# Math 2584, Fall 2016

If you need it, the following will be printed on the cover page of the exam:

**Theorem 2.4.2.** Consider the initial-value problem $\frac{dy}{dt} = f(t,y)$, $y(t_0) = y_0$. Suppose that $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous near $(t_0, y_0)$. Then there is some interval $(a,b)$ with $a < t_0 < b$ such that a solution to the initial-value problem exists and is unique on that interval.

**Theorem 2.4.1.** Consider the initial-value problem $\frac{dy}{dt} + p(t)y = g(t)$, $y(t_0) = y_0$. Suppose that there is some $a < t_0 < b$ such that $p(t)$ and $g(t)$ are continuous on $(a,b)$. Then there is a unique solution to the initial-value problem on the interval $(a,b)$.

**Theorem 3.2.1.** Consider the initial-value problem $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\,y = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_1$. Suppose that there is some $a < t_0 < b$ such that $p(t)$, $q(t)$ and $g(t)$ are all continuous on $(a,b)$. Then there is a unique solution to the initial-value problem on the interval $(a,b)$.

Euler's formula states that, if $\theta$ is any real number, then $e^{i\theta} = \cos\theta + i\sin\theta$.

The acceleration of gravity (on Earth) is 9.8 meters/second$^2$; alternatively, gravity exerts a force of 9.8 newtons per kilogram.

If an object of mass $M$ kilograms is subjected to a force of $F$ newtons, and its position is $x$ meters and its velocity is $v$ meters/second$^2$, then Newton's second law states that $F = M\frac{dv}{dt} = M\frac{d^2x}{dt^2}$, where $t$ denotes time (in seconds).

Here are some Laplace transforms:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \qquad s > a$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad s > 0, \quad n \geq 0$$

$$\mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}, \qquad s > a$$

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}, \qquad s > 0$$

$$\mathcal{L}\{e^{at}\cos(bt)\} = \frac{s-a}{(s-a)^2+b^2}, \qquad s > a$$

$$\mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}, \qquad s > 0$$

$$\mathcal{L}\{e^{at}\sin(bt)\} = \frac{b}{(s-a)^2+b^2}, \qquad s > a$$

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s}, \qquad s > 0, \quad c \geq 0$$

$$\mathcal{L}\{\delta(t-c)\} = e^{-cs}, \qquad s > 0, \quad c \geq 0$$

$$\mathcal{L}\{u_c(t)f(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}, \qquad c \geq 0$$

$$\mathcal{L}\{u_c(t)g(t-c)\} = e^{-cs}\mathcal{L}\{g(t)\}, \qquad c \geq 0$$

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0)$$

$$\mathcal{L}\Big\{\int_0^t f(r)\,g(t-r)\,dr\Big\} = \mathcal{L}\Big\{\int_0^t f(t-r)\,g(r)\,dr\Big\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a $2 \times 2$ matrix and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is a vector, then $A\vec{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a\,v_1 + b\,v_2 \\ c\,v_1 + d\,v_2 \end{bmatrix}$.

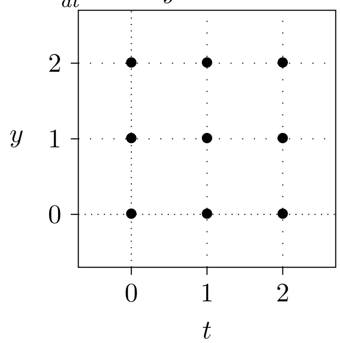
If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a $2 \times 2$ matrix then $\det A = |A| = ad - bc$.

If $A$ is a matrix, $\lambda$ is a number (real or complex), and $\vec{v}$ is a vector, $\vec{v} \neq 0$, and if $A\vec{v} = \lambda\vec{v}$, then

$$\det(A - \lambda I) = 0 \qquad \text{and} \qquad (A - \lambda I)\vec{v} = 0$$

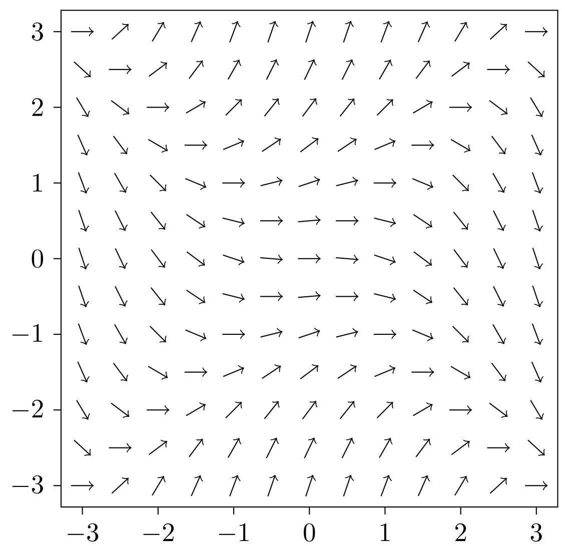
where $I$ is the identity matrix of the same size as $A$. (If $A$ is a $2 \times 2$ matrix then $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.)

**(Problem 1)** Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.

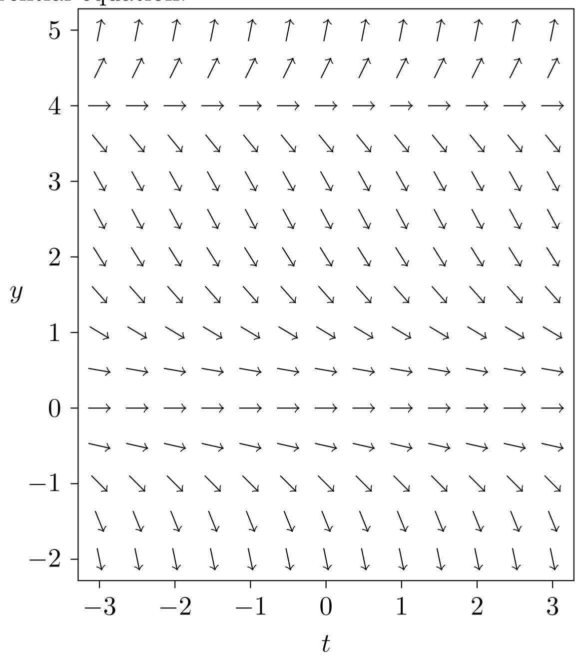


**(Problem 2)** Consider the differential equation $\frac{dy}{dx} = y^2 - x^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-3) = 0.$$



**(Problem 3)** Consider the differential equation $\frac{dy}{dt} = y^2(y-4)/5$. Here is the direction field for this differential equation.




Based on the direction field, determine the behavior of $y$ as $t \to \infty$. If this behavior depends on the initial value of $y$ at $t = 0$, describe this dependency.

**(Problem 4)** A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Set up, but **do not solve**, the differential equation and initial conditions that describe the amount of salt in the tank. Be sure to define your independent and dependent variables, as well as all unknown parameters, and be sure to include units.

**(Problem 5)** According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C. Set up, but **do not solve**, the differential equation and initial conditions that describe the temperature of a cup of coffee. Be sure to define your independent and dependent variables, as well as all unknown parameters, and be sure to include units.

**(Problem 6)** Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write a differential equation for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

**(Problem 7)** I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write a differential equation for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

**(Problem 8)** A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 $\text{cm}^3$ per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write a differential equation for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

**(Problem 9)** According to the Stefan-Boltzmann law, a black body in a dark vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write a differential equation for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

**(Problem 10)** A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write a differential equation for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

**(Problem 11)** A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write a differential equation for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

**(Problem 12)** Is $y = e^t$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

**(Problem 13)** Is $y = e^{2t}$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

**(Problem 14)** Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 1$?

**(Problem 15)** Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 2$?

**(Problem 16)** Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 1$?

**(Problem 17)** Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 2$?

**(Problem 18)** For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution on some interval. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + \arctan(t)\,y = e^t$, $y(3) = 7$.
(b) $\frac{dy}{dt} - t^5 y = \cos(6t)$, $y(2) = -1$, $y'(2) = 3$.
(c) $\frac{d^2y}{dt^2} - 5\sin(t)\,y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.
(d) $\frac{d^2y}{dt^2} + 3(t-4)\frac{dy}{dt} + 4t^6 y = 2$, $y(3) = 1$, $y'(3) = -1$.
(e) $\frac{d^2y}{dt^2} + \cos(t)\frac{dy}{dt} + 3\ln(1+t^2)\,y = 0$, $y(2) = 3$.

**(Problem 19)** For each of the following differential equations, determine whether it is linear, separable, homogeneous, or a Bernoulli equation. Then solve the differential equation.

(a) $\frac{dy}{dt} = \frac{t+\cos t}{\sin y - y}$
(b) $(t^2+1)\frac{dy}{dt} = ty - t^2 - 1$
(c) $t^2\frac{dy}{dt} = y^2 + t^2 - ty$
(d) $\frac{dy}{dt} = -y\tan 2t - y^3\cos 2t$
(e) $(t+y)\frac{dy}{dt} = 5y - 3t$
(f) $\frac{dy}{dt} = -y^3\cos(2t)$
(g) $4ty\frac{dy}{dt} = 3y^2 - 2t^2$
(h) $t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$
(i) $\frac{dy}{dt} = 8y - y^8$
(j) $t\frac{dy}{dt} = -\cos t - 3y$

**(Problem 20)** For each of the following differential equations, determine whether it is linear, separable, or homogeneous. Then solve the given initial-value problem.

(a) $2ty\frac{dy}{dt} = 4t^2 - y^2$, $y(1) = 3$.
(b) $t\frac{dy}{dt} = -1 - y^2$, $y(1) = 1$
(c) $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$.
(d) $\frac{dy}{dt} = -3y - \sin t\, e^{-3t}$, $y(0) = 2$

**(Problem 21)** Solve the initial-value problem $\frac{dy}{dt} = \frac{t-5}{y}$, $y(0) = 3$ and determine the range of $t$-values in which the solution is valid.

**(Problem 22)** Solve the initial-value problem $\frac{dy}{dt} = y^2$, $y(0) = 1/4$ and determine the range of $t$-values in which the solution is valid.

**(Problem 23)** Suppose that $\frac{d^2x}{dt^2} = 18x^3$, $x(0) = 1$, $x'(0) = 3$. Let $v = \frac{dx}{dt}$. Find a formula for $v$ in terms of $x$. Then find a formula for $x$ in terms of $t$.

