suppose that $f$ is holomorphic on $\Omega$ and continuous on $\overline{\Omega} \cap X$, and that $f$ is real on $\partial D(0,1) \cap X$. Find an open set $\Psi \supset \Omega$ such that $f$ may be extended to a holomorphic function $\widehat{f}$ on $\Psi$. Give a formula for $\widehat{f}(z)$ whenever $z \in \Psi \setminus \overline{\Omega}$.

**(Problem 46)** Suppose that $f$ is holomorphic on $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and continuous on $\overline{\mathbb{H}}$ and that $f(x) = 0$ for all $0 < x < 1$. Show that $f(z) = 0$ for all $z \in \mathbb{H}$.

### 7.6. Harnack's principle

**(Problem 47)** Recall that if $u$ is harmonic in $D(P,R)$ and continuous on $\overline{D}(P,R)$, then for any $0 \leq r < R$ and any $0 \leq \theta \leq 2\pi$,

$$u(P + re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(P + Re^{i\psi}) \frac{R^2 - r^2}{|Re^{i\psi} - re^{i\theta}|^2} \, d\psi.$$

Find $\displaystyle\min_{0 \leq \theta \leq 2\pi, 0 \leq \psi \leq 2\pi} \frac{R^2 - r^2}{|Re^{i\psi} - re^{i\theta}|^2}$ and $\displaystyle\max_{0 \leq \theta \leq 2\pi, 0 \leq \psi \leq 2\pi} \frac{R^2 - r^2}{|Re^{i\psi} - re^{i\theta}|^2}$.

**(Problem 48) The Harnack inequality.** Suppose that $u$ is nonnegative and harmonic in $D(P,R)$ and continuous on $\overline{D}(P,R)$. Let $z = P + re^{i\theta} \in D(P,R)$. Show that

$$\frac{R-r}{R+r} u(P) \leq u(z) \leq \frac{R+r}{R-r} u(P).$$

**(Problem 48a)** Did we need the assumption that $u$ was continuous on $\overline{D}(P,R)$?

**(Bonus problem 48b)** Suppose that $u$ is harmonic in $D(P,R)$ and continuous on $\overline{D}(P,R)$. Find formulas for $\partial_x u$ and $\partial_y u$ in terms of $u(P + Re^{i\theta})$, $0 \leq \theta \leq 2\pi$.

**(Bonus problem 48c)** Suppose that $u$ is harmonic $D(P,R)$ and that $|u| \leq M$ in $D(P,R)$. Find an upper bound on $|\nabla u(z)|$ for any $z \in D(P,R)$ in terms of $M$, $R$ and $|z - P|$.

In Problems 49–50, let $\{u_j\}_{j=1}^{\infty}$ be a sequence of real-valued functions harmonic in $D(P,R)$ such that $u_1(z) \leq u_2(z) \leq u_3(z) \leq \cdots$ for each $z \in D(P,R)$.

**(Problem 49)** Suppose that $\lim_{j \to \infty} u_j(P) = \infty$. Show that $u_j \to \infty$ uniformly on $D(P,r)$ for any $0 < r < R$.

**(Problem 50)** Suppose that $\lim_{j \to \infty} u_j(P) < \infty$. Show that $u_j$ converges to some (finite) harmonic function, uniformly on $D(P,r)$ for any $0 < r < R$.

In Problems 51–52, let $\Omega \subseteq \mathbb{C}$ be a connected open set and let $\{u_j\}_{j=1}^{\infty}$ be a sequence of real-valued functions harmonic in $\Omega$ such that $u_1(z) \leq u_2(z) \leq u_3(z) \leq \cdots$ for each $z \in \Omega$.

**(Problem 51)** Show that either $\lim_{j\to\infty} u_j(z) = \infty$ for all $z \in \Omega$ or $\lim_{j\to\infty} u_j(z) < \infty$ for all $z \in \Omega$. *Hint*: Show that $\{z : \lim_{j\to\infty} u_j(z) = \infty\}$ and $\{z : \lim_{j\to\infty} u_j(z) < \infty\}$ are both open.

**(Problem 52) Harnack's principle.** Show that either $\lim_{j\to\infty} u_j(z) = \infty$ for all $z \in \Omega$, uniformly on compact sets, or that there is some function $u_0$ harmonic in $\Omega$ such that $u_j \to u_0$ uniformly on compact sets.

### 7.7. Subharmonic functions

[**Definition: Subharmonic functions**] Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \mapsto \mathbb{R}$ be continuous. Suppose that for every $\overline{D}(P,r) \subset \Omega$, we have that

$$f(P) \leq \frac{1}{2\pi}\int_0^{2\pi} f(P + re^{i\theta})\,d\theta.$$

Then we say that $f$ is subharmonic in $\Omega$.

[**Definition: Superharmonic functions**] Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \mapsto \mathbb{R}$ be continuous. Suppose that for every $\overline{D}(P,r) \subset \Omega$, we have that

$$f(P) \geq \frac{1}{2\pi}\int_0^{2\pi} f(P + re^{i\theta})\,d\theta.$$

Then we say that $f$ is superharmonic in $\Omega$.

**(Problem 53)** Show that $f$ is subharmonic if and only if $-f$ is superharmonic.

**(Problem 54)** Suppose that $f$ is subharmonic in an open set $\Omega$ and that $\alpha \geq 0$ is a nonnegative real number. Show that $\alpha f$ is subharmonic in $\Omega$. Did we need the assumption $\alpha \geq 0$?

**(Problem 55)** Suppose that $f$ and $g$ are both subharmonic in an open set $\Omega$. Show that $f+g$ is subharmonic in $\Omega$. Is $f - g$ subharmonic in $\Omega$?

**(Problem 56)** Suppose that $f$ is subharmonic and $g$ is superharmonic in an open set $\Omega \subseteq \mathbb{C}$. Show that $f - g$ is subharmonic in $\Omega$.

**(Problem 57)** Suppose that $f$ is a continuous, real-valued function in an open set $\Omega \subseteq \mathbb{C}$. Show that $f$ is harmonic if and only if $f$ is both subharmonic and superharmonic.

**(Problem 58)** Suppose that $u$ and $v$ are both subharmonic in an open set $\Omega$. Let $f(z) = \max(u(z), v(z))$. Show that $f$ is subharmonic in $\Omega$. (In particular, if $u$ and $v$ are real and harmonic then $f$ is subharmonic.)

