Math 2584, Fall 2016

If you need it, the following will be printed on the cover page of the exam:

Theorem 2.4.2. Consider the initial-value problem $\frac{dy}{dt} = f(t,y)$, $y(t_0) = y_0$. Suppose that $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous near (t_0, y_0). Then there is some interval (a,b) with $a < t_0 < b$ such that a solution to the initial-value problem exists and is unique on that interval.

Theorem 2.4.1. Consider the initial-value problem $\frac{dy}{dt} + p(t)y = g(t)$, $y(t_0) = y_0$. Suppose that there is some $a < t_0 < b$ such that $p(t)$ and $g(t)$ are continuous on (a,b). Then there is a unique solution to the initial-value problem on the interval (a,b).

Theorem 3.2.1. Consider the initial-value problem $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\,y = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_1$. Suppose that there is some $a < t_0 < b$ such that $p(t)$, $q(t)$ and $g(t)$ are all continuous on (a,b). Then there is a unique solution to the initial-value problem on the interval (a,b).

Euler's formula states that, if θ is any real number, then $e^{i\theta} = \cos\theta + i\sin\theta$.

The acceleration of gravity (on Earth) is 9.8 meters/second^2; alternatively, gravity exerts a force of 9.8 newtons per kilogram.

If an object of mass M kilograms is subjected to a force of F newtons, and its position is x meters and its velocity is v meters/second^2, then Newton's second law states that $F = M\frac{dv}{dt} = M\frac{d^2x}{dt^2}$, where t denotes time (in seconds).

Here are some Laplace transforms:

$f(t)$	$\mathcal{L}\{f(t)\}$	
1	$\frac{1}{s}$,	$s > 0$
e^{at}	$\frac{1}{s-a}$,	$s > a$
t^n	$\frac{n!}{s^{n+1}}$,	$s > 0$, $n \geq 0$
$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$,	$s > a$

$f(t)$	$\mathcal{L}\{f(t)\}$	
$\cos(bt)$	$\frac{s}{s^2+b^2}$,	$s > 0$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$,	$s > a$
$\sin(bt)$	$\frac{b}{s^2+b^2}$,	$s > 0$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$,	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$,	$s > 0$, $c \geq 0$
$\delta(t-c)$	e^{-cs},	$s > 0$, $c \geq 0$

$$\mathcal{L}\{u_c(t)f(t)\} = e^{-cs}\mathcal{L}\{f(t+c)\}, \qquad c \geq 0$$

$$\mathcal{L}\{u_c(t)g(t-c)\} = e^{-cs}\mathcal{L}\{g(t)\}, \qquad c \geq 0$$

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0)$$

$$\mathcal{L}\Big\{\int_0^t f(r)\,g(t-r)\,dr\Big\} = \mathcal{L}\Big\{\int_0^t f(t-r)\,g(r)\,dr\Big\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a vector, then $A\vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a\,v_1 + b\,v_2 \\ c\,v_1 + d\,v_2 \end{pmatrix}$.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix then $\det A = |A| = ad - bc$.

If A is a matrix, λ is a number (real or complex), and $\vec{v}$ is a vector, $\vec{v} \neq 0$, and if $A\vec{v} = \lambda\vec{v}$, then

$$\det(A - \lambda I) = 0 \qquad \text{and} \qquad (A - \lambda I)\vec{v} = 0$$

where I is the identity matrix of the same size as A. (If A is a 2×2 matrix then $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.)

(Problem 1) An object weighing 5 lbs stretches a spring 4 inches. It is attached to a viscous damper with damping constant 16 lb $\cdot$ sec/ft. The object is pulled down an additional 2 inches and is released with initial velocity 3 ft/sec upwards.

Write the differential equation and initial conditions that describe the position of the object.

(Problem 2) A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 N $\cdot$ s/m and is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(20t)$ N upwards.

Write the differential equation and initial conditions that describe the position of the object.

(Problem 3) A series circuit contains a resistor of 20 Ω, an inductor of 2 H, a capacitor of 5×10^{-5} F, and a 12-volt battery. At time $t = 0$, there is no charge on the capacitor and no current; at this time the circuit is closed and current begins to flow. Write the differential equation and initial conditions that describe the charge on the capacitor.

