MATH 5523–5533

Theory of Functions of a Complex Variable I–II

Fall 2023–Spring 2024

(Fall) MEEG 216, MWF 2:00–2:50 p.m.
(Spring) AGRI 301A, MWF 12:55–1:45 p.m.

Next class day: Monday, April 29, 2024

Contents

[Definition: The complex numbers] The set of complex numbers is $\mathbb{R}^2$, denoted $\mathbb{C}$. (In this class, you may use everything you know about $\mathbb{R}$ and $\mathbb{R}^2$—in particular, that $\mathbb{R}^2$ is an abelian group and a normed vector space.)

[Definition: Real and imaginary parts] If (x, y) is a complex number, then $\mathrm{Re}(x, y) = x$ and $\mathrm{Im}(x, y) = y$.

[Definition: Addition and multiplication] If (x, y) and (ξ, η) are two complex numbers, we define

$$(x, y) + (\xi, \eta) = (x + \xi, y + \eta),$$
$$(x, y) \cdot (\xi, \eta) = (x\xi - y\eta, x\eta + y\xi).$$

(Problem 10) Show that multiplication in the complex numbers is commutative.

(Fact 20) This notion of addition and multiplication makes the complex numbers a ring—thus, multiplication is also associative and distributes over addition.

(Problem 30) What is the multiplicative identity?

(Problem 40) Let r be a real number. Recall that $\mathbb{C} = \mathbb{R}^2$ is a vector space over $\mathbb{R}$, so we can multiply vectors (complex numbers) by scalars (real numbers). Is there a complex number (ξ, η) such that $r(x, y) = (\xi, \eta) \cdot (x, y)$ for all $(x, y) \in \mathbb{C}$?

[Definition: Notation for the complex numbers]

- If $r \in \mathbb{R}$, we identify r with the number $(r, 0) \in \mathbb{C}$.
- We let i denote $(0, 1)$.

(Problem 50) If x, y are real numbers, what complex number is $x + iy$?

(Problem 60) If $z = x + iy$ for x, y real, what are $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$?

(Problem 70) If $z \in \mathbb{C}$ and r is real, what are $\mathrm{Re}(zr)$ and $\mathrm{Im}(zr)$?

(Problem 80) If z, $w \in \mathbb{C}$, what are $\mathrm{Re}(zw)$, $\mathrm{Im}(zw)$ in terms of $\mathrm{Re}\, z$, $\mathrm{Re}\, w$, $\mathrm{Im}\, z$, and $\mathrm{Im}\, w$?

[Definition: Conjugate] The conjugate to the complex number $x + iy$, where x, y are real, is $\overline{x + iy} = x - iy$.[1]

(Problem 90) If z and w are complex numbers, show that $\overline{z} + \overline{w} = \overline{z + w}$.

(Problem 100) Show that $\overline{z} \cdot \overline{w} = \overline{zw}$.

(Problem 110) Write $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$ in terms of z and $\overline{z}$.

(Problem 120) Show that $z\overline{z}$ is always real and nonnegative. If $z\overline{z} = 0$, what can you say about z?

(Problem 130) If z is a complex number with $z \neq 0$, show that there exists another complex number w such that $zw = 1$. Give a formula for w in terms of z. We will write $w = \frac{1}{z}$.

$z\overline{z}$ is a positive real number, and we know from real analysis that positive real numbers have reciprocals. Thus $\frac{1}{z\overline{z}} \in \mathbb{R}$. We can multiply complex numbers by real numbers, so $\frac{1}{z\overline{z}}\overline{z}$ is a complex number and it is the w of the problem statement.

[Definition: Modulus] If z is a complex number, we define its modulus $|z|$ as $|z| = \sqrt{z\overline{z}}$.

(Fact 140) $|\mathrm{Re}\, z| \leq |z|$ and $|\mathrm{Im}\, z| \leq |z|$ (where the first $|\cdot|$ denotes the absolute value in the real numbers and the second $|\cdot|$ denotes the modulus in the complex numbers.)

(Problem 150) If z and w are complex numbers, show that $|zw| = |z|\,|w|$.

(Problem 160) Give an example of a non-constant polynomial that has no roots (solutions) that are real numbers. Find a root (solution) to your polynomial that is a complex number.

[1] Some authors, especially in physics, write z^* instead of $\overline{z}$ for the complex conjugate of z.

(Fact 170) If $z = x + iy = (x, y)$, then the complex modulus $|z|$ is equal to the vector space norm $\|(x, y)\|$ in $\mathbb{R}^2$.

(Fact 180) $\mathbb{C}$ is complete as a metric space if we use the expected metric $d(z, w) = |z - w|$.

(Problem 190) Recall that $(\mathbb{R}^2, d)$ is a metric space, where $d(u, v) = \|u - v\|$. In particular, this metric satisfies the triangle inequality. Write the triangle inequality as a statement about moduli of complex numbers. Simplify your statement as much as possible.

The conclusion is that $|z + w| \leq |z| + |w|$ for all z, $w \in \mathbb{C}$. This is Proposition 1.2.3 in your textbook.

(Memory 200) If $\{a_n\}_{n=1}^{\infty}$ is a sequence of points in $\mathbb{R}^p$, $a \in \mathbb{R}^p$, and we write $a_n = (a_n^1, a_n^2, \ldots, a_n^p)$, $a = (a^1, \ldots a^p)$, then $a_n \to a$ (in the metric space sense) if and only if $a_n^k \to a^k$ for each $1 \leq k \leq p$.

(Problem 210) What does this tell you about the complex numbers?

[Definition: Maclaurin series] If $f : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable function, the Maclaurin series for f is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

with the convention that $0^0 = 1$.

(Memory 220) The Maclaurin series for the exp function is $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

(Memory 230) The Maclaurin series for the sin function is $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$.

(Memory 240) The Maclaurin series for the cos function is $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$.

(Memory 250) If x is real, then the Maclaurin series for $\exp x$, $\sin x$, or $\cos x$ converges to $\exp x$, $\sin x$, or $\cos x$, respectively.

(Memory 260) If x and t are real numbers then

$$\sin(x + t) = \sin x \cos t + \sin t \cos x,$$
$$\cos(x + t) = \cos x \cos t - \sin x \sin t.$$

(Memory 270) The Cauchy-Schwarz inequality for real numbers states that if $n \in \mathbb{N}$ is a positive integer, and if for each j with $1 \leq j \leq n$ the numbers x_j, ξ_j are real, then

$$\left(\sum_{j=1}^{n} x_j \, \xi_j\right)^2 \leq \left(\sum_{j=1}^{n} x_j^2\right)\left(\sum_{j=1}^{n} \xi_j^2\right).$$

1.2. Further Properties of the Complex Numbers

(Problem 280) State the Cauchy-Schwarz inequality for complex numbers and prove that it is valid.

This is Proposition 1.2.4 in your book. If $n \in \mathbb{N}$, and if z_1, $z_2, \ldots, z_n$ and w_1, $w_2, \ldots, w_n$ are complex numbers, then

$$\left|\sum_j z_j \, w_j\right|^2 \leq \sum_j |z_j|^2 \sum_j |w_j|^2.$$

(Problem 290) Let $z \in \mathbb{C}$. Consider the series $\sum_{j=0}^{\infty} \frac{z^j}{j!}$, that is, the sequence of complex numbers $\left\{\sum_{j=0}^{n} \frac{z^j}{j!}\right\}_{n=0}^{\infty}$. Show that this sequence is a Cauchy sequence.

(Problem 300) Since $\mathbb{C}$ is complete, the series converges. If $z = x$ is a real number, to what number does the series converge?

(Problem 310) If $z = iy$ is purely imaginary (that is, if $y \in \mathbb{R}$), show that $\sum_{j=0}^{\infty} \frac{(iy)^j}{j!}$ converges to $\cos y + i \sin y$.

(Bonus Problem 320) If $z = x + iy$, show that $\sum_{j=0}^{\infty} \frac{z^j}{j!}$ converges to the product $\left(\sum_{j=0}^{\infty} \frac{x^j}{j!}\right)\left(\sum_{j=0}^{\infty} \frac{(iy)^j}{j!}\right)$.

[Definition: The complex exponential] If x is real, we define

$$\exp(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} \quad \text{and} \quad \exp(ix) = \sum_{j=0}^{\infty} \frac{(ix)^j}{j!}.$$

If $z = x + iy$ is a complex number, we define

$$\exp(z) = \exp(x) \cdot \exp(iy).$$

(Problem 330) If y, η are real, show that $\exp(iy + i\eta) = \exp(iy) \cdot \exp(i\eta)$.

Using the sum angle identities for sine and cosine, we compute

$$\exp(iy + i\eta) = \exp(i(y + \eta)) = \cos(y + \eta) + i\sin(y + \eta) = \cos y \cos \eta - \sin y \sin \eta + i \sin y \cos \eta + i \cos y \sin \eta$$

and

$$\exp(iy)\exp(i\eta) = (\cos y + i \sin y)(\cos \eta + i \sin \eta) = \cos y \cos \eta - \sin y \sin \eta + i \sin y \cos \eta + i \cos y \sin \eta$$

and observe that they are equal.

(Problem 340) If z, w are any complex numbers, show that $\exp(z + w) = \exp(z) \cdot \exp(w)$.

There are real numbers x, y, ξ, η such that $z = x + iy$ and $w = \xi + i\eta$.
By definition

$$\exp(z) = \exp(x)\exp(iy), \qquad \exp(w) = \exp(\xi)\exp(i\eta).$$

Because multiplication in the complex numbers is associative and commutative,

$$\exp(z)\exp(w) = [\exp(x)\exp(iy)][\exp(\xi)\exp(i\eta)] = [\exp(x)\exp(\xi)][\exp(iy)\exp(i\eta)].$$

By properties of exponentials in the real numbers and by the previous problem, we see that

$$\exp(z)\exp(w) = [\exp(x)\exp(iy)][\exp(\xi)\exp(i\eta)] = \exp(x + \xi)\exp(iy + i\eta).$$

By definition of the complex exponential,

$$\exp(z)\exp(w) = [\exp(x)\exp(iy)][\exp(\xi)\exp(i\eta)] = \exp((x + \xi) + i(y + \eta)) = \exp(z + w)$$

as desired.

(Problem 350) Suppose that z is a complex number and that $|z| = 1$. Show that there is a number $\theta \in \mathbb{R}$ with $\exp(i\theta) = z$. How many such numbers θ exist?

We know from real analysis that, if (x, y) lies on the unit circle, then $(x, y) = (\cos\theta, \sin\theta)$ for some real number θ. By definition of complex modulus, if $|z| = 1$ and $z = x + iy$ then (x, y) lies on the unit circle. Thus $z = \cos\theta + i\sin\theta = \exp(i\theta)$ for some $\theta \in \mathbb{R}$.

Infinitely many such numbers θ exist.

[Chapter 1, Problem 25] If θ, $\varpi \in \mathbb{R}$, then $e^{i\theta} = e^{i\varpi}$ if and only if $(\theta - \varpi)/(2\pi)$ is an integer.

(Problem 360) Suppose that z is a complex number. Show that there exist numbers $r \in [0, \infty)$ and $\theta \in \mathbb{R}$ such that $z = r\exp(i\theta)$. How many possible values of r exist? How many possible values of θ exist?

Observe that $|re^{i\theta}| = r|e^{i\theta}|$ because $r \geq 0$ and because the modulus distributes over products. But $|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$, and so the only choice for r is $r = |z|$.

If $z = 0$ then we must have that $r = 0$ and can take any real number for θ.

If $z \neq 0$, let $r = |z|$. Then $w = \frac{1}{r}z$ is a complex number with $|z| = 1$, and so there exist infinitely many values θ with $e^{i\theta} = w$ and thus $z = re^{i\theta}$.

(Problem 370) Find all solutions to the equation $z^6 = i$.

Suppose that $z = re^{i\theta}$ for some $r \geq 0$, $\theta \in \mathbb{R}$.

Then $z^6 = r^6 e^{6i\theta}$. If $z^6 = i$, then $1 = |i| = |z^6| = r^6$ and so $r = 1$ because $r \geq 0$. We must then have that $i = e^{6i\theta}$. Observe that $i = e^{i\pi/2}$. By Homework 1.25, we must have that $6\theta = \pi/2 + 2\pi n$ for some $n \in \mathbb{Z}$, and so $(e^{i\theta})^6 = i$ if and only if $\theta = \pi/12 + n\pi/3$. Thus the solutions are

$$e^{\pi/12}, \quad e^{5\pi/12}, \quad e^{9\pi/12}, \quad e^{13\pi/12}, \quad e^{17\pi/12}, \quad e^{21\pi/12}.$$

Any other solution is of the form $e^{i\theta}$, where θ differs from one of the listed numbers by 2π.

1.3. Real Analysis

(Problem 380) Give an example of a function that can be written in two different ways.

[Definition: Ring of polynomials] Let $\mathbb{R}[z]$ be the ring of polynomials in one variable with real coefficients, that is,

$$\mathbb{R}[z] = \Big\{p : p(z) = \sum_{k=0}^{n} a_k\, z^k \text{ for some } n \in \mathbb{N}_0, a_k \in \mathbb{R}\Big\}.$$

Let $\mathbb{R}[x, y]$ be the ring of polynomials in two variables with real coefficients, that is,

$$\mathbb{R}[x, y] = \Big\{p : p(x, y) = \sum_{j=0}^{n}\sum_{k=0}^{n} a_{j,k}\, x^j\, y^k \text{ for some } n \in \mathbb{N}_0, a_{j,k} \in \mathbb{R}\Big\}.$$

[Definition: Degree] If $p \in \mathbb{R}[z]$ and $p(z) = \sum_{k=0}^{n} a_k\, z^k$, then the degree of p is the largest nonnegative integer m such that $a_m \neq 0$. (The degree of the zero polynomial $p(z) = 0$ is either undefined, -1, or $-\infty$.)

(Problem 390) Let $p(x) = \sum_{k=0}^{n} a_k\, x^k$ and let $q(x) = \sum_{k=0}^{n} b_k\, x^k$ be two polynomials in $\mathbb{R}[x]$, with a_k, $b_k \in \mathbb{R}$. Show that if $p(x) = q(x)$ for all $x \in \mathbb{R}$ then $a_k = b_k$ for all $k \in \mathbb{N}_0$.

p and q are infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}$, and because $p(x) = q(x)$ for all $x \in \mathbb{R}$, we must have that $p' = q'$, $p'' = q''$, $\dots$, $p^{(k)} = q^{(k)}$ for all $k \in \mathbb{N}$.

We compute $p^{(k)}(0) = k!a_k$ and $q^{(k)}(0) = k!b_k$. Setting them equal we see that $a_k = b_k$.

(Problem 400) Let $p \in \mathbb{R}[x]$ be a polynomial. Suppose that $x_0 \in \mathbb{R}$ and that $p(x_0) = 0$. Show that there exists a polynomial $q \in \mathbb{R}[x]$ such that $p(x) = (x - x_0)q(x)$ for all $x \in \mathbb{R}$. Further show that, if p is a polynomial of degree $m \geq 0$, then q is a polynomial of degree $m - 1$. *Hint*: Use induction.

If p is the zero polynomial we may take q to also be the zero polynomial. If p is a nonzero constant polynomial then no such x_0 can exist. We therefore need only consider the case where p is a polynomial of degree $m \geq 1$.

If $m = 1$, then $p(x) = a_1 x + a_0$ for some a_1, a_0; if $p(x_0) = 0$ then $a_0 = -a_1 x_0$ and so $p(x) = a_1(x - x_0)$. Then $q(x) = a_1$ is a polynomial of degree $0 = m - 1$.

Suppose that the statement is true for all polynomials of degree at most $m - 1$, $m \geq 2$. Let p be a polynomial of degree m. Then $p(x) = a_m x^m + r(x)$ where r is a polynomial of degree at most $m - 1$. We add and subtract $a_m x_0 x^{m-1}$ to see that

$$p(x) = a_m x^{m-1}(x - x_0) + a_m x_0 x^{m-1} + r(x).$$

Then $s(x) = a_m x_0 x^{m-1} + r(x)$ is a polynomial of degree at most $m-1$. If s is a constant then $0 = p(x_0) = a_m x_0^{m-1}(x - x_0) + s$ and so $s = 0$; taking $q(x) = a_m x^{m-1}$ we are done.

Otherwise, $s(x)$ is a polynomial of degree at least one and at most $m - 1$. Also, $s(x_0) = p(x_0) - a_m x_0^m(x_0 - x_0) = 0$, so by the induction hypothesis $s(x) = (x - x_0)t(x)$ for a polynomial t of degree at most $m - 2$. Taking $q(x) = a_m x^{m-1} + t(x)$ we are done.

(Problem 410) Let $p(x) = \sum_{k=0}^{n} a_k\, x^k$ and let $q(x) = \sum_{k=0}^{n} b_k\, x^k$ be two polynomials of degree at most n in $\mathbb{R}[x]$, with a_k, $b_k \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Suppose that there are $n + 1$ distinct numbers $x_0, x_1, \dots, x_n \in \mathbb{R}$ such that $p(x_j) = q(x_j)$ for all $0 \leq j \leq n$. Show that $a_k = b_k$ for all $k \in \mathbb{N}_0$. *Hint*: Consider the polynomial $r(x) = p(x) - q(x)$.

Let $r(x) = p(x) - q(x)$. Then $r(x_j) = p(x_j) - q(x_j) = 0$ for all $0 \leq j \leq n$ and r is a polynomial of degree at most n. Furthermore, $r(x_j) = 0$ for all $0 \leq j \leq n$.

Suppose for the sake of contradiction that r is not identically equal to zero. Then r is a polynomial of degree m, $0 \leq m \leq n$. By Problem 400,

$$r(x) = (x - x_1)(x - x_2) \dots (x - x_m) r_m(x)$$

where r_m is a polynomial of degree $m - m$, that is, a constant. But

$$0 = p(x_0) - q(x_0) = r(x_0) = (x_0 - x_1)(x_0 - x_2) \dots (x_0 - x_m) r_m(x_0).$$

Since $x_j \neq x_0$ for all $j \geq 1$ we must have that $r_m(x_0) = 0$; thus r_m is the constant function zero and so r is the constant function zero, as was to be proven. (This is technically a contradiction to the assumption $m \geq 0$ because if $m \geq 0$ then r is not the zero polynomial.)

(Problem 420) Let $p(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{n} a_{j,k} x^j y^k$ and let $q(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{n} b_{j,k} x^j y^k$ be two polynomials in $\mathbb{R}[x, y]$, with $a_{j,k}$, $b_{j,k} \in \mathbb{R}$. Show that if $p(x, y) = q(x, y)$ for all $(x, y) \in \mathbb{R}^2$ then $a_{j,k} = b_{j,k}$ for all j, $k \in \mathbb{N}_0$.

Fix a $y \in \mathbb{R}$. Then $p_y(x) = \sum_{j=0}^{n} \left(\sum_{k=0}^{n} a_{j,k} y^k \right) x^j$ and $q_y(x) = \sum_{j=0}^{n} \left(\sum_{k=0}^{n} b_{j,k} y^k \right) x^j$ are both polynomials in one variable that are equal for all x. So by Problem 390 their coefficients must be equal, so $\left(\sum_{k=0}^{n} a_{j,k} y^k \right) = \left(\sum_{k=0}^{n} b_{j,k} y^k \right)$. This is true for all $y \in \mathbb{R}$; another application of Problem 390 shows that $a_{j,k} = b_{j,k}$ for all j and k.

(Memory 421) If $\Omega \subseteq \mathbb{R}^2$ is both open and connected, then Ω is path connected: for every z, $w \in \Omega$ there is a continuous function $\gamma : [0, 1] \to \Omega$ such that $\gamma(0) = z$ and $\gamma(1) = w$.

(Memory 422) If $\Omega \subseteq \mathbb{R}^2$ is open and connected, we may require the paths in the definition of path connectedness to be C^1.

(Memory 423) If $\Omega \subseteq \mathbb{R}^2$ is open and connected, we may require the paths in the definition of path connectedness to consist of finitely many horizontal or vertical line segments.

Definition 1.3.1 (part 1). Let $\Omega \subseteq \mathbb{R}^2$ be open. Suppose that $f : \Omega \to \mathbb{R}$. We say that f is continuously differentiable, or $f \in C^1(\Omega)$, if the two partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere in Ω and f, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ are all continuous on Ω.

(Problem 424) Let $B = B(z, r)$ be a ball in $\mathbb{R}^2$. Let $f \in C^1(B)$ and suppose that $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} = 0$ everywhere in B. Show that f is a constant.

Let $z = (x, y)$. Let $(\xi, \eta) \in B((x, y), r)$.

We consider the case $\xi \geq x$ and $\eta \geq y$; the cases $\xi < x$ or $\eta < y$ are similar. Then $\{(t, y) : x \leq t \leq \xi\} \subset B((x, y), r)$, and if we let $F_y(x) = F(x, y)$, then F_y is a continuously differentiable function on $[x, \xi]$ with $F_y'(t) = 0$ for all $x \leq t \leq \xi$; by the Mean Value Theorem, $F_y(x) = F_y(\xi)$ and so $f(x, y) = f(\xi, y)$. Similarly, $\{(\xi, t) : y \leq t \leq \eta\} \subset B((x, y), r)$, and so $f(x, y) = f(\xi, y) = f(\xi, \eta)$.

Thus f is a constant in $B((x, y), r)$.

(Problem 430) Suppose that $\Omega \subseteq \mathbb{R}^2$ is open and connected. Let $f \in C^1(\Omega)$ and suppose that $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} = 0$ everywhere in Ω. Show that f is a constant.

Pick some $(x_0, y_0) \in \Omega$ and let $s = f(x_0, y_0)$.

If $(x, y) \in f^{-1}(s)$ then $f(x, y) = s$, and because Ω is open there is a $r > 0$ such that $B((x, y), r) \subset \Omega$. By the previous problem f is constant on $B((x, y), r)$ and so $B((x, y), r) \subset f^{-1}(\{s\})$. Thus $f^{-1}(\{s\})$ is open.

But $\{s\}$ is a closed set, and f is continuous, and so $f^{-1}(\{s\})$ must be relatively closed in Ω.