**(Problem 58a)** Let $\Omega \subset \mathbb{C}$ be open and let $f : \Omega \mapsto \mathbb{C}$ be holomorphic. Show that $u(z) = |f(z)|$ is subharmonic in $\Omega$.

**(Bonus problem 58b)** Let $\Omega \subset \mathbb{C}$ be open and let $u : \Omega \mapsto \mathbb{C}$ be subharmonic. Let $\varphi : \mathbb{R} \mapsto \mathbb{R}$ be nondecreasing and convex, so that if $0 < t < 1$ and $a$, $b \in \mathbb{R}$ then $\varphi(ta + (1-t)b) \leq t\varphi(a) + (1-t)\varphi(b)$. Show that $v(z) = \varphi(uf(z))$ is subharmonic in $\Omega$.

**(Problem 59)** Give eight examples of functions that are subharmonic in a domain $\Omega$ but are not harmonic in that domain.

**(Problem 60)** Prove the maximum principle for subharmonic functions.

**(Problem 61)** Is there a minimum principle for subharmonic functions?

**(Problem 69)** Let $\Omega \subseteq \mathbb{C}$ be open and connected. Suppose that $f : \Omega \mapsto \mathbb{R}$ is continuous and satisfies the small circle sub-mean-value property: for every $P \in \Omega$, there is some $\varepsilon_P > 0$ such that $D(P, \varepsilon_P) \subset \Omega$ and such that

$$f(P) \leq \frac{1}{2\pi}\int_0^{2\pi} f(P + \varepsilon e^{i\theta})\,d\theta \quad \text{for all } 0 < \varepsilon < \varepsilon_P.$$

Show that $f$ satisfies the maximum principle in $\Omega$.

**(Problem 62)** Suppose that $f$ is continuous on $\overline{D}(P,r)$ and subharmonic in $D(P,r)$. Suppose that $h$ is continuous on $\overline{D}(P,r)$ and harmonic in $D(P,r)$. Suppose that $f \leq h$ on $\partial D(P,r)$. Show that $f \leq h$ in $D(P,r)$.

**(Problem 63)** Suppose that $\Omega \subseteq \mathbb{C}$ is open and that $f : \Omega \mapsto \mathbb{R}$ is continuous. Suppose further that whenever $\overline{D}(P,r) \subset \Omega$, $h$ is harmonic in $D(P,r)$ and continuous on $\overline{D}(P,r)$, and $h \geq f$ on $\partial D(P,r)$, we have that $h \geq f$ in $D(P,r)$. Prove that $f$ is subharmonic.

**(Problem 70)** Let $\Omega \subseteq \mathbb{C}$ be open. Suppose that $f : \Omega \mapsto \mathbb{R}$ is continuous and satisfies the small circle sub-mean-value property in $\Omega$ (as in Problem 69). Use Problem 63 to show that $f$ is subharmonic in $\Omega$.

### 7.7. The Dirichlet problem

**[Definition: The Dirichlet problem]** Let $\Omega \subsetneq \mathbb{C}$ be a bounded open connected set. We say that the Dirichlet problem is well posed on $\Omega$ if, for every function $f$ defined and continuous on $\partial\Omega$, there is exactly one function $u$ that is harmonic in $\Omega$, continuous on $\overline{\Omega}$, and such that $u = f$ on $\partial\Omega$.

**(Problem 64)** Give an example of an unbounded domain $\Omega$ and two functions $u$ and $v$ that are harmonic in $\Omega$, continuous on $\overline{\Omega}$ and equal zero on $\partial\Omega$.

**(Problem 65)** Prove that we have uniqueness for the Dirichlet problem in any bounded domain; that is, show that if $\Omega \subsetneq \mathbb{C}$ is bounded, if $u$ and $v$ are both harmonic in $\Omega$ and continuous on $\overline{\Omega}$, and if $u = v$ on $\partial\Omega$, then $u = v$ in $\Omega$. Clearly explain how you used the fact that $\Omega$ is bounded.

**(Problem 66)** Let $0 < r < 1$. Find a function $u$ that is harmonic in the annulus $\Omega = D(0,1) \setminus D(0,r)$, continuous on $\overline{\Omega}$ and such that $u(e^{i\theta}) = 0$, $u(re^{i\theta}) = 1$ for any $0 \leq \theta \leq 2\pi$.

**(Problem 67)** Let $\Omega = D(0,1) \setminus \{0\}$. Let $u$ be harmonic in $\Omega$ and continuous on $\overline{\Omega}$. Suppose that $u(e^{i\theta})$ is constant for $0 \leq \theta \leq 2\pi$. Show that $u$ is radial; that is, for any fixed $r$ with $0 < r < 1$, $u(re^{i\theta}) = u(re^{i\psi})$ for any $0 \leq \theta \leq 2\pi$, $0 \leq \psi \leq 2\pi$.

**(Problem 68)** Let $\Omega = D(0,1) \setminus \{0\}$. Suppose that $u$ is harmonic in $\Omega$, continuous on $\overline{\Omega}$, and that $u = 0$ on $\partial D(0,1)$. Prove that $u(0) = 0$. Is the Dirichlet problem well posed in $\Omega$?

### 7.8. The Perròn method and the solution to the Dirichlet problem

Our goal is to use subharmonic functions to construct solutions to the Dirichlet problem.

**(Problem 71)** Let $\Omega \subset \mathbb{C}$ be a bounded open set. Let $f : \partial\Omega \mapsto \mathbb{R}$ be continuous. Let

$$S = \{\psi : \psi \text{ is subharmonic in } \Omega, \text{ continuous on } \overline{\Omega} \text{ and } \psi(w) \leq f(w) \text{ for all } w \in \partial\Omega.\}$$

Show that $S$ is nonempty.

**(Problem 72)** Let $S$ be as in Problem 71. For each $z \in \Omega$, let $u(z) = \sup\{\psi(z) : \psi \in S\}$. Show that $u$ is finite for all $z \in \Omega$ and, in fact, is bounded above and below.

**(Problem 73)** Suppose that $\Omega = D(0,1) \setminus \{0\}$ and that $f(e^{i\theta}) = 1$, $f(0) = 0$. Let $u$ be as in Problem 72. Show that $u(z) = 1$ for all $z \in \Omega$.

**(Problem 74)** Let $\Omega \subsetneq \mathbb{C}$ be open. Let $u$ be as in Problem 72. Show that $u$ is lower semicontinuous on $\overline{\Omega}$; that is, for each $P \in \overline{\Omega}$ and each $\varepsilon > 0$, show that there is some $\delta > 0$ such that if $z \in D(P,\delta) \cap \overline{\Omega}$, then $u(z) > u(P) - \varepsilon$.