(Problem 4) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ (**not** the table on the front of the exam), find the Laplace transforms of the following functions.

(a) $f(t) = t$

(b) $f(t) = e^{-11t}$

(c) $f(t) = \begin{cases} 3, & 0 < t < 4, \\ 0, & 4 \leq t \end{cases}.$

(Problem 5) Find the Laplace transforms of the following functions. You may use the table on the front of the exam.

(a) $f(t) = t^4 + 5t^2 + 4$

(b) $f(t) = (t+2)^3$

(c) $f(t) = 9e^{4t+7}$

(d) $f(t) = -e^{3(t-2)}$

(e) $f(t) = (e^t + 1)^2$

(f) $f(t) = 8\sin(3t) - 4\cos(3t)$

(g) $f(t) = t^2 e^{5t}$

(h) $f(t) = 7e^{3t}\cos 4t$

(i) $f(t) = 4e^{-t}\sin 5t$

(j) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$

(k) $f(t) = \begin{cases} 0, & t < 1, \\ t - 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$

(l) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

(m) $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$

(n) $f(t) = \begin{cases} 0, & t < \pi, \\ \sin t, & t \geq \pi \end{cases}$

(Problem 6) For each of the following problems, find y.

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$
(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$
(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$
(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$
(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$
(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$
(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$
(h) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$
(i) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$
(j) $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$
(k) $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$
(l) $\mathcal{L}\{y\} = \frac{s}{s^2-9}\mathcal{L}\{\sqrt{t}\}$

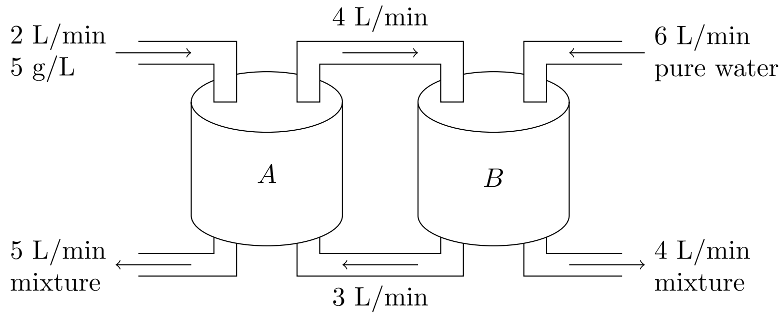
(Problem 7) Solve the following initial-value problems using the Laplace transform.

(a) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$
(b) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$
(c) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$
(d) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$
(e) $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$
(f) $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$
(g) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$
(h) $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$
(i) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$
(j) $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t \end{cases}$, $y(0) = 1$, $y'(0) = 0$
(k) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$

(Problem 8) Solve the following initial-value problems using the Laplace transform. Do the graphs of y, $\frac{dy}{dt}$, or $\frac{d^2y}{dt^2}$ have any corners or jump discontinuities? If so, where?

(a) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4u_2(t)$, $y(0) = 0$, $y'(0) = 1$.
(b) $\frac{d^2y}{dt^2} + 4y = -2\delta(t - 4\pi)$, $y(0) = 1/2$, $y'(0) = 0$
(c) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + u_2(t)$, $y(0) = 1$, $y'(0) = 0$

(Problem 9) Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



Write the differential equations and initial conditions that describe the amount of salt in each tank.

Answer key

(Answer 1) Let t denote time (in seconds) and let u denote the object's displacement above equilibrium (in feet).

Then
$$\frac{5}{32}\frac{d^2u}{dt^2} + 16\frac{du}{dt} + 15u = 0, \qquad u(0) = -\frac{1}{6}, \quad u'(0) = 3.$$

(Answer 2) Let t denote time (in seconds) and let u denote the object's displacement above equilibrium (in meters).

Then
$$3\frac{d^2u}{dt^2} + 27\frac{du}{dt} + 588u = 3\cos(20t), \qquad u(0) = 0, \quad u'(0) = 0.$$

(Answer 3) Let t denote time (in seconds), let Q denote the charge on the capacitor, and let I denote the current through the circuit.