Because Ω is connected, and $f^{-1}(\{s\})$ is both relatively open and relatively closed, $f^{-1}(\{s\})$ must be either empty or all of Ω; because $(x_0, y_0) \in f^{-1}(\{s\})$, $f^{-1}(\{s\})$ must be Ω and so $f(x, y) = s$ for all $(x, y) \in \Omega$.

[Definition: Ring of polynomials] Let $\mathbb{C}[z]$ be the ring of polynomials in one variable with complex coefficients, that is,

$$\mathbb{C}[z] = \Big\{p : p(z) = \sum_{k=0}^{n} a_k\, z^k \text{ for some } n \in \mathbb{N}_0,\, a_k \in \mathbb{C}\Big\}.$$

Let $\mathbb{C}[x, y]$ be the ring of polynomials in two variables with complex coefficients, that is,

$$\mathbb{C}[x, y] = \Big\{p : p(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{n} a_{j,k}\, x^j\, y^k \text{ for some } n \in \mathbb{N}_0,\, a_{j,k} \in \mathbb{C}\Big\}.$$

(Problem 440) Let $p(z) = \sum_{k=0}^{n} a_k\, z^k$ and let $q(z) = \sum_{k=0}^{n} b_k\, z^k$ be two polynomials in $\mathbb{C}[z]$. Show that if $p(x) = q(x)$ for all $x \in \mathbb{R}$ then $a_k = b_k$ for all k, and so $p(z) = q(z)$ for all $z \in \mathbb{C}$.

Define $p_r(x) = \sum_{k=0}^{n} (\operatorname{Re} a_k)\, x^k$ and $q_r(x) = \sum_{k=0}^{n} (\operatorname{Re} b_k)\, x^k$. Then p_r, $q_r \in \mathbb{R}[x]$. If $x \in \mathbb{R}$ then

$$p_r(x) = \sum_{k=0}^{n} (\operatorname{Re} a_k)\, x^k = \operatorname{Re}\Big(\sum_{k=0}^{n} a_k\, x^k\Big) = \operatorname{Re} p(x)$$
$$= \operatorname{Re} q(x) = \operatorname{Re}\Big(\sum_{k=0}^{n} b_k\, x^k\Big) = \sum_{k=0}^{n} (\operatorname{Re} b_k)\, x^k = q_r(x)$$

and so by Problem 390 we must have that $\operatorname{Re} a_k = \operatorname{Re} b_k$ for all k. Similarly, $\operatorname{Im} a_k = \operatorname{Im} b_k$ for all k and so $a_k = \operatorname{Re} a_k + i \operatorname{Im} a_k = \operatorname{Re} b_k + i \operatorname{Im} b_k = b_k$ for all k, as desired.

(Problem 450) Show that Problems 400 and 410 are valid for polynomials in $\mathbb{C}[z]$ with complex roots.

(Problem 460) Let $p(z, w) = \sum_{j=0}^{n} \sum_{k=0}^{n} a_{j,k}\, z^j\, w^k$ and let $q(z, w) = \sum_{j=0}^{n} \sum_{k=0}^{n} b_{j,k}\, z^j\, w^k$ be two polynomials in $\mathbb{C}[z, w]$, with $a_{j,k}$, $b_{j,k} \in \mathbb{C}$. Show that if $p(x, y) = q(x, y)$ for all $(x, y) \in \mathbb{R}^2$ then $a_{j,k} = b_{j,k}$ for all j, $k \in \mathbb{N}_0$.

(Problem 470) Let $p(z, w) = \sum_{j=0}^{n} \sum_{k=0}^{n} a_{j,k}\, z^j\, w^k$ and let $q(z, w) = \sum_{j=0}^{n} \sum_{k=0}^{n} b_{j,k}\, z^j\, w^k$ be two polynomials in $\mathbb{C}[z, w]$, with $a_{j,k}$, $b_{j,k} \in \mathbb{C}$. Show that if $p(z, \bar{z}) = q(z, \bar{z})$ for all $z \in \mathbb{C}$ then $a_{j,k} = b_{j,k}$ for all j, $k \in \mathbb{N}_0$.

(Problem 480) Let $p \in \mathbb{C}[z, w]$ satisfy $p(z, \bar{z}) = z^2 - \bar{z}^3$. Is there a polynomial $q \in \mathbb{C}[z]$ such that $q(z) = p(z, \bar{z})$ for all $z \in \mathbb{C}$?

No. Suppose for the sake of contradiction that such a q exists. Then $q(x) = p(x, x) = x^2 - x^3$ for all $x \in \mathbb{R}$. So we must have that $q(z) = z^2 - z^3$ for all $z \in \mathbb{C}$. In particular, $q(i) = i^2 - i^3 = -1 + i$, and $p(i, -i) = i^2 + i^3 = -1 - i$, and so we cannot have $q(z) = p(z, \bar{z})$ for all $z \in \mathbb{C}$.

Definition 1.3.1 (part 2). Let $\Omega \subseteq \mathbb{C}$ be an open set. Recall $\mathbb{C} = \mathbb{R}^2$. Let $f : \Omega \to \mathbb{C}$ be a function. Then $f \in C^1(\Omega)$ if $\operatorname{Re} f$, $\operatorname{Im} f \in C^1(\Omega)$.

[Definition: Derivative of a complex function] Let $f \in C^1(\Omega)$. Let $u(z) = \operatorname{Re} f(z)$ and let $v(z) = \operatorname{Im} f(z)$. Then

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \qquad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}.$$

(Problem 490) Establish the Leibniz rules

$$\frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x} g + f \frac{\partial g}{\partial x}, \qquad \frac{\partial}{\partial y}(fg) = \frac{\partial f}{\partial y} g + f \frac{\partial g}{\partial y}$$

for f, $g \in C^1(\Omega)$.

Let $f = u + iv$, $g = w + i\varpi$, where u, v, w, and ϖ are real-valued functions in $C^1(\Omega)$.

Then $fg = (uw - v\varpi) + i(vw + u\varpi)$, where $(uw - v\varpi)$ and $(vw + u\varpi)$ are both real-valued C^1 functions.

Then

$$\frac{\partial}{\partial x}(fg) = \frac{\partial}{\partial x}[(uw - v\varpi) + i(vw + u\varpi)]$$
$$= \frac{\partial}{\partial x}(uw - v\varpi) + i\frac{\partial}{\partial x}(vw + u\varpi).$$

Applying the Leibniz (product) rule for real-valued functions, we see that

$$\frac{\partial}{\partial x}(fg) = \frac{\partial u}{\partial x}w + u\frac{\partial w}{\partial x} - \frac{\partial v}{\partial x}\varpi - v\frac{\partial \varpi}{\partial x}$$
$$+ i\frac{\partial v}{\partial x}w + iv\frac{\partial w}{\partial x} + i\frac{\partial u}{\partial x}\varpi + iu\frac{\partial \varpi}{\partial x}.$$

Furthermore,

$$f\frac{\partial g}{\partial x} + \frac{\partial f}{\partial x}g = (u + iv)\left(\frac{\partial w}{\partial x} + i\frac{\partial \varpi}{\partial x}\right)$$
$$+ \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(w + i\varpi)$$
$$= u\frac{\partial w}{\partial x} + iu\frac{\partial \varpi}{\partial x} + iv\frac{\partial w}{\partial x} - v\frac{\partial \varpi}{\partial x}$$
$$+ \frac{\partial u}{\partial x}w + i\frac{\partial u}{\partial x}\varpi + i\frac{\partial v}{\partial x}w - \frac{\partial v}{\partial x}\varpi.$$

Rearranging, we see that the two terms are the same.

1.3. The complex derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$

[Definition: Complex derivative] Let $f \in C^1(\Omega)$. Then

$$\frac{\partial f}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}.$$

(Problem 500) Let $f(z) = z$. Show that $\frac{\partial f}{\partial z} = 1$ and $\frac{\partial f}{\partial \bar{z}} = 0$.

Recall that $z = x + iy$. Thus,

$$\frac{\partial}{\partial z}(z) = \frac{1}{2}\frac{\partial}{\partial x}(x + iy) + \frac{1}{2i}\frac{\partial}{\partial y}(x + iy) = \frac{1}{2} + \frac{i}{2i} = 1$$

and

$$\frac{\partial}{\partial \bar{z}}(z) = \frac{1}{2}\frac{\partial}{\partial x}(x + iy) - \frac{1}{2i}\frac{\partial}{\partial y}(x + iy) = \frac{1}{2} - \frac{i}{2i} = 0.$$

(Problem 510) Let $g(z) = \bar{z}$. Show that $\frac{\partial g}{\partial z} = 0$ and $\frac{\partial g}{\partial \bar{z}} = 1$.

Recall that $\bar{z} = x - iy$. Thus,

$$\frac{\partial}{\partial z}(\bar{z}) = \frac{1}{2}\frac{\partial}{\partial x}(x - iy) + \frac{1}{2i}\frac{\partial}{\partial y}(x - iy) = \frac{1}{2} - \frac{i}{2i} = 0$$

and

$$\frac{\partial}{\partial \bar{z}}(\bar{z}) = \frac{1}{2}\frac{\partial}{\partial x}(x - iy) - \frac{1}{2i}\frac{\partial}{\partial y}(x - iy) = \frac{1}{2} + \frac{i}{2i} = 1.$$

(Problem 520) Show that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are linear operators.

This follows immediately from linearity of the differential operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

(Problem 530) Show that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ commute in the sense that, if $\Omega \subseteq \mathbb{C}$ is open and $f \in C^2(\Omega)$, then $\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}}f\right) = \frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial z}f\right)$.

(Problem 540) Establish the Leibniz rules

$$\frac{\partial}{\partial z}(fg) = \frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}, \qquad \frac{\partial}{\partial \overline{z}}(fg) = \frac{\partial f}{\partial \overline{z}}g + f\frac{\partial g}{\partial \overline{z}}.$$

We have that

$$\frac{\partial}{\partial z}(fg) = \frac{1}{2}\frac{\partial}{\partial x}(fg) + \frac{1}{2i}\frac{\partial}{\partial y}(fg).$$

Using the Leibniz rules for $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, we see that

$$\begin{aligned}\frac{\partial}{\partial z}(fg) &= \frac{1}{2}f\frac{\partial g}{\partial x} + \frac{1}{2}\frac{\partial f}{\partial x}g + \frac{1}{2i}f\frac{\partial g}{\partial y} + \frac{1}{2i}\frac{\partial f}{\partial y}g \\ &= f\left(\frac{1}{2}\frac{\partial g}{\partial x} + \frac{1}{2i}\frac{\partial g}{\partial y}\right) + \left(\frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}\right)g \\ &= f\frac{\partial g}{\partial z} + \frac{\partial f}{\partial z}g.\end{aligned}$$

The argument for $\frac{\partial}{\partial \overline{z}}$ is similar.

(Problem 550) Show that $\frac{\partial}{\partial z}(z^\ell \overline{z}^m) = \ell z^{\ell-1}\overline{z}^m$ and $\frac{\partial}{\partial \overline{z}}(z^\ell \overline{z}^m) = m z^\ell \overline{z}^{m-1}$ for all nonnegative integers m and ℓ.

(Problem 560) Let j, k, ℓ, and m be nonnegative integers. Find $\frac{\partial^j}{\partial z^j}\frac{\partial^k}{\partial \overline{z}^k}(z^\ell \overline{z}^m)$.

A straightforward (but long and tedious) induction argument shows that

$$\frac{\partial^j}{\partial z^j}\frac{\partial^k}{\partial \overline{z}^k}(z^\ell \overline{z}^m) = \frac{\ell! m!}{(\ell-j)!(m-k)!} z^{\ell-j}\overline{z}^{m-k}$$

(Problem 570) Let $p \in \mathbb{C}[z,w]$. Show that there is a $q \in \mathbb{C}[z]$ such that $p(z,\overline{z}) = q(z)$ for all $z \in \mathbb{C}$ if and only if $\frac{\partial}{\partial \overline{z}}(p(z,\overline{z})) = 0$ everywhere in $\mathbb{C}$.

(Problem 580) Let $p \in \mathbb{C}[x,y]$. Show that there is a $q \in \mathbb{C}[z]$ such that $p(x,y) = q(x+iy)$ for all x, $y \in \mathbb{R}$ if and only if $\frac{\partial}{\partial \overline{z}}(p(x,y)) = 0$ everywhere in $\mathbb{C}$.

As in the previous problem, if $p(x,y) = q(x+iy)$ for some $q \in \mathbb{C}[z]$, then $\frac{\partial}{\partial \overline{z}}q(z) = 0$ for all $z \in \mathbb{C}$ and so $\frac{\partial}{\partial \overline{z}}p(x,y) = 0$ for all x, $y \in \mathbb{R}$.

Conversely, suppose $\frac{\partial}{\partial \overline{z}}p = 0$. There is a $r \in \mathbb{C}[z,w]$ such that $p(x,y) = r(x+iy, x-iy)$ for all x, $y \in \mathbb{R}$. Furthermore, $\frac{\partial}{\partial \overline{z}}r(z,\overline{z}) = \frac{\partial}{\partial \overline{z}}p(x,y) = 0$ for all $z \in \mathbb{C}$, and so by the previous problem $r(z,\overline{z}) = q(z)$ for some $q \in \mathbb{C}[z]$. But then $p(x,y) = r(x+iy, x-iy) = q(x+iy)$ for all x, $y \in \mathbb{R}$, as desired.

(Problem 590) Suppose that $\Omega \subseteq \mathbb{C}$ is open and connected, that $f \in C^1(\Omega)$, and that $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \overline{z}} = 0$ in Ω. Show that f is constant in Ω.

We observe that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \overline{z}}, \qquad \frac{\partial f}{\partial y} = i\frac{\partial f}{\partial z} - i\frac{\partial f}{\partial \overline{z}}.$$

Thus $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ in Ω and the result follows from Problem 430.

(Problem 600) Suppose that $\Omega \subseteq \mathbb{C}$ is open and that $f \in C^1(\Omega)$. Show that

$$\frac{\partial f}{\partial z} = \overline{\left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)}.$$

We write $f = u + iv$, where u and v are real-valued functions. By definition, u and v are in $C^1(\Omega)$.

Then

$$\frac{\partial}{\partial \bar z}(\bar f) = \left(\frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}\right)(u - iv)$$
$$= \frac{1}{2}\frac{\partial u}{\partial x} - \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial v}{\partial y}$$

and

$$\frac{\partial}{\partial z} f = \left(\frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y}\right)(u + iv)$$
$$= \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} + \frac{1}{2i}\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial v}{\partial y}$$

which we see is the complex conjugate of the previously found value.

(Problem 610) Show that $\frac{\partial}{\partial z}\frac{1}{z} = -\frac{1}{z^2}$ if $z \neq 0$.

(Problem 620) Show that $\frac{\partial}{\partial \bar z}\frac{1}{z} = 0$ if $z \neq 0$.

Observe that $\frac{1}{z} = \frac{\bar z}{z\bar z}$ and so if $z = x + iy$, x, $y \in \mathbb{R}$, then $\frac{1}{z} = \frac{x-iy}{x^2+y^2}$. We compute

$$\frac{\partial}{\partial x}\frac{1}{z} = \frac{y^2 - x^2 + 2ixy}{(x^2+y^2)^2}, \qquad \frac{\partial}{\partial y}\frac{1}{z} = \frac{-ix^2 + iy^2 - 2xy}{(x^2+y^2)^2}.$$

Thus

$$\frac{\partial}{\partial z}\frac{1}{z} = \frac{1}{2}\frac{\partial}{\partial x}\frac{1}{z} - \frac{i}{2}\frac{\partial}{\partial y}\frac{1}{z} = \frac{y^2 - x^2 + 2ixy}{(x^2+y^2)^2}$$
$$= \frac{-(x-iy)^2}{(x+iy)^2(x-iy)^2} = -\frac{1}{z^2}$$

and

$$\frac{\partial}{\partial z}\frac{1}{z} = \frac{1}{2}\frac{\partial}{\partial x}\frac{1}{z} + \frac{i}{2}\frac{\partial}{\partial y}\frac{1}{z} = 0.$$

(Problem 630) Find $\frac{\partial}{\partial z}\frac{1}{z^n}$ and $\frac{\partial}{\partial \bar z}\frac{1}{z^n}$ for any positive integer n.

Using the previous problem as our base case and the Leibniz rule for the inductive step, a straightforward induction argument shows that

$$\frac{\partial}{\partial z}\frac{1}{z^n} = -\frac{n}{z^{n+1}}, \qquad \frac{\partial}{\partial \bar z}\frac{1}{z^n} = 0.$$

[Chapter 1, Problem 49] Let Ω, $W \subseteq \mathbb{C}$ be open and let $g : \Omega \to W$, $f : W \to \mathbb{C}$ be two C^1 functions. The following chain rules are valid:

$$\frac{\partial}{\partial z}(f \circ g) = \frac{\partial f}{\partial g}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar g}\frac{\partial \bar g}{\partial z},$$
$$\frac{\partial}{\partial \bar z}(f \circ g) = \frac{\partial f}{\partial g}\frac{\partial g}{\partial \bar z} + \frac{\partial f}{\partial \bar g}\frac{\partial \bar g}{\partial \bar z},$$

where $\frac{\partial f}{\partial g} = \frac{\partial f}{\partial z}\big|_{z \to g(z)}$, $\frac{\partial f}{\partial \bar g} = \frac{\partial f}{\partial \bar z}\big|_{z\to g(z)}$.

In particular, if f and g are both holomorphic then so is $f \circ g$.

Definition 1.4.1. Let $\Omega \subseteq \mathbb{C}$ be open and let $f \in C^1(\Omega)$. We say that f is holomorphic in Ω if

$$\frac{\partial f}{\partial \bar{z}} = 0$$

everywhere in Ω.

Lemma 1.4.2. Let $f \in C^1(\Omega)$, let $u = \operatorname{Re} f$, and let $v = \operatorname{Im} f$. Then f is holomorphic in Ω if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

everywhere in Ω. (These equations are called the Cauchy-Riemann equations.)

(Problem 640) Prove the "only if" direction of Lemma 1.4.2: Suppose that f is holomorphic in Ω, $\Omega \subsetneq \mathbb{C}$ open, then the Cauchy-Riemann equations hold for $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$.

(Problem 650) Prove the "if" direction of Lemma 1.4.2: suppose that $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are C^1 in Ω and satisfy the Cauchy-Riemann equations. Show that f is holomorphic in Ω.

Recall that

$$2\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}$$

by definition of $\frac{\partial}{\partial \bar{z}}$. Applying the fact that $f = u + iv$, we see that

$$\begin{aligned} 2\frac{\partial f}{\partial \bar{z}} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \\ &= \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \end{aligned}$$

Because u and v are real-valued, so are their derivatives. Thus, the real and imaginary parts of the right hand side, respectively, are $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$.

Thus, $\frac{\partial f}{\partial \bar{z}} = 0$ if and only if the Cauchy-Riemann equations hold.

Proposition 1.4.3. [Slight generalization.] Let $f \in C^1(\Omega)$. Then f is holomorphic at $p \in \Omega$ if and only if $\frac{\partial f}{\partial x}(p) = \frac{1}{i}\frac{\partial f}{\partial y}(p)$ and that in this case

$$\frac{\partial f}{\partial z}(p) = \frac{\partial f}{\partial x}(p) = \frac{1}{i}\frac{\partial f}{\partial y}(p).$$

(Problem 660) Begin the proof of Proposition 1.4.3 by showing that if f is holomorphic then $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$.

By definition of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$, if $f \in C^1(\Omega)$ then $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial \bar{z}}$ and $\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial z} - i\frac{\partial f}{\partial \bar{z}}$. Thus, if $\frac{\partial f}{\partial \bar{z}}(p) = 0$ then $\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial z}(p)$ and $\frac{\partial f}{\partial y}(p) = i\frac{\partial f}{\partial z}(p) = i\frac{\partial f}{\partial x}(p)$, as desired.

(Problem 670) Complete the proof of Proposition 1.4.3 by showing that if $f \in C^1(\Omega)$ and $\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$, then f is holomorphic.

Recall

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right).$$

Thus, if $\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$ then $\frac{\partial f}{\partial \bar{z}} = 0$, as desired.

Definition 1.4.4. We let $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. If $\Omega \subseteq \mathbb{C}$ is open and $u \in C^2(\Omega)$, then u is harmonic if

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

everywhere in Ω.

(Problem 671) Show that if $f \in C^1(\Omega)$ then $\triangle f = 4\frac{\partial}{\partial z}\frac{\partial f}{\partial \bar z} = 4\frac{\partial}{\partial \bar z}\frac{\partial f}{\partial z}$.

We compute that

$$\begin{aligned}\frac{\partial}{\partial z}\frac{\partial f}{\partial \bar z} &= \frac{1}{4}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right) \\ &= \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{1}{i}\frac{\partial^2 f}{\partial y \partial x} - \frac{1}{i}\frac{\partial^2 f}{\partial x \partial y}\right).\end{aligned}$$

If $f \in C^1$ then $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ and the proof is complete. The argument for $\frac{\partial}{\partial \bar z}\frac{\partial f}{\partial z}$ is similar.

(Problem 680) Suppose that f is holomorphic and C^2 in an open set Ω and that $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. Compute $\triangle u$ and $\triangle v$.

Because f is holomorphic,

$$\triangle f = 4\frac{\partial}{\partial z}\frac{\partial f}{\partial \bar z} = 4\frac{\partial}{\partial z}0 = 0.$$

But

$$\triangle f = (\triangle u) + i(\triangle v)$$

and $\triangle u$ and $\triangle v$ are both real-valued, so because $\triangle f = 0$ we must have $\triangle u = 0 = \triangle v$ as well.

(Problem 690) Let $f \in \mathbb{C}[z]$ be a holomorphic polynomial. Show that there is a polynomial $F \in \mathbb{C}[z]$ such that $\frac{\partial F}{\partial z} = f$. How many such polynomials are there?

Lemma 1.4.5. Let u be harmonic and real valued in $\mathbb{C}$. Suppose in addition that $u \in \mathbb{R}[x, y]$, that is, that u is a polynomial. Then there is a holomorphic polynomial $f \in \mathbb{C}[z]$ such that $u(x, y) = \operatorname{Re} f(x + iy)$.

(Problem 700) Prove Lemma 1.4.5. *Hint*: Start by computing $\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}$ and $\frac{\partial}{\partial \bar z}\frac{\partial}{\partial z}$.