**(Problem 75)** Can we show that $u$ is continuous on $\overline{\Omega}$?

**(Problem 76)** Let $\Omega \subsetneq \mathbb{C}$ be open and let $f$ be subharmonic in $\Omega$. Suppose that $\overline{D}(P,r) \subset \Omega$. Let $h$ be harmonic in $D(P,r)$ with $h = f$ on $\partial D(P,r)$; we may construct $h$ using the Poisson integral. Let

$$\psi(z) = \begin{cases} h(z), & z \in D(P,r) \\ f(z), & z \in \Omega \setminus D(P,r). \end{cases}$$

Show that $\psi$ is subharmonic in $\Omega$.

**(Problem 77)** Let $u$, $\Omega$ and $S$ be as in Problems 71–72. Let $w \in \Omega$. Show that there is a sequence of functions $\{\psi_j^w\}_{j=1}^\infty \subset S$ such that $u(w) = \lim_{j\to\infty} \psi_j^w(w)$.

**(Problem 78)** Show that there is a sequence of functions $\{\varphi_j^w\}_{j=1}^\infty \subset S$ such that $u(w) = \lim_{j\to\infty} \varphi_j^w(w)$ and such that $\varphi_1^w(z) \leq \varphi_2^w(z) \leq \varphi_3^w(z) \leq \cdots$ for all $z \in \Omega$.

**(Problem 79)** Let $w \in D(P,r)$ for some $\overline{D}(P,r) \subset \Omega$. Show that there is a sequence of functions $\{\eta_j^w\}_{j=1}^\infty \subset S$ such that $u(w) = \lim_{j\to\infty} \eta_j^w(w)$, such that $\eta_1^w(z) \leq \eta_2^w(z) \leq \eta_3^w(z) \leq \cdots$ for all $z \in \Omega$, and such that $\eta_j^w$ is harmonic in $D(P,r)$.

**(Problem 80)** Suppose that $w \in D(P,r)$ and $\overline{D}(P,r) \subset \Omega$. Let $\eta^w = \lim_{j\to\infty} \eta_j^w$. Prove that $\eta^w$ is harmonic in $D(P,r)$.

**(Problem 80a)** Suppose that $w_1$, $w_2 \in D(P,r)$ and $\overline{D}(P,r) \subset \Omega$. Let $\eta^{w_1} = \lim_{j\to\infty} \eta_j^{w_1}$ and let $\eta^{w_2} = \lim_{j\to\infty} \eta_j^{w_2}$. Prove that $\eta^{w_1}(z) = \eta^{w_2}(z)$ for all $z \in D(P,r)$. *Hint*: Let $\varphi_j^{w_1,w_2}(z) = \max(\varphi_j^{w_1}(z), \varphi_j^{w_2}(z))$ and construct $\eta_j^{w_1,w_2}$ from $\varphi_j^{w_1,w_2}$ as before. What can you say about $\eta_j^{w_1,w_2}(z)$ and $\eta^{w_1,w_2}(z) = \lim_{j\to\infty} \eta_j^{w_1,w_2}(z)$ for arbitrary $z \in D(P,r)$, and for $z = w_1$ and $z = w_2$ in particular?

**(Problem 81)** Let $u$ be as in Problem 72. Prove that $u$ is harmonic in $\Omega$.

[**Definition: Barriers**] Let $\Omega \subsetneq \mathbb{C}$ be open and let $P \in \partial\Omega$. We say that $b : \overline{\Omega} \mapsto \mathbb{R}$ is a barrier for $\Omega$ at $P$ if:

(i) $b$ is continuous on $\overline{\Omega}$,
(ii) $b$ is subharmonic in $\Omega$,
(iii) $b(P) > b(z)$ for all $z \in \overline{\Omega} \setminus \{P\}$. (Often we take $b(P) = 0$.)

**(Problem 84)** Let $\Omega$, $f$ and $S$ be as in Problem 71. Let $P \in \partial\Omega$.
Suppose that a barrier $b$ at $P$ exists.
Let $\varepsilon > 0$. Use $b$ to construct a function $w_\varepsilon$ such that $w_\varepsilon(P) = f(P) - \varepsilon$ and such that $w_\varepsilon \in S$.

**(Problem 85)** Let $u$ be as in Problem 72. Use the functions $w_\varepsilon$ to show that $u(P) = f(P)$.

**(Problem 82)** Let $\Omega$, $f$ and $S$ be as in Problem 71. Let $P \in \partial\Omega$.
Suppose that a barrier $b$ at $P$ exists.
Let $\varepsilon > 0$. Use $b$ to construct a function $g_\varepsilon$ that is continuous on $\overline{\Omega}$, superharmonic in $\Omega$, and satisfies $g_\varepsilon \geq f$ on $\partial\Omega$, and such that $g_\varepsilon(P) = f(P) + \varepsilon$.

**(Problem 83)** Let $u$ be as in Problem 72. Use the functions $g_\varepsilon$ to show that $u$ is upper semicontinuous at $P$.

**(Problem 86)** Let $\Omega \subsetneq \mathbb{C}$ be open and bounded. Give a condition on $\Omega$ that ensures that the Dirichlet problem is well-posed in $\Omega$.

**(Problem 86a)** Let $\Omega \subsetneq \mathbb{C}$ be open and bounded. Suppose that the Dirichlet problem is well-posed in $\Omega$. Show that for any $P \in \partial\Omega$, there exists a function $b$ that is a barrier at $P$.

**(Problem 87)** Let $\Omega = D(0,1)$ and let $P = e^{i\theta} \in \partial\Omega$. Give an example of a function $b$ that is a barrier at $P$.

**(Problem 93)** Suppose that $\Omega \subsetneq \mathbb{C}$ and $\Psi \subsetneq \mathbb{C}$ are two open connected sets. Suppose that $\varphi : \overline{\Omega} \mapsto \overline{\Psi}$ is continuous and that $\varphi : \Omega \mapsto \Psi$ is holomorphic. Suppose that there is some $P \in \partial\Psi$ and some function $b : \overline{\Psi} \mapsto \mathbb{R}$ that is a barrier for $\Psi$ at $P$. Show that if there is exactly one $Q \in \bar{\Omega}$ with $\varphi(Q) = P$, then $\tilde{b} = b \circ \varphi$ is a barrier for $\Omega$ at $Q$.