**(Problem 24)** Consider a hanging pendulum. If $\theta$ denotes the angle between the pendulum and the vertical, then the pendulum satisfies the equation of motion $\frac{d^2\theta}{dt^2} = -\frac{mg}{\ell}\sin\theta$, where $m$ is the mass of the pendulum bob, $\ell$ is the length of the pendulum and $g$ is the acceleration of gravity.

Find a formula for $\omega = \frac{d\theta}{dt}$ in terms of $\theta$. Then find a formula relating $\theta$ and $t$. (Your answer may involve an indefinite integral.)

**(Problem 25)** Are we guaranteed a unique solution to the following initial value problems?

(a) $x^2\frac{dy}{dx} = y^2$, $y(0) = 1$.
(b) $x^2\frac{dy}{dx} = y^2$, $y(1) = 0$.
(c) $\frac{dy}{dx} = \sqrt[3]{y}$, $y(8) = 0$.
(d) $\frac{dy}{dx} = \sqrt[3]{y}$, $y(8) = 1$.

**(Problem 26)** For each of the following initial-value problems, tell me the longest open interval on which we are guaranteed that a unique solution exists.

(a) $\frac{dy}{dt} + 5y = \sec t$, $y(\pi) = 0$.
(b) $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 5y = 0$, $y(-3) = 0$, $y'(-3) = 2$.
(c) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \frac{1}{t^3-6t^2+8t}$, $y(1) = 0$, $y'(1) = 2$.
(d) $\frac{dy}{dt} = e^{y\cos t}$, $y(0) = 1$.

**(Problem 27)**

(a) Suppose that $y(t) = t^3+8$ and $y(t) = 2t^2+4t$ are both solutions to the differential equation $\frac{dy}{dt}+p(t)y = g(t)$. Are there any numbers $t_0$ such that you may be sure that $p(t)$ or $g(t)$ is discontinuous near $t_0$?

(b) Suppose that $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are both solutions to some differential equation $\frac{dy}{dt} = f(t, y)$. What can you say about $f(t, y)$ and $\partial f/\partial y$?

(c) Suppose that $y_1(t) = t^3 + 8$ and $y(t) = 2t^2 + 4t$ are both solutions to the differential equation $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\,y = g(t)$. Are there any numbers $t_0$ such that you may be sure that $p(t)$, $q(t)$ or $g(t)$ is discontinuous near $t_0$?

**(Problem 28)**

(a) Suppose that $y(t) = \ln|t^2 - 4|$ is a solution to the differential equation $\frac{dy}{dt} + p(t)y = g(t)$. Are there any numbers $t_0$ such that you may be sure that at least one of $p(t)$ and $g(t)$ is discontinuous near $t_0$?

(b) Suppose that $y(t) = \ln|t^2-4|$ is a solution to the nonlinear differential equation $\frac{dy}{dt} = f(t, y)$. Are there any numbers $t_0$, $y_0$ such that you may be sure that at least one of $f(t, y)$ and $\partial f/\partial y$ is discontinuous near $t = t_0$, $y = y_0$?

**(Problem 29)** Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y - 2)(y + 1)^2$. By hand, sketch some typical solutions.

**(Problem 30)** Find the critical points and draw the phase line of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

**(Problem 31)** Find the critical points and draw the phase line of the differential equation $\frac{dy}{dx} = y^2(y - 2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

**(Problem 32)** I owe my bank a debt of $B$ dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

(a) Formulate a differential equation for the amount of money I owe.

(b) Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

**(Problem 33)** A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

(a) Write the differential equation for the amount of salt in the tank.

(b) Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

**(Problem 34)** A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where $v$ is her velocity in meters per second.

(a) Write a differential equation for her velocity. Assume her velocity is always negative.

(b) Find the (negative) critical points of this differential equation. Be sure to include units.

(c) What is the real-world meaning of these critical points?

**(Problem 35)** Suppose that the population $y$ of catfish in a certain lake, absent human intervention, satisfies the logistic equation $\frac{dy}{dt} = ry(1 - y/K)$, where $r$ and $K$ are constants and $t$ denotes time.

(a) Assuming that $r > 0$ and $K > 0$, find the critical points of this equation and classify them as to stability.

(b) Suppose that humans intervene by harvesting fish at a rate proportional to the fish population. Write a new differential equation for the fish population. Find the critical points of your new equation and classify them as to stability.

(c) Suppose that humans intervene by harvesting fish at a constant rate. Write a new differential equation for the fish population. Find the critical points of your new equation and classify them as to stability.

**(Problem 36)** The function $y_1(t) = e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2} - (1+2t)\frac{dy}{dt} + (t+1)y = 0$. Find the general solution to this differential equation.

**(Problem 37)** The function $y_1(t) = t$ is a solution to the differential equation $t^2\frac{d^2y}{dt^2} - (t^2+2t)\frac{dy}{dt} + (t+2)y = 0$. Find the general solution to this differential equation.

**(Problem 38)** The function $y_1(x) = x^3$ is a solution to the differential equation $x^2\frac{d^2y}{dx^2} + (x^2\tan x - 6x)\frac{dy}{dx} + (12 - 3x\tan x)y = 0$. Find the general solution to this differential equation on the interval $0 < x < \pi/2$.

**(Problem 39)** Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $16\frac{d^2y}{dt^2} - y = 0$, $y(0) = 1$, $y'(0) = 0$.
(b) $\frac{d^2y}{dt^2} + 49y = 0$, $y(\pi) = 3$, $y'(\pi) = 4$.
(c) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 0$, $y(2) = 0$, $y'(2) = 3$.

**(Problem 40)** Solve the following differential equations. Express your answers in terms of real functions.

(a) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$.
(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.

**(Problem 41)** Find the general solution to the following differential equations.

(a) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$.
(b) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$.
(c) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}$.
(d) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t)$.
(e) $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t)$.
(f) $\frac{d^2y}{dt^2} + 9y = 5\sin(3t)$.
(g) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$.
(h) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$.
(i) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$.
(j) $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$.
(k) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$.
(l) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.

**(Problem 42)** Solve the initial-value problem $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$.

**(Problem 43)** Find the general solution to the equation $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = \frac{e^t}{1+t^2}$.

**(Problem 44)** Solve the initial-value problem $\frac{d^2y}{dt^2} + 9y = 9\sec^2 3t$, $y(0) = 4$, $y'(0) = 6$, on the interval $-\pi/6 < t < \pi/6$.

**(Problem 45)** The general solution to the differential equation $t^2\frac{d^2y}{dt^2} - 2y = 0$, $t > 0$, is $y(t) = C_1t^2 + C_2t^{-1}$. Solve the initial-value problem $t^2\frac{d^2y}{dt^2} - 2y = 7t^3$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

**(Problem 46)** The general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 0$, $t > 0$, is $y(t) = C_1t^3 + C_2t^{-1}$. Find the general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

**(Problem 47)** An object weighing 5 lbs stretches a spring 4 inches. It is attached to a viscous damper with damping constant 16 lb · sec/ft. The object is pulled down an additional 2 inches and is released with initial velocity 3 ft/sec upwards.

Write the differential equation and initial conditions that describe the position of the object.

**(Problem 48)** A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 N · s/m and is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time $t$ seconds, the force is $3\cos(20t)$ N upwards.

Write the differential equation and initial conditions that describe the position of the object.

**(Problem 49)** A series circuit contains a resistor of 20 Ω, an inductor of 2 H, a capacitor of $5 \times 10^{-5}$ F, and a 12-volt battery. At time $t = 0$, there is no charge on the capacitor and no current; at this time the circuit is closed and current begins to flow. Write the differential equation and initial conditions that describe the charge on the capacitor.

**(Problem 50)** Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ (**not** the table on the front of the exam), find the Laplace transforms of the following functions.

(a) $f(t) = t$

(b) $f(t) = e^{-11t}$

(c) $f(t) = \begin{cases} 3, & 0 < t < 4, \\ 0, & 4 \le t \end{cases}.$

**(Problem 51)** Find the Laplace transforms of the following functions. You may use the table on the front of the exam.

(a) $f(t) = t^4 + 5t^2 + 4$

(b) $f(t) = (t + 2)^3$

(c) $f(t) = 9e^{4t+7}$

(d) $f(t) = -e^{3(t-2)}$

(e) $f(t) = (e^t + 1)^2$

(f) $f(t) = 8\sin(3t) - 4\cos(3t)$

(g) $f(t) = t^2 e^{5t}$

(h) $f(t) = 7e^{3t}\cos 4t$

(i) $f(t) = 4e^{-t}\sin 5t$

(j) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \ge 1, \end{cases}$

(k) $f(t) = \begin{cases} 0, & t < 1, \\ t - 2, & 1 \le t < 2, \\ 0, & t \ge 2 \end{cases}$

(l) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \ge 3 \end{cases}$

(m) $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \ge 3 \end{cases}$

(n) $f(t) = \begin{cases} 0, & t < \pi, \\ \sin t, & t \ge \pi \end{cases}$

**(Problem 52)** For each of the following problems, find $y$.

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$
(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$
(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$
(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$
(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$
(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$
(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$
(h) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$
(i) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$
(j) $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$
(k) $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$
(l) $\mathcal{L}\{y\} = \frac{s}{s^2-9}\mathcal{L}\{\sqrt{t}\}$

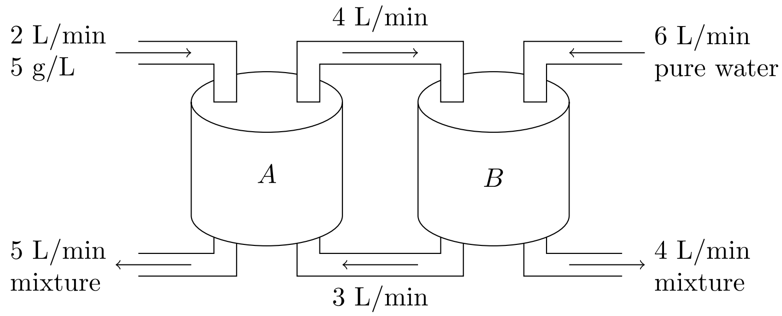
**(Problem 53)** Solve the following initial-value problems using the Laplace transform.

