Math 2584, Spring 2016

If you need it, the following will be printed on the cover page of the exam:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C, \qquad \int \tan x\,dx = \ln|\sec x| + C,$$
$$\int \frac{1}{1+x^2}\,dx = \arctan x + C, \qquad \int \cot x\,dx = \ln|\sin x| + C,$$
$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec} x + C. \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C,$$
$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C.$$

Theorem 2.4.2. Consider the initial-value problem $\frac{dy}{dt} = f(t,y)$, $y(t_0) = y_0$. Suppose that $f(t,y)$ and $\frac{\partial f}{\partial y}$ are continuous near (t_0, y_0). Then there is some interval (a,b) with $a < t_0 < b$ such that a solution to the initial-value problem exists and is unique on that interval.

Theorem 2.4.1. Consider the initial-value problem $\frac{dy}{dt} + p(t)y = g(t)$, $y(t_0) = y_0$. Suppose that there is some $a < t_0 < b$ such that $p(t)$ and $g(t)$ are continuous on (a,b). Then there is a unique solution to the initial-value problem on the interval (a,b).

Theorem 3.2.1. Consider the initial-value problem $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\,y = g(t)$, $y(t_0) = y_0$, $y'(t_0) = y_1$. Suppose that there is some $a < t_0 < b$ such that $p(t)$, $q(t)$ and $g(t)$ are all continuous on (a,b). Then there is a unique solution to the initial-value problem on the interval (a,b).

Euler's formula states that, if θ is any real number, then $e^{i\theta} = \cos\theta + i\sin\theta$.

The acceleration of gravity (on Earth) is 9.8 meters/second^2; alternatively, gravity exerts a force of 9.8 newtons per kilogram.

If an object of mass M kilograms is subjected to a force of F newtons, and its position is x meters and its velocity is v meters/second^2, then Newton's second law states that $F = M\frac{dv}{dt} = M\frac{d^2x}{dt^2}$, where t denotes time (in seconds).

(Problem 1) Are we guaranteed a unique solution to the following initial value problems?
(a) $x^2\frac{dy}{dx} = y^2$, $y(0) = 1$.
(b) $x^2\frac{dy}{dx} = y^2$, $y(1) = 0$.
(c) $\frac{dy}{dx} = \sqrt[3]{y}$, $y(8) = 0$.
(d) $\frac{dy}{dx} = \sqrt[3]{y}$, $y(8) = 1$.

(Problem 2) For each of the following initial-value problems, tell me the longest open interval on which we are guaranteed that a unique solution exists.
(a) $\frac{dy}{dt} + 5y = \sec t$, $y(\pi) = 0$.
(b) $t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + 5y = 0$, $y(-3) = 0$, $y'(-3) = 2$.
(c) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \frac{1}{t^3-6t^2+8t}$, $y(1) = 0$, $y'(1) = 2$.
(d) $\frac{dy}{dt} = e^{y\cos t}$, $y(0) = 1$.

(Problem 3)

(a) Suppose that $y(t) = t^3+8$ and $y(t) = 2t^2+4t$ are both solutions to the differential equation $\frac{dy}{dt}+p(t)y = g(t)$. Are there any numbers t_0 such that you may be sure that $p(t)$ or $g(t)$ is discontinuous near t_0?

(b) Suppose that $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are both solutions to some differential equation $\frac{dy}{dt} = f(t, y)$. What can you say about $f(t, y)$ and $\partial f/\partial y$?

(c) Suppose that $y_1(t) = t^3 + 8$ and $y(t) = 2t^2 + 4t$ are both solutions to the differential equation $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\,y = g(t)$. Are there any numbers t_0 such that you may be sure that $p(t)$, $q(t)$ or $g(t)$ is discontinuous near t_0?

(Problem 4)

(a) Suppose that $y(t) = \ln|t^2 - 4|$ is a solution to the differential equation $\frac{dy}{dt} + p(t)y = g(t)$. Are there any numbers t_0 such that you may be sure that at least one of $p(t)$ and $g(t)$ is discontinuous near t_0?