**(Problem 88)** Let $\Omega = \mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and let $P = x \in \partial\Omega$. Give an example of a function $b$ that is a barrier at $P$.

**(Problem 89)** Let $\Omega = \mathbb{C} \setminus \{x + 0i : 0 \leq x \leq \infty\}$ denote the complex plane with an infinite slit removed. Let $P = 0 \in \partial\Omega$. Give an example of a function $b$ that is a barrier at $P$.

**(Problem 90)** Let $\Omega = \mathbb{C} \setminus \{x + 0i : 0 \leq x \leq 1\}$ denote the complex plane with a finite slit removed. Let $P = 0 \in \partial\Omega$. Give an example of a function $b$ that is a barrier at $P$.

**(Problem 91)** Let $\Omega \subsetneq \mathbb{C}$ and $\Psi \subsetneq \mathbb{C}$ be two open sets, and suppose $\Omega \subset \Psi$. Suppose that $P \in \partial\Omega \cap \partial\Psi$. Suppose that there is a barrier for $\Psi$ at $P$. Show that there is a barrier for $\Omega$ at $\Psi$.

**(Problem 91a)** Let $\Omega \subsetneq \mathbb{C}$ be open. Suppose that for every point $P \in \partial\Omega$, there is some point $Q_P$ such that the line segment from $P$ to $Q_P$ is contained in $\mathbb{C} \setminus \Omega$. Show that the Dirichlet problem is well-posed in $\Omega$.

**(Problem 92)** Show that having a barrier is a local property. That is, let $\Omega \subsetneq \mathbb{C}$ and $\Psi \subsetneq \mathbb{C}$ be two open sets, and suppose that $P \in \partial\Omega \cap \partial\Psi$ and that for some $\varepsilon > 0$, it is the case that $\Omega \cap D(P, \varepsilon) = \Psi \cap D(P, \varepsilon)$. Suppose that $b$ is a barrier for $\Omega$ at $P$. Construct a barrier for $\Psi$ at $P$.

**(Problem 94)** Give an example of a domain $\Omega \subsetneq \mathbb{C}$ and a point $P \in \partial\Omega$ such that there is no function $b$ that is a barrier at $P$.

### 7.9. Conformal mappings of annuli

**(Problem 95)** Let $0 < r_1 < R_1 < \infty$ and $0 < r_2 < R_2 < \infty$. Let $P_1 \in \mathbb{C}$ and $P_2 \in \mathbb{C}$. Show that, if $R_1/r_1 = R_2/r_2$, then $A_1 = \{z \in \mathbb{C} : r_1 < |z - P_1| < R_1\}$ and $A_2 = \{z \in \mathbb{C} : r_2 < |z - P_2| < R_2\}$ are conformally equivalent; that is, there is a holomorphic bijection $\varphi : A_1 \mapsto A_2$.

**(Problem 96)** Let $\Omega \subsetneq \mathbb{C}$ and $\Psi \subsetneq \mathbb{C}$ be two open sets, and suppose that $\varphi : \Omega \mapsto \Psi$ is a conformal mapping (holomorphic bijection).

Suppose that $\{z_n\}_{n=1}^{\infty} \subset \Omega$ and that $z_n \to z_\infty$ for some $z_\infty \in \partial\Omega$. Suppose that $\varphi(z_n)$ converges to some $w_\infty \in \overline{\Psi}$. Show that $w_\infty \in \partial\Psi$.

**(Problem 97)** Let $\Omega \subsetneq \mathbb{C}$ and $\Psi \subsetneq \mathbb{C}$ be two bounded open sets, and suppose that $\varphi : \Omega \mapsto \Psi$. Suppose that $\{z_n\}_{n=1}^{\infty} \subset \Omega$. Show that there is a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ such that $\varphi(z_{n_k})$ converges as $k \to \infty$.

**(Problem 98)** Suppose that $1 < R_1 < \infty$ and that $1 < R_2 < \infty$. Let $A_1 = \{z \in \mathbb{C} : 1/R_1 < |z| < R_1\}$ and $A_2 = \{z \in \mathbb{C} : 1/R_2 < |z| < R_2\}$. Let $\varphi : A_1 \mapsto A_2$ be a holomorphic bijection.

Let $1/R_2 < \rho < R_2$. Let $r_\rho = \inf\{|\varphi^{-1}(\rho e^{i\theta})| : 0 \leq \theta \leq 2\pi\}$ and let $s_\rho = \sup\{|\varphi^{-1}(\rho e^{i\theta})| : 0 \leq \theta \leq 2\pi\}$. Show that $1/R_1 < r_\rho \leq s_\rho < R_1$.

**(Problem 99)** Let $z_\infty \in \partial A_1$ with $|z_\infty| = R_1$. Suppose there is some $\{w_n\}_{n=1}^{\infty} \subset A_1$ such that $w_n \to z_\infty$ and $|\varphi(w_n)| \to R_2$ as $n \to \infty$.

Suppose further that $\{z_n\}_{n=1}^{\infty} \subset A_1$, that $z_n \to z_\infty$ as $n \to \infty$, and that $\varphi(z_n)$ converges. Show that $|\varphi(z_n)| \to R_2$ as $n \to \infty$. *Hint*: Use the fact that $\{z \in A_1 : |\varphi(z)| < 1\}$ and $\{z \in A_1 : |\varphi(z)| > 1\}$ are disjoint open sets and $\{z \in A_1 : |z| > s_1\}$ is connected.

**(Problem 100)** If $z_n$, $w_n \in A_1$, $|z_\infty| = R_1$, $z_n \to z_\infty$, $w_n \to z_\infty$, $|\varphi(w_n)| \to 1/R_2$, and $\varphi(z_n)$ converges, what can you say about $\lim\limits_{n\to\infty} |\varphi(z_n)|$?

**(Problem 100a)** If $z_n$, $w_n \in A_1$, $|z_\infty| = 1/R_1$, $z_n \to z_\infty$, $w_n \to z_\infty$, $|\varphi(w_n)| \to R_2$, and $\varphi(z_n)$ converges, what can you say about $\lim\limits_{n\to\infty} |\varphi(z_n)|$?

**(Problem 100b)** If $z_n$, $w_n \in A_1$, $|z_\infty| = 1/R_1$, $z_n \to z_\infty$, $w_n \to z_\infty$, $|\varphi(w_n)| \to 1/R_2$, and $\varphi(z_n)$ converges, what can you say about $\lim\limits_{n\to\infty} |\varphi(z_n)|$?