(a) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$
(b) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$
(c) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$
(d) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$
(e) $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$
(f) $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$
(g) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$
(h) $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$
(i) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$
(j) $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t \end{cases}$, $y(0) = 1$, $y'(0) = 0$
(k) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$

**(Problem 54)** Solve the following initial-value problems using the Laplace transform. Do the graphs of $y$, $\frac{dy}{dt}$, or $\frac{d^2y}{dt^2}$ have any corners or jump discontinuities? If so, where?

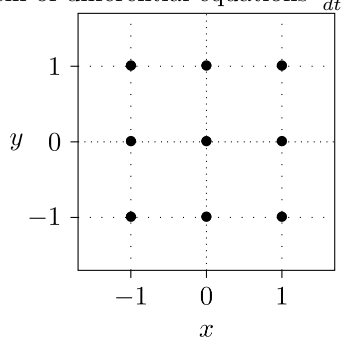
(a) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4u_2(t)$, $y(0) = 0$, $y'(0) = 1$.
(b) $\frac{d^2y}{dt^2} + 4y = -2\delta(t - 4\pi)$, $y(0) = 1/2$, $y'(0) = 0$
(c) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + u_2(t)$, $y(0) = 1$, $y'(0) = 0$

**(Problem 55)** Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



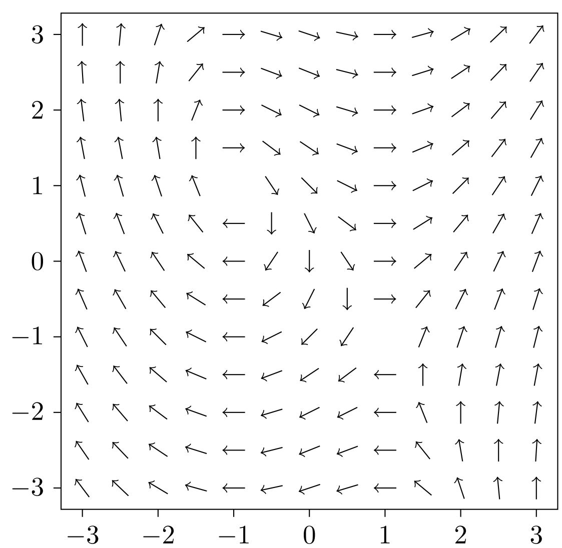

Write the differential equations and initial conditions that describe the amount of salt in each tank.

**(Problem 56)** Here is a grid. Draw a small direction field (with nine slanted segments) for the autonomous system of differential equations $\frac{dx}{dt} = x - y$, $\frac{dy}{dt} = y + x$




**(Problem 57)** Consider the system of differential equations $\frac{dx}{dt} = x + y$, $\frac{dy}{dt} = x^2 - 1$. Sketch, approximately, the solution to

$$\frac{dx}{dt} = x + y, \quad \frac{dy}{dt} = x^2 - 1, \qquad x(0) = 1, \quad y(0) = -2.$$



**(Problem 58)** Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

**(Problem 59)** Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

**(Problem 60)** Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

**(Problem 61)** Find the solution to the initial-value problem

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

**(Problem 62)** Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

**(Problem 63)** Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

**(Problem 64)** Find the solution to the initial-value problem

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

**(Problem 65)** Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

**(Problem 66)** Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y + e^{2t}\cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

**(Problem 67)** Find the solution to the initial-value problem

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

**(Problem 68)** Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

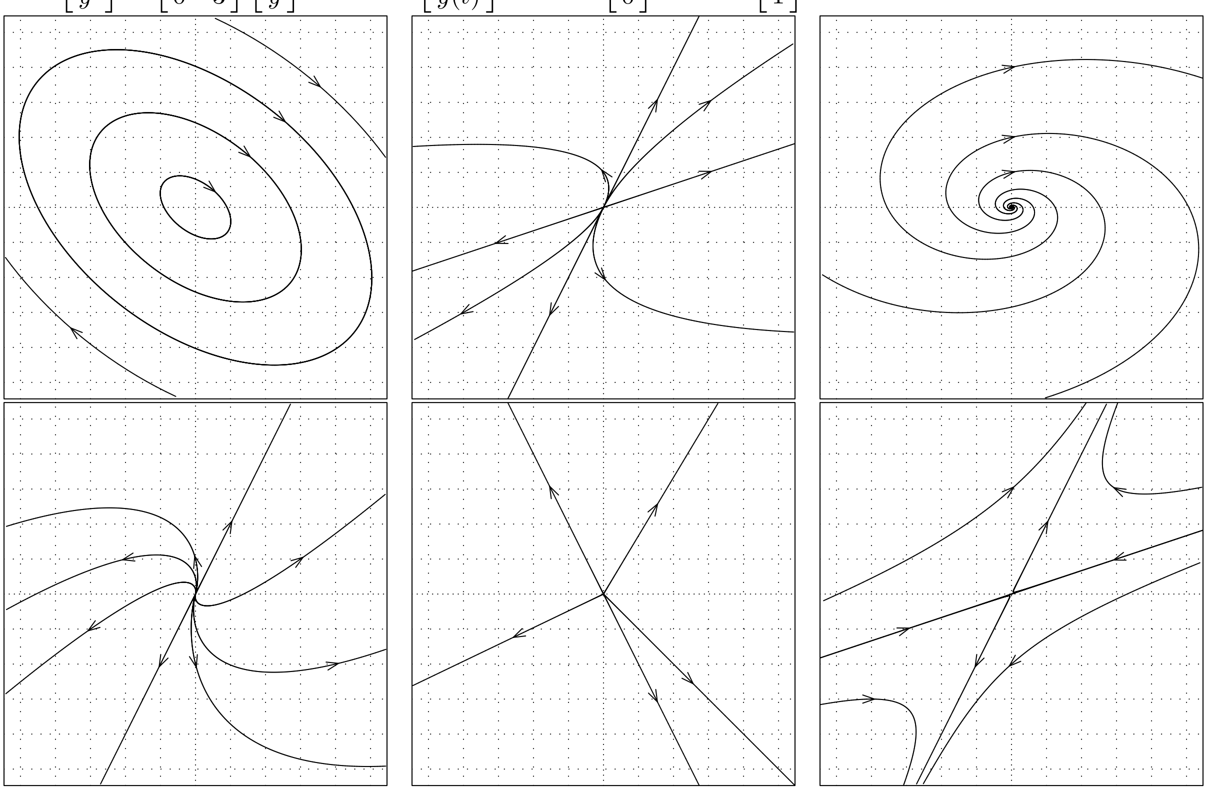
(b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2.6 & 1.8 \\ -1.2 & 1.6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4/3 & -1/6 \\ 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^t \begin{bmatrix} t+3 \\ 2t \end{bmatrix}$

(d) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 4.5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^t \begin{bmatrix} 3\sin 3t \\ 2\cos 3t \end{bmatrix} + C_2 e^t \begin{bmatrix} -3\cos 3t \\ 2\sin 3t \end{bmatrix}$

(e) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 5\sin 4t \\ 4\cos 4t - 2\sin 4t \end{bmatrix} + C_2 \begin{bmatrix} 5\cos 4t \\ -4\sin 4t - 2\cos 4t \end{bmatrix}$

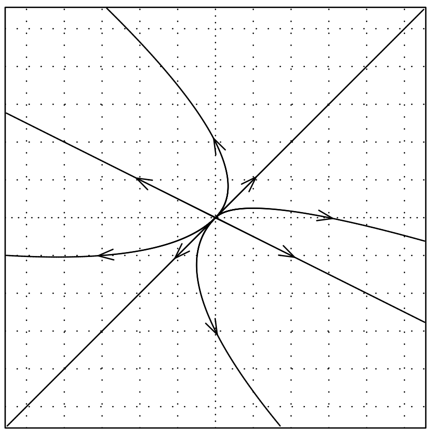
(f) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

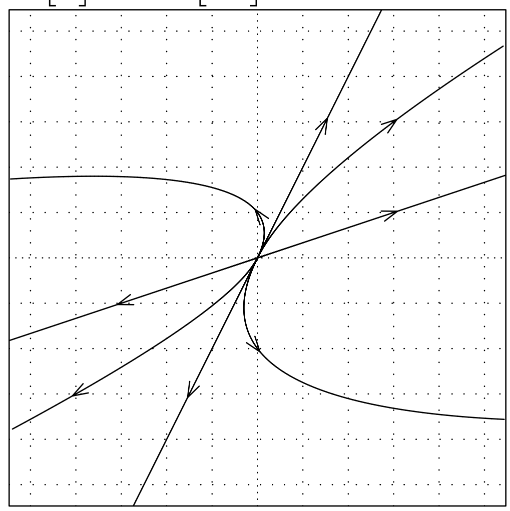


**(Problem 69)** Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

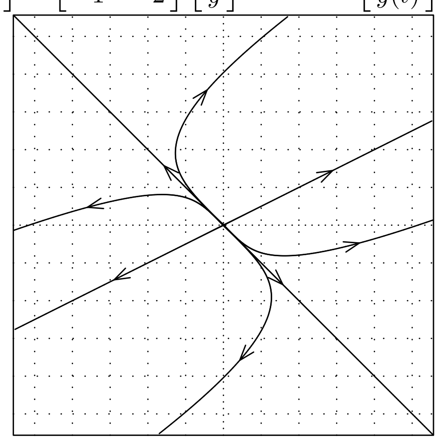


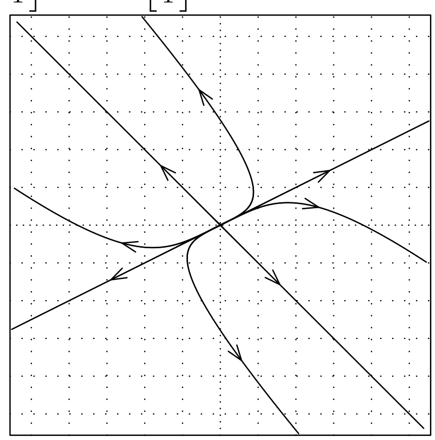


**(Problem 70)** Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

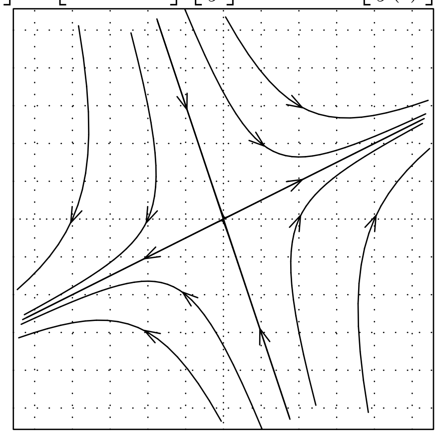


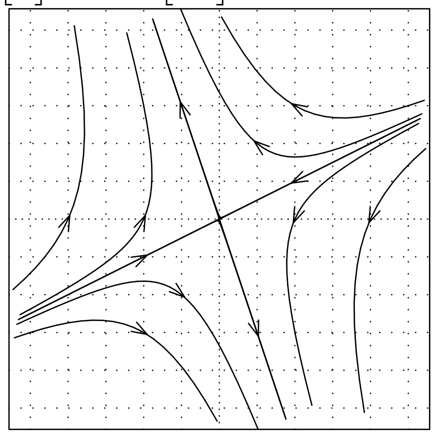


**(Problem 71)** Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 9 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-14t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