1.5. Real Analysis

(Memory 710) Let $a < c < b$ and let $f : (a, b) \to \mathbb{R}$ be continuous. Show that $\lim_{t\to 0} \frac{1}{t}\int_c^{c+t} f(x)\,dx = f(c)$.

(Memory 720) State Green's theorem.

(Memory 721) State the Mean Value Theorem.

(Memory 722) If $a < b$, if each f_n is bounded and Riemann integrable on $[a, b]$, and if $f_n \to f$ uniformly on $[a, b]$, then f is also Riemann integrable on $[a, b]$, $\lim_{n\to\infty}\int_a^b f_n$ exists, and $\int_a^b f = \lim_{n\to\infty}\int_a^b f_n$.

(Problem 723) Let $f : [a, b] \times [c, d] \to \mathbb{R}$. Suppose that f is continuous on $[a, b] \times [c, d]$. Define $F : [a, b] \to \mathbb{R}$ by $F(x) = \int_c^d f(x, y)\,dy$. Show that F is continuous on $[c, d]$.

(Problem 730) Let $f : (a, b) \times [c, d] \to \mathbb{R}$. Suppose that f is continuous on $(a, b) \times [c, d]$ and the function $\partial_x f = \frac{\partial f}{\partial x}$ is continuous on $(a, b) \times [c, d]$. Show that

$$\frac{d}{dx}\int_c^d f(x, y)\,dy = \int_c^d \frac{\partial}{\partial x} f(x, y)\,dy$$

for all $a < x < b$. In particular, note that the derivative exists and the function $F(x) = \int_c^d f(x, y)\,dy$ is continuous on (a, b).

(Fact 731) This is still true if f is continuous on $[x, b)$ or $(a, x]$ and we extend $\partial_1 f$ to $(a, x] \times [c, d]$ or $[x, b) \times [c, d]$ by taking one-sided derivatives.

(Memory 740) Let f be a C^2 function in an open set in $\mathbb{R}^2$. Show that $\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$.

(Problem 750) Prove the converse. That is, suppose that there are two C^1 functions g and h defined in an open rectangle or disc $\mathcal{R}$ such that $\frac{\partial}{\partial x}g = \frac{\partial}{\partial y}h$ everywhere in $\mathcal{R}$. Show that there is a function $f \in C^2(\mathcal{R})$ such that $\frac{\partial f}{\partial y} = g$ and $\frac{\partial f}{\partial x} = h$.

Let $P = (x_0, y_0)$ and let $f(x,y) = \int_{x_0}^{x} h(s, y_0)\,ds + \int_{y_0}^{y} g(x,t)\,dt$. Observe that if $(x,y) \in \mathcal{R}$ then so is (s, y_0) and (x,t) for all s between x_0 and x and all t between y_0 and y. Thus g and h are defined at all required values. Because g and h are continuous, the integrals exist.

Furthermore, I claim f is continuous. Let $(x,y) \in \mathcal{R}$ and let $\delta_1 > 0$ be such that $B((x,y),\delta_1) \subset \mathcal{R}$. By continuity of g and h and compactness of $\overline{B}((x,y),\delta_1/2)$, g and h are bounded on $\overline{B}((x,y),\delta_1/2)$. If $(\xi,\eta) \in B((x,y),\delta_1/2)$, then

$$|f(\xi,\eta) - f(x,y)| = \left|\int_x^{\xi} h(s,y_0)\,ds + \int_{y_0}^{y} g(\xi,t) - g(x,t)\,dt + \int_y^{\eta} g(\xi,t)\,dt\right|$$
$$\leq |\xi - x| \sup_{\overline{B}((x,y),\delta_1/2)} |h| + |\eta - y| \sup_{\overline{B}((x,y),\delta_1/2)} |g| + \left|\int_{y_0}^{y} g(x,t) - g(\xi,t)\,dt\right|.$$

Furthermore, g must be uniformly continuous on $\overline{B}((x,y),\delta_1/2)$. Choose $\varepsilon > 0$ and let δ_2 be such that if $|(x,t) - (\xi,t)| < \delta_2$ then $|g(x,t) - g(\xi,t)| < \varepsilon$. We then have that

$$|f(x,y) - f(\xi,\eta)| \leq |\xi - x| \sup_{\overline{B}((x,y),\delta_1/2)} |h| + |\eta - y| \sup_{\overline{B}((x,y),\delta_1/2)} |g| + |y_0 - y|\varepsilon.$$

There is then a $\delta_3 > 0$ such that if $|(\xi,\eta) - (x,y)| < \delta_3$ then $|f(x,y) - f(\xi,\eta)| < (1 + |y_0 - y|)\varepsilon$, and so f is continuous at (x,y), as desired.

By the fundamental theorem of calculus, we have that $\frac{\partial f}{\partial y} = g$ everywhere in $\mathcal{R}$, including at $(x,y) = (x_0,y_0) = P$.

Furthermore, by 730 and the fundamental theorem of calculus, we have that

$$\frac{\partial f}{\partial x}(x,y) = h(x,y_0) + \int_{y_0}^{y} \frac{\partial g}{\partial x}(x,t)\,dt.$$

We have that $\frac{\partial g}{\partial x}(x,t) = \frac{\partial h}{\partial t}(x,t)$ and so by the fundamental theorem of calculus $\frac{\partial f}{\partial x} = h$.

(Bonus Problem 760) State the definition of a simply connected set and then generalize Problem 750 to any simply connected open set.

(Problem 770) Let $\mathcal{R} = \mathbb{R}^2 \setminus \{(0,0)\}$. Let $g(x,y) = \frac{x}{x^2+y^2}$ and $h(x,y) = \frac{-y}{x^2+y^2}$. Show that $\frac{\partial}{\partial x}g = \frac{\partial}{\partial y}h$.

This is routine calculation. By the quotient rule of undergraduate calculus,

$$\frac{\partial}{\partial x}g = \frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

and

$$\frac{\partial}{\partial y}h = \frac{-1(x^2+y^2) - (-y)(2y)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

which are equal.

(Problem 780) Show that there is no function $f \in C^1(\mathcal{R})$ such that $\frac{\partial f}{\partial y} = g$ and $\frac{\partial f}{\partial x} = h$.

(Problem 790) Why doesn't this contradict Problem 760?

The domain $\mathcal{R}$ is not simply connected.

(Problem 800) Suppose that u is real-valued and harmonic (and not necessarily a polynomial) in an open rectangle or disc $\mathcal{R}$. Show that there is a function f that is holomorphic in $\mathcal{R}$ such that $u = \operatorname{Re} f$.

Let $g = \frac{\partial u}{\partial x}$ and let $h = -\frac{\partial u}{\partial y}$. Then

$$\frac{\partial g}{\partial x} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} = \frac{\partial h}{\partial y}$$

by definition of g and h and because u is harmonic. Thus by Problem 760 there is a $v : \mathcal{R} \to \mathbb{R}$ such that $\frac{\partial v}{\partial y} = g = \frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x} = h = -\frac{\partial u}{\partial y}$.

Then u and v satisfy the Cauchy-Riemann equations, and so by Problem 650, $f = u + iv$ is holomorphic in $\mathcal{R}$.

(Problem 810) Suppose that f is holomorphic in an open rectangle or disc $\mathcal{R}$. Show that there is a function F that is holomorphic in $\mathcal{R}$ such that $f = \frac{\partial F}{\partial z}$.

2.1. Real Analysis

(Memory 820) State the Intermediate Value Theorem.

(Memory 830) State the change of variables theorem for integrals over real intervals.

(Memory 840) Let $a < b$ and let $\varphi : [a, b] \to \mathbb{R}$ be continuous. Then $\left|\int_a^b \varphi\right| \leq \int_a^b |\varphi| \leq (b - a) \sup_{[a,b]} |\varphi|$.

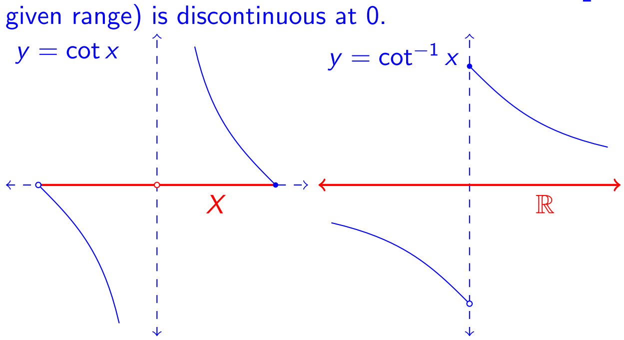
[Definition: Continuous] Let (X, d) and (Z, ρ) be two metric spaces and let $f : X \to Z$. We say that f is continuous at $x \in X$ if, for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $d(x, y) < \delta$ and $y \in X$ then $\rho(f(x), f(y)) < \varepsilon$.

(Memory 850) Let X be a compact metric space and let $f : X \to Z$ be a continuous function. Then $f(X)$ is compact.

(Memory 860) Let X be a compact metric space and let $f : X \to Z$ be a continuous bijection. Then f^{-1} is also continuous.

(Problem 870) Is the previous problem true if X is not compact?

No. Let $X = (-\pi/2, 0) \cup (0, \pi/2] \subset \mathbb{R}$ with the usual metric on $\mathbb{R}$. Then the function $\cot : X \to \mathbb{R}$ is continuous on X and is a bijection, but $\cot^{-1}(0) = \frac{\pi}{2}$ and $\lim_{x \to 0^-} \cot^{-1}(x) = -\frac{\pi}{2}$, and so $\cot^{-1}$ (with the given range) is discontinuous at 0.



(Memory 880) If $\gamma : X \to \mathbb{R}^2$ and $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ for all $t \in X$, then γ is continuous if and only if γ_1 and γ_2 are continuous.

Definition 2.1.1. (C^1 on a closed set.) Let $[a, b] \subseteq \mathbb{R}$ be a closed bounded interval and let $f : [a, b] \to \mathbb{R}$. We say that $f \in C^1([a, b])$, or f is continuously differentiable on $[a, b]$, if

(a) f is continuous on $[a, b]$,
(b) f is differentiable on (a, b),
(c) The derivative f' is continuous on (a, b),
(d) $\lim_{t \to a^+} f'(t)$ and $\lim_{t \to b^-} f'(t)$ both exist and are finite.

(Memory 890) If the conditions (a), (b) and (c) hold, then the condition (d) holds if and only if the two limits $\lim_{t\to a^+} \frac{f(t)-f(a)}{t-a}$ and $\lim_{t\to b^-} \frac{f(b)-f(t)}{b-t}$ exist, and in this case $\lim_{t\to a^+} \frac{f(t)-f(a)}{t-a} = \lim_{t\to a^+} f'(t)$ and $\lim_{t\to b^-} \frac{f(b)-f(t)}{b-t} = \lim_{t\to b^-} f'(t)$.

[Definition: One-sided derivative] If $f : [a, b] \to \mathbb{R}$, we define $f'(a) = \lim_{t\to a^+} \frac{f(t)-f(a)}{t-a}$ and $f'(b) = \lim_{t\to b^-} \frac{f(b)-f(t)}{b-t}$, if these limits exist.

(Memory 900) Suppose that $a < p < b$ and let $H : (a, b) \to \mathbb{R}$ be continuous. Suppose that H is differentiable on both (a, p) and (p, b), and that $\lim_{x\to p} H'(x) = h$ for some $h \in \mathbb{R}$. Then $H'(p)$ exists and that $H'(p) = \lim_{x\to p} H'(x)$.

[Definition: Curve] A curve in $\mathbb{R}^2$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$, where $[a, b] \subseteq \mathbb{R}$ is a closed and bounded interval. The trace (or image) of γ is $\widetilde{\gamma} = \gamma([a, b]) = \{\gamma(t) : t \in [a, b]\}$.

[Definition: Closed; simple] A curve $\gamma : [a, b] \to \mathbb{R}^2$ is closed if $\gamma(a) = \gamma(b)$. A closed curve is simple if $\gamma(b) = \gamma(a)$ and γ is injective on $[a, b)$ (equivalently on $(a, b]$).

[Definition: C^1 curve in $\mathbb{R}^2$] A curve $\gamma : [a, b] \to \mathbb{R}^2$ is C^1 (or continuously differentiable) if $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ for all $t \in [a, b]$ and both γ_1, γ_2 are C^1. We write

$$\gamma'(t) = \frac{d\gamma}{dt} = \left(\frac{d\gamma_1}{dt}, \frac{d\gamma_2}{dt}\right).$$

[Definition: Arc length] If $\gamma : [a, b] \to \mathbb{R}^2$ is a C^1 curve, then its length (or arc length) is $\int_a^b \|\gamma'(t)\|\, dt$.

Proposition 2.1.4. Let $\gamma \in C^1([a, b])$, $\gamma : [a, b] \to \Omega$ for some open set $\Omega \subseteq \mathbb{R}^2$ and let $f : \Omega \to \mathbb{R}$ with $f \in C^1(\Omega)$. Then

$$f(\gamma(b)) - f(\gamma(a)) = \int_a^b \frac{\partial f}{\partial x}\bigg|_{(x,y)=\gamma(t)} \frac{\partial \gamma_1}{\partial t} + \frac{\partial f}{\partial y}\bigg|_{(x,y)=\gamma(t)} \frac{\partial \gamma_2}{\partial t}\, dt.$$

(Problem 910) Prove Proposition 2.1.4. *Hint*: Start by computing $\frac{d(f\circ\gamma)}{dt}$.

By the multivariable chain rule,

$$\frac{d(f\circ\gamma)}{dt} = \frac{\partial f}{\partial x}\bigg|_{(x,y)=\gamma(t)} \frac{\partial \gamma_1}{\partial t} + \frac{\partial f}{\partial y}\bigg|_{(x,y)=\gamma(t)} \frac{\partial \gamma_2}{\partial t}.$$

The result then follows from the fundamental theorem of calculus.

[Definition: Real line integral] Let $\gamma \in C^1([a, b])$, $\gamma : [a, b] \to \Omega$ for some open set $\Omega \subseteq \mathbb{R}^2$ and $F : \Omega \to \mathbb{R}$ be continuous on Ω. We define

$$\int_\gamma F\, ds = \int_a^b F(\gamma(t))\, \|\gamma'(t)\|\, dt.$$

Let $\vec{F} : \Omega \to \mathbb{R}^2$ be continuous on Ω. We define

$$\int_\gamma \vec{F} \cdot \tau\, ds = \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt$$

where we use a dot product in the second integral.

2.1. Real and Complex Line Integrals

Definition 2.1.3. (Integral of a complex function.) If $f : [a, b] \to \mathbb{C}$, and both $\operatorname{Re} f$ and $\operatorname{Im} f$ are integrable on $[a, b]$, we define $\int_a^b f = \int_a^b \operatorname{Re} f + i \int_a^b \operatorname{Im} f$.

Proposition 2.1.7. Suppose that $a < b$ and that $f : [a, b] \to \mathbb{C}$ is continuous. Then $|\int_a^b f| \le \int_a^b |f| \le (b - a) \sup_{[a,b]} |f|$.

(Problem 920) Prove Proposition 2.1.7. *Hint*: Start by showing that the integral is finite.

(Problem 930) [Redacted]

Definition 2.1.4. (C^1 curve in $\mathbb{C}$.) A curve $\gamma : [a, b] \to \mathbb{C}$ is a C^1 curve (in $\mathbb{C}$) if $(\operatorname{Re}\gamma, \operatorname{Im}\gamma)$ is a C^1 curve (in $\mathbb{R}^2$). We write

$$\gamma'(t) = \frac{d\gamma}{dt} = \frac{d(\operatorname{Re}\gamma)}{dt} + i\frac{d(\operatorname{Im}\gamma)}{dt}.$$

(Problem 940) If $t \in (a, b)$ and $\gamma : [a, b] \to \mathbb{C}$ is C^1, show that $\gamma'(t) = \lim_{s\to t} \frac{\gamma(s)-\gamma(t)}{s-t}$.

We compute that

$$\lim_{s\to t} \frac{\gamma(s)-\gamma(t)}{s-t} = \lim_{s\to t} \left(\frac{\operatorname{Re}\gamma(s) - \operatorname{Re}\gamma(t)}{s-t} + i\frac{\operatorname{Im}\gamma(s) - \operatorname{Im}\gamma(t)}{s-t} \right).$$

By linearity of limits

$$\lim_{s\to t} \frac{\gamma(s)-\gamma(t)}{s-t} = \left(\lim_{s\to t} \frac{\operatorname{Re}\gamma(s) - \operatorname{Re}\gamma(t)}{s-t} \right) + i\left(\lim_{s\to t} \frac{\operatorname{Im}\gamma(s) - \operatorname{Im}\gamma(t)}{s-t} \right) = (\operatorname{Re}\gamma)'(t) + i(\operatorname{Im}\gamma)'(t)$$

as desired.

Definition 2.1.5. (Complex line integral.) Let $\gamma \in C^1([a, b])$, $\gamma : [a, b] \to \Omega$ for some open set $\Omega \subseteq \mathbb{C}$ and $F : \Omega \to \mathbb{C}$ be continuous on Ω. We define

$$\oint_\gamma F(z)\,dz = \int_a^b F(\gamma(t))\,\gamma'(t)\,dt$$

where we use complex multiplication in the second integral.

(Problem 941) Let $\gamma : [0, 1] \to \Omega \subset \mathbb{R}^2$ be a C^1 curve and let $\vec{F} : \Omega \to \mathbb{R}^2$ be a vector-valued function. Recall that we identify $\mathbb{R}^2$ with $\mathbb{C}$, so that we identify $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1 + i\gamma_2$ and $\vec{F} = (F_1, F_2)$ with $F = F_1 + iF_2$.

Show that

$$\int_\gamma \vec{F} \cdot \tau\,ds = \operatorname{Re} \oint_\gamma \overline{F}(z)\,dz, \qquad \int_\gamma \vec{F} \cdot \nu\,ds = \operatorname{Im} \oint_\gamma \overline{F}(z)\,dz,$$

where $\nu = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tau$ is the unit rightward normal vector to γ.

(Problem 950) [Redacted]

Proposition 2.1.6. Let $\gamma : [a, b] \to \Omega \subseteq \mathbb{C}$ be C^1, where Ω is open, and let f be holomorphic in Ω. Show that

$$\oint_\gamma \frac{\partial f}{\partial z}\,dz = f(\gamma(b)) - f(\gamma(a)).$$

(Problem 960) Prove Proposition 2.1.6. *Hint*: Start by computing $(f\circ\gamma)'(t)$ and the integrand in the definition of $\oint_\gamma \frac{\partial f}{\partial z}\,dz$.

By definition

$$\oint_\gamma \frac{\partial f}{\partial z}\,dz = \int_a^b \frac{\partial f}{\partial z}\bigg|_{z=\gamma(t)} \gamma'(t)\,dt.$$

Let $f = u + iv$ where u and v are real-valued C^1 functions on Ω, and let $g = \operatorname{Re}\gamma$, $h = \operatorname{Im}\gamma$. We let $\frac{\partial}{\partial g} = \frac{\partial}{\partial x}|_{(x,y)=\gamma(t)}$ and $\frac{\partial}{\partial h} = \frac{\partial}{\partial y}|_{(x,y)=\gamma(t)}$.

Then

$$\begin{aligned}\frac{\partial f}{\partial z}\bigg|_{z=\gamma(t)} \gamma'(t) &= \frac{1}{2}\left(\frac{\partial f}{\partial g} + \frac{1}{i}\frac{\partial f}{\partial h}\right)(g'(t)+ih'(t))\\ &= \frac{1}{2}\left(\frac{\partial u}{\partial g} + i\frac{\partial v}{\partial g} - i\frac{\partial u}{\partial h} + \frac{\partial v}{\partial h}\right)(g'(t)+ih'(t))\\ &= \frac{1}{2}\left(\frac{\partial u}{\partial g}g'(t) + \frac{\partial v}{\partial h}g'(t) - \frac{\partial v}{\partial g}h'(t) + \frac{\partial u}{\partial h}h'(t)\right)\\ &\qquad + \frac{i}{2}\left(\frac{\partial v}{\partial g}g'(t) - \frac{\partial u}{\partial h}g'(t) + \frac{\partial u}{\partial g}h'(t) + \frac{\partial v}{\partial h}h'(t)\right).\end{aligned}$$

By the Cauchy-Riemann equations, $\frac{\partial u}{\partial g} = \frac{\partial v}{\partial h}$ and $\frac{\partial u}{\partial h} = -\frac{\partial v}{\partial g}$, so we may simplify to

$$\frac{\partial f}{\partial z}\bigg|_{z=\gamma(t)} \gamma'(t) = \left(\frac{\partial u}{\partial g}g'(t) + \frac{\partial u}{\partial h}h'(t)\right) + i\left(\frac{\partial v}{\partial g}g'(t) + \frac{\partial v}{\partial h}h'(t)\right).$$

By the real multivariable chain rule

$$\frac{d}{dt}u(g(t),h(t)) = \frac{\partial u}{\partial g}\frac{dg}{dt} + \frac{\partial u}{\partial h}\frac{dh}{dt}$$

and

$$\frac{d}{dt}v(g(t),h(t)) = \frac{\partial v}{\partial g}\frac{dg}{dt} + \frac{\partial v}{\partial h}\frac{dh}{dt}$$

and so

$$\begin{aligned}\frac{d}{dt}f\circ\gamma(t) = \frac{d}{dt}f(g(t),h(t)) &= \left(\frac{\partial u}{\partial g}g'(t) + \frac{\partial u}{\partial h}h'(t)\right) + i\left(\frac{\partial v}{\partial g}g'(t) + \frac{\partial v}{\partial h}h'(t)\right)\\ &= \frac{\partial f}{\partial z}\bigg|_{z=\gamma(t)} \gamma'(t).\end{aligned}$$

Thus

$$\oint_\gamma \frac{\partial f}{\partial z}\,dz = \int_a^b \frac{d}{dt}(f\circ\gamma(t))\,dt.$$

Dividing into real and imaginary parts and applying the fundamental theorem of calculus completes the proof.

Proposition 2.1.8. If $\gamma : [a,b] \to \Omega \subseteq \mathbb{C}$ is a C^1 curve and $f : \Omega \to \mathbb{C}$ is continuous, then $\left|\oint_\gamma f(z)\,dz\right| \leq \sup_{[a,b]} |f\circ\gamma| \cdot \ell(\gamma) = \sup_{\widetilde{\gamma}} |f| \cdot \ell(\gamma)$, where $\ell(\gamma) = \int_a^b |\gamma'|$.

