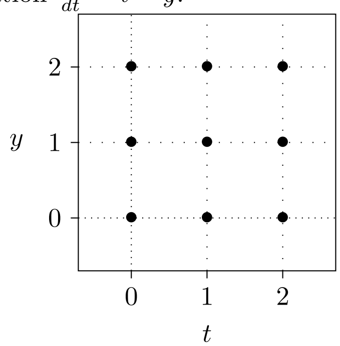
Math 2584, Spring 2016

If you need it, the following will be printed on the cover page of the exam:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C,$$
$$\int \frac{1}{1+x^2}\,dx = \arctan x + C,$$
$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec} x + C.$$

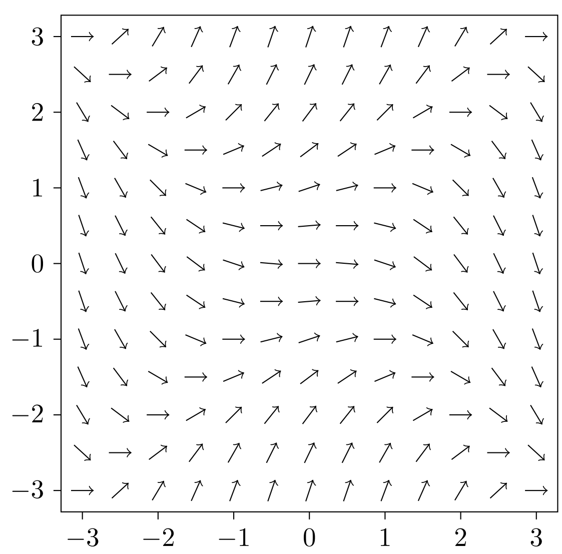
$$\int \tan x\,dx = \ln|\sec x| + C,$$
$$\int \cot x\,dx = \ln|\sin x| + C,$$
$$\int \sec x\,dx = \ln|\sec x + \tan x| + C,$$
$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C.$$

(Problem 1) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.

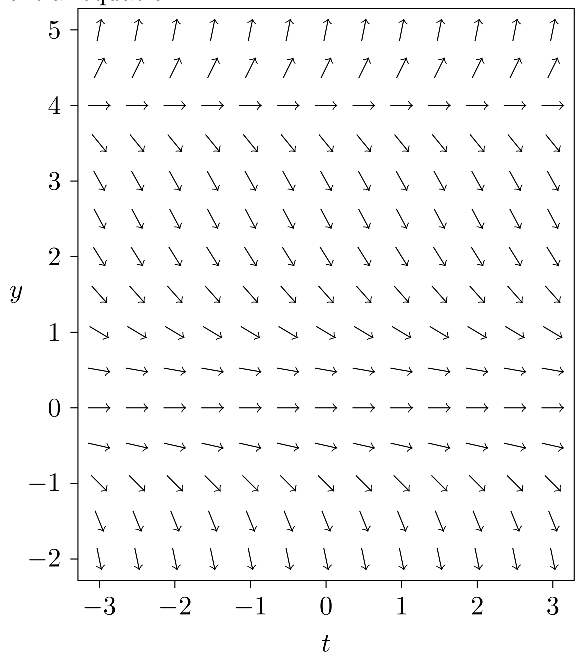


(Problem 2) Consider the differential equation $\frac{dy}{dx} = y^2 - x^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-3) = 0.$$



(Problem 3) Consider the differential equation $\frac{dy}{dt} = y^2(y-4)/5$. Here is the direction field for this differential equation.



Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency.

(Problem 4) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Set up, but **do not solve**, the differential equation and initial conditions that describe the amount of salt in the tank. Be sure to define your independent and dependent variables, as well as all unknown parameters, and be sure to include units.

(Problem 5) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C. Set up, but **do not solve**, the differential equation and initial conditions that describe the temperature of a cup of coffee. Be sure to define your independent and dependent variables, as well as all unknown parameters, and be sure to include units.

(Problem 6) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write a differential equation for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Problem 7) I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write a differential equation for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Problem 8) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write a differential equation for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Problem 9) According to the Stefan-Boltzmann law, a black body in a vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write a differential equation for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Problem 10) A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write a differential equation for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Problem 11) A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write a differential equation for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Problem 12) Is $y = e^t$ a solution to the differential equation $y'' + \frac{4t}{1-2t}y' - \frac{4}{1-2t}y = 0$?

(Problem 13) Is $y = e^{2t}$ a solution to the differential equation $y'' + \frac{4t}{1-2t}y' - \frac{4}{1-2t}y = 0$?