(b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 & -6 \\ -9 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{-7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{14t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

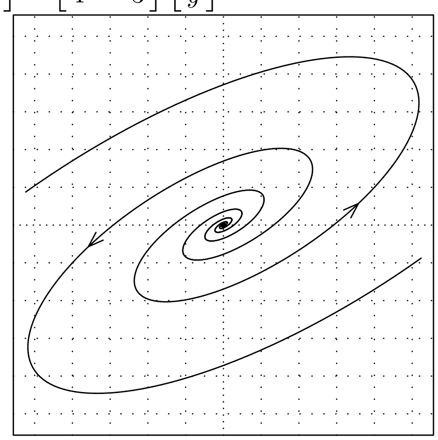


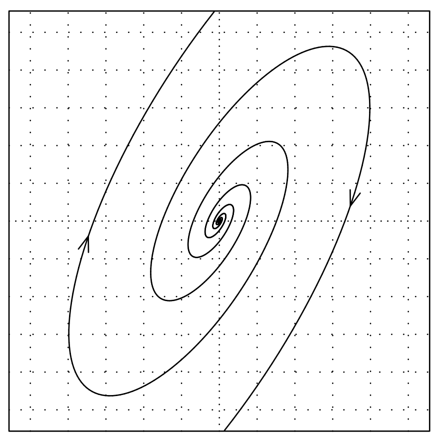


**(Problem 72)** Here are some phase planes. To which of the following systems do these phase planes correspond? How do you know?

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, eigenvalues $r = 1 \pm 4i$

(b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, eigenvalues $r = 1 \pm 4i$





**(Problem 73)** Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y - 15, \qquad \frac{dy}{dt} = -5x + 6y - 8, \qquad x(0) = 2, \quad y(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

**(Problem 74)** Find and classify all the critical points of the nonlinear system

$$\frac{dx}{dt} = 4x - 2x^2 - xy, \quad \frac{dy}{dt} = 3y - xy - y^2.$$

**(Problem 75)** Find and classify all the critical points of the nonlinear system

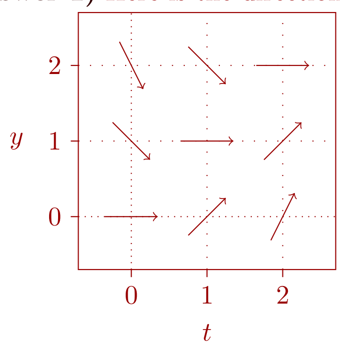
$$\frac{dx}{dt} = 1 - 2y, \quad \frac{dy}{dt} = x^2 - y^2.$$

**(Problem 76)** What can you say about the critical points of the nonlinear system

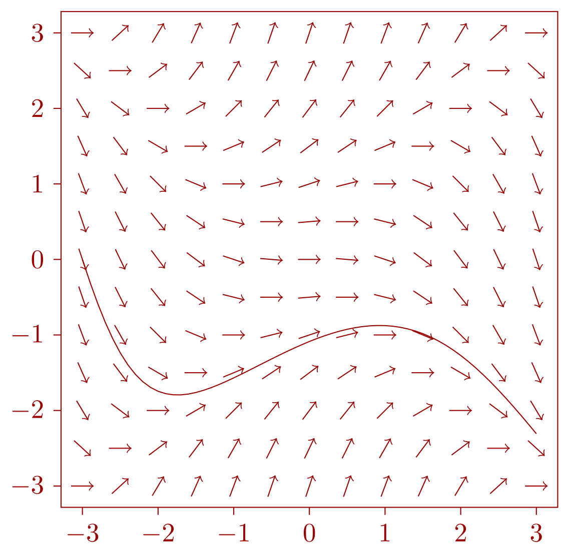
$$\frac{dx}{dt} = x - 2xy, \quad \frac{dy}{dt} = xy - y?$$

# Answer key

**(Answer 1)** Here is the direction field for the differential equation $\frac{dy}{dt} = t - y$.




**(Answer 2)**




**(Answer 3)**

If $y(0) > 4$, then $y \to \infty$ as $t \to \infty$.
If $y(0) = 4$, then $y(t) = 4$ for all $t$.
If $0 \le y(0) < 4$, then $y \to 0$ as $t \to \infty$. In particular, if $y(0) = 0$, then $y(t) = 0$ for all $t$.
If $y(0) < 0$, then $y \to -\infty$ as $t \to \infty$.

**(Answer 4)** Independent variable: $t$ = time (in minutes).
Dependent variable: $Q$ = amount of dissolved salt (in kilograms).
Initial condition: $Q(0) = 3$.
Differential equation: $\frac{dQ}{dt} = 0.01 - \frac{3Q}{600-t}$.

**(Answer 5)** Independent variable: $t$ = time (in seconds).
Dependent variable: $T$ = Temperature of the cup (in degrees Celsius)
Initial condition: $T(0) = 95$.
Differential equation: $\frac{dT}{dt} = -\alpha(T - 25)$, where $\alpha$ is a parameter (constant of proportionality) with units of 1/seconds.

**(Answer 6)** Independent variable: $t$ = time (in years).
Dependent variable: $P$ = Number of birds on the island.
Initial condition: $P(0) = 5$.
Differential equation: $\frac{dP}{dt} = \alpha P - \beta P^2$, where $\alpha$ and $\beta$ are parameters.

**(Answer 7)** Independent variable: $t$ = time (in years).
Dependent variable: $B$ = balance of my loan (in dollars).
Initial condition: $B(0) = 300{,}000$.
Differential equation: $\frac{dB}{dt} = 0.04B - 12 \cdot 1600$

**(Answer 8)** Independent variable: $t$ = time (in minutes).
Dependent variables:
$h$ = depth of water in the hole (in centimeters)
$V$ = volume of water in the hole (in cubic centimeters); notice that $V = \frac{1}{3}\pi(5h)^2 h$
Initial condition: $V(0) = 0$.
Differential equation: $\frac{dV}{dt} = 1 - \alpha\pi(5h)^2 = 1 - 25\alpha\pi(3V/25\pi)^{2/3}$, where $\alpha$ is a proportionality constant with units of cm/s.

**(Answer 9)** Independent variable: $t$ = time (in seconds).
Dependent variable: $T$ = object's temperature (in kelvins)
Parameter: $\sigma$ = proportionality constant (in 1/(seconds·kelvin$^3$))
Initial condition: $T(0) = 400$.
Differential equation: $\frac{dT}{dt} = -\sigma T^4$.

**(Answer 10)** Independent variable: $t$ = time (in seconds).
Dependent variable: $v$ = velocity of the ball (in meters/second, where a positive velocity denotes upward motion).
Parameters: $\alpha$ = proportionality constant of the drag force (in newton·seconds/meter)
$m$ = mass of the ball (in kilograms)
Initial condition: $v(0) = 10$.
Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/m)v$.

**(Answer 11)** Independent variable: $t$ = time (in seconds).
Dependent variable: $v$ = velocity of the ball (in meters/second, where a positive velocity denotes upward motion).
Parameter: $\alpha$ = proportionality constant of the drag force (in newton·seconds/meter).
Initial condition: $v(0) = 20$.
Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/0.3)v|v|$, or $\frac{dv}{dt} = \begin{cases} -9.8 - (\alpha/0.3)v^2, & v \geq 0, \\ -9.8 + (\alpha/0.3)v^2, & v < 0. \end{cases}$

**(Answer 12)** No, $y = e^t$ is not a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

**(Answer 13)** Yes, $y = e^{2t}$ is a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

**(Answer 14)** No.

**(Answer 15)** No.

**(Answer 16)** Yes.

**(Answer 17)** No.

**(Answer 18)**

(a) We expect a unique solution.
(b) We do not expect any solutions.
(c) We do not expect any solutions.
(d) We expect a unique solution.
(e) We expect an infinite family of solutions.

**(Answer 19)**

(a) $\frac{dy}{dt} = \frac{t+\cos t}{\sin y - y}$ is separable and has solution $\frac{1}{2}y^2 + \cos y = -\frac{1}{2}t^2 - \sin t + C$.
(b) $(t^2+1)\frac{dy}{dt} = ty - t^2 - 1$ is linear and has solution $y = -\sqrt{t^2+1}\ln(t+\sqrt{t^2+1}) + C\sqrt{t^2+1}$.
(c) $t^2\frac{dy}{dt} = y^2 + t^2 - ty$ is homogeneous and has solution $y = \frac{t}{C-\ln|t|} + t$.
(d) Let $v = y^{-2}$. Then $\frac{dv}{dt} = 2v\tan 2t + 2\cos 2t$, so $v = \frac{1}{2}\sin(2t) + t\sec(2t) + C\sec(2t)$ and $y = \frac{1}{\sqrt{\frac{1}{2}\sin(2t)+t\sec(2t)+C\sec(2t)}}$.
(e) $(t+y)\frac{dy}{dt} = 5y - 3t$ is homogeneous and has solution $(y-3t)^2 = C(y-t)$.
(f) $\frac{dy}{dt} = -y^3\cos(2t)$ is separable and has solution $y = \pm\frac{1}{\sqrt{\sin(2t)+C}}$.
(g) $4ty\frac{dy}{dt} = 3y^2 - 2t^2$ is homogeneous and has solution $2\ln(y^2/t^2+2) = -\ln|t| + C$.
(h) Let $v = y^6$. Then $t\frac{dv}{dt} = 18v - 6t^2$, so $v = Ct^{18} + \frac{3}{8}t^2$ and $y = \sqrt[6]{Ct^{18}+\frac{3}{8}t^2}$.
(i) Make the substitution $v = y^{-7}$. Then $\frac{dv}{dt} = -56v + 7$, so $v = Ce^{-56t} + \frac{1}{8}$ and $y = \frac{1}{\sqrt[7]{Ce^{-56t}+1/8}}$.
(j) $t\frac{dy}{dt} = -\cos t - 3y$ is linear and has solution $y = \frac{1}{t}\sin t + \frac{2}{t^2}\cos t - \frac{2}{t^3}\sin t + \frac{C}{t^3}$.