(Problem 970) Prove Proposition 2.1.8.

By definition

$$\oint_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

By Problem 920

$$\left|\oint_\gamma f(z)\,dz\right| \leq \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt. \leq \sup_{[a,b]}|f\circ\gamma|\int_a^b |\gamma'(t)|\,dt.$$

Recalling that the complex modulus $|\gamma'(t)|$ is equal to the real vector length $\|((\operatorname{Re}\gamma)'(t), (\operatorname{Im}\gamma)'(t))\|$ and the definition of arc length completes the proof.

Proposition 2.1.9. Let $\Omega\subseteq\mathbb{C}$ be open, let $F:\Omega\to\mathbb{C}$ be continuous, let $\gamma_1:[a,b]\to\Omega$ be a C^1 curve, and let $\varphi:[c,d]\to[a,b]$ be C^1 and satisfy $\varphi(c)=a$, $\varphi(d)=b$. Define $\gamma_2=\gamma_1\circ\varphi$. Then $\oint_{\gamma_1} F(z)\,dz=\oint_{\gamma_2} F(z)\,dz$.

(Problem 980) In this problem we begin the proof of Proposition 2.1.9. If $\varphi:[c,d]\to[a,b]$ is C^1 and $\gamma_1:[a,b]\to\mathbb{C}$ is C^1, show that $\gamma_2'(t)=\gamma_1'(\varphi(t))\,\varphi'(t)$ where $\gamma_2=\gamma_1\circ\varphi$.

We have that $\operatorname{Re}\gamma_2(t)=(\operatorname{Re}\gamma_1)\circ\varphi$ and $\operatorname{Im}\gamma_2=(\operatorname{Im}\gamma_1)\circ\varphi$. By the chain rule of real analysis and the definition of γ',

$$\gamma_2'(t)=(\operatorname{Re}\gamma_2)'(t)+i(\operatorname{Im}\gamma_2)'(t)=(\operatorname{Re}\gamma_1)'(\varphi(t))\varphi'(t)+i(\operatorname{Im}\gamma_1)'(\varphi(t))\varphi'(t)=\gamma_1'(\varphi(t))\varphi'(t)$$

as desired.

(Problem 990) Let $\gamma_1:[a,b]\to\mathbb{C}$ be a C^1 curve. Let $\varphi:[c,d]\to[a,b]$ be continuous and satisfy $\varphi(c)=a$, $\varphi(d)=b$. Define $\gamma_2=\gamma_1\circ\varphi$. Compute $\gamma_2'(t)$ in terms of γ_1, γ_1', φ, and φ'. Then show that $\widetilde{\gamma}_1=\widetilde{\gamma}_2$. (Recall $\widetilde{\gamma}$ denotes the image of γ.)

(Problem 1000) Prove Proposition 2.1.9.

(Problem 1010) Let $\gamma_1:[-a,a]\to\mathbb{C}$. Let $\gamma_2:[-a,a]\to\mathbb{C}$ be given by $\gamma_2(t)=\gamma_1(-t)$. Show that if F is continuous in a neighborhood of $\widetilde{\gamma}_1$, then $\oint_{\gamma_1}F(z)\,dz=-\oint_{\gamma_2}F(z)\,dz$.

Let $\varphi(t)=-t$, so that $\gamma_2=\gamma_1\circ\varphi$. We compute

$$\oint_{\gamma_2}F(z)\,dz=\int_{-a}^a F(\gamma_2(t))\gamma_2'(t)\,dt=\int_{-a}^a F(\gamma_1(\phi(t)))\gamma_1'(\phi(t))\phi'(t)\,dt$$

by definition of $\oint$, γ_2, and by Problem 980. By the change of variables theorem,

$$\oint_{\gamma_2}F(z)\,dz=\int_{-a}^a F(\gamma_2(t))\gamma_2'(t)\,dt=\int_{\varphi(-a)}^{\varphi(a)}F(\gamma_1(s))\gamma_1'(s)\,ds.$$

Because $\varphi(a)=-a$ and $\varphi(-a)=a$, we have that

$$\oint_{\gamma_2}F(z)\,dz=\int_a^{-a}F(\gamma_1(s))\gamma_1'(s)\,ds=-\int_{-a}^a F(\gamma_1(s))\gamma_1'(s)\,ds=-\oint_{\gamma_1}F(z)\,dz$$

as desired.

(Problem 1020) Let $\gamma_1:[a,b]\to\mathbb{C}$ and $\gamma_2:[c,d]\to\mathbb{C}$ be two curves. Suppose further that $\widetilde{\gamma}_1=\widetilde{\gamma}_2$, $\gamma_1(a)=\gamma_2(c)$, $\gamma_1(b)=\gamma_2(d)$, and that γ_1 and γ_2 are injective. Show that there is a continuous strictly increasing function $\varphi:[c,d]\to[a,b]$ such that $\gamma_2=\gamma_1\circ\varphi$.

(Problem 1030) If $\gamma_1:[a,b]\to\mathbb{C}$ and $\gamma_2:[c,d]\to\mathbb{C}$ are simple closed curves rather than injective functions, with $\widetilde{\gamma}_1=\widetilde{\gamma}_2$ and $\gamma_1(a)=\gamma_1(b)=\gamma_2(c)=\gamma_2(d)$, is it necessarily the case that $\gamma_2=\gamma_1\circ\varphi$ for a continuous strictly increasing function $\varphi:[c,d]\to[a,b]$?

No. Let $\gamma_1(t)=e^{it}$ and let $\gamma_2=e^{-it}$, both with domain $[-\pi,\pi]$. If $\gamma_2(t)=\gamma_1(\varphi(t))$, then $\varphi(t)=-t$ for all $t\in(-\pi,\pi)$, and so in particular φ cannot be strictly increasing.

(Bonus Problem 1040) Let $\gamma_1:[a,b]\to\mathbb{C}$ and $\gamma_2:[c,d]\to\mathbb{C}$ be two curves. Suppose further that $\widetilde{\gamma}_1=\widetilde{\gamma}_2$, $\gamma_1(a)=\gamma_2(c)$, $\gamma_1(b)=\gamma_2(d)$, and that γ_1 and γ_2 are injective. Show that if F is continuous in a neighborhood of $\widetilde{\gamma}_1$, then $\oint_{\gamma_1}F(z)\,dz=\oint_{\gamma_2}F(z)\,dz$. (This does not follow immediately from Problems 1000 and 1020 because φ may not be continuously differentiable.)

(Problem 1050) Let $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ be two C^1 curves. Suppose that $\gamma_1(b) = \gamma_2(c)$. Show that there is a C^1 curve $\gamma_3 : [-1,1] \to \mathbb{C}$ such that $\gamma_3\big|_{[-1,0]}$ is a reparameterization of γ_1 and $\gamma_3\big|_{[0,1]}$ is a reparameterization of γ_2. (This means that $\widetilde{\gamma}_3 = \widetilde{\gamma}_1 \cup \widetilde{\gamma}_2$ and $\oint_{\gamma_3} F(z)\,dz = \oint_{\gamma_1} F(z)\,dz + \oint_{\gamma_2} F(z)\,dz$ for all F continuous in a neighborhood of $\widetilde{\gamma}_3$.)

2.2. Real Analysis

[Definition: Limit in metric spaces] If (X,d) and (Z,ρ) are metric spaces, $p \in Z$, and $f : Z \setminus \{p\} \to X$, we say that $\lim_{z\to p} f(z) = \ell$ if, for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $z \in Z$ and $0 < \rho(z,p) < \delta$, then $d(f(z), f(p)) < \varepsilon$.

[Definition: Continuous function on metric spaces] If (X,d) and (Z,ρ) are metric spaces and $f : Z \to X$, we say that f is continuous at $p \in Z$ if $f(p) = \lim_{z\to p} f(z)$.

(Fact 1060) Let $\Omega \subseteq \mathbb{R}^2$ be open and let $f : \Omega \to \mathbb{R}$ be C^1. Let $\vec{p} \in \Omega$ and let $\vec{\eta} \in \mathbb{R}^2$. Define $g(t)$ by $g(t) = f(\vec{p} + t\vec{\eta})$. Suppose that $(\vec{p} + t\vec{\eta}) \in \Omega$. Then $g'(t) = \vec{\eta} \cdot \nabla f(\vec{p} + t\vec{\eta})$.

(Problem 1070) Suppose that $\overline{B}(\vec{p}, \|\vec{\eta}\|) \subseteq \Omega$. Show that $f(\vec{p}+\vec{\eta}) - f(\vec{p}) = \vec{\eta} \cdot \nabla f(\vec{w})$ for some $\vec{w} \in B(\vec{p}, r)$.

As in Fact 1060, let $g(t) = f(\vec{p} + t\vec{\eta})$. g is then continuous on $[0,1]$ because f is continuous on $\overline{B}(\vec{p}, \|\vec{\eta}\|)$. By Fact 1060, we have that $g'(t) = \vec{\eta} \cdot \nabla f(\vec{p} + t\vec{\eta})$ exists for all $t \in (0,1)$. Thus by the Mean Value Theorem, there is a $t \in (0,1)$ such that

$$f(\vec{p}+\vec{\eta}) - f(\vec{p}) = g(1) - g(0) = \frac{g(1)-g(0)}{1-0} = g'(t) = \vec{\eta} \cdot \nabla f(\vec{p} + t\vec{\eta}).$$

Choosing $\vec{w} = \vec{p} + t\vec{\eta}$ and observing that $\|\vec{w} - \vec{p}\| < \|\vec{\eta}\|$ completes the proof.

(Fact 1080) Let $\Omega \subseteq \mathbb{R}^d$ be open and let $\vec{f} : \Omega \to \mathbb{R}^n$ be C^1 (that is, the n components $f_1, f_2, \dots, f_n$ of $\vec{f}$ are all C^1). Let $\vec{p} \in \Omega$. Define $L : \mathbb{R}^d \to \mathbb{R}^n$ by

$$L(a_1, \dots, a_d) = \sum_{k=1}^{d} \frac{\partial \vec{f}}{\partial x_k}\Big|_{\vec{x}=\vec{p}} a_k.$$

Then $\lim_{\vec{x}\to\vec{p}} \frac{\|f(\vec{x}) - f(\vec{p}) - L(\vec{x}-\vec{p})\|}{\|\vec{x}-\vec{p}\|} = 0$. We often write $L = D\vec{f}(\vec{p})$.

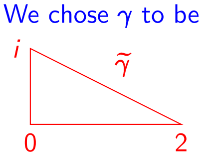
(Bonus Problem 1090) State and prove the chain rule for this form of derivative.

2.2. Complex Differentiability and Conformality

(Problem 1100) Let $\gamma : [0,1] \to \mathbb{C}$ be a parameterization of a nondegenerate scalene triangle of your choice. Sketch the trace of γ and of $f \circ \gamma$ for the following choices of f:

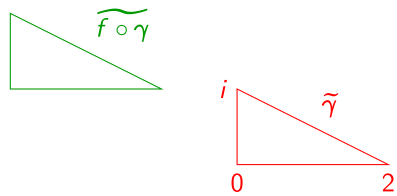
(a) $f(z) = z - 3 + i$
(b) $f(z) = (\frac{1}{2} + \frac{\sqrt{3}}{2} i)z$
(c) $f(z) = 2z$
(d) $f(z) = (1+i)z$
(e) $f(z) = (1+i)z - 3 + i$
(f) $f(z) = \overline{z}$
(g) $f(z) = z + 2\overline{z}$

We chose γ to be a parameterization of:

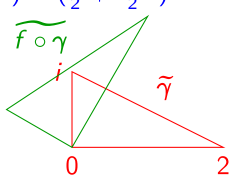


Then:

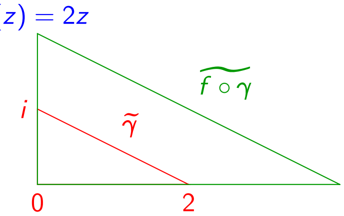
(a) $f(z) = z - 3 + i$



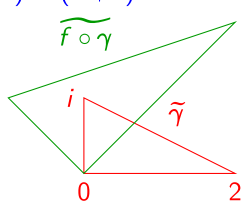
(b) $f(z) = (\frac{1}{2} + \frac{\sqrt{3}}{2} i)z$



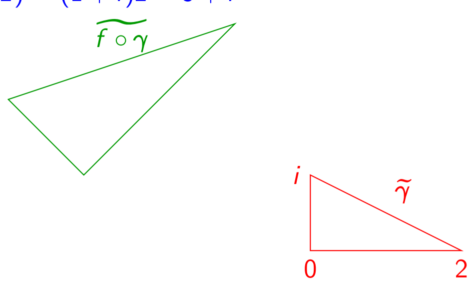
(c) $f(z) = 2z$



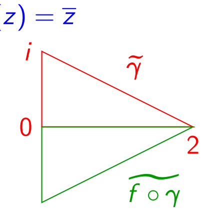
(d) $f(z) = (1 + i)z$



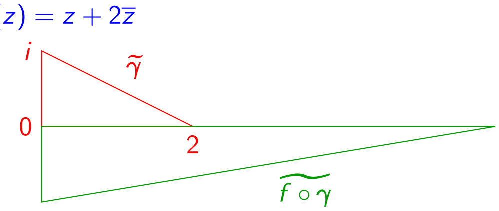
(e) $f(z) = (1 + i)z - 3 + i$



(f) $f(z) = \overline{z}$



(g) $f(z) = z + 2\overline{z}$



(Problem 1110) [Redacted]

[Definition: Complex derivative] Let $p \in \Omega \subseteq \mathbb{C}$, where Ω is open. Let $f : \Omega \to \mathbb{C}$. Suppose that $\lim_{z\to p} \frac{f(z)-f(p)}{z-p}$ exists. Then we say that f has a complex derivative at p and write $f'(p) = \lim_{z\to p} \frac{f(z)-f(p)}{z-p}$.

(Fact 1120) If $\Omega \subseteq \mathbb{C}$ is open, $p \in \Omega$, and f, $g : \Omega \setminus \{p\} \to \mathbb{C}$ are such that $\lim_{z\to p} f(z)$ and $\lim_{z\to p} g(z)$ exist (as complex numbers), then we have the usual formulas

$$\lim_{z\to p}\bigl(f(z)+g(z)\bigr) = \bigl(\lim_{z\to p} f(z)\bigr) + \bigl(\lim_{z\to p} g(z)\bigr),$$
$$\lim_{z\to p}\bigl(f(z)-g(z)\bigr) = \bigl(\lim_{z\to p} f(z)\bigr) - \bigl(\lim_{z\to p} g(z)\bigr),$$
$$\lim_{z\to p}\bigl(f(z)g(z)\bigr) = \bigl(\lim_{z\to p} f(z)\bigr)\bigl(\lim_{z\to p} g(z)\bigr)$$

and (if $\lim_{z\to p} g(z) \neq 0$)

$$\lim_{z\to p}\frac{f(z)}{g(z)} = \frac{\lim_{z\to p} f(z)}{\lim_{z\to p} g(z)}.$$

(Fact 1130) If $\Omega \subseteq \mathbb{C}$ and $W \subseteq \mathbb{C}$ are open, $p \in \Omega$, $f : \Omega \setminus \{p\} \to W$ is such that $L = \lim_{z\to p} f(z)$ exists, $L \in W$, and $g : W \to \mathbb{C}$ is continuous at L, then

$$\lim_{z\to p} g(f(z)) = g(L).$$

Observe that we do require $g(L)$ to exist, not only $\lim_{w\to L} g(w)$.

[Chapter 2, Problem 8] If $f'(p)$ exists, show that $\frac{\partial f}{\partial x}\big|_{x+iy=p} = \frac{1}{i}\frac{\partial f}{\partial y}\big|_{x+iy=p} = \frac{\partial f}{\partial z}\big|_{z=p} = f'(p)$.

[Chapter 2, Problem 10] If f has a complex derivative at p, then f is continuous at p.

(Problem 1140) (Note: If you are presenting this problem, do either part (a) or part (b), at your option. If you are citing this problem, you may use either part.)

Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be continuous.

(a) Let $a, b \in \mathbb{R}$ with $a < b$, and let $\gamma : [a, b] \to \Omega$ be a continuous curve. Then $f \circ \gamma : [a, b] \to \mathbb{C}$ is also a continuous curve. Let $a < t_0 < b$. Suppose that $\gamma'(t_0)$ and $f'(\gamma(t_0))$ exist (in both cases in the sense of limits). Show that $(f \circ \gamma)'(t_0)$ exists and that $(f \circ \gamma)'(t_0) = f'(\gamma(t_0))\,\gamma'(t_0)$.

(b) Let $W \subseteq \mathbb{C}$ be open. Let $g : W \to \Omega$ be continuous. Then $f \circ g : W \to \mathbb{C}$ is continuous. Suppose that $z_0 \in W$ and that $g'(z_0)$ and $f'(g(z_0))$ exist (in the sense of limits as above). Show that $(f \circ g)'(z_0)$ exists and that $(f \circ g)'(z_0) = f'(g(z_0))\,g'(z_0)$.

Theorem 2.2.2. Suppose that f has a complex derivative at p. Then $\frac{\partial f}{\partial z}\big|_{z=p} = f'(p)$.

(Problem 1150) Suppose that f has a complex derivative at p. Prove Theorem 2.2.2 and also show that $\frac{\partial f}{\partial \bar z}\big|_{z=p} = 0$. *Hint*: Start by writing the ε-δ definition of a limit of a function from $\Omega \setminus \{p\}$ to $\mathbb{C}$, where $p \in \Omega \subseteq \mathbb{C}$.

First, observe that

$$\frac{\partial f}{\partial x}\Big|_{x+iy=p} = \lim_{\substack{s\to 0\\ s\in\mathbb{R}}} \frac{f(p+s)-f(p)}{s}.$$

Because $f'(p) = \lim_{z\to p} \frac{f(z)-f(p)}{z-p}$, by definition of limit, for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $z \in \mathbb{C}$ and $0 < |z| < \delta$ then $\left|\frac{f(p+z)-f(p)}{z} - f'(p)\right| < \varepsilon$. This in particular is true if s is real and $0 < |s| < \delta$, as real numbers are complex numbers. Thus we must have that $f'(p) = \frac{\partial f}{\partial x}\big|_{x+iy=p}$.

Similarly,

$$\frac{\partial f}{\partial y}\Big|_{x+iy=p} = \lim_{\substack{s\to 0\\ s\in\mathbb{R}}} \frac{f(p+is)-f(p)}{s} = i\lim_{\substack{s\to 0\\ s\in\mathbb{R}}} \frac{f(p+is)-f(p)}{is} = if'(p).$$

We thus compute

$$\frac{\partial f}{\partial z}\Big|_{z=p} = \frac{1}{2}\left(\frac{\partial f}{\partial x}\Big|_{x+iy=p} + \frac{1}{i}\frac{\partial f}{\partial y}\Big|_{x+iy=p}\right) = f'(p)$$

and
$$\left.\frac{\partial f}{\partial \bar z}\right|_{z=p} = \frac{1}{2}\left(\left.\frac{\partial f}{\partial x}\right|_{x+iy=p} - \frac{1}{i}\left.\frac{\partial f}{\partial y}\right|_{x+iy=p}\right) = 0.$$

[Definition: Disc] The open disc (or ball) in $\mathbb{C}$ of radius r and center p is $D(p,r) = B(p,r) = \{z \in \mathbb{C} : |z-p| < r\}$. The closed disc (or ball) in $\mathbb{C}$ of radius r and center p is $\overline{D}(p,r) = \overline{B}(p,r) = \{z \in \mathbb{C} : |z-p| \leq r\}$.

Theorem 2.2.1. (Generalization.) Suppose that $\Omega \subseteq \mathbb{C}$ is open and that f is C^1 on Ω. Let $p \in \Omega$ and suppose $\left.\frac{\partial f}{\partial \bar z}\right|_{z=p} = 0$. Then f has a complex derivative at p and $f'(p) = \left.\frac{\partial f}{\partial z}\right|_{z=p}$.

(Problem 1160) Prove this generalization of Theorem 2.2.1.

Let $p \in \Omega$ and let $D(p,r) \subset \Omega$; by definition of open set this is true for some $r > 0$. By Fact 1080 and Fact 170, we have that
$$\lim_{w\to p} \frac{|f(w) - f(p) - \mathrm{Re}(w-p)\partial_x f(p) - \mathrm{Im}(w-p)\partial_y f(p)|}{|w-p|} = 0.$$
A standard ε-δ argument yields that
$$\lim_{w\to p} \frac{f(w) - f(p) - \mathrm{Re}(w-p)\partial_x f(p) - \mathrm{Im}(w-p)\partial_y f(p)}{w-p} = 0.$$
Because f is holomorphic at p, by Proposition 1.4.3 (Problem 660) $\partial_x f(p) = \frac{\partial}{\partial z} f(p) = -i\partial_y f(p)$ and so
$$\begin{aligned}0 &= \lim_{w\to p} \frac{f(w) - f(p) - \mathrm{Re}(w-p)\frac{\partial}{\partial z} f(p) - i\,\mathrm{Im}(w-p)\frac{\partial}{\partial z} f(p)}{w-p}\\ &= \lim_{w\to p} \frac{f(w) - f(p) - (w-p)\frac{\partial}{\partial z} f(p)}{w-p}\\ &= \lim_{w\to p} \frac{f(w)-f(p)}{w-p} - \frac{\partial}{\partial z} f(p).\end{aligned}$$
Because $\frac{\partial}{\partial z} f(p) = \left.\frac{\partial f}{\partial z}\right|_{z=p}$ is independent of w, we must have that $\lim_{w\to p} \frac{f(w)-f(p)}{w-p}$ exists and equals $\frac{\partial}{\partial z} f(p)$.

(Problem 1170) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that ∇F_1 and ∇F_2 are constants. Show that $F(x,y) = F(0,0) + (\partial_1 F_1, \partial_1 F_2)x + (\partial_2 F_1, \partial_2 F_2)y$ for all $(x,y) \in \mathbb{R}^2$.