(b) Suppose that $y(t) = \ln|t^2-4|$ is a solution to the nonlinear differential equation $\frac{dy}{dt} = f(t, y)$. Are there any numbers t_0, y_0 such that you may be sure that at least one of $f(t, y)$ and $\partial f/\partial y$ is discontinuous near $t = t_0$, $y = y_0$?

(Problem 5) Solve the following differential equations.

(a) $\frac{dy}{dt} = 8y - y^8$.

(b) $t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$.

(c) $\frac{dy}{dt} = -y\tan 2t - y^3\cos 2t$.

(Problem 6) Solve the initial-value problem $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$.

(Problem 7) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y - 2)(y + 1)^2$. By hand, sketch some typical solutions.

(Problem 8) Find the critical points and draw the phase line of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(Problem 9) Find the critical points and draw the phase line of the differential equation $\frac{dy}{dx} = y^2(y - 2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(Problem 10) I owe my bank a debt of B dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

(a) Formulate a differential equation for the amount of money I owe.

(b) Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

(Problem 11) A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

(a) Write the differential equation for the amount of salt in the tank.

(b) Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

(Problem 12) A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where v is her velocity in meters per second.

(a) Write a differential equation for her velocity. Assume her velocity is always negative.

(b) Find the (negative) critical points of this differential equation. Be sure to include units.

(c) What is the real-world meaning of these critical points?

(Problem 13) Suppose that the population y of catfish in a certain lake, absent human intervention, satisfies the logistic equation $\frac{dy}{dt} = ry(1 - y/K)$, where r and K are constants and t denotes time.

(a) Assuming that $r > 0$ and $K > 0$, find the critical points of this equation and classify them as to stability.

(b) Suppose that humans intervene by harvesting fish at a rate proportional to the fish population. Write a new differential equation for the fish population. Find the critical points of your new equation and classify them as to stability.

(c) Suppose that humans intervene by harvesting fish at a constant rate. Write a new differential equation for the fish population. Find the critical points of your new equation and classify them as to stability.

(Problem 14) The function $y_1(t) = e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2} - (1+2t)\frac{dy}{dt} + (t+1)y = 0$. Find the general solution to this differential equation.

(Problem 15) The function $y_1(t) = t$ is a solution to the differential equation $(1-t)\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = 0$. Find the general solution to this differential equation.

(Problem 16) The function $y_1(x) = x^3$ is a solution to the differential equation $x^2\frac{d^2y}{dx^2} + (x^2\tan x - 6x)\frac{dy}{dx} + (12 - 3x\tan x)y = 0$. Find the general solution to this differential equation on the interval $0 < x < \pi/2$.

(Problem 17) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $16\frac{d^2y}{dt^2} - y = 0$, $y(0) = 1$, $y'(0) = 0$.

(b) $\frac{d^2y}{dt^2} + 49y = 0$, $y(\pi) = 3$, $y'(\pi) = 4$.

(c) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 0$, $y(2) = 0$, $y'(2) = 3$.

(Problem 18) Solve the following differential equations. Express your answers in terms of real functions.

(a) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$.

(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.

(Problem 19) Find the general solution to the following differential equations.

(a) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$.

(b) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$.

(c) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}$.

(d) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t)$.

(e) $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t)$.

(f) $\frac{d^2y}{dt^2} + 9y = 5\sin(3t)$.

(g) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$.

(h) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$.

(i) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$.

(j) $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$.

(k) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$.

(l) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.

(Problem 20) Solve the initial-value problem $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$.

(Problem 21) Find the general solution to the equation $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = \frac{e^t}{1+t^2}$.

(Problem 22) Solve the initial-value problem $\frac{d^2y}{dt^2} + 9y = 9\sec^2 3t$, $y(0) = 4$, $y'(0) = 6$, on the interval $-\pi/6 < t < \pi/6$.

(Problem 23) The general solution to the differential equation $t^2 \frac{d^2y}{dt^2} - 2y = 0$, $t > 0$, is $y(t) = C_1 t^2 + C_2 t^{-1}$. Solve the initial-value problem $t^2 \frac{d^2y}{dt^2} - 2y = 7t^3$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(Problem 24) The general solution to the differential equation $t^2 \frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 0$, $t > 0$, is $y(t) = C_1 t^3 + C_2 t^{-1}$. Find the general solution to the differential equation $t^2 \frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

Answer key

(Answer 1)

(a) We are not guaranteed a unique solution to the problem $x^2y' = y^2$, $y(0) = 1$.
(b) We are guaranteed a unique solution to the problem $x^2y' = y^2$, $y(1) = 0$.
(c) We are not guaranteed a unique solution to the problem $y' = \sqrt[3]{y}$, $y(8) = 0$.
(d) We are guaranteed a unique solution to the problem $y' = \sqrt[3]{y}$, $y(8) = 1$.