**(Answer 20)**

(a) If $2ty\frac{dy}{dt} = 4t^2 - y^2$, $y(1) = 3$, then $y = \sqrt{\frac{4}{3}t^2 + \frac{23}{3t}}$.
(b) If $t\frac{dy}{dt} = -1 - y^2$, $y(1) = 1$, then $y = \tan(\pi/4 - \ln t)$.
(c) If $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$, then $y = \sqrt[3]{340e^{6t}+3}$.
(d) If $\frac{dy}{dt} = -3y - \sin t\, e^{-3t}$, $y(0) = 2$, then $y = e^{-3t}\cos t + e^{-3t}$.

**(Answer 21)** $y = \sqrt{(t-5)^2 - 16} = \sqrt{(t-1)(t-9)} = \sqrt{t^2 - 10t + 9}$. The solution is valid for all $t < 1$.

**(Answer 22)** $y = \frac{1}{4-t}$. The solution is valid for all $t < 4$.

**(Answer 23)** We have that $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ and also $\frac{dv}{dt} = \frac{d^2x}{dt^2}$. Thus $v\frac{dv}{dx} = 18x^3$, $v(1) = 3$. Solving, we see that $v = 3x^2$.

But then $\frac{dx}{dt} = 3x^2$, $x(0) = 1$, and so $x = \frac{1}{1-3t}$.

**(Answer 24)** We have that $\frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}$ and also $\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$. Thus $\omega\frac{d\omega}{d\theta} = -\frac{mg}{\ell}\sin\theta$. Solving, we see that $\frac{1}{2}\omega^2 = \frac{mg}{\ell}\cos\theta + C_1$, or $\omega = \pm\sqrt{\frac{2mg}{\ell}\cos\theta + C_1}$.

We are left with the differential equation $\frac{d\theta}{dt} = \pm\sqrt{\frac{2mg}{\ell}\cos\theta + C_1}$. Solving, we see that

$$\int \sqrt{\frac{2mg}{\ell}\cos\theta + C_1}\,d\theta = \pm t + C_2.$$

(Unfortunately, we cannot evaluate that integral and find a simple formula for $\theta$ in terms of $t$.)

**(Answer 25)**

(a) We are not guaranteed a unique solution to the problem $x^2 y' = y^2$, $y(0) = 1$.

(b) We are guaranteed a unique solution to the problem $x^2 y' = y^2$, $y(1) = 0$.

(c) We are not guaranteed a unique solution to the problem $y' = \sqrt[3]{y}$, $y(8) = 0$.

(d) We are guaranteed a unique solution to the problem $y' = \sqrt[3]{y}$, $y(8) = 1$.

**(Answer 26)**

(a) The interval $\pi/2 < t < 3\pi/2$.

(b) The interval $t < 0$.

(c) The interval $0 < t < 2$.

(d) We are not guaranteed a solution on any open interval. (We are guaranteed a unique solution on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, but $\varepsilon$ could be arbitrarily small.)

**(Answer 27)**

(a) The solutions cross at $(-2, 0)$ and $(2, 16)$, and so we may be sure that at least one of $p(t)$ and $g(t)$ is discontinuous near $t = -2$ and $t = 2$.

That is, there are two solutions to the initial value problem $\frac{dy}{dt} + p(t)y = g(t)$, $y(-2) = 0$, and so at least one of $p(t)$ and $g(t)$ is discontinuous near $t = -2$. Similarly, there are two solutions to the initial value problem $\frac{dy}{dt} + p(t)y = g(t)$, $y(2) = 16$, and so at least one of $p(t)$ and $g(t)$ is discontinuous near $t = 2$.

(b) The solutions cross at $(0, 1)$, and so we may be sure that either $f(t, y)$ or $\partial f/\partial y$ is discontinuos near $(t, y) = (0, 1)$.

(c) The solutions cross and are tangent at $t = 2$, and so at least one of $p(t)$, $q(t)$, $g(t)$ are discontinuous near $t = 2$.

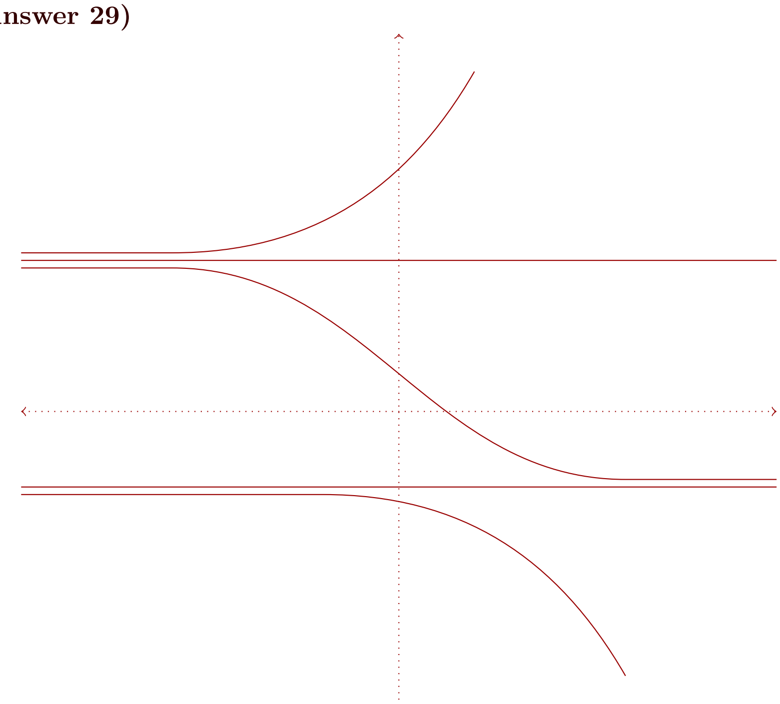
Put another way, there are two solutions to the initial value problem $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\,y = g(t)$, $y(2) = 16$, $y'(2) = 12$, and so at least one of $p(t)$, $q(t)$, $g(t)$ must be discontinuous near $t = 2$.

**(Answer 28)**

(a) Yes; $p(t)$ or $g(t)$ must be discontinuous near $t = 2$ and $t = -2$.

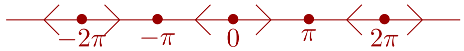
(b) No, we cannot conclude that $f(t, y)$ or $\partial f/\partial y$ is discontinuous anywhere.

**(Answer 29)**



**(Answer 30)**

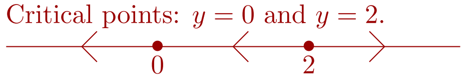
Critical points: $y = k\pi$ for any integer $k$.




If $k$ is even then $y = k\pi$ is unstable.
If $k$ is odd then $y = k\pi$ is stable.

**(Answer 31)**

Critical points: $y = 0$ and $y = 2$.




$y = 0$ is semistable. $y = 2$ is unstable.

**(Answer 32)**

(a) $\frac{dB}{dt} = 0.05B - 19200$, where $t$ denotes time in years.

(b) The critical point is $B = \$384{,}000$. It is unstable. If my initial debt is less than \$384,000, then I will eventually pay it off (have a debt of zero dollars), but if my initial debt is greater than \$384,000, my debt will grow exponentially. The critical point $B = 384{,}000$ corresponds to the balance that will allow me to make interest-only payments on my debt.

**(Answer 33)**

(a) $\frac{dQ}{dt} = 10 - Q/300$, where $Q$ denotes the amount of salt in grams and $t$ denotes time in minutes.

(b) The critical point is $Q = 3000$. It is stable. No matter the initial amount of salt in the tank, as time passes the amount of salt will approach 3000 g or 3 kg.

**(Answer 34)**

(a) If $v \leq 0$ then $70\frac{dv}{dt} = -70 * 9.8 + 2v^2$. (If $v > 0$ then $70\frac{dv}{dt} = -70 * 9.8 - 2v^2$.)

(b) $v = -\sqrt{343}$ meters/second.

(c) As $t \to \infty$, her velocity will approach the stable critical point of $v = -\sqrt{343}$ meters/second.

**(Answer 35)**

(a) The critical points are $y = 0$ (unstable) and $y = K$ (stable).

(b) $\frac{dy}{dt} = ry(1-y/K) - Ey$, where $E$ is a proportionality constant. The critical points are $y = 0$ (unstable) and $y = K - KE/r$ (stable).

(c) $\frac{dy}{dt} = ry(1 - y/K) - h$, where $h$ is the harvesting rate. If $h < Kr/4$, then the critical points are $y = K/2 - \sqrt{(K/2)^2 - Kh/r}$ (unstable) and $y = K/2 + \sqrt{(K/2)^2 - Kh/r}$ (stable). Notice that both critical points are positive (i.e., correspond to the physically meaningful case of at least zero fish in the lake.) If $h = Kr/4$, then there is one critical point at $y = K/2$; it is semistable. Finally, if $h > Kr/4$, then there are no critical points and the fish population will decrease until it goes extinct.

**(Answer 36)** $y(t) = C_1\, e^t + C_2\, t^2 e^t$.

**(Answer 37)** $y(t) = C_1\, t + C_2\, te^t$.

**(Answer 38)** $y(x) = C_1\, x^3 + C_2\, x^3 \sin x$.

**(Answer 39)**

(a) If $16\frac{d^2y}{dt^2} - y = 0$, $y(0) = 1$, $y'(0) = 0$, then $y(t) = \frac{1}{2}e^{t/4} + \frac{1}{2}e^{-t/4}$.