We compute that (for $j = 1$ or $j = 2$)
$$\begin{aligned}F_j(x,y) &= F_j(x,y) - F_j(x,0) + F_j(x,0) - F_j(0,0) + F_j(0,0)\\ &= F_j(0,0) + \int_0^x \frac{\partial}{\partial s} F_j(s,0)\,ds + \int_0^y \frac{\partial}{\partial t} F_j(x,t)\,dt.\end{aligned}$$
Because the integrands are constants, they may easily be evaluated to yield the desired result.

(Problem 1180) Suppose that $f : \mathbb{C} \to \mathbb{C}$. Suppose that f' exists everywhere and is a constant. Show that $f(z) = f(0) + f'(0)z$ for all $z \in \mathbb{C}$. Conclude that if z, ω, $w \in \mathbb{C}$ with $\omega \neq z \neq w$, then $\frac{|f(\omega)-f(z)|}{|\omega-z|} = \frac{|f(w)-f(z)|}{|w-z|}$.

Define $\gamma : [0,1] \to \Omega$ by $\gamma(t) = p + t(z-p)$. Then by Proposition 2.1.6 (Problem 960), we have that
$$f(\zeta) - f(p) = \oint_\gamma \frac{\partial f}{\partial z}\,dz.$$
By Theorem 2.2.2 (Problem 1150), $\frac{\partial f}{\partial z} = f'(z) = f'(0)$ because f' is constant. The result follows by definition of line integral.

(Problem 1190) Let $F : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that ∇F_1 and ∇F_2 are constants. If C is a circle, what is $F(C)$? If S is a square, what is $F(S)$? Now suppose that $f : \mathbb{C} \to \mathbb{C}$ and that f' exists everywhere and is a constant. If C is a circle, what is $f(C)$? If S is a square, what is $f(S)$?

Theorem 2.2.3.1. Let $z_0 \in \Omega \subseteq \mathbb{C}$ for some open set Ω. Let $f : \Omega \to \mathbb{C}$. Let $w_1, w_2 \in \mathbb{C}$ with $w_1, w_2 \neq 0$. Suppose that $f'(z_0)$ exists. Then $\lim_{t\to 0} \frac{|f(z_0+tw_1)-f(z_0)|}{|tw_1|} = \lim_{t\to 0} \frac{|f(z_0+tw_2)-f(z_0)|}{|tw_2|}$.

(Problem 1200) Prove Theorem 2.2.3.1. How does this relate to the result of Problem 1180?

Using an ε-δ argument and the limit definition of $f'(z_0)$, we may show that

$$\lim_{t\to 0} \frac{f(z_0+tw_1)-f(z_0)}{tw_1} = f'(z_0) = \lim_{t\to 0} \frac{f(z_0+tw_2)-f(z_0)}{tw_2}.$$

Because the modulus is a continuous function from $\mathbb{C}$ to $\mathbb{C}$, this implies that

$$\lim_{t\to 0} \frac{|f(z_0+tw_1)-f(z_0)|}{|tw_1|} = |f'(z_0)| = \lim_{t\to 0} \frac{|f(z_0+tw_2)-f(z_0)|}{|tw_2|}$$

as desired.

This implies that for small displacements from z_0, the function f stretches approximately equally in all directions, with the approximation becoming better and better as the displacements get smaller.

[Chapter 2, Problem 12] Let $z_0 \in \Omega \subseteq \mathbb{C}$ for some open set Ω. Let $f : \Omega \to \mathbb{C}$. Suppose that $\lim_{t\to 0} \frac{|f(z_0+tw_1)-f(z_0)|}{|tw_1|} = \lim_{t\to 0} \frac{|f(z_0+tw_2)-f(z_0)|}{|tw_2|}$ for all $w_1, w_2 \in \mathbb{C}\setminus\{0\}$. Then either $f'(z_0)$ exists or $(\overline{f})'(z_0)$ exists.

[Definition: Angle preserving] Let $z_0 \in \Omega \subseteq \mathbb{C}$ for some open set Ω. Let $f : \Omega \to \mathbb{C}$. We say that f preserves angles at z_0 if, for all $w_1, w_2 \in \mathbb{C}\setminus\{0\}$, we have that

$$\lim_{t\to 0^+} \frac{f(z_0+tw_1)-f(z_0)}{f(z_0+tw_2)-f(z_0)} \left| \frac{f(z_0+tw_2)-f(z_0)}{f(z_0+tw_1)-f(z_0)} \right| = \frac{w_1/|w_1|}{w_2/|w_2|}.$$

and in particular that the denominators $f(z_0+tw_1)-f(z_0)$ and $f(z_0+tw_2)-f(z_0)$ are not zero when t is sufficiently close to 0. [This is not the definition in the book.]

Theorem 2.2.3.2. If $f'(z_0)$ exists and is not zero, then f preserves angles at z_0.

(Problem 1210) Prove Theorem 2.2.3.2.

Because $f'(z_0) \neq 0$, we have that

$$\lim_{w\to z_0} \frac{w}{f(w)-f(z)}$$

and in particular the denominator is not zero for all w sufficiently close to but not equal to z_0.

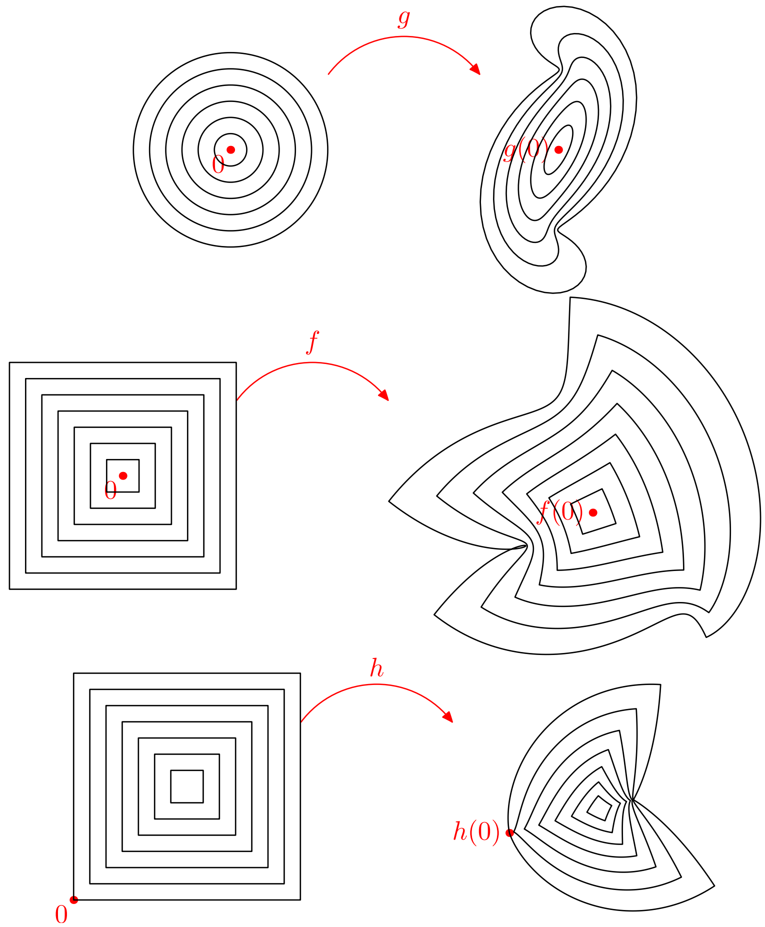
Now, observe that

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{f(z_0+tw_1)-f(z_0)}{f(z_0+tw_2)-f(z_0)} \left| \frac{f(z_0+tw_2)-f(z_0)}{f(z_0+tw_1)-f(z_0)} \right| \\
&\qquad = \lim_{t\to 0^+} \frac{w_1}{w_2} \frac{f(z_0+tw_1)-f(z_0)}{tw_1} \frac{tw_2}{f(z_0+tw_2)-f(z_0)} \\
&\qquad\qquad \times \left| \frac{w_2}{w_1} \frac{tw_1}{f(z_0+tw_1)-f(z_0)} \frac{f(z_0+tw_2)-f(z_0)}{tw_2} \right|.
\end{aligned}$$

The result follows from the limit definition of derivative and from the fact that the limit of a quotient is the quotient of the limits.

[Chapter 2, Problem 9a] If f is C^1 and preserves angles at z_0, then $f'(z_0)$ exists.

(Problem 1220) Consider the following figures. On the left is shown the traces of γ_j for several values of j. On the right is shown the traces of $f \circ \gamma_j$, $g \circ \gamma_j$, or $h \circ \gamma_j$ for the same γ_j. You are given that exactly two of the quantities $f'(0)$, $g'(0)$, and $h'(0)$ exist and that exactly one of those quantities is zero. Based on the images, which function do you think has nonzero derivative, which has zero derivative, and which does not have a derivative?



2.3. Real Analysis

(Memory 1221) Let $f(x) = x^2 \sin(1/x)$ if $x \neq 0$ and let $f(0) = 0$. Then f is continuous on $(-\infty, \infty)$, continuously differentiable on $(-\infty, 0)$ and $(0, \infty)$, and $f'(0)$ exists, but the limit $\lim_{x \to 0} f'(x)$ does not exist.

Lemma 2.3.1. Suppose that $a < p < b$ and let $H : (a, b) \to \mathbb{R}$ be continuous. Suppose that H is differentiable on both (a, p) and (p, b), and that $\lim_{x \to p} H'(x) = h$ for some $h \in \mathbb{R}$. Then $H'(p)$ exists and $H'(p) = \lim_{x \to p} H'(x)$.

2.3. Antiderivatives Revisited

(Memory 1230) Recall Problem 750: Suppose that there are two C^1 functions g and h defined in an open rectangle or disc $\mathcal{R}$ such that $\frac{\partial}{\partial x} g = \frac{\partial}{\partial y} h$. Then there is a function $f \in C^2(\mathcal{R})$ such that $\frac{\partial f}{\partial y} = g$ and $\frac{\partial f}{\partial x} = h$.

Theorem 2.3.2. Let $\mathcal{R} \subset \mathbb{R}^2$ be an open rectangle or disc and let $P \in \mathcal{R}$. Suppose that there are two functions g and h that are continuous on $\mathcal{R}$, continuously differentiable on $\mathcal{R} \setminus \{P\}$, and such that $\frac{\partial}{\partial x} g = \frac{\partial}{\partial y} h$ on $\mathcal{R} \setminus \{P\}$ for some $P \in \mathcal{R}$. Then there is a function $f \in C^1(\mathcal{R})$ such that $\frac{\partial f}{\partial y} = g$ and $\frac{\partial f}{\partial x} = h$ everywhere in $\mathcal{R}$ (including at P).

(Problem 1240) Prove Theorem 2.3.2.

Let $P = (x_0, y_0)$ and let $f(x, y) = \int_{x_0}^{x} h(s, y_0)\,ds + \int_{y_0}^{y} g(x, t)\,dt$. Observe that if $(x, y) \in \mathcal{R}$ then so is (s, y_0) and (x, t) for all s between x_0 and x and all t between y_0 and y. Thus g and h are defined at all required values. Because g and h are continuous, the integrals exist.

Furthermore, I claim f is continuous. Let $(x, y) \in \mathcal{R}$ and let $\delta_1 > 0$ be such that $B((x, y), \delta_1) \subset \mathcal{R}$. By continuity of g and h and compactness of $\overline{B}((x, y), \delta_1/2)$, g and h are bounded on $\overline{B}((x, y), \delta_1/2)$. If $(\xi, \eta) \in B((x, y), \delta_1/2)$, then

$$\begin{aligned}|f(\xi, \eta) - f(x, y)| &= \left|\int_{x}^{\xi} h(s, y_0)\,ds + \int_{y_0}^{y} g(\xi, t) - g(x, t)\,dt + \int_{y}^{\eta} g(\xi, t)\,dt\right| \\ &\leq |\xi - x| \sup_{\overline{B}((x,y),\delta_1/2)} |h| + |\eta - y| \sup_{\overline{B}((x,y),\delta_1/2)} |g| + \left|\int_{y_0}^{y} g(x, t) - g(\xi, t)\,dt\right|.\end{aligned}$$

Furthermore, g must be uniformly continuous on $\overline{B}((x, y), \delta_1/2)$. Choose $\varepsilon > 0$ and let δ_2 be such that if $|(x, t) - (\xi, t)| < \delta_2$ then $|g(x, t) - g(\xi, t)| < \varepsilon$. We then have that

$$|f(x, y) - f(\xi, \eta)| \leq |\xi - x| \sup_{\overline{B}((x,y),\delta_1/2)} |h| + |\eta - y| \sup_{\overline{B}((x,y),\delta_1/2)} |g| + |y_0 - y|\varepsilon.$$

There is then a $\delta_3 > 0$ such that if $|(\xi, \eta) - (x, y)| < \delta_3$ then $|f(x, y) - f(\xi, \eta)| < (1 + |y_0 - y|)\varepsilon$, and so f is continuous at (x, y), as desired.

By the fundamental theorem of calculus, we have that $\frac{\partial f}{\partial y} = g$ everywhere in $\mathcal{R}$, including at $(x, y) = (x_0, y_0) = P$.

Furthermore, by 730 and the fundamental theorem of calculus, we have that if $x \neq x_0$ then

$$\frac{\partial f}{\partial x}(x, y) = h(x, y_0) + \int_{y_0}^{y} \frac{\partial g}{\partial x}(x, t)\,dt.$$

Again because $x \neq x_0$, we have that $\frac{\partial g}{\partial x}(x, t) = \frac{\partial h}{\partial t}(x, t)$ and so by the fundamental theorem of calculus $\frac{\partial f}{\partial x} = h$ provided $x \neq x_0$.

We need only show that $\frac{\partial f}{\partial x} = h$ even if $x = x_0$. Fix a y and let $F_y(x) = f(x, y)$. Then $I = \{x \in \mathbb{R} : (x, y) \in \mathcal{R}\}$ is an open interval. We need only consider the case where $I \neq \emptyset$. Then F_y is continuous on I, $F_y'(x) = h(x, y)$ for all $x \neq x_0$, and $\lim_{x \to x_0} F_y'(x) = h(x_0, y)$ because h is continuous. Thus by Lemma 2.3.1, $F_y'(x_0) = h(x_0, y)$ and so $\frac{\partial f}{\partial x} = h$ even if $x = x_0$.

Theorem 2.3.3. Let $P \in \mathcal{R}$, where $\mathcal{R}$ is an open rectangle or disc. Suppose that f is continuous on $\mathcal{R}$ and holomorphic on $\mathcal{R} \setminus \{P\}$. Then there is a function F that is holomorphic on all of $\mathcal{R}$ (including P) such that $\frac{\partial F}{\partial z} = f$.

(Problem 1250) Prove Theorem 2.3.3.

Let $f = u + iv$ where u and v are real valued; then by definition u, v are continuous on $\mathcal{R}$ and C^1 on $\mathcal{R} \setminus \{P\}$.

Then by the Cauchy-Riemann equations, we have that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$. Therefore, by Theorem 2.3.2, there is a $V \in C^1(\mathcal{R})$ such that $\frac{\partial V}{\partial y} = u$ and $\frac{\partial V}{\partial x} = v$ in all of $\mathcal{R}$. Similarly, $\frac{\partial u}{\partial y} = \frac{\partial(-v)}{\partial x}$, and so there is a $U \in C^1(\mathcal{R})$ such that $\frac{\partial U}{\partial x} = u$ and $\frac{\partial U}{\partial y} = -v$.

Let $F = U + iV$. Then

$$\frac{\partial F}{\partial z} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + \frac{1}{i}\frac{\partial F}{\partial y}\right) = \frac{1}{2}\left(u + iv + \frac{1}{i}(-v + iu)\right) = f$$

and

$$\frac{\partial F}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} - \frac{1}{i}\frac{\partial F}{\partial y}\right) = \frac{1}{2}\left(u + iv - \frac{1}{i}(-v + iu)\right) = 0$$

in $\mathcal{R}$, as desired.

(Problem 1260) Is the previous problem true if we relax the assumption that f is continuous at P (and that $\frac{\partial F}{\partial z} = f$ at P)?

No. Let $\mathcal{R} = D(0,2)$ and let $P = 0$. Then $f(z) = 1/z$ is holomorphic on $\mathcal{R} \setminus \{P\}$ (see Problem 630) but $f(z)$ has no holomorphic antiderivative on $D(0,2) \setminus \overline{D}(0,1)$, and so it certainly (see Problem 1.52 in the book).

2.4. Real Analysis

(Memory 1270) In $\mathbb{C}$, $\overline{D}(P,r)$ is the closure of $D(P,r)$.

(Memory 1280) In $\mathbb{C}$, $\partial D(P,r) = \partial\overline{D}(P,r) = \{z \in \mathbb{C} : |z-P| = r\}$.

2.4. The Cauchy Integral Formula and the Cauchy Integral Theorem

Theorem 2.4.3. [The Cauchy integral theorem.] Let f be holomorphic in $D(P,R)$. Let $\gamma : [a,b] \to D(P,R)$ be a closed curve. Then $\oint_\gamma f(z)\,dz = 0$.

[Chapter 2, Problem 1] Prove the Cauchy integral theorem.

Theorem 2.4.2. [The Cauchy integral formula.] Let $\Omega \subseteq \mathbb{C}$ be open and let $\overline{D}(z_0,r) \subset \Omega$. Let f be holomorphic in Ω and let $z \in D(z_0,r)$. Then

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Lemma 2.4.1. The Cauchy integral formula is true in the special case where $f(\zeta) = 1$ for all $\zeta \in \mathbb{C}$.

(Problem 1281) Let $\gamma : [a,b] \to K$ be a C^1 curve for some set $K \subseteq \mathbb{C}$ (not necessarily open). Let $V \subseteq \mathbb{C}$ be compact and let $f : K \times V \to \mathbb{C}$ be continuous. Let $F : V \to \mathbb{C}$ be defined by

$$F(z) = \oint_\gamma f(\zeta, z)\,d\zeta.$$

Show that F is continuous on V.

(Problem 1290) In this problem we will begin the proof of Lemma 2.4.1 (and thus ultimately of Theorem 2.4.2). Let $\gamma : [a,b] \to K$ be a C^1 curve for some set $K \subseteq \mathbb{C}$ (not necessarily open). Let $W \subseteq \mathbb{C}$ be open and let $f : K \times W \to \mathbb{C}$ be continuous. Suppose that the functions $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ given by $\frac{\partial f}{\partial x}(\zeta, x+iy) = \frac{\partial}{\partial x} f(\zeta, x+iy)$ and $\frac{\partial f}{\partial y}(\zeta, x+iy) = \frac{\partial}{\partial y} f(\zeta, x+iy)$ are continuous on $K \times W$. Show that

$$\begin{aligned}
\frac{\partial}{\partial x}\oint_\gamma f(\zeta, x+iy)\,d\zeta &= \oint_\gamma \frac{\partial}{\partial x} f(\zeta, x+iy)\,d\zeta,\\
\frac{\partial}{\partial y}\oint_\gamma f(\zeta, x+iy)\,d\zeta &= \oint_\gamma \frac{\partial}{\partial y} f(\zeta, x+iy)\,d\zeta.,\\
\frac{\partial}{\partial z}\oint_\gamma f(\zeta, z)\,d\zeta &= \oint_\gamma \frac{\partial}{\partial z} f(\zeta, z)\,d\zeta.,\\
\frac{\partial}{\partial \bar z}\oint_\gamma f(\zeta, z)\,d\zeta &= \oint_\gamma \frac{\partial}{\partial \bar z} f(\zeta, z)\,d\zeta
\end{aligned}$$

for all $z = x+iy \in W$.

(Problem 1300) Prove Lemma 2.4.1. *Hint*: Start by proving Lemma 2.4.1 in the special case $z = z_0$. Computing $\operatorname{Re}\oint_\gamma \frac{1}{\zeta - z}\,d\zeta$ and $\operatorname{Im}\oint_\gamma \frac{1}{\zeta - z}\,d\zeta$ directly from the definition of line integral is very difficult if $z \neq z_0$. Instead compute $\frac{\partial}{\partial z}\oint_\gamma \frac{1}{\zeta - z}\,d\zeta$ and $\frac{\partial}{\partial \bar z}\oint_\gamma \frac{1}{\zeta - z}\,d\zeta$ and use the known value of $\oint_\gamma \frac{1}{\zeta - z_0}\,d\zeta$.

$\oint_\gamma \frac{1}{\zeta - z_0}\,d\zeta = 1$ (this is a routine computation).

If $\zeta \neq x+iy$, then

$$\begin{aligned}\frac{\partial}{\partial x}\frac{1}{\zeta-(x+iy)} &= \frac{\partial}{\partial x}\frac{\bar{\zeta}-x+iy}{(\operatorname{Re}\zeta-x)^2+(\operatorname{Im}\zeta-y)^2} \\ &= \frac{-|\zeta-(x+iy)|^2+(\bar{\zeta}-x+iy)(2(\operatorname{Re}\zeta-x))}{|\zeta-(x+iy)|^4} \\ &= \frac{-|\zeta-(x+iy)|^2+(\bar{\zeta}-x+iy)((\bar{\zeta}-x+iy)+(\zeta-x-iy))}{(\zeta-x-iy)^2(\bar{\zeta}-x+iy)^2} \\ &= \frac{1}{(\zeta-(x+iy))^2}.\end{aligned}$$

Now, by the previous problem, if $x+iy \in D(P,r)$, then

$$\begin{aligned}\frac{\partial}{\partial x}\oint_\gamma \frac{1}{\zeta-(x+iy)}\,d\zeta &= \oint_\gamma \frac{\partial}{\partial x}\frac{1}{\zeta-(x+iy)}\,d\zeta \\ &= \oint_\gamma \frac{1}{(\zeta-(x+iy))^2}\,d\zeta.\end{aligned}$$

Let $F(\zeta) = \frac{-1}{\zeta-(x+iy)}$. By Problem 630 and the chain rule we have that $\frac{\partial}{\partial\zeta}F(\zeta) = \frac{1}{(\zeta-(x+iy))^2}$. Thus

$$\frac{\partial}{\partial x}\oint_\gamma \frac{1}{\zeta-(x+iy)}\,d\zeta = \oint_\gamma \frac{\partial}{\partial\zeta}\frac{-1}{\zeta-(x+iy)}\,d\zeta$$

which by Problem 960 is zero because γ is a closed curve.