**(Problem 101)** Let $z_\infty \in \partial A_1$. Show that there is a sequence of points $\{w_n\}_{n=1}^{\infty} \subset A_1$ such that $w_n \to z_\infty$ and such that either $|\varphi(w_n)| \to R_2$ or $|\varphi(w_n)| \to 1/R_2$ as $n \to \infty$.

**(Problem 101a)** Let $z_\infty \in \partial A_1$. Show that one of the following is true:

- For every sequence $\{z_n\}_{n=1}^{\infty} \subset A_1$ that satisfies $z_n \to z_\infty$, we have that $|\varphi(z_n)| \to R_2$ as $n \to \infty$.
- For every sequence $\{z_n\}_{n=1}^{\infty} \subset A_1$ that satisfies $z_n \to z_\infty$, we have that $|\varphi(z_n)| \to 1/R_2$ as $n \to \infty$.

**(Problem 102)** Let $h(z) = |\varphi(z)|$. Show that there is a function $\tilde{h}$ that is continuous on $\overline{A_1}$ and satisfies $h = \tilde{h}$ on $A_1$.

**(Problem 102a)** What values can $\tilde{h}$ take on $\partial A_1$?

**(Problem 103)** Show that $\tilde{h}(z)$ is constant on each of the two boundary components of $A_1$.

**(Problem 104)** Show that $g(z) = \log h(z) = \log|\varphi(z)|$ is harmonic in $A_1$.

**(Problem 104a)** Can $\tilde{h} = |\varphi(z)|$ be equal on the two boundary components of $A_1$?

**(Problem 105)** For fixed $R_1$ and $R_2$, there are two possible values of $g(z) = \log|\varphi(z)|$. Find them.

**(Problem 106)** Suppose $f : A_1 \mapsto A_2$ is holomorphic and $\log|f(z)| = \beta \log|z|$ for some real number $\beta$. Find $f(z)$. Are there any restrictions on $\beta$? If we require that $f$ be one-to-one, are there any additional restrictions on $\beta$?

**(Problem 107)** Let $0 < r_1 < R_1 < \infty$ and $0 < r_2 < R_2 < \infty$. Let $P_1 \in \mathbb{C}$ and $P_2 \in \mathbb{C}$. Suppose that $A_1 = \{z \in \mathbb{C} : r_1 < |z - P_1| < R_1\}$ and $A_2 = \{z \in \mathbb{C} : r_2 < |z - P_2| < R_2\}$ are conformally equivalent. Show that $R_1/r_1 = R_2/r_2$.

**(Problem 108)** Let $A = \{z \in \mathbb{C} : 1/R < |z| < R\}$ be an annulus for some $R > 1$. Find all conformal self-maps of $A$.

### 8.1. Basic concepts for infinite products

**(Problem 109)** Let $\{A_j\}_{j=1}^{\infty}$ be a sequence of complex numbers. Suppose $A_j \neq 0$ for all $j$ and that $\lim_{N\to\infty} \prod_{j=1}^{N} A_j$ exists and is nonzero. Show that $\lim_{j\to\infty} A_j = 1$.

**(Problem 110)** Let $\{A_j\}_{j=1}^{\infty}$ be a sequence of complex numbers. Suppose $A_j \neq 0$ for all $j$ and that $\lim_{N\to\infty} \prod_{j=1}^{N} A_j = 0$. Can we conclude that $\lim_{j\to\infty} A_j = 1$?

**(Problem 111)** Let $\{A_j\}_{j=1}^{\infty}$ be a sequence of complex numbers. Suppose $A_k = 0$ for some $k$. What is $\lim_{N\to\infty} \prod_{j=1}^{N} A_j$? If the sequence of partial products converges, can we conclude that $\lim_{j\to\infty} A_j = 1$?

**(Problem 112)** Show that if $0 \leq x \leq 1$, then $1 + x \leq e^x \leq 1 + 2x$.

**(Problem 113)** Show that if $a_j \in \mathbb{C}$ with $|a_j| < 1$, then $\exp\left(\frac{1}{2}\sum_{j=1}^{n}|a_j|\right) \leq \prod_{j=1}^{n} 1 + |a_j| \leq \exp\left(\sum_{j=1}^{n}|a_j|\right)$.

**(Problem 114)** Show that if $a_j \in \mathbb{C}$ and $\sum_{j=1}^{\infty}|a_j|$ converges, then $\lim_{N\to\infty} \prod_{j=1}^{N} 1 + |a_j|$ exists. Can the limit be zero?

**(Problem 115)** Show that if $a_j \in \mathbb{C}$ and $\lim_{N\to\infty} \prod_{j=1}^{N} 1 + |a_j|$ exists, then $\sum_{j=1}^{\infty}|a_j|$ converges.

**(Problem 116)** Suppose that $\lim_{N\to\infty} \prod_{j=1}^{N} 1 + |a_j|$ exists. Show that there is some $N_0 > 0$ such that $1 + a_j \neq 0$ for any $j \geq N_0$.

**(Problem 117)** Suppose that $a_j \in \mathbb{C}$ and $N \geq M$. Show that $\left|\left(\prod_{j=M}^{N} 1 + a_j\right) - 1\right| \leq \left|\left(\prod_{j=M}^{N} 1 + |a_j|\right) - 1\right|$. *Hint*: Use induction.

**(Problem 118)** Suppose that $a_j \in \mathbb{C}$ and $a_j \neq -1$. Suppose that $\lim_{N\to\infty} \prod_{j=1}^{N} 1 + |a_j|$ exists. Show that $\lim_{N\to\infty} \prod_{j=1}^{N} 1 + a_j$ exists and is nonzero.

**(Problem 119)** Suppose that $a_j \in \mathbb{C}$ and $\sum_{j=1}^{\infty}|a_j|$ converges. Show that $\lim_{N\to\infty} \prod_{j=1}^{N} 1 + a_j$ exists; if $a_j \neq -1$ for all $j$, the limit is nonzero.

**(Problem 120)** Let $K \subset \mathbb{C}$ and let $\{f_j\}_{j=1}^{\infty}$ be a sequence of functions $f_j : K \mapsto \mathbb{C}$. Suppose that $|f_j(z)| < 1$ for all $j \geq 1$ and all $z \in K$ and that $\displaystyle\sum_{j=1}^{\infty} |f_j(z)|$ converges uniformly for all $z \in K$. Show that $\displaystyle\prod_{j=1}^{N} 1 + f_j(z)$ converges as $N \to \infty$ to a function $F(z)$, uniformly for all $z \in K$.