(Answer 2)

(a) The interval $\pi/2 < t < 3\pi/2$.
(b) The interval $t < 0$.
(c) The interval $0 < t < 2$.
(d) We are not guaranteed a solution on any open interval. (We are guaranteed a unique solution on $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, but ε could be arbitrarily small.)

(Answer 3)

(a) The solutions cross at $(-2, 0)$ and $(2, 16)$, and so we may be sure that at least one of $p(t)$ and $g(t)$ is discontinuous near $t = -2$ and $t = 2$.

That is, there are two solutions to the initial value problem $\frac{dy}{dt} + p(t)y = g(t)$, $y(-2) = 0$, and so at least one of $p(t)$ and $g(t)$ is discontinuous near $t = -2$. Similarly, there are two solutions to the initial value problem $\frac{dy}{dt} + p(t)y = g(t)$, $y(2) = 16$, and so at least one of $p(t)$ and $g(t)$ is discontinuous near $t = 2$.

(b) The solutions cross at $(0, 1)$, and so we may be sure that either $f(t, y)$ or $\partial f/\partial y$ is discontinuos near $(t, y) = (0, 1)$.
(c) The solutions cross and are tangent at $t = 2$, and so at least one of $p(t)$, $q(t)$, $g(t)$ are discontinuous near $t = 2$.

Put another way, there are two solutions to the initial value problem $\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)\,y = g(t)$, $y(2) = 16$, $y'(2) = 12$, and so at least one of $p(t)$, $q(t)$, $g(t)$ must be discontinuous near $t = 2$.

(Answer 4)

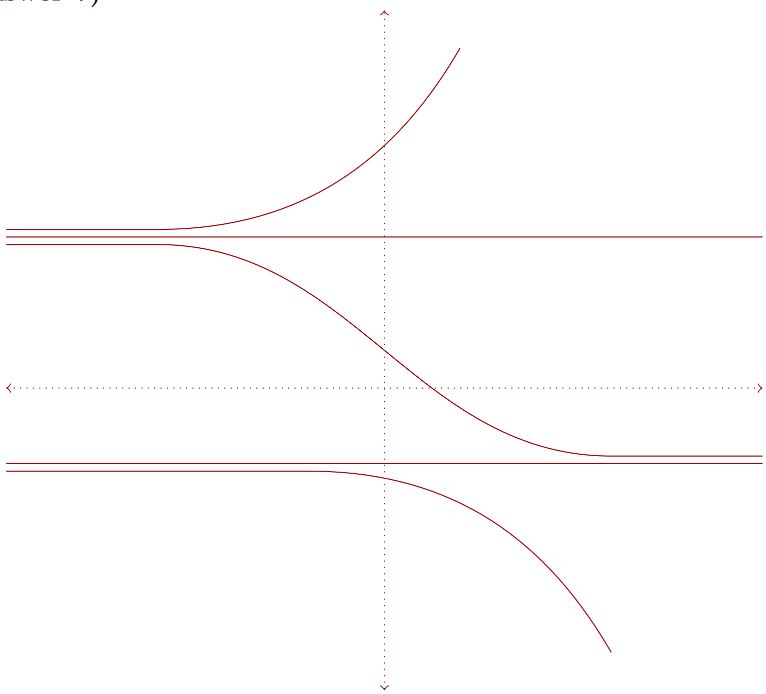
(a) Yes; $p(t)$ or $g(t)$ must be discontinuous near $t = 2$ and $t = -2$.
(b) No, we cannot conclude that $f(t, y)$ or $\partial f/\partial y$ is discontinuous anywhere.