Similarly

$$\frac{\partial}{\partial y}\oint_\gamma \frac{1}{\zeta-(x+iy)}\,d\zeta = 0$$

for all $x+iy \in D(z_0,r)$. Thus $\oint_\gamma \frac{1}{\zeta-(x+iy)}\,d\zeta$ (regarded as a function of $x+iy$) must be a constant; since it equals 1 at $x+iy=z_0$, it must be 1 everywhere.

2.5. The Cauchy Integral Formula: Some Examples

(Problem 1310) Let $\gamma(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$. Let n be an integer. Let $z \in D(z_0,r)$. Show that $\oint_\gamma (\zeta-z)^n\,d\zeta = 0$ if $n \neq -1$.

(Problem 1320) Show that if $n \geq 0$ and γ is as in the previous problem, then

$$\frac{1}{2\pi i}\oint_\gamma \frac{\zeta^n}{\zeta-z}\,d\zeta = z^n.$$

(As we have not yet proven the Cauchy integral formula, do not cite the Cauchy integral formula to perform this computation.)

If $n=0$ then the result follows from Lemma 2.4.1. Otherwise, by the binomial theorem

$$\zeta^n = [(\zeta-z)+z]^n = \sum_{k=0}^{n}\binom{n}{k}z^k(\zeta-z)^{n-k}.$$

Thus

$$\oint_\gamma \frac{\zeta^n}{\zeta-z}\,d\zeta = \sum_{k=0}^{n}\binom{n}{k}z^k\oint_\gamma (\zeta-z)^{n-k-1}\,d\zeta.$$

If $k<n$ then the integral is zero, while if $k=n$ then the integral is $2\pi i$ by Lemma 2.4.1, as desired.

(Problem 1330) Let $p \in \mathbb{C}[z]$ be a polynomial. Find $\frac{1}{2\pi i}\oint_\gamma \frac{p(\zeta)}{\zeta-z}\,d\zeta$. (As we have not yet proven the Cauchy integral formula, do not cite the Cauchy integral formula to perform this computation.)

Theorem 2.4.2. [The Cauchy integral formula.] Let $\Omega \subseteq \mathbb{C}$ be open and let $\overline{D}(z_0, r) \subset \Omega$. Let f be holomorphic in Ω and let $z \in D(z_0, r)$. Then

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

(Problem 1340) Prove Theorem 2.4.2. *Hint*: Let $h(\zeta) = \frac{f(\zeta)-f(z)}{\zeta - z}$ if $\zeta \neq z$. How should you define $h(z)$? What can you say about the behavior of h at z and in $\Omega \setminus \{z\}$?

(Bonus Problem 1350) Prove the previous result without using Problem 1250.

[Definition: Integral over a circle] We define $\oint_{\partial D(P,r)} f(z)\, dz = \oint_\gamma f(z)\, dz$, where γ is a counterclockwise simple parameterization of $\partial D(P, r)$.

[Chapter 2, Problem 20] Let f be continuous on $\overline{D}(P, r)$ and holomorphic in $D(P, r)$. Show that $f(z) = \oint_{\partial D(P,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$ for all $z \in D(P, r)$.

(Problem 1360) Let f and g be holomorphic in $D(P, r)$ and continuous on $\overline{D}(P, r)$. Suppose that $f(\zeta) = g(\zeta)$ for all $\zeta \in \partial D(P, r)$. Show that $f(z) = g(z)$ for all $z \in D(P, r)$.

CODA: CHANGE OF VARIABLES

(Problem 1370) Let $\Omega \subseteq \mathbb{C}$ be open, let $\gamma : [0, 1] \to \Omega$ be a C^1 curve, and let $u : \Omega \to \mathbb{C}$ be holomorphic. Show that $u \circ \gamma$ is also a C^1 curve and that $(u \circ \gamma)'(t) = u'(\gamma(t))\, \gamma'(t)$.

(Problem 1380) Let Ω, $W \subseteq \mathbb{C}$ be open and let $u : \Omega \to W$ be holomorphic. Let $\gamma : [0, 1] \to \Omega$ be a C^1 closed curve. Let $f : W \to \mathbb{C}$ be continuous. Show that

$$\oint_{u\circ\gamma} f(w)\, dw = \oint_\gamma f(u(z))\, \frac{du}{dz}\, dz.$$

(Problem 1390) I want to compute $\int_{-1}^{1} \frac{(t+i)^3}{(t+i)^4+1} dt$. A naïve student uses the u-substitution $u = (t + i)^4$ and converts the integral to $\int_{-4}^{-4} \frac{1}{4} \frac{1}{u+1} du = 0$. But when I compute $\int_{-1}^{1} \frac{(t+i)^3}{(t+i)^4+1} dt$ using a numerical solver, I get $-i\pi/2$. What went wrong?

2.6. AN INTRODUCTION TO THE CAUCHY INTEGRAL THEOREM AND THE CAUCHY INTEGRAL FORMULA FOR MORE GENERAL CURVES

Proposition 2.6.6. Let $\Omega = D(P, \tau) \setminus \overline{D}(P, \sigma)$ for some $P \in \mathbb{C}$ and some $0 < \sigma < \tau$. Let $\sigma < r < R < \tau$ and let γ_r, γ_R be the counterclockwise parameterizations of $\partial D(P, r)$, $\partial D(P, R)$. Suppose that f is holomorphic in Ω. Then $\oint_{\gamma_r} f = \oint_{\gamma_R} f$.

(Problem 1400) Prove Proposition 2.6.6. *Hint*: Define γ_s in the natural way and find a function h such that $\frac{d}{ds} \oint_{\gamma_s} f = \oint_{\gamma_s} h$.

[Definition: Homotopic curves] Let $a < b$, $c < d$. Let $\Omega \subseteq \mathbb{C}$ be open. Let γ_c, $\gamma_d : [a, b] \to \Omega$ be two C^1 curves with the same endpoints (so $\gamma_c(a) = \gamma_d(a)$, $\gamma_c(b) = \gamma_d(b)$).

We say that γ_c and γ_d are C^1-homotopic in Ω if there is a function Γ such that:

- $\Gamma : [a, b] \times [c, d] \to \Omega$,
- $\Gamma(t, c) = \gamma_c(t)$, $\Gamma(t, d) = \gamma_d(t)$ for all $t \in [a, b]$,
- $\Gamma(a, s) = \gamma_c(a) = \gamma_d(a)$, $\Gamma(b, s) = \gamma_c(b) = \gamma_d(b)$ for all $s \in [c, d]$,
- Γ is continuous on $[a, b] \times [c, d]$,
- Γ is C^1 in the first variable in the sense that if $\gamma_s(t) = \Gamma(t, s)$, then $\gamma_s \in C^1[a, b]$ for all $c \leq s \leq d$.

[Definition: Homotopic closed curves] Let $a < b$, $c < d$. Let $\Omega \subseteq \mathbb{C}$ be open. Let γ_c, $\gamma_d : [a, b] \to \Omega$ be two closed C^1 curves.

We say that γ_c and γ_d are C^1-homotopic in Ω if there is a function Γ such that:

- $\Gamma : [a, b] \times [c, d] \to \Omega$,

- $\Gamma(t,c)=\gamma_c(t)$, $\Gamma(t,d)=\gamma_d(t)$ for all $t\in[a,b]$,
- $\Gamma(a,s)=\Gamma(b,s)$ for all $s\in[c,d]$,
- Γ is continuous on $[a,b]\times[c,d]$,
- Γ is C^1 in the first variable on $[a,b]\times[c,d]$.

(Bonus Problem 1401) Show that the assumption that Γ be C^1 in the first variable is unnecessary: if γ_c and γ_d are C^1 and there is a function Γ satisfying all of the above conditions except that Γ is not C^1 in the first variable, then Γ may be perturbed slightly to yield a C^1 function.

(Problem 1410) Let $\Omega=D(P,\tau)\setminus\overline{D}(P,\sigma)$ for some $P\in\mathbb{C}$ and some $0<\sigma<\tau$. Let $\sigma<r<R<\tau$ and let γ_r, γ_R be the counterclockwise parameterizations of $\partial D(P,r)$, $\partial D(P,R)$. Show that γ_r and γ_R are homotopic in Ω.

(Problem 1420) Let Ω be an open set, and let γ_c, $\gamma_d:[a,b]\to\Omega$ be two curves with the same endpoints that are homotopic in Ω. Let Γ be the homotopy.

Suppose that $\nabla\Gamma$ exists and is continuous on $[a,b]\times[c,d]$ (with the derivatives on the boundary defined as one-sided limits, as in Problem 890). Suppose further that $\nabla\Gamma$ is continuously differentiable on $[a,b]\times[c,d]$.

Let $\gamma_s(t)=\Gamma(t,s)$. Let f be holomorphic on Ω. Show that $\oint_{\gamma_c}f=\oint_{\gamma_d}f$. *Hint:* Start by computing $\frac{d}{ds}\oint_{\gamma_s}f$ and then rewrite the result as as $\int_a^b\frac{\partial}{\partial t}h(s,t)\,dt$ for some function h.

By definition of line integral,

$$\oint_{\gamma_s}f=\int_a^b f(\Gamma(t,s))\,\frac{\partial}{\partial t}\Gamma(t,s)\,dt.$$

By Problem 730 if $c\leq s\leq d$ then $\oint_{\gamma_s}f$ is continuous (as a function of s) at s and

$$\frac{d}{ds}\oint_{\gamma_s}f=\int_a^b\frac{\partial}{\partial s}\left(f(\Gamma(t,s))\,\frac{\partial}{\partial t}\Gamma(t,s)\right)dt.$$

By the product rule and problem 1140,

$$\frac{d}{ds}\oint_{\gamma_s}f=\int_a^b f'(\Gamma(t,s))\,\frac{\partial}{\partial s}\Gamma(t,s)\,\frac{\partial}{\partial t}\Gamma(t,s)+f(\Gamma(t,s))\,\frac{\partial^2}{\partial s\,\partial t}\Gamma(t,s)\,dt.$$

By Clairaut's theorem and the product rule,

$$\frac{d}{ds}\oint_{\gamma_s}f=\int_a^b\frac{\partial}{\partial t}\left(f(\Gamma(t,s))\,\frac{\partial}{\partial s}\Gamma(t,s)\right)dt.$$

By the fundamental theorem of calculus,

$$\frac{d}{ds}\oint_{\gamma_s}f=\left(f(\Gamma(b,s))\,\frac{\partial}{\partial s}\Gamma(b,s)\right)-\left(f(\Gamma(a,s))\,\frac{\partial}{\partial s}\Gamma(a,s)\right).$$

By definition of homotopy between curves with the same endpoints, $\Gamma(b,s)=\gamma_c(b)$ for all s, and so $\frac{\partial}{\partial s}\Gamma(b,s)=0$. Similarly $\frac{\partial}{\partial s}\Gamma(a,s)=0$ and so

$$\frac{d}{ds}\oint_{\gamma_s}f=0$$

for all $c<s<d$. Thus by the mean value theorem, $\oint_{\gamma_c}f=\oint_{\gamma_d}f$.

(Bonus Problem 1430) Let $\Omega\subseteq\mathbb{C}$ be an open set, let γ_c and γ_d be two curves homotopic in Ω with the same endpoints, and let $f:\Omega\to\mathbb{C}$ be holomorphic. Show that $\oint_{\gamma_c}f=\oint_{\gamma_d}f$ even if the homotopy is merely continuous and C^1 in the first variable.

(Problem 1440) Let $\Omega\subseteq\mathbb{C}$ be any open set, let γ_c and γ_d be any two closed curves that are homotopic in Ω, and let $f:\Omega\to\mathbb{C}$ be holomorphic. Assume the homotopy is C^2. Show that $\oint_{\gamma_c}f=\oint_{\gamma_d}f$. *Hint:* Start by computing $\frac{d}{ds}\oint_{\gamma_s}f$ and then rewrite the result as as $\int_a^b\frac{\partial}{\partial t}h(s,t)\,dt$ for some function h.

As in Problem 1420, $\oint_{\gamma_s} f$ is a continuous function of s and

$$\frac{d}{ds}\oint_{\gamma_s} f = \left(f(\Gamma(b,s))\,\frac{\partial}{\partial s}\Gamma(b,s)\right) - \left(f(\Gamma(a,s))\,\frac{\partial}{\partial s}\Gamma(a,s)\right).$$

By definition of homotopy between closed curves, $\Gamma(b,s) = \Gamma(a,s)$ for all s, and so in particular $\frac{\partial}{\partial s}\Gamma(b,s) = \frac{\partial}{\partial s}\Gamma(a,s)$. Thus

$$\frac{d}{ds}\oint_{\gamma_s} f = \left(f(\Gamma(a,s))\,\frac{\partial}{\partial s}\Gamma(a,s)\right) - \left(f(\Gamma(a,s))\,\frac{\partial}{\partial s}\Gamma(a,s)\right) = 0.$$

By the mean value theorem, $\oint_{\gamma_c} f = \oint_{\gamma_d} f$.

(Problem 1450) Let Ω be open and let $f : \Omega \to \mathbb{C}$ be holomorphic.

(a) Let $\gamma : [0,1] \to \Omega$ be homotopic in Ω to a point (constant function). Show that $\oint_\gamma f = 0$.

(b) Suppose that $\overline{D}(z,r) \subset \Omega$ and that γ is homotopic in $\Omega \setminus \{z\}$ to $\partial D(z,r)$ (traversed once with counter-clockwise orientation). Show that $\frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z)$.

3.1. Differentiability Properties of Holomorphic Functions

Theorem 3.1.3. [Generalization.] Let $D(P,r) \subset \mathbb{C}$. Let $\varphi : \partial D(P,r) \to \mathbb{C}$ be continuous. Let k be a nonnegative integer. Define $f : D(P,r) \to \mathbb{C}$ by

$$f(z) = \frac{k!}{2\pi i}\oint_{\partial D(P,r)} \frac{\varphi(\zeta)}{(\zeta - z)^{k+1}}\,d\zeta.$$

Then f is C^1 and holomorphic in $D(P,r)$, and

$$\frac{\partial f}{\partial z} = \frac{(k+1)!}{2\pi i}\oint_{\partial D(P,r)} \frac{\varphi(\zeta)}{(\zeta - z)^{k+2}}\,d\zeta.$$

(Problem 1460) Let f be as in Theorem 3.1.3. Begin the proof of Theorem 3.1.3 by showing that $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar z}$ exist and satisfy

$$\frac{\partial f}{\partial z} = \frac{(k+1)!}{2\pi i}\oint_{\partial D(P,r)} \frac{\varphi(\zeta)}{(\zeta - z)^{k+2}}\,d\zeta, \qquad \frac{\partial f}{\partial \bar z} = 0.$$

(Problem 1470) Complete the proof of Theorem 3.1.3 by showing that $f \in C^1(\Omega)$ and so is holomorphic.

[Chapter 2, Problem 21] If $z \in \partial D(P,r)$, is it necessarily true that $\lim_{w\to z} f(w) = \varphi(z)$?

Theorem 3.1.1. Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be holomorphic. Then $f \in C^\infty(\Omega)$. Moreover, if $\overline{D}(P,r) \subset \Omega$, then

$$\frac{\partial^k f}{\partial z^k}(z) = \frac{k!}{2\pi i}\oint_{\partial D(P,r)} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\,d\zeta.$$

Corollary 3.1.2. Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be holomorphic. Then $\frac{\partial^k f}{\partial z^k}$ is holomorphic in Ω for all $k \in \mathbb{N}$.

(Problem 1480) Prove Theorem 3.1.1 and Corollary 3.1.2.

Let $\overline{D}(P,r) \subset \Omega$. By the Cauchy integral formula, if $z \in D(P,r)$ then

$$f(z) = \frac{1}{2\pi i}\oint_{\partial D(P,r)} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Because f is holomorphic, it must be continuous, and so we may apply Theorem 3.1.3.

We perform an induction argument. Suppose that f, $f' = \frac{\partial f}{\partial z}, \dots, f^{(k-1)} = \frac{\partial^{k-1} f}{\partial z^{k-1}}$ exist and are C^1 and holomorphic in $D(P,r)$, and that $f^{(k)} = \frac{\partial^k f}{\partial z^k}$ exists and satisfies

$$f^{(k)}(z) = \frac{k!}{2\pi i}\oint_{\partial D(P,r)} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\,d\zeta$$

in $D(P,r)$. We have shown that this is true for $k=0$. By Theorem 3.1.3, we have that $f^{(k)}$ is also C^1 and holomorphic in $D(P,r)$, and that

$$f^{(k+1)} = \frac{(k+1)!}{2\pi i}\oint_{\partial D(P,r)} \frac{f(\zeta)}{(\zeta-z)^{k+2}}\,d\zeta.$$

Because $f^{(k)}$ is holomorphic, $f^{(k+1)} = \frac{\partial^{k+1} f}{\partial z^{k+1}}$. Thus by induction this must be true for all nonnegative integers k.

This proves Corollary 3.1.2 and part of Theorem 3.1.1. To prove that $f \in C^\infty(\Omega)$, observe that

$$\frac{\partial^{j+\ell} f}{\partial x^j \partial y^\ell} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar z}\right)^j \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar z}\right)^\ell f$$

We can write all partial derivatives of f (in terms of x and y) as linear combinations of $\frac{\partial^j f}{\partial z^j}$ and $\frac{\partial^{m+n+\ell} f}{\partial x^m \partial y^n \partial \bar z^\ell}$ for various values of j, ℓ, m, and n, and so

(Problem 1481) (Problem 1281) Let $\gamma : [a,b] \to K$ be a C^1 curve for some set $K \subseteq \mathbb{C}$ (not necessarily open). Let $V \subseteq \mathbb{C}$ be compact and let $f : K \times V \to \mathbb{C}$ be continuous. Let $F : V \to \mathbb{C}$ be defined by

$$F(z) = \oint_\gamma f(\zeta, z)\,d\zeta.$$

Show that F is continuous on V.

(Problem 1490) Suppose that $P \in \Omega \subsetneq \mathbb{C}$ for some open set Ω. Suppose that f is continuous on Ω and holomorphic on $\Omega \setminus \{P\}$. Show that f is holomorphic on Ω.

We must show that $\nabla f(P)$ exists, that ∇f is continuous at P, and that $\frac{\partial f}{\partial \bar z}(P) = 0$. That is, we only need to work at P. Let $\mathcal{R}$ be an open disc centered at P and contained in Ω; by definition of open set $\mathcal{R}$ must exist. Then by Theorem 2.3.3 (Problem 1250) there is a function $F : \mathcal{R} \to \mathbb{C}$ that is holomorphic in $\mathcal{R}$ such that $f = \frac{\partial F}{\partial z}$ in $\mathcal{R}$ (including at P). By Theorem 3.1.1 and Corollary 3.1.2 (Problem 1480), $f = \frac{\partial F}{\partial z}$ is C^1 and holomorphic in $\mathcal{R}$, and in particular at P.

Notes on Homework 5

[Chapter AB, Problem 5] The function $\log : \mathbb{C} \setminus (-\infty, 0]$ given by

$$\log z = \log |z| + i\theta$$

if $z = |z|e^{i\theta}$ and $-\pi < \theta < \pi$ is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ and satisfies

$$\frac{\partial}{\partial z} \log z = \frac{1}{z}.$$

Theorem 2.2.1. (Generalization.) Suppose that $\Omega \subseteq \mathbb{C}$ is open and that f is C^1 on Ω. Let $p \in \Omega$ and suppose $\frac{\partial f}{\partial \bar z}\big|_{z=p} = 0$. Then f has a complex derivative at p and $f'(p) = \frac{\partial f}{\partial z}\big|_{z=p}$.

(Problem 1491) Let $\Omega = \mathbb{R}^2 \setminus (-\infty, 0]$. Define a function $F : \Omega \to \mathbb{R}$ such that if $x + iy = re^{i\theta}$ for some real numbers x, y, r, θ with $r > 0$ and $-\pi < \theta < \pi$, then we have that $F(x,y) = \theta$.

(Problem 1492) Let $W \subset \mathbb{R}^2 \setminus \{(0,0)\}$ be open. Let $\Omega = \{re^{i\theta} : (r,\theta) \in W\}$ and suppose that Ω is open. Let $F : \Omega \to \mathbb{C}$ be holomorphic. Define $f : W \to \mathbb{C}$ by $f(r,\theta) = F(re^{i\theta})$.

(a) Suppose that $(r,\theta) \in W$. Show that

$$\begin{pmatrix} \partial_r f(r,\theta) \\ \partial_\theta f(r,\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\cos\theta & r\sin\theta \end{pmatrix} \begin{pmatrix} \partial_x F \\ \partial_y F \end{pmatrix}\Bigg|_{x+iy=re^{i\theta}}.$$

(b) Suppose that $x + iy \in \Omega$. Then there is an $(r,\theta) \in W$ such that $re^{i\theta} = x + iy$. Show that

$$\begin{pmatrix} \partial_x F(x+iy) \\ \partial_y F(x+iy) \end{pmatrix} = \begin{pmatrix} x/\sqrt{x^2+y^2} & -y/(x^2+y^2) \\ y/\sqrt{x^2+y^2} & x/(x^2+y^2) \end{pmatrix} \begin{pmatrix} \partial_r f(r,\theta) \\ \partial_\theta f(r,\theta) \end{pmatrix}.$$

(Memory 1500) Let $\Omega \subseteq \mathbb{C}$ be open and connected. Show that Ω is path connected and that the paths may be taken to be C^1; that is, if z, $w \in \Omega$ then there is a $\gamma : [0, 1] \to \Omega$ with γ a C^1 function such that $\gamma(0) = z$ and $\gamma(1) = w$.

3.1. Morera's theorem

Theorem 3.1.4. (Morera's theorem.) Let $\Omega \subseteq \mathbb{C}$ be open and connected. Let $f \in C(\Omega)$ be such that $\oint_\gamma f = 0$ for all closed curves γ. Then f is holomorphic in Ω.

(Problem 1510) Prove Morera's theorem. Furthermore, show that there is a function F holomorphic in Ω such that $F' = f$.

Fix some $z_0 \in \Omega$. Suppose that $z \in \Omega$. By Problem 1500, there is a C^1 curve $\psi = \psi_z : [0, 1] \to \Omega$ such that $\psi(0) = z_0$ and $\psi(1) = z$.