Then
$$12 = 2\frac{dI}{dt} + 20I + 2000000Q, \quad \frac{dQ}{dt} = 0, \qquad Q(0) = 0, \quad I(0) = 0.$$

(Answer 4)

(a) $\mathcal{L}\{t\} = \frac{1}{s^2}$.
(b) $\mathcal{L}\{e^{-11t}\} = \frac{1}{s+11}$.
(c) $\mathcal{L}\{f(t)\} = \frac{3-3e^{-4s}}{s}$.

(Answer 5)

(a) $\mathcal{L}\{t^4 + 5t^2 + 4\} = \frac{96}{s^5} + \frac{10}{s^3} + \frac{4}{s}$.
(b) $\mathcal{L}\{(t+2)^3\} = \mathcal{L}\{t^3 + 6t^2 + 12t + 8\} = \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{8}{s}$.
(c) $\mathcal{L}\{9e^{4t+7}\} = \mathcal{L}\{9e^7 e^{4t}\} = \frac{9e^7}{s-4}$.
(d) $\mathcal{L}\{-e^{3(t-2)}\} = \mathcal{L}\{-e^{-6}e^{3t}\} = -\frac{e^{-6}}{s-3}$.
(e) $\mathcal{L}\{(e^t+1)^2\} = \mathcal{L}\{e^{2t} + 2e^t + 1\} = \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s}$.
(f) $\mathcal{L}\{8\sin(3t) - 4\cos(3t)\} = \frac{24}{s^2+9} - \frac{4s}{s^2+9}$.
(g) $\mathcal{L}\{t^2 e^{5t}\} = \frac{2}{(s-5)^3}$.
(h) $\mathcal{L}\{7e^{3t}\cos 4t\} = \frac{7s-21}{(s-3)^2+16}$.
(i) $\mathcal{L}\{4e^{-t}\sin 5t\} = \frac{20}{(s+1)^2+25}$.
(j) If $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$ then $\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{1}{s} + \frac{2}{s^3}\right)$.
(k) If $f(t) = \begin{cases} 0, & t < 1, \\ t-2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2}$.
(l) If $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{5}{s-2} - \frac{5e^6 e^{-3s}}{s-2}$.
(m) If $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{14}{(s+1)^3} - \frac{14e^{-3s-3}}{(s+1)^3} - \frac{42e^{-3s-3}}{(s+1)^2} - \frac{63e^{-3s-3}}{s+1}$.
(n) If $f(t) = \begin{cases} 0, & t < \pi, \\ \sin t, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s}\frac{1}{s^2+1}$.

(Answer 6)

(a) If $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$, then $y = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$

(b) If $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$, then $y = \frac{5}{2}t^2 - \frac{7}{6}t^3$

(c) If $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$, then $y = \frac{1}{2}t^2e^{-t} + \frac{1}{6}t^3e^{-t}$

(d) If $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$, then $y = (1/4)e^{2t} + (7/4)e^{-2t}$

(e) If $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$, then $y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{1}{9}t\,e^{3t}$

(f) If $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$, then $y = \frac{1}{2} - \frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t)$

(g) If $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$, then $y = \frac{1}{8}\cos t - \frac{1}{8}\cos 3t$

(h) If $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$, then $y = 2\,u_2(t)\,e^{t-2}\,\cos(t-2) + u_2(t)\,e^{t-2}\,\sin(t-2)$

(i) If $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$, then $y = \frac{1}{2}u_1(t)\,e^{3(t-1)} + \frac{1}{2}u_1(t)\,e^{t-1}$

(j) If $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$, then $y = \int_0^t \frac{1}{3}\sin 3r\cos(3t-3r)\,dr$

(k) If $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$, then $y = e^{-2t}\int_0^t \sin(2r)\sin(2t-2r)\,dr$

(l) If $\mathcal{L}\{y\} = \frac{s}{s^2-9}\mathcal{L}\{\sqrt{t}\}$, then $y = \int_0^t \frac{e^{3r}+e^{-3r}}{2}\sqrt{t-r}\,dr$.

(Answer 7)

(a) If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$, then $y(t) = e^{2t} - te^{2t}$.