(Answer 5)

(a) Make the substitution $v = y^{-7}$. Then $\frac{dv}{dt} = -56v + 7$, so $v = Ce^{-56t} + \frac{1}{8}$ and $y = \frac{1}{\sqrt[7]{Ce^{-56t}+1/8}}$.
(b) Let $v = y^6$. Then $t\frac{dv}{dt} = 18v - 6t^2$, so $v = Ct^{18} + \frac{3}{8}t^2$ and $y = \sqrt[6]{Ct^{18} + \frac{3}{8}t^2}$.
(c) Let $v = y^{-2}$. Then $\frac{dv}{dt} = 2v\tan 2t + 2\cos 2t$, so $v = \frac{1}{2}\sin(2t) + t\sec(2t) + C\sec(2t)$ and $y = \frac{1}{\sqrt{\frac{1}{2}\sin(2t)+t\sec(2t)+C\sec(2t)}}$.

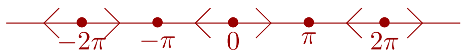
(Answer 6) $y = \sqrt[3]{340e^{6t} + 3}$

(Answer 7)



(Answer 8)

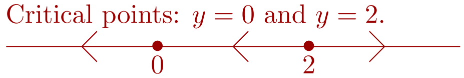
Critical points: $y = k\pi$ for any integer k.



If k is even then $y = k\pi$ is unstable.
If k is odd then $y = k\pi$ is stable.

(Answer 9)

Critical points: $y = 0$ and $y = 2$.



$y = 0$ is semistable. $y = 2$ is unstable.

(Answer 10)

(a) $\frac{dB}{dt} = 0.05B - 19200$, where t denotes time in years.

(b) The critical point is $B = \$384{,}000$. It is unstable. If my initial debt is less than \$384,000, then I will eventually pay it off (have a debt of zero dollars), but if my initial debt is greater than \$384,000, my debt will grow exponentially. The critical point $B = 384{,}000$ corresponds to the balance that will allow me to make interest-only payments on my debt.

(Answer 11)

(a) $\frac{dQ}{dt} = 10 - Q/300$, where Q denotes the amount of salt in grams and t denotes time in minutes.

(b) The critical point is $Q = 3000$. It is stable. No matter the initial amount of salt in the tank, as time passes the amount of salt will approach 3000 g or 3 kg.

(Answer 12)

(a) If $v \leq 0$ then $70\frac{dv}{dt} = -70 * 9.8 + 2v^2$. (If $v > 0$ then $70\frac{dv}{dt} = -70 * 9.8 - 2v^2$.)

(b) $v = -\sqrt{343}$ meters/second.

(c) As $t \to \infty$, her velocity will approach the stable critical point of $v = -\sqrt{343}$ meters/second.

(Answer 13)

(a) The critical points are $y = 0$ (unstable) and $y = K$ (stable).

(b) $\frac{dy}{dt} = ry(1-y/K)-Ey$, where E is a proportionality constant. The critical points are $y = 0$ (unstable) and $y = K - KE/r$ (stable).

(c) $\frac{dy}{dt} = ry(1 - y/K) - h$, where h is the harvesting rate. If $h < Kr/4$, then the critical points are $y = K/2 - \sqrt{(K/2)^2 - Kh/r}$ (unstable) and $y = K/2 + \sqrt{(K/2)^2 - Kh/r}$ (stable). Notice that both critical points are positive (i.e., correspond to the physically meaningful case of at least zero fish in the lake.) If $h = Kr/4$, then there is one critical point at $y = K/2$; it is semistable. Finally, if $h > Kr/4$, then there are no critical points and the fish population will decrease until it goes extinct.

(Answer 14) $y(t) = C_1\, e^t + C_2\, t^2 e^t$.

(Answer 15) $y(t) = C_1\, t + C_2\, e^t$.

(Answer 16) $y(x) = C_1\, x^3 + C_2\, x^3 \sin x$.

(Answer 17)

(a) If $16\frac{d^2y}{dt^2} - y = 0$, $y(0) = 1$, $y'(0) = 0$, then $y(t) = \frac{1}{2}e^{t/4} + \frac{1}{2}e^{-t/4}$.

(b) If $\frac{d^2y}{dt^2} + 49y = 0$, $y(\pi) = 3$, $y'(\pi) = 4$, then $y(t) = 3\cos(7t) + \frac{4}{7}\sin(7t)$.