(b) If $\frac{d^2y}{dt^2} + 49y = 0$, $y(\pi) = 3$, $y'(\pi) = 4$, then $y(t) = 3\cos(7t) + \frac{4}{7}\sin(7t)$.

(c) If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 0$, $y(2) = 0$, $y'(2) = 3$, then $y(t) = \frac{3}{7}e^{2t-4} - \frac{3}{7}e^{-5t+10}$.

**(Answer 40)**

(a) If $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$, then $y = C_1 e^{-6t}\cos(7t) + C_2 e^{-6t}\sin(7t)$.

(b) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$, then $y = C_1 e^{(-2+\sqrt{2})t} + C_2 e^{(-2-\sqrt{2})t}$.

**(Answer 41)**

(a) The general solution to $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 0$ is $y_g = C_1 + C_2 e^{4t}$. To solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$ we make the guess $y_p = Ae^{3t}$. The solution is $y = C_1 + C_2 e^{4t} - (4/3)e^{3t}$.

(b) The general solution to $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 0$ is $y_g = C_1 e^{2t} + C_2 e^{-5t}$. To solve $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$ we make the guess $y_p = Ate^{-5t}$. The solution is $y = C_1 e^{2t} + C_2 e^{-5t} - (1/7)te^{-5t}$.

(c) The general solution to $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$ is $y_g = C_1 e^{-5t} + C_2 te^{-5t}$. To solve $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}$ we make the guess $y_p = At^2 e^{-5t}$. The solution is $y = C_1 e^{-5t} + C_2 te^{-5t} + (3/2)t^2 e^{-5t}$.

(d) The general solution to $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$ is $y_g = C_1 e^{-2t} + C_2 e^{-3t}$. To solve $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t)$, we make the guess $y_p = A\cos(2t) + B\sin(2t)$.

(e) The general solution to $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$ is $y_g = C_1 e^{-3t} + C_2 te^{-3t}$. To solve $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t)$, we make the guess $y_p = A\cos(4t) + B\sin(4t)$.

(f) The general solution to $\frac{d^2y}{dt^2} + 9y = 0$ is $y_g = C_1\cos(3t) + C_2\sin(3t)$. To solve $\frac{d^2y}{dt^2} + 9y = 5\sin(3t)$, we make the guess $y_p = C_1 t\cos(3t) + C_2 t\sin(3t)$. The solution is $y = C_1\cos(3t) + C_2\sin(3t) - \frac{5}{6}t\cos(3t)$.

(g) The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ is $y_g = C_1 e^{-t} + C_2 te^{-t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$, we make the guess $y_p = At^2 + Bt + C$. The solution is $y = C_1 e^{-t} + C_2 te^{-t} + 2t^2 - 8t + 12$.

(h) The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 0$ is $y_g = C_1 + C_2 e^{-2t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$, we make the guess $y_p = At^2 + Bt$. The solution is $y = C_1 + C_2 e^{-2t} + \frac{3}{4}t^2 - \frac{3}{4}t$.

(i) The general solution to $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 0$ is $y_g = C_1 e^{3t} + C_2 e^{4t}$. To solve $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$, we make the guess $y_p = At + B + C\cos(2t) + D\sin(2t)$.

(j) The general solution to $\frac{d^2y}{dt^2} - 9y = 0$ is $y_g = C_1 e^{3t} + C_2 e^{-3t}$. To solve $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$, we make the guess $y_p = Ae^t + Be^{-t} + Ct + D$.

(k) The general solution to $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$ is $y_g = C_1 e^{2t} + C_2 te^{2t}$. To solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$, we make the guess $y_p = At^2 e^{2t} + B\cos t + C\sin t$.

(l) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$ then $y = C_1 e^{(-2-\sqrt{2})t} + C_2 e^{(-2+\sqrt{2})t}$.

**(Answer 42)** If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$, then $y(t) = -\frac{4}{3}e^{3t} + \frac{7}{4}e^{4t} + \frac{7}{12}$.

**(Answer 43)** $y(t) = c_1\, e^t + c_2\, t\, e^t - \frac{1}{2}e^t\, \ln(1+t^2) + t\, e^t \arctan t$.

**(Answer 44)** $y(t) = 5\cos(3t) + 2\sin(3t) + (\sin(3t))\ln(\tan(3t) + \sec(3t)) - 1$.

**(Answer 45)** $y(t) = \frac{7}{4}t^3 - \frac{4}{3}t^2 + \frac{7}{12}t^{-1}$.

**(Answer 46)** $y(t) = -\frac{3}{2}t^{-1}\ln t + C_1 t^3 + C_2 t^{-1}$.

**(Answer 47)** Let $t$ denote time (in seconds) and let $u$ denote the object's displacement above equilibrium (in feet).

Then

$$\frac{5}{32}\frac{d^2u}{dt^2} + 16\frac{du}{dt} + 15u = 0, \qquad u(0) = -\frac{1}{6}, \quad u'(0) = 3.$$

**(Answer 48)** Let $t$ denote time (in seconds) and let $u$ denote the object's displacement above equilibrium (in meters).

Then

$$3\frac{d^2u}{dt^2} + 27\frac{du}{dt} + 588u = 3\cos(20t), \qquad u(0) = 0, \quad u'(0) = 0.$$

**(Answer 49)** Let $t$ denote time (in seconds), let $Q$ denote the charge on the capacitor, and let $I$ denote the current through the circuit.

Then

$$12 = 2\frac{dI}{dt} + 20I + 2000000Q, \quad \frac{dQ}{dt} = 0, \qquad Q(0) = 0, \quad I(0) = 0.$$

**(Answer 50)**

(a) $\mathcal{L}\{t\} = \frac{1}{s^2}$.

(b) $\mathcal{L}\{e^{-11t}\} = \frac{1}{s+11}$.

(c) $\mathcal{L}\{f(t)\} = \frac{3-3e^{-4s}}{s}$.

**(Answer 51)**

(a) $\mathcal{L}\{t^4 + 5t^2 + 4\} = \frac{96}{s^5} + \frac{10}{s^3} + \frac{4}{s}$.

(b) $\mathcal{L}\{(t+2)^3\} = \mathcal{L}\{t^3 + 6t^2 + 12t + 8\} = \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{8}{s}$.

(c) $\mathcal{L}\{9e^{4t+7}\} = \mathcal{L}\{9e^7e^{4t}\} = \frac{9e^7}{s-4}$.

(d) $\mathcal{L}\{-e^{3(t-2)}\} = \mathcal{L}\{-e^{-6}e^{3t}\} = -\frac{e^{-6}}{s-3}$.

(e) $\mathcal{L}\{(e^t+1)^2\} = \mathcal{L}\{e^{2t} + 2e^t + 1\} = \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s}$.

(f) $\mathcal{L}\{8\sin(3t) - 4\cos(3t)\} = \frac{24}{s^2+9} - \frac{4s}{s^2+9}$.

(g) $\mathcal{L}\{t^2e^{5t}\} = \frac{2}{(s-5)^3}$.

(h) $\mathcal{L}\{7e^{3t}\cos 4t\} = \frac{7s-21}{(s-3)^2+16}$.

(i) $\mathcal{L}\{4e^{-t}\sin 5t\} = \frac{20}{(s+1)^2+25}$.

(j) If $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$ then $\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{1}{s} + \frac{2}{s^3}\right)$.

(k) If $f(t) = \begin{cases} 0, & t < 1, \\ t-2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2}$.

(l) If $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{5}{s-2} - \frac{5e^6e^{-3s}}{s-2}$.

(m) If $f(t) = \begin{cases} 7t^2e^{-t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{14}{(s+1)^3} - \frac{14e^{-3s-3}}{(s+1)^3} - \frac{42e^{-3s-3}}{(s+1)^2} - \frac{63e^{-3s-3}}{s+1}$.

(n) If $f(t) = \begin{cases} 0, & t < \pi, \\ \sin t, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s}\frac{1}{s^2+1}$.

**(Answer 52)**

(a) If $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$, then $y = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$

(b) If $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$, then $y = \frac{5}{2}t^2 - \frac{7}{6}t^3$

(c) If $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$, then $y = \frac{1}{2}t^2e^{-t} + \frac{1}{6}t^3e^{-t}$

(d) If $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$, then $y = (1/4)e^{2t} + (7/4)e^{-2t}$

(e) If $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$, then $y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{1}{9}t\,e^{3t}$

(f) If $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$, then $y = \frac{1}{2} - \frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t)$

(g) If $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$, then $y = \frac{1}{8}\cos t - \frac{1}{8}\cos 3t$

(h) If $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$, then $y = 2\,u_2(t)\,e^{t-2}\,\cos(t-2) + u_2(t)\,e^{t-2}\,\sin(t-2)$

($i$) If $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$, then $y = \frac{1}{2}u_1(t)\,e^{3(t-1)} + \frac{1}{2}u_1(t)\,e^{t-1}$
($j$) If $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$, then $y = \int_0^t \frac{1}{3}\sin 3r\cos(3t-3r)\,dr$
($k$) If $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$, then $y = e^{-2t}\int_0^t \sin(2r)\sin(2t-2r)\,dr$
($l$) If $\mathcal{L}\{y\} = \frac{s}{s^2-9}\mathcal{L}\{\sqrt{t}\}$, then $y = \int_0^t \frac{e^{3r}+e^{-3r}}{2}\sqrt{t-r}\,dr$.

**(Answer 53)**

($a$) If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$, then $y(t) = e^{2t} - te^{2t}$.
($b$) If $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$, then $y(t) = \frac{1}{5}(e^{-t} - e^t\cos t + 7e^t\sin t)$.
($c$) If $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$, then $y(t) = 4e^{3t} - 2e^{4t} - te^{3t}$.
($d$) If $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$, then $y(t) = \frac{1}{5}\cos 2t + \cos 3t + \frac{5}{3}\sin 3t$.
($e$) If $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t r\sin 3r\sin(3t-3r)\,dr$.
($f$) If $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t \sin 3r\sin(3t-3r)\,dr = \frac{1}{6}\sin 3t - \frac{1}{2}t\cos 3t$.
($g$) If $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$, then $y(t) = 7e^{-2t} - 5e^{-3t} + \int_0^t (e^{-2r} - e^{-3r})\sqrt{t-r+1}\,dr$.