**(Problem 120a)** Let $K \subset \mathbb{C}$ and let $\{f_j\}_{j=1}^{\infty}$ be a sequence of functions $f_j : K \mapsto \mathbb{C}$. Suppose that the $f_j$s are uniformly bounded (although not necessarily by 1) and that $\displaystyle\sum_{j=1}^{\infty} |f_j(z)|$ converges uniformly for all $z \in K$. Show that $\displaystyle\prod_{j=1}^{N} 1 + f_j(z)$ converges as $N \to \infty$ to a function $F(z)$, uniformly for all $z \in K$.

**(Problem 121)** Let $f_j$, $K$ and $F$ be as in Problem 120a. Let $z_0 \in K$. Show that $1 + f_j(z_0) = 0$ for at most finitely many numbers $j$.

**(Problem 122)** Let $\Omega \subsetneq \mathbb{C}$ be open and let $f_j : \Omega \mapsto \mathbb{C}$ be holomorphic. Suppose that $\displaystyle\sum_{j=1}^{\infty} |f_j(z)|$ converges normally (that is, uniformly on compact sets). Show that $\displaystyle\prod_{j=1}^{N} 1 + f_j(z)$ converges normally as $N \to \infty$ to a holomorphic function $F$.

**(Problem 123)** Let $\Omega$, $f_j$ and $F$ be as in Problem 122. Let $z_0 \in \Omega$. Show that $F(z_0) = 0$ if and only if $f_j(z_0) = -1$ for some $j \geq 1$.

**(Problem 124)** Let $\Omega$, $f_j$ and $F$ be as in Problem 122. Suppose that $F$ is not identically equal to zero. Let $z_0 \in \Omega$ and suppose that $F(z_0) = 0$. Show that the multiplicity of the zero of $F$ at $z_0$ is equal to the sum of the multiplicities of the zeros of $1 + f_j$ at $z_0$.

**(Problem 124a)** Let $\Omega$, $f_j$ and $F$ be as in Problem 122. Show that if $F$ is identically equal to zero then $f_j$ is identically equal to $-1$ for some $j$.

### 8.2. The Weierstrauss factorization theorem

**(Problem 125)** Let $f : \mathbb{C} \mapsto \mathbb{C}$ be an entire function. Suppose that $f$ has finitely many zeros. Show that there is an entire function $g$, an integer $N$, and complex numbers $a_n$ such that

$$f(z) = e^{g(z)} \prod_{n=1}^{N} (z - a_n).$$

**(Problem 126)** Let $f : \mathbb{C} \mapsto \mathbb{C}$ be an entire function that is not identically zero. Show that $f$ can have at most countably many zeros (counted with multiplicity).

[**Definition: Elementary factors**] If $p \geq 0$ is an integer, we let $E_p(z) = (1 - z) \exp\left(z + \dfrac{z^2}{2} + \ldots + \dfrac{z^p}{p}\right)$.

**(Problem 127)** Show that $E_p$ is an entire function and that $E_p(z) = 0$ if and only if $z = 1$. What is the multiplicity of the zero of $E_p(z)$ at 1?

**(Problem 128)** Let $b_n$ be the components of the power series for $E_p$ centered at zero, so $E_p(z) = \displaystyle\sum_{n=0}^{\infty} b_n z^n$. Find $b_0$.

**(Problem 129)** Compute $E_p'(z)$ using the definition given above and also using the power series expansion.

**(Bonus problem 129a)** Write a recurrence relation for the coefficients $b_n$. That is, write the power series for $(1 - z)E_p'(z) - z^p E_p(z)$ in terms of $b_n$, and then use your formula for $(1 - z)E_p'(z) - z^p E_p(z)$ to find a formula for $b_n$ in terms of $b_0, b_1, \ldots, b_{n-1}$.

**(Problem 130)** What can you say about $b_n$ for $1 \leq n \leq p$?

**(Problem 131)** Show that $b_n$ is real and that $b_n \leq 0$ for any $n > p$.

**(Problem 132)** Compute $\displaystyle\sum_{n=p+1}^{\infty} |b_n|$. *Hint*: Start by computing $E_p(1)$.

**(Problem 133)** Show that if $|z| \leq 1$ then $|E_p(z) - 1| \leq |z|^{p+1}$.

**(Problem 134)** Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence of nonzero complex numbers. Suppose that the $a_n$s have no accumulation point in the sense that no subsequence converges. (We do not require that the $a_n$s be distinct.) Show that $\displaystyle\lim_{n\to\infty} |a_n| = \infty$.

**(Problem 134a)** Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence of nonzero complex numbers. Suppose that the $a_n$s have no accumulation point.

Fix an $r > 0$. Show that $\displaystyle\sum_{n=1}^{\infty} |1 - E_n(z/a_n)|$ converges uniformly for all $|z| < r$.

**(Problem 135)** Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence of complex numbers. Suppose that the $a_n$s have no accumulation point. (We do not require that the $a_n$s be distinct; we also allow $a_n = 0$ for finitely many $n$.)

Show that there is an entire function $F$ whose zero set is precisely equal to $\{a_n\}_{n=1}^{\infty}$ (counting multiplicities).

**(Problem 136) The Weierstrauss factorization theorem.** Let $f : \mathbb{C} \mapsto \mathbb{C}$ be an entire function that is not identically equal to zero. Show that there is an entire function $g(z)$, an integer $m \geq 0$, and complex numbers $a_n \in \mathbb{C}$ such that

$$f(z) = e^{g(z)}\, z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right)$$

where the infinite product converges uniformly on compact sets.

### 8.3. The Theorems of Weierstrauss and Mittag-Leffler: interpolation problems

**(Problem 137)** Let $\Omega \subsetneq \mathbb{C}$ be an open set, let $R > 0$, and let $\{a_n\}_{n=1}^{\infty} \subset D(0, R) \cap \Omega$ be a sequence with no accumulation points in $\Omega$. Show that $\displaystyle\lim_{n\to\infty} \operatorname{dist}(a_n, \partial\Omega) = 0$.

**(Problem 137a)** Show that there exists a sequence of points $\{\widehat{a}_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \Omega$ and such that $|a_n - \widehat{a}_n| \to 0$ as $n \to \infty$.