(c) If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 0$, $y(2) = 0$, $y'(2) = 3$, then $y(t) = \frac{3}{7}e^{2t-4} - \frac{3}{7}e^{-5t+10}$.

(Answer 18)

(a) If $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$, then $y = C_1e^{-6t}\cos(7t) + C_2e^{-6t}\sin(7t)$.

(b) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$, then $y = C_1e^{(-2+\sqrt{2})t} + C_2e^{(-2-\sqrt{2})t}$.

(Answer 19)

(a) The general solution to $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 0$ is $y_g = C_1 + C_2e^{4t}$. To solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$ we make the guess $y_p = Ae^{3t}$. The solution is $y = C_1 + C_2e^{4t} - (4/3)e^{3t}$.

(b) The general solution to $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 0$ is $y_g = C_1e^{2t} + C_2e^{-5t}$. To solve $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$ we make the guess $y_p = Ate^{-5t}$. The solution is $y = C_1e^{2t} + C_2e^{-5t} - (1/7)te^{-5t}$.

(c) The general solution to $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$ is $y_g = C_1e^{-5t} + C_2te^{-5t}$. To solve $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}$ we make the guess $y_p = At^2e^{-5t}$. The solution is $y = C_1e^{-5t} + C_2te^{-5t} + (3/2)t^2e^{-5t}$.

(d) The general solution to $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$ is $y_g = C_1e^{-2t} + C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t)$, we make the guess $y_p = A\cos(2t) + B\sin(2t)$.

(e) The general solution to $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0$ is $y_g = C_1e^{-3t} + C_2te^{-3t}$. To solve $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t)$, we make the guess $y_p = A\cos(4t) + B\sin(4t)$.

(f) The general solution to $\frac{d^2y}{dt^2} + 9y = 0$ is $y_g = C_1\cos(3t) + C_2\sin(3t)$. To solve $\frac{d^2y}{dt^2} + 9y = 5\sin(3t)$, we make the guess $y_p = C_1t\cos(3t) + C_2t\sin(3t)$. The solution is $y = C_1\cos(3t) + C_2\sin(3t) - \frac{5}{6}t\cos(3t)$.

(g) The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ is $y_g = C_1e^{-t} + C_2te^{-t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$, we make the guess $y_p = At^2 + Bt + C$. The solution is $y = C_1e^{-t} + C_2te^{-t} + 2t^2 - 8t + 12$.

(h) The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 0$ is $y_g = C_1 + C_2e^{-2t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$, we make the guess $y_p = At^2 + Bt$. The solution is $y = C_1 + C_2e^{-2t} + \frac{3}{4}t^2 - \frac{3}{4}t$.

(i) The general solution to $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 0$ is $y_g = C_1e^{3t} + C_2e^{4t}$. To solve $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$, we make the guess $y_p = At + B + C\cos(2t) + D\sin(2t)$.

(j) The general solution to $\frac{d^2y}{dt^2} - 9y = 0$ is $y_g = C_1e^{3t} + C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$, we make the guess $y_p = Ae^t + Be^{-t} + Ct + D$.

(k) The general solution to $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$ is $y_g = C_1e^{2t} + C_2te^{2t}$. To solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$, we make the guess $y_p = At^2e^{2t} + B\cos t + C\sin t$.

(l) If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$ then $y = C_1e^{(-2-\sqrt{2})t} + C_2e^{(-2+\sqrt{2})t}$.

(Answer 20) If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$, then $y(t) = -\frac{4}{3}e^{3t} + \frac{7}{4}e^{4t} + \frac{7}{12}$.

(Answer 21) $y(t) = c_1\, e^t + c_2\, t\, e^t - \frac{1}{2}e^t\, \ln(1+t^2) + t\, e^t \arctan t$.

(Answer 22) $y(t) = 5\cos(3t) + 2\sin(3t) + (\sin(3t))\ln(\tan(3t) + \sec(3t)) - 1$.

(Answer 23) $y(t) = \frac{7}{4}t^3 - \frac{4}{3}t^2 + \frac{7}{12}t^{-1}$.

(Answer 24) $y(t) = -\frac{3}{2}t^{-1}\ln t + C_1t^3 + C_2t^{-1}$.