Suppose that τ is another such curve, that is, a C^1 function $\tau : [0, 1] \to \Omega$ such that $\tau(0) = z_0$ and $\tau(1) = z$. Let $\tau_{-1}(t) = \tau(1 - t)$. Then by Problem 1010, we have that $\oint_{\tau_{-1}} f = -\oint_\tau f$. Furthermore, by Problem 1050, there is a C^1 curve $\gamma : [0, 1] \to \mathbb{C}$ such that $\gamma(0) = \psi(0) = z_0$, $\gamma(1) = \tau_{-1}(1) = \tau(0) = z_0$ and such that

$$\oint_\gamma f = \oint_\psi f + \oint_{\tau_{-1}} f = \oint_\psi f - \oint_\tau f.$$

But γ is closed and so $\oint_\gamma f = 0$, and so $\oint_\psi f = \oint_\tau f$.

Thus, if we define $F(z) = \oint_{\psi_z} f$, then F is well defined, as $F(z)$ is independent of our choice of path ψ_z from z_0 to z.

Now, let $z \in \Omega$ and let $r > 0$ be such that $D(z, r) \subseteq \Omega$; such an r must exist by definition of Ω. If $w \in D(z, r) \setminus \{z\}$, then

$$F(w) - F(z) = \oint_{\psi_w} f - \oint_{\psi_z} f.$$

Let $\varphi(t) = z + t(w - z)$, so $\varphi : [0, 1] \to D(z, r)$ is a C^1 path from z to w. We may assume without loss of generality that ψ_w is generated from ψ_z and φ by Problem 1050; thus,

$$\frac{F(w) - F(z)}{w - z} = \frac{1}{w - z} \oint_\varphi f = \int_0^1 f(z + t(w - z))\, dt.$$

A straightforward ε-δ argument yields that

$$\lim_{w \to z} \frac{F(w) - F(z)}{w - z} = f(z)$$

so F possesses a complex derivative at z. Furthermore, $F' = f$ is continuous on Ω. Thus $F \in C^1(\Omega)$ and is holomorphic on Ω by Problem 1160, and so by Theorem 3.1.1 and Corollary 3.1.2 $f = F'$ is C^∞ (in particular C^1) and holomorphic in Ω.

(Problem 1520) Can you rewrite Morera's theorem to involve a statement true for all holomorphic functions (can you write it with the phrase "if and only if")?

3.2. Real Analysis

(Memory 1530) State the Root Test and Ratio Test from undergraduate real analysis.

[Definition: Taylor series] Let $f \in C^\infty(a, b)$ and let $a < c < b$. The Taylor series for f at c is $\sum_{n=0}^\infty \frac{f^{(n)}(c)}{n!}(x-c)^n$ (with the convention $0^0 = 1$).

(Memory 1540) Let $P_{m,c}(x) = \sum_{n=0}^m \frac{f^{(n)}(c)}{n!}(x - c)^n$ be the mth partial sum of the Taylor series at c. Suppose that $x \in (a, b)$, $x \neq c$, $m \in \mathbb{N}$. Show that there is a $y_m \in (a, b)$ with $|y_m - c| < |x - c|$ such that

$$f(x) = P_{m-1,c}(x) + \frac{1}{m!} f^{(m)}(y_m)\,(x - c)^m.$$

(Memory 1550) The Taylor series for sin, cos, and exp converge to the parent function on all of $\mathbb{R}$.

(Problem 1560) Give an example of a function $f \in C^\infty(\mathbb{R})$ such that the Taylor series for f converges for all $x \in \mathbb{R}$ but such that $f(x) \neq \sum_{n=0}^\infty \frac{f^{(n)}(c)}{n!}(x-c)^n$ for all $x \neq c$.

The function

$$f(x) = \begin{cases} \exp(-1/x^2), & x \neq 0, \\ 0, & x = 0 \end{cases}$$

satisfies $f^{(n)}(0) = 0$ for all nonnegative integers n, and thus the Taylor series is zero everywhere; however, $f(x) \neq 0$ if $x \neq 0$ and so the Taylor series never converges to the function.

(Problem 1570) Give an example of a function $f \in C^\infty(-2,\infty)$ such that the Taylor series for f at 2 diverges for all $|x-2| > 2$. Can we do this for a function $f \in C^\infty(\mathbb{R})$?

(Bonus Problem 1580) Give an example of a function $f \in C^\infty(\mathbb{R})$ such that the Taylor series for f at 0 diverges for all $x \neq 0$.

[Definition: Absolute convergence] Let $\sum_{n=0}^\infty a_n$ be a series of real numbers. If $\sum_{n=0}^\infty |a_n|$ converges, then we say $\sum_{n=0}^\infty a_n$ converges absolutely.

[Definition: Uniform convergence] Let E be a set, let (X,d) be a metric space, and let f_k, $f : E \to X$. We say that $f_k \to f$ uniformly on E if for every $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that if $k \geq N$, then $d(f_k(z), f(z)) < \varepsilon$ for all $z \in E$.

[Definition: Uniformly Cauchy] Let E be a set, let (X,d) be a metric space, and let $f_k : E \to X$. We say that $\{f_k\}_{k=1}^\infty$ is uniformly Cauchy on E if for every $\varepsilon > 0$ there is a $N \in \mathbb{N}$ such that if $n > m \geq N$, then $d(f_n(z), f-m(z)) < \varepsilon$ for all $z \in E$.

[Definition: Uniform convergence and Cauchy for series] If E is a set, V is a vector space, and $f_k : E \to V$ for each $k \in \mathbb{N}$, then the series $\sum_{k=1}^\infty f_k$ converges uniformly to $f : E \to V$ or is uniformly Cauchy, respectively, if the sequences of partial sums $\{\sum_{k=1}^n f_k\}_{n=1}^\infty$ converge uniformly or are uniformly Cauchy.

(Memory 1590) Suppose that (X,d) is a complete metric space. Then any uniformly Cauchy sequence is uniformly convergent.

(Memory 1600) Suppose that (E,ρ) and (X,d) are two metric spaces. Let f_k, $f : E \to X$. Suppose $f_k \to f$ uniformly on E and that each f_k is continuous. Then f is also continuous.

(Problem 1601) Give an example of a compact metric space (X,d) and a sequence of continuous functions from X to $\mathbb{R}$ that converge pointwise, but not uniformly, to a continuous function.

(Memory 1610) Let f_k, $f : [a,b] \to \mathbb{R}$. Suppose that each f_k is Riemann integrable and that $f_k \to f$ uniformly on $[a,b]$. Then f is Riemann integrable, $\lim_{k\to\infty} \int_a^b f_k$ exists, and $\lim_{k\to\infty} \int_a^b f_k = \int_a^b f$.

(Memory 1611) (The Weierstraß M-test.) Suppose that A is a set and that for each n, $f_n : A \to \mathbb{C}$ is a bounded function. Suppose that there is a sequence $\{M_n\}_{n=0}^\infty \subset [0,\infty)$ such that $|f_n(x)| \leq M_n$ for all $x \in A$ and $\sum_{n=0}^\infty M_n < \infty$. Then the series $\sum_{n=0}^\infty f_n(x)$ converges absolutely and uniformly on A.

3.2. Complex Power Series

(Problem 1620) Let $\sum_{n=0}^\infty a_n$ be a series of complex numbers. Show that if $\sum_{n=0}^\infty |a_n|$ converges then $\sum_{n=0}^\infty a_n$ converges (that is, that in the complex numbers, we still have that absolute convergence implies convergence).

Let $a_n = x_n + iy_n$ where x_n, $y_n \in \mathbb{R}$. Then $|x_n| \leq |a_n|$ and $|y_n| \leq |a_n|$. We recall from real analysis that a nondecreasing sequence converges if and only if it is bounded. Therefore $\{\sum_{n=0}^m |a_n|\}_{m\in\mathbb{N}}$ is bounded. Because $|x_n| \leq |a_n|$, we have that $\sum_{n=0}^m |x_n| \leq \sum_{n=0}^m |a_n| \leq \sup_{m\in\mathbb{N}} \sum_{n=0}^m |a_n|$ and so $\{\sum_{n=0}^m |x_n|\}_{m\in\mathbb{N}}$. Thus $\sum_{n=0}^\infty x_n$ converges absolutely, and therefore $\sum_{n=0}^\infty x_n$ converges. Similarly, $\sum_{n=0}^\infty y_n$ converges. By Problem 210, $\sum_{n=0}^\infty a_n$ converges.

Definition 3.2.2. (Complex power series.) A complex power series is a formal sum $\sum_{k=0}^\infty a_k(z-P)^k$ for some $\{a_k\}_{k=1}^\infty \subseteq \mathbb{C}$. The series converges at z if $\lim_{n\to\infty} \sum_{k=0}^n a_k(z-P)^k$ exists.

Lemma 3.2.3. Suppose that the series $\sum_{k=0}^{\infty} a_k(z-P)^k$ converges at $z=w$ for some $w \in \mathbb{C}$. Then the series converges absolutely at z for all z with $|z-P|<|w-P|$.

Proposition 3.2.9. Suppose that the series $\sum_{k=0}^{\infty} a_k(z-P)^k$ converges at $z=w$ for some $w \in \mathbb{C}$. If $0<r<|w-P|$, then the series converges uniformly on $\overline{D}(P,r)$.

(Problem 1630) Prove Lemma 3.2.3.

(Problem 1640) Prove Proposition 3.2.9.

Because $\sum_{k=0} a_k(w-P)^k$ converges, we have that $\lim_{k\to\infty} a_k(w-P)^k=0$. In particular, $\{a_k(w-P)^k\}_{k=0}^{\infty}$ is bounded. Let $A=\sup_{k\geq 0}|a_k(w-P)^k|$.

Because $0<r/|w-P|<1$, the geometric series $\sum_{k=0}^{\infty} A(r/|w-P|)^k$ converges. Thus for every $\varepsilon>0$ there is an $N>0$ such that $\sum_{k=N}^{\infty} A(r/|w-P|)^k<\varepsilon$.

If $z\in\overline{D}(P,r)$, then $|z-P|\leq r$ and so $|a_k(z-P)^k|\leq |a_k(w-P)^k|(r/|w-P|)^k\leq A(r/|w-P|)^k$. We then have that $\sum_{k=N}^{\infty}|a_k(z-P)^k|\leq\sum_{k=N}^{\infty}A(r/|w-P|)^k<\varepsilon$ for all $z\in\overline{D}(P,r)$. Furthermore, by Lemma 3.2.3 and Problem 1620 $\sum_{k=0}^{\infty} a_k(z-P)^k$ exists, and if $m\geq N$ then

$$\left|\sum_{k=0}^{\infty} a_k(z-P)^k-\sum_{k=0}^{m} a_k(z-P)^k\right|=\left|\sum_{k=m+1}^{\infty} a_k(z-P)^k\right|\leq\sum_{k=m+1}^{\infty}|a_k(z-P)^k|<\varepsilon.$$

Thus the series converges uniformly on $\overline{D}(P,r)$.

(Problem 1650) Suppose that the series diverges at w for some $w\in\mathbb{C}$. Show that the series diverges at z for all z with $|z-P|>|w-P|$.

Suppose for the sake of contradiction that the series converges at z. By Lemma 3.2.3 with z and w interchanged, we know that the series converges at w. But we assumed that the series diverged at w, a contradiction. Therefore the series must diverge at z.

Definition 3.2.4. (Radius of convergence.) The radius of convergence of $\sum_{k=0}^{\infty} a_k(z-P)^k$ is $\sup\{|w-P| : \sum_{k=0}^{\infty} a_k(w-P)^k \text{ converges}\}$.

(Problem 1660) Show that the radius of convergence is also $\inf\{|w-P| : \sum_{k=0}^{\infty} a_k(w-P)^k \text{ diverges}\}$.

Let $R_1=\sup\{|w-P| : \sum_{k=0}^{\infty} a_k(w-P)^k \text{ converges}\}$, $R_2=\inf\{|\zeta-P| : \sum_{k=0}^{\infty} a_k(\zeta-P)^k \text{ diverges}\}$.

If $r\in\{|w-P| : \sum_{k=0}^{\infty} a_k(w-P)^k \text{ converges}\}$, then $r=|w-P|$ for some w such that the series converges. If the series diverges at ζ, then $|\zeta-P|\geq|w-P|$ by Lemma 3.2.3, and so r is a lower bound on $\{|\zeta-P| : \sum_{k=0}^{\infty} a_k(\zeta-P)^k \text{ diverges}\}$. Thus $r\leq R_2$. So R_2 is an upper bound on $\{|w-P| : \sum_{k=0}^{\infty} a_k(w-P)^k \text{ converges}\}$, and so $R_2\geq R_1$.

If $R_2>R_1$, let $z\in\mathbb{C}$ be such that $R_1<|z-P|<R_2$. Then the series either converges or diverges at z. If it converges, then $|z-P|\in\{|w-P| : \sum_{k=0}^{\infty} a_k(w-P)^k \text{ converges}\}$, and so $|z-P|\leq R_1$, a contradiction. We similarly derive a contradiction if the series diverges at z, and so we must have that $R_2=R_1$, as desired.

(Problem 1670) (Lemma 3.2.6.) State the root test from undergraduate real analysis. What does the root test say about complex power series?

The root test for real numbers states that if $\{b_n\}_{n=0}^{\infty}\subset\mathbb{R}$, then

- If $\limsup_{n\to\infty}\sqrt[n]{|b_n|}<1$, then $\sum_{n=0}^{\infty} b_n$ converges absolutely.
- If $\limsup_{n\to\infty}\sqrt[n]{|b_n|}>1$, then the sequence $\{b_n\}_{n=0}^{\infty}$ is unbounded (and in particular the series $\sum_{n=0}^{\infty} b_n$ diverges).

Let $\sum_{k=0}^{\infty} a_k(z-P)^k$ be a complex power series. Fix a $z \in \mathbb{C}$ and observe that

$$\limsup_{k\to\infty} \sqrt[k]{|a_k(z-P)^k|} = |z-P| \limsup_{k\to\infty} \sqrt[k]{|a_k|}.$$

Thus the series $\sum_{k=0}^{\infty} a_k(z-P)^k$ converges if $|z-P|\limsup_{k\to\infty} \sqrt[k]{|a_k|} < 1$ and diverges if $|z-P|\limsup_{k\to\infty} \sqrt[k]{|a_k|} > 1$. Thus the radius of convergence must be

$$\frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}$$

with the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$; that is, if $\limsup_{k\to\infty} \sqrt[k]{|a_k|} = 0$ then the series converges everywhere and if $\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \infty$ then the series diverges unless $z = P$.

(Problem 1680) State the ratio test from undergraduate real analysis. What does the ratio test say about power series?

The ratio test for real numbers states that if $\{b_n\}_{n=0}^{\infty} \subset \mathbb{R}$, then

- If $\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|}$ exists and is less than 1, then $\sum_{n=0}^{\infty} b_n$ converges absolutely.
- If $\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|}$ exists and is greater than 1, then the sequence $\{b_n\}_{n=0}^{\infty}$ is unbounded (and in particular the series $\sum_{n=0}^{\infty} b_n$ diverges).

Let $\sum_{k=0}^{\infty} a_k(z-P)^k$ be a complex power series. Fix a $z \in \mathbb{C} \setminus \{P\}$ and observe that if either $\lim_{k\to\infty} \frac{|a_{k+1}(z-P)^{k+1}|}{|a_k(z-P)^k|}$ or $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}$ exists, then the other must exist and

$$\lim_{k\to\infty} \frac{|a_{k+1}(z-P)^{k+1}|}{|a_k(z-P)^k|} = |z-P| \lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|}.$$

Thus, the series converges absolutely if $|z-P| < \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|}$ and diverges if $|z-P| > \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|}$, so the radius of convergence must be $\lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|}$.

Lemma 3.2.10. Let $\sum_{k=0}^{\infty} a_k(z-P)^k$ be a power series with radius of convergence $R > 0$. Define $f : D(P,R) \to \mathbb{C}$ by $f(z) = \sum_{k=0}^{\infty} a_k(z-P)^k$.

Then f is C^∞ and holomorphic in $D(P,R)$, and if $n \in \mathbb{N}$ then the series

$$\sum_{k=n}^{\infty} \frac{k!}{(k-n)!} a_k(z-P)^{k-n}$$

has radius of convergence at least R and converges to $f^{(n)}(z) = \frac{\partial^n f}{\partial z^n}$.

(Problem 1690) Begin the proof of Lemma 3.2.10 by showing that f is continuous on $D(P,R)$.

If $m \in \mathbb{N}$, define $f_m(z) = \sum_{k=0}^{m} a_k(z-P)^k$. Then each f_m is continuous. By Proposition 3.2.9, if $0 < r < R$ then $f_m \to f$ uniformly on $\overline{D}(P,r)$.

By Memory (1600), we have that f must be continuous on $\overline{D}(P,r)$. But if $z \in D(P,R)$, then $|z-P| < R$ and so there is a $\varepsilon > 0$ and an $r < r$ such that $D(z,\varepsilon) \subset \overline{D}(P,r)$, and so f is continuous at z for all $z \in D(P,R)$. Thus f is continuous on $D(P,R)$.

(Problem 1700) Continue the proof of Lemma 3.2.10 by showing that f is holomorphic in $D(P,R)$.

(Problem 1710) Complete the proof of Lemma 3.2.10 by showing that

$$\sum_{k=n}^{\infty} \frac{k!}{(k-n)!} a_k(z-P)^{k-n}$$

indeed converges to $f^{(n)}(z)$. *Hint*: Use Theorem 3.1.1 (Problem 1480) and Memory 1610.

[Definition: Taylor series] Let $P \in \Omega \subseteq \mathbb{C}$ where Ω is open, and let f be holomorphic in Ω. By Theorem 3.1.1 (Problem 1480), $f^{(n)}$ exists everywhere in Ω. The Taylor series for f at P is the power series $\sum_{k=0}^{\infty} \frac{f^{(k)}(P)}{k!}(z-P)^k$.

(Problem 1720) Let f be as in Lemma 3.2.10. Show that the Taylor series for f at P is simply $\sum_{k=0}^{\infty} a_k(z-P)^k$.

Proposition 3.2.11. Suppose that the two power series $\sum_{k=0}^{\infty} a_k(z-P)^k$ and $\sum_{k=0}^{\infty} b_k(z-P)^k$ both have positive radius of convergence and that there is some $r>0$ such that $\sum_{k=0}^{\infty} a_k(z-P)^k = \sum_{n=0}^{\infty} b_k(z-P)^k$ (and both sums converge) whenever $|z-P|<r$. Then $a_k = b_k$ for all k.

(Problem 1730) Prove Proposition 3.2.11.

Let $f(z) = \sum_{k=0}^{\infty} a_k(z-P)^k = \sum_{k=0}^{\infty} b_k(z-P)^k$ in $D(P,r)$. Then by Lemma 3.2.10, we must have that f is holomorphic in $D(P,r)$, and if $n \geq 0$ is an integer then

$$a_n = \frac{f^{(n)}(P)}{n!} = b_n$$

implying $a_n = b_n$ for all n.

[Definition: Analytic function] Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be a function. If for every $P \in \Omega$ there is a $r>0$ with $D(P,r) \subseteq \Omega$ and a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} a_n(z-P)^n$ for all $z \in D(P,r)$, we say that f is analytic.

(Problem 1740) Show that analytic functions are holomorphic.

(Problem 1750) Recall that if $x \in \mathbb{R}$, then $\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$, $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$. Show that the functions $\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$, and $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$ are holomorphic on $\mathbb{C}$ and take the correct values at all real numbers.

3.3. The Power Series Expansion for a Holomorphic Function

Theorem 3.3.1. Let $\Omega \subseteq \mathbb{C}$ be an open set and let f be holomorphic in Ω. Let $D(P,r) \subseteq \Omega$ for some $r>0$.

Then the Taylor series for f at P has radius of convergence at least r and converges to $f(z)$ for all $z \in D(P,r)$.

(Problem 1760) Let f be holomorphic in $D(P,R)$ and let $0<r<R$. Begin the proof of Theorem 3.3.1 by showing that there is a power series with radius of convergence at least r that converges to f in $D(P,r)$.

Without loss of generality we may assume $P=0$. By the Cauchy integral formula (Problem 1340), if $z \in D(0,r) = D(P,r)$ and $|z|<\rho<r$, then

$$f(z) = \oint_{\partial D(0,\rho)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Because $|z|<|\zeta|$ all $\zeta \in \partial D(0,\rho)$, we have that

$$\frac{1}{\zeta - z} = \frac{1/\zeta}{1-z/\zeta} = \sum_{k=0}^{\infty} \frac{z^k}{\zeta^{k+1}}.$$

Thus

$$f(z) = \oint_{\partial D(0,\rho)} \sum_{k=0}^{\infty} \frac{f(\zeta)}{\zeta^{k+1}} z^k\, d\zeta.$$

Because f is continuous on the compact set $\partial D(0,\rho)$, it is bounded there. Let $m = \sup_{\partial D(0,\rho)} |f|$. Then $|\frac{f(\zeta)}{\zeta^{k+1}} z^k| \leq m|z|^k/\rho^{k+1}$. But $\sum_{k=0}^{\infty} m|z|^k/\rho^{k+1}$ is a convergent geometric series, so by the Weierstraß M-test we have that the series converges uniformly on $\partial D(0,r)$. Thus by Memory 1610 we may interchange the sum with the integral and see that

$$f(z) = \sum_{k=0}^{\infty} z^k \oint_{\partial D(0,\rho)} \frac{f(\zeta)}{\zeta^{k+1}}\, d\zeta$$

and the sum converges.

Let $a_k = \oint_{\partial D(0,\rho)} \frac{f(\zeta)}{\zeta^{k+1}}\, d\zeta$ for any $\rho \in (0,R)$; we have shown that $\sum_{k=0}^{\infty} a_k z^k$ converges in $D(0,\rho)$ to $f(z)$. But by Proposition 2.6.6 (Problem 1400) we have that a_k is independent of ρ, and so we must have that $\sum_{k=0}^{\infty} a_k z^k$ converges in $D(0,r)$ to $f(z)$, as desired.

(Problem 1770) Complete the proof of Theorem 3.3.1 by show that the power series for f in $D(P,r)$ must be the Taylor series for f at P and that the radius of convergence of the Taylor series for f at P must be at least R.