($h$) If $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = (1/6)(1 - u_{2\pi}(t))(2\sin t - \sin 2t).$$

($i$) If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - u_{10}(t)\left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right].$$

($j$) If $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t \end{cases}$, $y(0) = 1$, $y'(0) = 0$, then

$$\begin{aligned} y(t) = {} & e^{-t/2}\cos t + \frac{1}{2}e^{-t/2}\sin t \\ & - u_\pi(t)\left(\frac{16}{17}\cos(t-\pi) - \frac{4}{17}\sin(t-\pi)\right) \\ & + u_\pi(t)\left(\frac{4}{17}e^{-(t-\pi)/2}\sin(t-\pi) + \frac{16}{17}e^{-(t-\pi)/2}\cos(t-\pi)\right). \end{aligned}$$

($k$) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$, then $y(t) = 2e^{-2t} + 5te^{-2t} + \frac{3}{4}u_2(t) - \frac{3}{4}e^{-2t+4}u_2(t) - \frac{3}{2}(t-2)e^{-2t+4}u_2(t)$.

**(Answer 54)**

(a) If $6\frac{d^2y}{dt^2}+5\frac{dy}{dt}+y=4u_2(t)$, $y(0)=0$, $y'(0)=1$, then

$$y=6e^{-t/3}-6e^{-t/2}+4u_2(t)-12u_2(t)e^{-(t-2)/3}+8u_2(t)e^{-(t-2)/2}.$$

The graph of $y'(t)$ has a corner at $t=2$, and the graph of $y''(t)$ has a jump at $t=2$.

(b) If $\frac{d^2y}{dt^2}+4y=\delta(t-4\pi)$, $y(0)=1/2$, $y'(0)=0$, then

$$y=\frac{1}{2}\cos(2t)-u_{4\pi}(t)\sin(2t).$$

The graph of $y(t)$ has a corner at $t=4\pi$, and graph of $y'(t)$ has a jump at $t=4\pi$. ($y''(t)$ is hard to graph, because the impulse function $\delta(t-4\pi)$ is part of $y''(t)$.)

(c) If $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+3y=2\delta(t-1)+u_2(t)$, $y(0)=1$, $y'(0)=0$, then

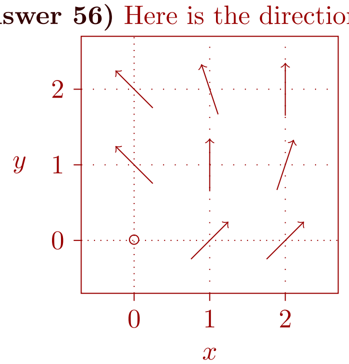
$$y=\frac{3}{2}e^{-t}-\frac{1}{2}e^{-3t}+\frac{1}{2}u_1(t)e^{-t+1}-\frac{1}{2}u_1(t)e^{-3t+3}+\frac{1}{3}u_2(t)-\frac{1}{2}e^{-t+2}u_2(t)+\frac{1}{6}u_2(t)e^{-3t+6}.$$

The graph of $y(t)$ has a corner at $t=1$. The graph of $y'(t)$ has a corner at $t=2$, and a jump at $t=1$. $y''(t)$ has an impulse at $t=1$, and a jump at $t=2$.

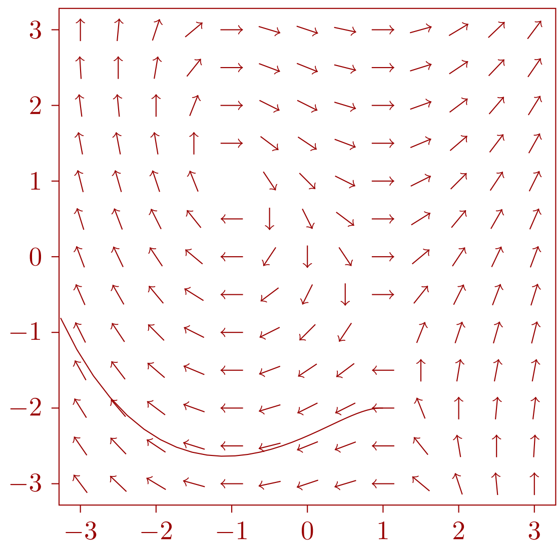
**(Answer 55)** Let $t$ denote time (in minutes).
Let $x$ denote the amount of salt (in grams) in tank A.
Let $y$ denote the amount of salt (in grams) in tank B.
Then $x(0)=3000$ and $y(0)=2000$.
If $t<50$, then

$$\frac{dx}{dt}=-\frac{9x}{200-4t}+\frac{3y}{300+3t},\qquad \frac{dy}{dt}=\frac{4x}{200-4t}-\frac{7y}{300+3t}.$$

**(Answer 56)** Here is the direction field for the differential equation $\frac{dx}{dt}=x-y$, $\frac{dy}{dt}=y+x$



**(Answer 57)**



**(Answer 58)**

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{2t} \begin{bmatrix} 4t-5 \\ -2t+3 \end{bmatrix}.$$

**(Answer 59)**

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{2t} \begin{bmatrix} -3\cos t + 16\sin t \\ 2\cos t + 7\sin t \end{bmatrix}.$$

**(Answer 60)**

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{3}{20} e^{-20t} \begin{bmatrix} 1 \\ 6 \end{bmatrix} + \frac{1}{20} e^{10t} \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

**(Answer 61)**

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{-5t} \begin{bmatrix} 5\cos 12t + \sin 12t \\ 2\cos 12t + 2\sin 12t \end{bmatrix}.$$

**(Answer 62)**

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos 3t - 11\sin 3t \\ -3\cos 3t - (23/3)\sin 3t \end{bmatrix}.$$

**(Answer 63)**

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{-5t} \begin{bmatrix} 15t+1 \\ (15/4)t+4 \end{bmatrix}.$$

**(Answer 64)**

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{-t} \begin{bmatrix} 4/5 \\ 2/5 \end{bmatrix} + e^{-6t} \begin{bmatrix} 1/5 \\ -2/5 \end{bmatrix}.$$

**(Answer 65)** If

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 0, \quad y(0) = 0$$

then

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{2t} \begin{bmatrix} (38/45)t^{10} + (1/9)t^9 \\ -(29/45)t^{10} \end{bmatrix}.$$

**(Answer 66)** If

$$\frac{dx}{dt} = 5y + e^{2t}\cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 0, \quad y(0) = 0$$

then

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{2t} \begin{bmatrix} -t\sin t + (1/2)t\cos t + (1/2)\sin t \\ -(1/2)t\sin t \end{bmatrix}.$$

**(Answer 67)** If

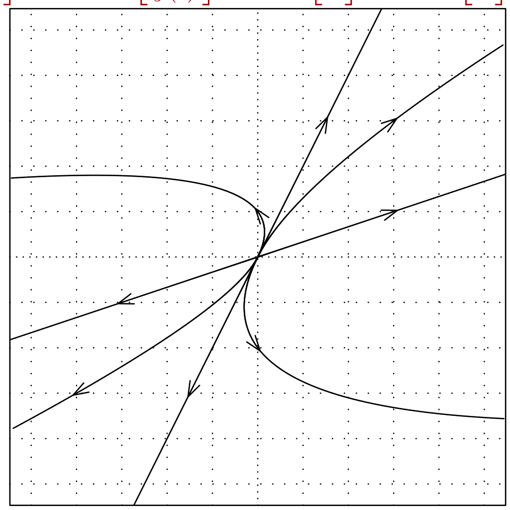
$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0$$

then

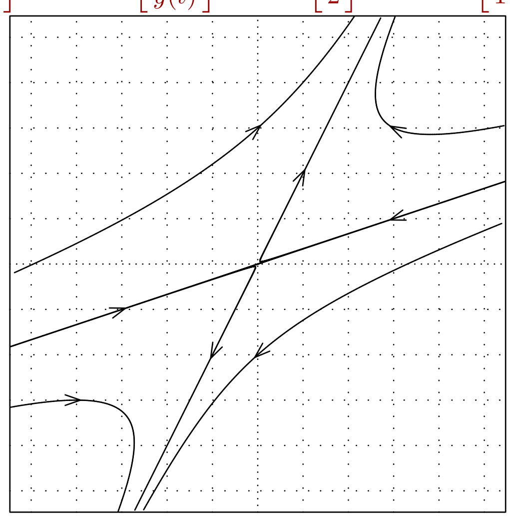
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = e^{-t} \begin{bmatrix} 4 - 4\cos(e^t) \\ 4\cos(e^t) - 4 \end{bmatrix} + e^{-2t} \begin{bmatrix} 3\sin(e^t) - 3e^t\cos(e^t) - 3\sin 1 + 3\cos 1 \\ 4e^t\cos(e^t) - 4\sin(e^t) - 4\cos 1 + 4\sin 1 \end{bmatrix}.$$

**(Answer 68)**

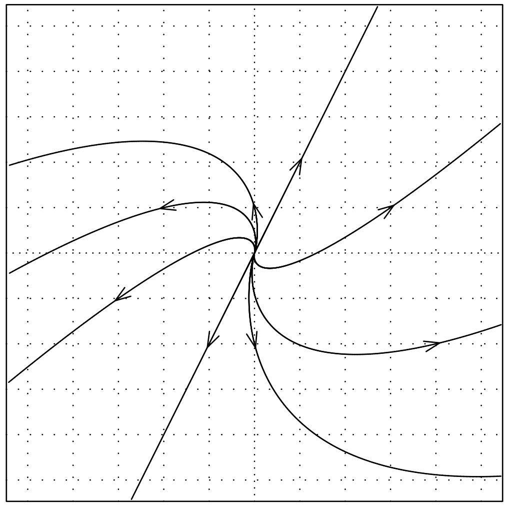
(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$