(b) If $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$, then $y(t) = \frac{1}{5}(e^{-t} - e^t\cos t + 7e^t\sin t)$.

(c) If $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$, then $y(t) = 4e^{3t} - 2e^{4t} - te^{3t}$.

(d) If $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$, then $y(t) = \frac{1}{5}\cos 2t + \cos 3t + \frac{5}{3}\sin 3t$.

(e) If $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t r\sin 3r\sin(3t-3r)\,dr$.

(f) If $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t \sin 3r\sin(3t-3r)\,dr = \frac{1}{6}\sin 3t - \frac{1}{2}t\cos 3t$.

(g) If $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$, then $y(t) = 7e^{-2t}-5e^{-3t}+\int_0^t(e^{-2r}-e^{-3r})\sqrt{t-r+1}\,dr$.

(h) If $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = (1/6)(1 - u_{2\pi}(t))(2\sin t - \sin 2t).$$

(i) If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - u_{10}(t)\left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right].$$

(j) If $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = \begin{cases} \sin t, & 0 \le t < \pi, \\ 0, & \pi \le t \end{cases}$, $y(0) = 1$, $y'(0) = 0$, then

$$\begin{aligned} y(t) &= e^{-t/2}\cos t + \frac{1}{2}e^{-t/2}\sin t \\ &\quad - u_\pi(t)\left(\frac{16}{17}\cos(t-\pi) - \frac{4}{17}\sin(t-\pi)\right) \\ &\quad + u_\pi(t)\left(\frac{4}{17}e^{-(t-\pi)/2}\sin(t-\pi) + \frac{16}{17}e^{-(t-\pi)/2}\cos(t-\pi)\right). \end{aligned}$$

(k) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$, then $y(t) = 2e^{-2t} + 5te^{-2t} + \frac{3}{4}u_2(t) - \frac{3}{4}e^{-2t+4}u_2(t) - \frac{3}{2}(t-2)e^{-2t+4}u_2(t)$.

(Answer 8)

(a) If $6\frac{d^2y}{dt^2}+5\frac{dy}{dt}+y=4u_2(t)$, $y(0)=0$, $y'(0)=1$, then

$$y=6e^{-t/3}-6e^{-t/2}+4u_2(t)-12u_2(t)e^{-(t-2)/3}+8u_2(t)e^{-(t-2)/2}.$$

The graph of $y'(t)$ has a corner at $t=2$, and the graph of $y''(t)$ has a jump at $t=2$.

(b) If $\frac{d^2y}{dt^2}+4y=\delta(t-4\pi)$, $y(0)=1/2$, $y'(0)=0$, then

$$y=\frac{1}{2}\cos(2t)-u_{4\pi}(t)\sin(2t).$$

The graph of $y(t)$ has a corner at $t=4\pi$, and graph of $y'(t)$ has a jump at $t=4\pi$. ($y''(t)$ is hard to graph, because the impulse function $\delta(t-4\pi)$ is part of $y''(t)$.)

(c) If $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+3y=2\delta(t-1)+u_2(t)$, $y(0)=1$, $y'(0)=0$, then

$$y=\frac{3}{2}e^{-t}-\frac{1}{2}e^{-3t}+\frac{1}{2}u_1(t)e^{-t+1}-\frac{1}{2}u_1(t)e^{-3t+3}+\frac{1}{3}u_2(t)-\frac{1}{2}e^{-t+2}u_2(t)+\frac{1}{6}u_2(t)e^{-3t+6}.$$

The graph of $y(t)$ has a corner at $t=1$. The graph of $y'(t)$ has a corner at $t=2$, and a jump at $t=1$. $y''(t)$ has an impulse at $t=1$, and a jump at $t=2$.

(Answer 9) Let t denote time (in minutes).

Let x denote the amount of salt (in grams) in tank A.
Let y denote the amount of salt (in grams) in tank B.
Then $x(0)=3000$ and $y(0)=2000$.
If $t<50$, then

$$\frac{dx}{dt}=-\frac{9x}{200-4t}+\frac{3y}{300+3t},\qquad \frac{dy}{dt}=\frac{4x}{200-4t}-\frac{7y}{300+3t}.$$