**(Problem 138)** Let $\Omega$, $\{a_n\}_{n=1}^{\infty}$ and $\{\widehat{a}_n\}_{n=1}^{\infty}$ be as in Problem 137a. Let $N \geq 1$ be an integer. Show that $\displaystyle\prod_{n=1}^{N} E_n\left(\frac{a_j - \widehat{a}_j}{z - \widehat{a}_j}\right)$ is holomorphic in $\Omega$.

**(Problem 139)** Let $\Omega$, $\{a_n\}_{n=1}^{\infty}$ and $\{\widehat{a}_n\}_{n=1}^{\infty}$ be as in Problem 137a. Show that $\displaystyle\prod_{n=1}^{N} E_n\left(\frac{a_j - \widehat{a}_j}{z - \widehat{a}_j}\right)$ converges as $N \to \infty$ for all $z \in \Omega$, uniformly on compact sets.

**(Problem 140)** Let $\Omega$, $\{a_n\}_{n=1}^{\infty}$ and $\{\widehat{a}_n\}_{n=1}^{\infty}$ be as in Problem 137a. What can you say about $f(z) = \displaystyle\lim_{N\to\infty} \prod_{n=1}^{N} E_n\left(\frac{a_j - \widehat{a}_j}{z - \widehat{a}_j}\right)$?

**(Problem 141) Weierstrauss's theorem.** Let $\Omega \subseteq \mathbb{C}$ be open and let $\{a_n\}_{n=1}^{\infty} \subset \Omega$ have no accumulation points in $\Omega$. We do not require that $\{a_n\}$ be bounded. Show that there is a function $f$ that is holomorphic in $\Omega$ and such that the zero set of $f$ (with multiplicity) is precisely equal to $\{a_n\}_{n=1}^{\infty}$.

**(Problem 142)** Let $\Omega \subseteq \mathbb{C}$ be open. Let $m$ be meromorphic on $\Omega$. Show that there are functions $f$ and $g$ that are holomorphic in $\Omega$ and such that $m(z) = f(z)/g(z)$ for all $z \in \Omega \setminus A$, where $A$ is the set of poles of $m$.

**(Problem 143)** Let $f(e^{i\theta}) = \sum_{n=0}^{\infty} \frac{5^n}{6^n} \cos(7^n\theta)$. (This is a special case of the *Weierstrauss function.*) Show that $f$ is well-defined (the sum converges) for all $0 \leq \theta \leq 2\pi$ and that $f$ is continuous on $\partial D(0,1)$.

**(Problem 144)** Plot the first few partial sums for the Weierstrauss function.

**(Problem 145)** Let $u$ be the function that is harmonic in $D(0,1)$, continuous on $\overline{D}(0,1)$ and with $u(e^{i\theta}) = f(e^{i\theta})$ for $0 \leq \theta \leq 2\pi$. Let $H$ be the function that is holomorphic in $D(0,1)$ with real part $u$.

Show that $D(0,1)$ is the domain of existence of $H$; that is, if $\widetilde{H} = H$ in $D(0,1)$ and $\widetilde{H}$ is holomorphic on some open set $\Omega \supseteq D(0,1)$, then $\Omega = D(0,1)$. *Hint*: Use the fact (proven by Weierstauss in 1872) that $f(\theta)$ is nowhere differentiable.

**(Problem 146)** Let $\mathcal{Q}_j = \{[k2^j, (k+1)2^j) \times [\ell 2^j, (\ell+1)2^j) : k, \ell \text{ are integers}\}$ be the grid of squares in $\mathbb{C}$ with side-length $2^j$ aligned with the axes. Sketch $\mathcal{Q}_j$.

**(Problem 147)** Suppose that $S \in \mathcal{Q}_j$. Let $P(S)$ be the "dyadic parent" of $S$, so $S \subsetneq P(S) \in \mathcal{Q}_{j+1}$. Let $2S$ be the square concentric to $S$ of side-length $2^{j+1}$.

Sketch $S$, $2S$ and the four possibilities for $P(S)$.

**(Problem 151)** If $S \in \mathcal{Q}_j$, let $\ell(S) = 2^j$ be the side-length of $S$. Show that if $S \in \mathcal{Q}_j$ and $z \in S$, then $D(z, \ell(S)/2) \subset 2S$ and $2P(S) \subset D(z, 3\sqrt{2}\ell(S))$.

**(Problem 148)** Let $\mathcal{Q} = \cup_{j=-\infty}^{\infty} \mathcal{Q}_j$. Let $\Omega \subsetneq \mathbb{C}$ be open. Let $\mathcal{G} = \{S \in \mathcal{Q} : 2S \subset \Omega,\ 2P(S) \not\subset \Omega\}$. We call $\mathcal{G}$ a *dyadic Whitney decomposition* of $\Omega$. Show that $\cup_{S\in\mathcal{G}} S \subseteq \Omega$.

**(Problem 148a)** Show that if $z \in \Omega$, then there is some $S \in \mathcal{G}$ with $z \in S$.

**(Problem 149)** Show that if $S \in \mathcal{G}$ and $T \in \mathcal{G}$, then either $S = T$ or $S \cap T = \emptyset$.

**(Problem 149a)** If $z \in \Omega$, then how many cubes $S \in \mathcal{G}$ can satisfy $z \in S$?

**(Problem 150)** Show that $\mathcal{G}$ is a countable set.

**(Problem 152)** Suppose that $S, T \in \mathcal{G}$ and that $\operatorname{dist}(S,T) = 0$; that is, the closures of $S$ and $T$ intersect. Show that $\ell(S) \leq 8\ell(T)$ and that $\ell(T) \leq 8\ell(S)$.

**(Problem 152a)** If $S \in \mathcal{G}$, let $z_S$ be the midpoint of $S$. Let $A = \{z_S : S \in \mathcal{G}\}$.

Let $z \in \Omega$. Show that $z$ is not an accumulation point for $A$. *Hint*: if $z \in T \in \mathcal{G}$, then how many midpoints $z_S$ can appear in $D(z, \ell(T)/16)$?

**(Problem 153)** Let $z \in \partial\Omega$. Show that $z$ is an accumulation point for $A$.

**(Problem 154)** Show that there is a function $f$ that is holomorphic in $\Omega$ and such that $f(z) = 0$ if and only if $z \in A$.

**(Problem 155)** Show that the domain of existence of $f$ is $\Omega$; that is, if $\widetilde{f} = f$ in $\Omega$ and $\widetilde{f}$ is holomorphic on some open set $\Psi \supseteq \Omega$, then $\Psi = \Omega$.