(Problem 14) For each of the following initial-value problems, tell me whether I am guaranteed an infinite family of solutions, am not guaranteed any solutions, or am guaranteed a unique solution on some interval. Do not find the solution to the differential equation.

(a) $t\frac{d^2y}{dt^2} - 5y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.
(b) $t(t-4)\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + 4y = 2$, $y(3) = 1$, $y'(3) = -1$.
(c) $\frac{d^2y}{dt^2} + (\cos t)\frac{dy}{dt} + 3(\ln|t|)y = 0$, $y(2) = 3$.

(Problem 15) For each of the following differential equations, determine whether it is linear, separable, or homogeneous. Then solve the differential equation.

(a) $\frac{t+\cos t}{y-\sin y} + \frac{dy}{dt} = 0$
(b) $1 + t^2 - ty + (t^2+1)\frac{dy}{dt} = 0$
(c) $ty - t^2 - y^2 + t^2\frac{dy}{dt} = 0$
(d) $3t - 5y + (t+y)\frac{dy}{dt} = 0$
(e) $y^3\cos(2t) + \frac{dy}{dt} = 0$
(f) $\cos t + 3y + t\frac{dy}{dt} = 0$

(Problem 16) For each of the following differential equations, determine whether it is linear, separable, or homogeneous. Then solve the given initial-value problem.

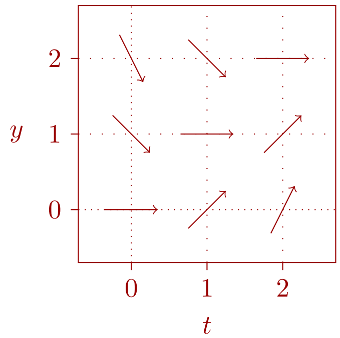
(a) $1 + y^2 + t\frac{dy}{dt} = 0$, $y(1) = 1$
(b) $\sin t\, e^{-3t} + 3y + \frac{dy}{dt} = 0$, $y(0) = 2$
(c) $y^2 - 4t^2 + 2ty\frac{dy}{dt} = 0$, $y(1) = 3$.

(Problem 17) Solve the initial-value problem $\frac{dy}{dt} = \frac{t-5}{y}$, $y(0) = 3$ and determine the range of t-values in which the solution is valid.

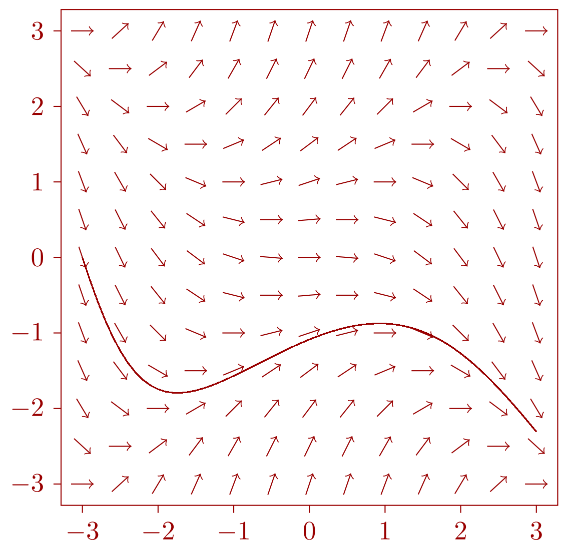
(Problem 18) Solve the initial-value problem $\frac{dy}{dt} = y^2$, $y(0) = 1/4$ and determine the range of t-values in which the solution is valid.

Answer key

(Answer 1) Here is the direction field for the differential equation $\frac{dy}{dt} = t - y$.



(Answer 2)



(Answer 3)

If $y(0) > 4$, then $y \to \infty$ as $t \to \infty$.
If $y(0) = 4$, then $y(t) = 4$ for all t.
If $0 \le y(0) < 4$, then $y \to 0$ as $t \to \infty$. In particular, if $y(0) = 0$, then $y(t) = 0$ for all t.
If $y(0) < 0$, then $y \to -\infty$ as $t \to \infty$.

(Answer 4) Independent variable: t = time (in minutes).
Dependent variable: Q = amount of dissolved salt (in kilograms).
Initial condition: $Q(0) = 3$.
Differential equation: $\frac{dQ}{dt} = 0.01 - \frac{3Q}{600-t}$.

(Answer 5) Independent variable: t = time (in seconds).
Dependent variable: T = Temperature of the cup (in degrees Celsius)
Initial condition: $T(0) = 95$.
Differential equation: $\frac{dT}{dt} = -\alpha(T - 25)$, where α is a parameter (constant of proportionality) with units of 1/seconds.