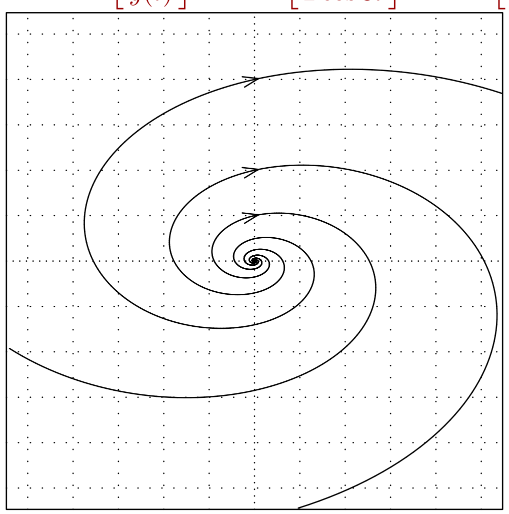
(b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2.6 & 1.8 \\ -1.2 & 1.6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$



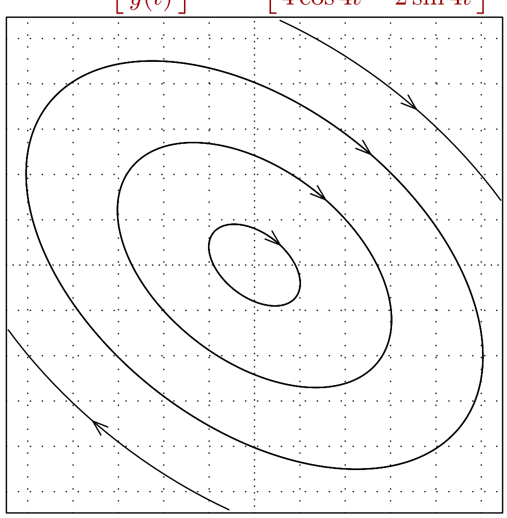
(c) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4/3 & -1/6 \\ 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} t+3 \\ 2t \end{bmatrix}$



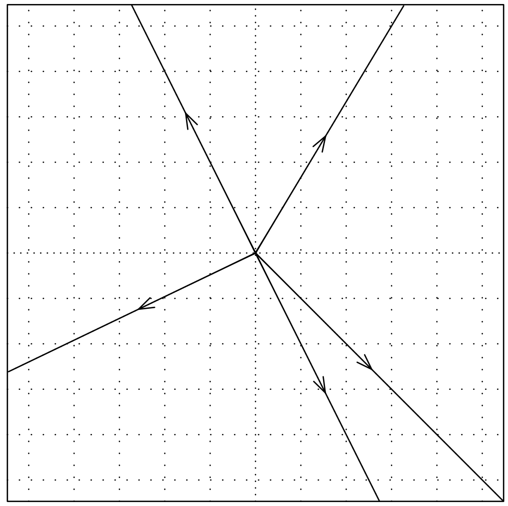
(d) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 4.5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^t \begin{bmatrix} 3\sin 3t \\ 2\cos 3t \end{bmatrix} + C_2 e^t \begin{bmatrix} -3\cos 3t \\ 2\sin 3t \end{bmatrix}$



(e) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 5\sin 4t \\ 4\cos 4t - 2\sin 4t \end{bmatrix} + C_2 \begin{bmatrix} 5\cos 4t \\ -4\sin 4t - 2\cos 4t \end{bmatrix}$

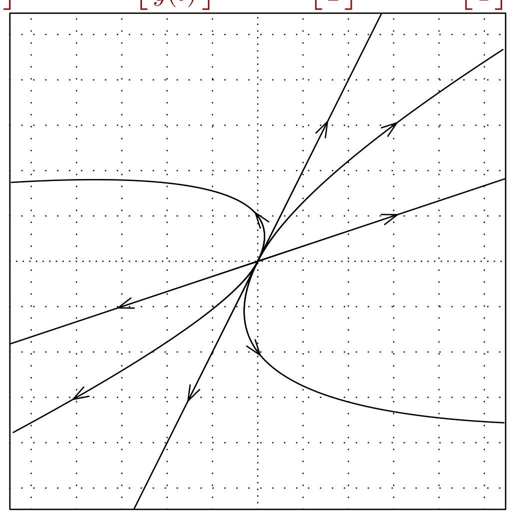


(f) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{3t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

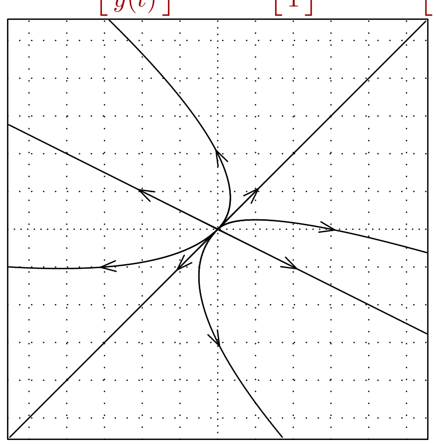


**(Answer 69)**

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

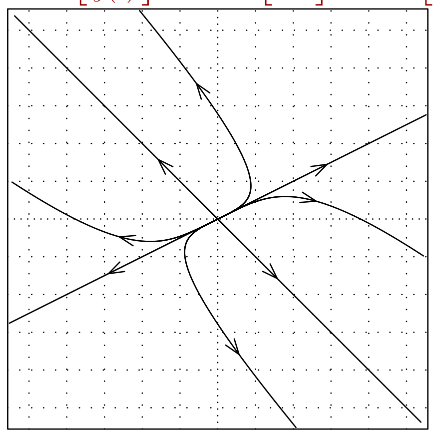


(b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2.2 & -0.6 \\ 0.4 & 0.8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

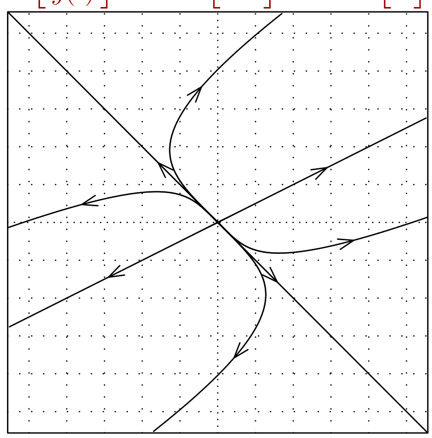


**(Answer 70)**

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{4t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

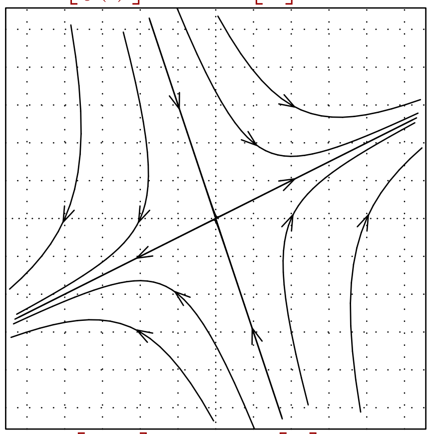


$(b)$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

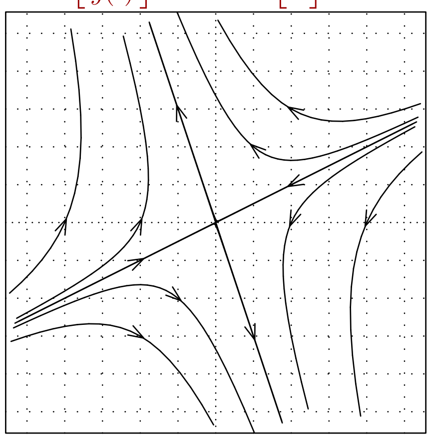


**(Answer 71)**

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 9 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-14t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

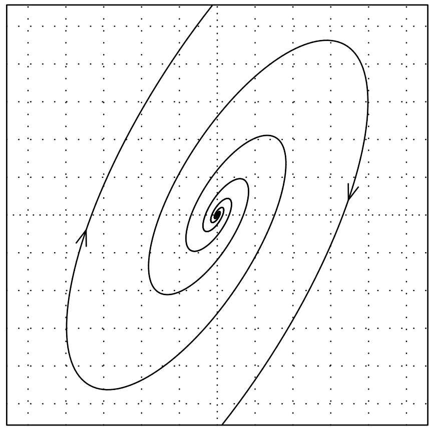


$(b)$ $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 & -6 \\ -9 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 e^{-7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{14t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

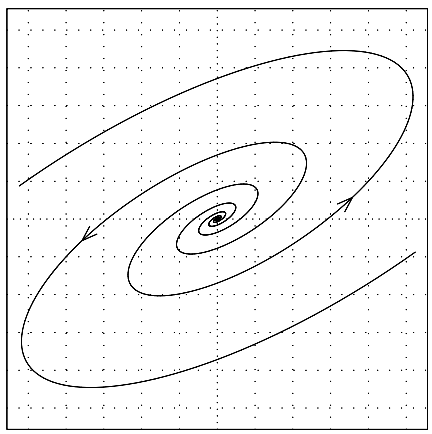


**(Answer 72)**

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -8 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, eigenvalues $r = 1 \pm 4i$



(b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, eigenvalues $r = 1 \pm 4i$



**(Answer 73)**

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 6\sin 3t \\ 4\sin 3t + 2\cos 3t \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

**(Answer 74)** The critical points for

$$\frac{dx}{dt} = 4x - 2x^2 - xy, \quad \frac{dy}{dt} = 3y - xy - y^2$$

are $(x, y) = (0, 0)$, $(0, 3)$, $(2, 0)$ and $(1, 2)$.

The point $(x, y) = (0, 0)$ is an unstable node.
The point $(x, y) = (0, 3)$ is an unstable saddle point.
The point $(x, y) = (2, 0)$ is an unstable saddle point.
The point $(x, y) = (1, 2)$ is an asymptotically stable node.

**(Answer 75)** The critical points for

$$\frac{dx}{dt} = 1 - 2y, \quad \frac{dy}{dt} = x^2 - y^2.$$

are $(x, y) = (1/2, 1/2)$, $(-1/2, 1/2)$.

The point $(x, y) = (1/2, 1/2)$ is asymptotically stable spiral point.

The point $(x, y) = (-1/2, 1/2)$ is an unstable saddle point.

**(Answer 76)** The critical points for

$$\frac{dx}{dt} = x - 2xy, \quad \frac{dy}{dt} = xy - y?$$

are $(x, y) = (0, 0)$, $(1, 1/2)$.

The point $(x, y) = (0, 0)$ is an unstable saddle point.

The point $(x, y) = (1, 1/2)$ is either a center or a spiral point; we cannot determine which using the tools we now have.