In Problems 156–159, let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence of distinct points with no accumulation points.

**(Problem 156)** Let $\beta \in \mathbb{C}$ and let $k > 0$ be an integer. Find an entire function $f$ such that $f(a_k) = \beta$ and such that $f(a_n) = 0$ for all $n \neq k$.

**(Problem 157)** Let $\beta \in \mathbb{C}$ and let $k > 0$ be an integer. Find an entire function $f$ such that $f(a_k) = \beta$ and such that $f(a_n) = 0$ for all $n \neq k$, and such that $|f(z)| < 2^{-k}$ for all $|z| < \frac{1}{2}|a_k|$.

**(Problem 158)** Let $\{\beta_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence of complex numbers. Find an entire function $f$ such that $f(a_n) = \beta_n$ for all $n \geq 1$.

**(Problem 159)** Let $\{\beta_n\}_{n=1}^{\infty} \subset \mathbb{C}$ and $\{\gamma_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be two sequences of complex numbers. Find an entire function $f$ such that $f(a_n) = \beta_n$ and $f'(a_n) = \gamma_n$ for all $n \geq 1$.

### 10.1. Definition of an analytic function element

**(Problem 160)** Let $\Omega \subseteq \mathbb{C}$ be open. Let $\Psi \subsetneq \Omega$ be open and nonempty. Suppose that $f$ is holomorphic in $\Psi$. Show that there is at most one function $F$ that is holomorphic in $\Omega$ and such that $F = f$ in $\Psi$.

**(Problem 161)** Let $\Psi$ be the open sector $\{re^{i\theta} : r > 0,\ 0 < \theta < \pi/2\}$. Let $f(z)$ be the branch of the logarithm given by $f(re^{i\theta}) = \log r + i\theta$ whenever $0 < \theta < \pi/2$.

Find two functions $F$ and $\widetilde{F}$ and domains $\Omega \supsetneq \Psi$ and $\widetilde{\Omega} \supsetneq \Psi$ such that $F$ is holomorphic in $\Omega$, $\widetilde{F}$ is holomorphic in $\widetilde{\Omega}$, $\Omega \cap \widetilde{\Omega}$ is nonempty, $F \neq \widetilde{F}$ on $\Omega \cap \widetilde{\Omega}$, and $F = f = \widetilde{F}$ in $\Psi$.

**(Problem 162)** The gamma function $\Gamma(z)$ is defined by $\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt$. Find $\Gamma(n)$, where $n$ is a positive integer.

**(Problem 163)** For what values of $z$ does the integral in the definition of the gamma function converge?

**(Problem 164)** Integrate by parts to find a new formula for $\Gamma(z)$ that converges for $z$ in a larger set.

[**Definition: Function elements**] A function element is an ordered pair $(f, D(P, r))$ where $P \in \mathbb{C}$, $r > 0$ and $f$ is a holomorphic function defined on $D(P, r)$.

[**Definition: Direct analytic continuation**] If $(f, D(P, r))$ and $(g, D(Q, s))$ are function elements, if $D(P, r) \cap D(Q, s) \neq \emptyset$, and if $f = g$ on $D(P, r) \cap D(Q, s)$, we say that $(g, D(Q, s))$ is a direct analytic continuation of $(f, D(P, r))$.

[**Definition: Analytic continuation**] Suppose that we have a finite sequence of function elements $\{(f_j, D(P_j, r_j))\}_{j=1}^k$ such that $(f_j, D(P_j, r_j))$ is a direct analytic continuation of $(f_{j-1}, D(P_{j-1}, r_{j-1}))$ for all $1 < j \leq k$. Then $(f_k, D(P_k, r_k))$ is an analytic continuation of $(f_1, D(P_1, r_1))$.

**(Problem 165)** Find a function element $(f, D(P, r))$ and two distinct function elements $(g, D(Q, s))$ and $(\tilde{g}, D(Q, s))$, with the same disc $D(Q, s)$, such that $(g, D(Q, s))$ and $(\tilde{g}, D(Q, s))$ are both analytic continuations of $(f, D(P, r))$.

**(Problem 166)** Can you do this for a direct analytic continuation?

### 10.2. Analytic continuation along a curve

[**Definition: Analytic continuation along a curve**] Let $\gamma : [0, 1] \mapsto \mathbb{C}$ be a continuous function (we will call $\gamma$ a curve). Let $(f, D(\gamma(0), r))$ be a function element. An analytic continuation of $(f, D(\gamma(0), r))$ along $\gamma$ is a collection of function elements $\{(f_t, D(\gamma(t), r_t))\}_{0 \leq t \leq 1}$ such that $(f_0, D(\gamma(0), r_0)) = (f, D(\gamma(0), r))$ and such that if $0 \leq t \leq 1$, then there is an $\varepsilon > 0$ such that, if $0 \leq s \leq 1$ and $|t - s| < \varepsilon$, then $(f_s, D(\gamma(s), r_s))$ is a direct analytic continuation of $(f_t, D(\gamma(t), r_t))$.

**(Problem 167)** Let $\gamma : [0, 1] \mapsto \mathbb{C}$ be a curve and let $(f, D(\gamma(0), r))$ be a function element. Suppose that $\{(f_t, D(\gamma(t), r_t))\}_{0 \leq t \leq 1}$ and $\{(\tilde{f}_t, D(\gamma(t), \tilde{r}_t))\}_{0 \leq t \leq 1}$ are two analytic continuations of $(f, D(\gamma(0), r))$ along $\gamma$.

Let $S = \{s : 0 \leq s \leq 1,\ f_s = \tilde{f}_s \text{ on } D(\gamma(s), \min(r_s, \tilde{r}_s))\}$. Let $T = \{t : 0 \leq t \leq 1,\ s \in S \text{ for all } 0 \leq s \leq t\}$. Show that $T$ is not empty.

**(Problem 168)** Show that $T$ is closed.

**(Problem 169)** Show that $T$ is open in $[0, 1]$.

**(Problem 170)** Is there a sense in which an analytic continuation along a curve is unique?

**(Problem 171)** Suppose that $\gamma : [0, 1] \mapsto \mathbb{C}$ is a *closed* curve (so $\gamma(1) = \gamma(0)$). Let $\{(f_t, D(\gamma(t), r_t))\}_{0 \leq t \leq 1}$ be an analytic continuation of $(f, D(\gamma(0), r))$ along $\gamma$. Is it necessarily true that $f_1 = f_0$ on $D(\gamma(0), \min(r_0, r_1))$?