(Answer 6) Independent variable: t = time (in years).
Dependent variable: P = Number of birds on the island.
Initial condition: $P(0) = 5$.
Differential equation: $\frac{dP}{dt} = \alpha P - \beta P^2$, where α and β are parameters.

(Answer 7) Independent variable: t = time (in years).
Dependent variable: B = balance of my loan (in dollars).
Initial condition: $B(0) = 300{,}000$.
Differential equation: $\frac{dB}{dt} = 0.04B - 12 \cdot 1600$

(Answer 8) Independent variable: t = time (in minutes).
Dependent variables:
h = depth of water in the hole (in centimeters)
V = volume of water in the hole (in cubic centimeters); notice that $V = \frac{1}{3}\pi(5h)^2 h$
Initial condition: $V(0) = 0$.
Differential equation: $\frac{dV}{dt} = 1 - \alpha\pi h^2 = 1 - \alpha\pi(3V/25\pi)^{2/3}$, where α is a proportionality constant with units of cm/s.

(Answer 9) Independent variable: t = time (in seconds).
Dependent variable: T = object's temperature (in kelvins)
Parameter: σ = proportionality constant (in 1/(seconds·kelvin^3))
Initial condition: $T(0) = 400$.
Differential equation: $\frac{dT}{dt} = -\sigma T^4$.

(Answer 10) Independent variable: t = time (in seconds).
Dependent variable: v = velocity of the ball (in meters/second, where a positive velocity denotes upward motion).
Parameter: α = proportionality constant of the drag force (in newton·seconds/meter).
Initial condition: $v(0) = 10$.
Differential equation: $\frac{dv}{dt} = -9.8 - \alpha v$.

(Answer 11) Independent variable: t = time (in seconds).
Dependent variable: v = velocity of the ball (in meters/second, where a positive velocity denotes upward motion).
Parameter: α = proportionality constant of the drag force (in newton·seconds/meter).
Initial condition: $v(0) = 20$.
Differential equation: $\frac{dv}{dt} = -9.8 - \alpha v^2$.

(Answer 12) No, $y = e^t$ is not a solution to the differential equation $y'' + \frac{4t}{1-2t}y' - \frac{4}{1-2t}y = 0$.

(Answer 13) Yes, $y = e^{2t}$ is a solution to the differential equation $y'' + \frac{4t}{1-2t}y' - \frac{4}{1-2t}y = 0$.

(Answer 14)
(a) We are not guaranteed any solutions.
(b) We are guaranteed a unique solution.
(c) We are guaranteed an infinite family of solutions.

(Answer 15)

(a) $\frac{t+\cos t}{y-\sin y}+\frac{dy}{dt}=0$ is separable and has solution $\frac{1}{2}y^2+\cos y=-\frac{1}{2}t^2-\sin t+C$.

(b) $1+t^2-ty+(t^2+1)\frac{dy}{dt}=0$ is linear and has solution $y=-\sqrt{t^2+1}\ln(t+\sqrt{t^2+1})+C\sqrt{t^2+1}$.

(c) $ty-t^2-y^2+t^2\frac{dy}{dt}=0$ is homogeneous and has solution $y=\frac{t}{C-\ln|t|}+t$.

(d) $3t-5y+(t+y)\frac{dy}{dt}=0$ is homogeneous and has solution $(y-3t)^2=C(y-t)$.

(e) $y^3\cos(2t)+\frac{dy}{dt}=0$ is separable and has solution $y=\pm\frac{1}{\sqrt{\sin(2t)+C}}$.

(f) $\cos t+3y+t\frac{dy}{dt}=0$ is linear and has solution $y=\frac{1}{t}\sin t+\frac{2}{t^2}\cos t-\frac{2}{t^3}\sin t+\frac{C}{t^3}$.

(Answer 16)

(a) If $1+y^2+t\frac{dy}{dt}=0$, $y(1)=1$, then $y=\tan(\pi/4-\ln t)$.

(b) If $\sin t\,e^{-3t}+3y+\frac{dy}{dt}=0$, $y(0)=2$, then $y=e^{-3t}\cos t+e^{-3t}$.

(c) If $y^2-4t^2+2ty\frac{dy}{dt}=0$, $y(1)=3$, then $y=\sqrt{\frac{4}{3}t^2+\frac{23}{3t}}$.

(Answer 17) $y=\sqrt{(t-5)^2-16}$. The solution is valid for all $t<1$.

(Answer 18) $y=\frac{1}{4-t}$. The solution is valid for all $t<4$.