By the previous problem, we know that if $P \in \Omega$ then there are complex numbers a_k such that $\sum_{k=0}^\infty a_k(z-P)^k$ has radius of convergence at least r and $f(z) = \sum_{k=0}^\infty a_k(z-P)^k$ for all $z \in D(P,r)$.

But by Problem 1720, we have that the Taylor series for f is simply $\sum_{k=0}^\infty a_k(z-P)^k$, and by assumption $\sum_{k=0}^\infty a_k(z-P)^k$ converges to f in $D(P,r)$, and so the Taylor series converges to f in $D(P,r)$.

(Problem 1780) Let f be holomorphic in $D(P,r)$. Let R be the radius of convergence of the Taylor series for f at P. Observe that $R \geq r$. Suppose $R > r$. Show that there is a unique function F that is holomorphic in $D(P,R)$ with $F = f$ in $D(P,r)$.

Define $F : D(P,R) \to \mathbb{C}$ by $F(z) = \sum_{k=0}^\infty \frac{f^{(k)}(P)}{k!}(z-P)^k$. By assumption the power series does converge in $D(P,R)$, and by Lemma 3.2.10 (Problem 1700) F is holomorphic in $D(P,R)$.

Suppose for the sake of contradiction that G is holomorphic in $D(P,R)$ and that $G = f = F$ in $D(P,r)$. A straightforward induction argument shows that if $n \in \mathbb{N}$ then $G^{(n)} = f^{(n)} = F^{(n)}$ in $D(P,r)$, and so in particular $G^{(n)}(P) = F^{(n)}(P)$. But by Theorem 3.3.1 (Problem 1770) we have that

$$F(z) = \sum_{n=0}^\infty \frac{F^{(n)}(P)}{n!}(z-P)^n, \qquad G(z) = \sum_{n=0}^\infty \frac{G^{(n)}(P)}{n!}(z-P)^n$$

for all $z \in D(P,R)$, and so we must have that $G = F$ in $D(P,R)$.

(Problem 1790) Let f be an analytic function in a neighborhood of P. Show that the Taylor series for f' at P has the same radius of convergence as the Taylor series for f at P.

(Problem 1800) Let $\Omega = \{re^{i\theta} : 0 < r < \infty, -\pi < \theta < \pi\} = \mathbb{C} \setminus (-\infty, 0]$. Define $F : \Omega \to \mathbb{C}$ by $F(re^{i\theta}) = \ln r + i\theta$ whenever $-\pi < \theta < \pi$. Recall that F is holomorphic and that $F'(z) = \frac{1}{z}$ for all $z \in \Omega$.

(a) Show that the Taylor series for F at $-3+4i$ has radius of convergence 5.
(b) Can we extend F to a function that is holomorphic on $\Omega \cup D(-3+4i, 5)$?

A straightforward induction argument yields that $F^{(n)}(z) = \frac{(-1)^{n-1}(n-1)!}{z^n}$ for all $n \geq 1$. We may thus compute that the Taylor series for F at $-3+4i$ is

$$F(-3+4i) + \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n(-3+4i)^n}(z+3-4i)^n.$$

By the ratio test, the radius of convergence is

$$\lim_{n\to\infty} \left| \frac{(-1)^{n-1}}{n(-3+4i)^n} \frac{(n+1)(-3+4i)^{n+1}}{(-1)^n} \right| = |-3+4i| = 5.$$

However, observe that $-3 \in D(-3+4i, 5)$ and so $\Omega \cup D(-3+4i, 5)$ contains the circle $\partial D(0,3)$.

Furthermore, suppose $G(z)$ is holomorphic in an open set W containing $D(-3+4i, 5)$ and coincides with F on $W \cap \Omega$. Then $G'(z)$ is holomorphic in W and coincides with $F'(z)$ in $D(-3+4i, 3) \subset W \cap \Omega$. Thus, $G'(z) = 1/z$ in $D(-3+4i, 3)$, and so by Problem 1780 we must have that $G'(z) = 1/z$ in $D(-3+4i, 5)$. The function $1/z$ has no antiderivative on any open set containing $\partial D(0,3)$, and so W cannot be all of $\Omega \cup D(-3+4i, 5)$.

(Fact 1801) Recall from Proposition 1.4.3 (Problem 660) that if f is holomorphic in Ω then $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}$ in Ω. By Corollary 3.1.2 (Problem 1480), $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}$ is holomorphic in Ω. A straightforward induction argument yields that $\frac{\partial^n f}{\partial z^n} = \frac{\partial^n f}{\partial x^n}$ in Ω for all $n \in \mathbb{N}$.

(Problem 1810) Show that the functions exp, sin, and cos in Problem 1750 are the only functions that are holomorphic on all of $\mathbb{C}$ and take the correct values for all real numbers.

Let $g : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable and suppose that there exists a function f that is holomorphic on all of $\mathbb{C}$ and satisfies $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Note that $g'(x)$ denotes the *real* derivative $\lim_{\substack{y\to x\\ y\in\mathbb{R}}} \frac{g(y)-g(x)}{y-x}$, while $f'(z)$ denotes the *complex* derivative $\lim_{\substack{w\to z\\ w\in\mathbb{C}}} \frac{f(w)-f(z)}{w-z}$.

We then have that $\frac{\partial^n}{\partial x^n} f(x) = \frac{d^n}{dx^n} g(x) = g^{(n)}(x)$ for all $x \in \mathbb{R}$. By the previous fact, $\frac{\partial^n}{\partial z^n} f(z)\big|_{z=x} = \frac{d^n}{dx^n} \cos x$ for all $x \in \mathbb{R}$. By Theorems 2.2.1 and 2.2.2 (Problems 1160 and 1150), we have that

$$f^{(n)}(x) = \frac{\partial^n}{\partial z^n} f(z)\bigg|_{z=x} = \frac{d^n}{dx^n} \cos x$$

for all $x \in \mathbb{R}$. By Theorem 3.3.1, we must have that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n$$

for all $z \in \mathbb{C}$. In particular, for a given g there is at most one function f that satisfies the given conditions, and so the functions given in Problem 1750 are the unique functions that satisfy these conditions with $g(x) = \exp x$, $g(x) = \cos x$ or $g(x) = \sin x$.

(Problem 1820) Suppose that $\sum_{n=0}^\infty a_n z^n$ and $\sum_{n=0}^\infty b_n z^n$ are two power series with radius of convergence at least r. Show that

$$\sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k\, b_{n-k}\Big) z^n$$

has radius of convergence at least r and that

$$\sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} a_k\, b_{n-k}\Big) z^n = \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right)$$

for all $|z| < r$.

Let $f(z) = \sum_{n=0}^\infty a_n z^n$ and let $g(z) = \sum_{n=0}^\infty b_n z^n$; by Problem 1700, f and g are both holomorphic in $D(0,r)$. Thus $h = fg$ is holomorphic in $D(0,r)$. By Problems 1760 and 1770, we must have that

$$\sum_{n=0}^{\infty} \frac{h^{(n)}(0)}{n!} z^n$$

has radius of convergence at least r and must converge to $h(z) = f(z)g(z)$. A straightforward induction argument and the Leibniz rule (Problem 490) shows that

$$h^{(n)}(0) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} f^{(k)}(0)\, g^{(n-k)}(0)$$

and so by Lemma 3.2.10 (Problem 1710) we have that

$$\frac{h^{(n)}(0)}{n!} = \sum_{k=0}^{n} a_k\, b_{n-k}.$$

3.4. The Cauchy Estimates and Liouville's Theorem

Theorem 3.4.1. (The Cauchy estimates) Let $f : \Omega \to \mathbb{C}$ be holomorphic and let $\overline{D}(P,r) \subseteq \Omega$. Let $k \in \mathbb{N}_0$. Then

$$\left|\frac{\partial^k f}{\partial z^k}\bigg|_{z=P}\right| \leq \frac{k!}{r^k} \sup_{z\in\partial D(P,r)} |f(z)|.$$

(Problem 1830) Prove Theorem 3.4.1.

Let $\gamma : [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = e^{it}$, so γ is a parameterization of $\partial D(P,r)$. By Theorem 3.1.1 (Problem 1480),

$$f^{(k)}(P) = \frac{k!}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - P)^{k+1}}\, d\zeta.$$

By Problem 970,

$$\left|\oint_{\gamma} \frac{f(\zeta)}{(\zeta-P)^{k+1}}\,d\zeta\right| \leq \ell(\gamma) \sup_{\zeta\in\widetilde{\gamma}} \frac{|f(\zeta)|}{|\zeta-P|^{k+1}}$$

where $\ell(\gamma) = \int_0^{2\pi} |\gamma'(t)|\,dt = 2\pi r$. But $\widetilde{\gamma} = \partial D(P,r)$, and so if $\zeta \in \widetilde{\gamma}$ then $|\zeta - P| = r$. Thus this reduces to

$$|f^{(k)}(P)| \leq \frac{k!}{2\pi} 2\pi r \sup_{\zeta\in\widetilde{\gamma}} \frac{|f(\zeta)|}{r^{k+1}} = \frac{k!}{r^k} \sup_{\zeta\in\widetilde{\gamma}} |f(\zeta)|$$

as desired.

(Problem 1840) Let k, $n \in \mathbb{N}$. Show that there is a function $f \in C^\infty(\mathbb{R})$ with $\sup_{x\in\mathbb{R}} |f(x)| \leq 1$ but with $|f^{(k)}(0)| \geq n$.

We let $f(x) = \sin(nx) + \cos(nx)$. Then $|f(x)| \leq |\sin(nx)| + |\cos(nx)| \leq 2$ and so f is bounded. But $f^{(k)}(0) = \pm n^k$. Because $k \geq 1$, this implies $|f^{(k)}(0)| = n^k \geq n$, as desired.

Lemma 3.4.2. If f is holomorphic on a connected open set Ω and $\frac{\partial f}{\partial z} = 0$ in Ω, then f is constant. (This was proven in Problem 590.)

(Problem 1850) Let $P \in \mathbb{C}$, $r > 0$, $k \in \mathbb{N}$, and let $f : D(P,r) \to \mathbb{C}$ be holomorphic. Suppose that $\frac{\partial^{k+1} f}{\partial z^{k+1}} = 0$ in $D(P,r)$. Show that f is a polynomial of degree at most k.

By Theorem 3.3.1 (Problems 1760 and 1770), we have that

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(P)}{n!}(z-P)^n.$$

But $f^{(k+1)} \equiv 0$ in $D(P,r)$, and therefore all derivatives of order higher than k are zero. We then have that

$$f(z) = \sum_{n=0}^{k} \frac{f^{(n)}(P)}{n!}(z-P)^n$$

which is a polynomial of degree at most k.

(Bonus Problem 1860) Show that this is still true in an arbitrary connected open set.

[Definition: Entire] A function $f : \mathbb{C} \to \mathbb{C}$ is entire if f is holomorphic on all of $\mathbb{C}$.

Theorem 3.4.3. [Liouville's theorem.] A bounded entire function is constant.

(Problem 1870) Prove Liouville's theorem.

[Exercise: Theorem 3.4.4]. If f is entire and there is a constant $C \in \mathbb{R}$ and a $k \in \mathbb{N}_0$ such that $|f(z)| \leq C + C|z|^k$ for all $z \in \mathbb{C}$, then f is a polynomial of degree at most k.

Theorem 3.4.5. (The fundamental theorem of algebra.) Let p be a nonconstant (holomorphic) polynomial. Prove that p has a root; that is, prove that there is an $\alpha \in \mathbb{C}$ with $p(\alpha) = 0$.

[Chapter 3, Problem 36] Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial. Show that there is an $R \in (0,\infty)$ such that if $|z| \geq R$, then $|p(z)| \geq |a_n|\,|z|^n/2$.

(Problem 1880) Prove the fundamental theorem of algebra.

(Problem 1890) (Corollary 3.4.6.) Let p be a polynomial of degree $k > 0$. Can p necessarily be factored completely?

Yes. Clearly every linear polynomial ($k = 1$) can be factored completely.

Suppose that all polynomials of degree at most k can be factored completely. Let p be a polynomial of degree $k + 1$.

By the Fundamental Theorem of Algebra, there is a root $r_1 \in \mathbb{C}$ such that $p(r_1) = 0$. By the standard division algorithm, this means that there is a polynomial q of degree k such that $p(z) = (z - r_1)q(z)$. By our induction hypothesis q may be factored completely, and so p may be factored completely.

[Definition: Domain] A domain is a connected open subset of $\mathbb{C}$.

Theorem 3.5.1. Let $\Omega \subseteq \mathbb{C}$ be a domain. Let f_j, $f : \Omega \to \mathbb{C}$. Suppose that each f_j is holomorphic in Ω and that if $K \subset \Omega$ is compact, then $f_j \to f$ uniformly on K. Then f is holomorphic in Ω.

(Problem 1900) Use Theorem 3.1.3 (Problem 1460) to prove Theorem 3.5.1.

Suppose that $P \in \mathbb{C}$, $r > 0$, and $\overline{D}(P, r) \subset \Omega$. Then by the Cauchy integral formula,

$$f_j(z) = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f_j(\zeta)}{\zeta - z}\, d\zeta$$

for all $z \in D(P, r)$ and all j.

But $\partial D(P, r) \subset \Omega$ is compact, and so by assumption $f_j \to f$ uniformly on $\partial D(P, r)$. For any fixed $z \in D(P, r)$, $\frac{1}{\zeta - z}$ is bounded on $\partial D(P, r)$, and so $\frac{f_j(\zeta)}{\zeta - z} \to \frac{f(\zeta)}{\zeta - z}$ uniformly on $\partial D(P, r)$. By assumption, if $z \in D(P, r)$ then $f_j(z) \to f(z)$, while by Problem (1610)

$$\lim_{j\to\infty} \oint_{\partial D(P,r)} \frac{f_j(\zeta)}{\zeta - z}\, d\zeta = \oint_{\partial D(P,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

In particular,

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

Because each f_j is continuous on Ω and $f_j \to f$ uniformly on $\partial D(P, r)$, we have that f is continuous on $\partial D(P, r)$. By Theorem 3.1.3, this implies that f is holomorphic in $D(P, r)$, as desired.

[Chapter 3, Problem 4] Prove Theorem 3.5.1 using Morera's theorem.

(Problem 1910) Give an example of a sequence of functions in $C^\infty(\mathbb{R})$ that converge uniformly to a function that is not differentiable.

Let $f_j(x) = \frac{x}{\sqrt{x^2+1/j^2}}$. f_j is infinitely differentiable on $\mathbb{R}$, but $\{f_j\}_{j=1}^\infty$ converges uniformly to $|x|$.

Corollary 3.5.2. Let $\Omega \subseteq \mathbb{C}$ be a domain. Let f_j, $f : \Omega \to \mathbb{C}$. Suppose that each f_j is holomorphic in Ω and that if $K \subset \Omega$ is compact, then $f_j \to f$ uniformly on K. Then $\frac{\partial}{\partial z} f_j \to \frac{\partial}{\partial z} f$ on Ω and the convergence is uniform on all compact subsets X of Ω.

(Problem 1920) Prove Corollary 3.5.2.

Let $X \subset \Omega$ be compact. Because Ω is open, a standard real analysis argument yields that there is a $r > 0$ and finitely many points z_1, $z_2, \dots z_N$ such that $X \subset \cup_{n=1}^N D(z_n, r/3)$.

Let $K = \cup_{n=1}^N \overline{D}(z_n, 2r/3)$. Then K is the union of finitely many compact sets, and so is compact. Furthermore, if $z \in X$ then $z \in D(z_\ell, r/3)$ for some ℓ and so $\overline{D}(z, r/3) \subset D(z_\ell, 2r/3) \subset K$.

By assumption $f_j \to f$ uniformly on K. That is, for every $\varepsilon > 0$ there is a $M \in \mathbb{N}$ such that if $j \geq M$ then $|f_j - f| < \varepsilon$ on K.

By Theorem 3.5.1, f and thus $f_j - f$ is holomorphic, and so by Theorem 3.4.1 (Problem 1830) we have that if $j \geq M$ and $z \in X$ then

$$|f'(z) - f_j'(z)| \leq \frac{3}{r} \sup_{\partial D(z,r/3)} |f - f_j| \leq \frac{3\varepsilon}{r}.$$

Thus $f_j' \to f'$ uniformly on X, as desired.

(Problem 1930) Give an example of a sequence of functions $\{f_n\}_{n=1}^\infty$ in $C^\infty(\mathbb{R})$ that converge uniformly to a differentiable function f but where f_n' does not converge to f'.

(Problem 1940) Show that the real Taylor series for $f(x) = \ln x$ at any point $c \in (0, \infty)$ has a positive radius of convergence and converges to $\ln x$.

[Definition: Relatively open and closed] Let (X, d) be a metric space and let $Y \subseteq X$. Then (Y, d) is also a metric space. If $F \subseteq Y$ is closed in (Y, d), then we say that F is relatively closed. If $G \subseteq Y$ is open in (Y, d), then we say that G is relatively open.

(Memory 1950) Suppose that $F \subseteq X$ is closed. Then $F \cap Y$ is relatively closed in Y. In particular, if $F \subseteq Y$ and F is closed in (X, d), then F is relatively closed in (Y, d).

(Memory 1960) Suppose that $G \subseteq X$ is open. Then $G \cap Y$ is relatively open in Y.

(Problem 1970) Give an example of a metric space (X, d), a subset $Y \subset X$, and a set $F \subseteq Y$ such that F is relatively closed in (Y, d) but not closed in (X, d).

(Problem 1980) Give an example of a metric space (X, d), a subset $Y \subset X$, and a set $G \subseteq Y$ such that G is relatively open in (Y, d) but not open in (X, d).

3.6. The Zeros of a Holomorphic Function

Corollary 3.6.2. Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f : \Omega \to \mathbb{C}$ be holomorphic. Suppose that there is a $P \in \Omega$ and an $r > 0$ such that $D(P, r) \subseteq \Omega$ and $f = 0$ in $D(P, r)$. Then $f = 0$ everywhere in Ω.

Corollary 3.6.5. Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f : \Omega \to \mathbb{C}$ be holomorphic. Suppose that there is a $P \in \Omega$ such that $f^{(k)}(P) = 0$ for all $k \in \mathbb{N}_0$ (that is, all integers k such that $k \geq 0$). Then $f = 0$ everywhere in Ω.

(Problem 1990) [Redacted]

(Problem 2000) [Redacted]

(Problem 2010) In this problem we begin the proof of Corollaries 3.6.2 and 3.6.3. Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be holomorphic. Let

$$E = \{z \in \Omega : \text{ there is a } r > 0 \text{ such that } D(z, r) \subseteq \Omega \text{ and } f = 0 \text{ in } D(z, r)\},$$
$$F = \{z \in \Omega : f^{(k)}(z) = 0 \text{ for all } k \in \mathbb{N}_0\}.$$

Show that $E = F$.

If $\zeta \in E$, let $r > 0$ be such that $D(\zeta, r) \subseteq \Omega$ and $f = 0$ in $D(\zeta, r)$. Then all first partial derivatives of f are zero in $D(\zeta, r)$; by induction, all derivatives of f (of any order) are zero in $D(\zeta, r)$, and in particular at ζ. Thus $\zeta \in F$ and so $E \subseteq F$.

Conversely, suppose that $\zeta \in F$, so $\zeta \in \Omega$ and $\frac{\partial^k f}{\partial z^k}(\zeta) = 0$ for all $k \in \mathbb{N}_0$. Because Ω is open there is an $r > 0$ such that $D(\zeta, r) \subset \Omega$. By Theorem 3.3.1 (Problem 1760), if $z \in D(\zeta, r)$ then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}\zeta}{k!}(z - \zeta)^k.$$

But because $f^{(k)}\zeta = 0$ for all k, we have that $f(z) = 0$ for all $z \in D(\zeta, r)$, as desired.

(Problem 2020) E is clearly open. Complete the proof of Corollaries 3.6.2 and 3.6.5 by showing that F is relatively closed in Ω and then drawing appropriate conclusions if Ω is connected.

Let $\{z_n\}_{n=1}^{\infty} \subseteq F \subseteq \Omega$ and suppose that $z_n \to z$ for some $z \in \Omega$.

If $k \in \mathbb{N}_0$, then $f^{(k)}(z_n) = 0$ for all n because $z_n \in E$. But each $f^{(k)}$ is continuous on Ω by Theorem 3.1.1 (Problem 1480), and so we must have that $f^{(k)}(z) = 0$ because $z_n \to z$ and $f^{(k)}(z_n) \to 0$. Thus $z \in F$.

We have showed that E is open in $\mathbb{C}$ (and therefore relatively open in Ω) and equals F, which relatively closed in Ω. By definition of connected set, we must have that either $E = F = \Omega$ or $E = F = \emptyset$. Recalling the definitions of E and F completes the proof of both corollaries.

Theorem 3.6.1. Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f : \Omega \to \mathbb{C}$ be holomorphic.
Suppose that there is a sequence $\{z_n\}_{n=1}^{\infty}$ such that

- $z_n \in \Omega$ for all $n \in \mathbb{N}$,
- $f(z_n) = 0$ for all $n \in \mathbb{N}$,
- The sequence $\{z_n\}_{n=1}^{\infty}$ is convergent,
- $z_0 = \lim_{n\to\infty} z_n$ is in Ω,
- $z_0 \neq z_n$ for all $n \geq 1$.

Then $f(z) = 0$ for all $z \in \Omega$.

(Problem 2030) Prove Theorem 3.6.1. You may use Corollary 3.6.2.

Let $r > 0$ be such that $D(z_0, r) \subseteq \Omega$; r must exist because Ω is open. By Theorem 3.3.1, there are constants a_k such that if $z \in D(z_0, r)$, then $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ converges absolutely to $f(z)$.

We claim that $a_k = 0$ for all k; this immediately implies that $f = 0$ in $D(z_0, r)$, and so by Corollary 3.6.2 we have that $f = 0$ in Ω.

We have that, if $m \geq 0$ is an integer, then $\sum_{k=m}^{\infty} a_k (z - z_0)^k$ also converges absolutely. Define $f_m : D(z_0, r) \to \mathbb{C}$ by

$$f_m(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^k.$$

We observe that for any fixed $z \in D(z_0, r)$, $\lim_{m\to\infty} f_m(z) = 0$.

If $z = z_0$ then the series $\sum_{k=m}^{\infty} a_k (z - z_0)^{k-m}$ converges trivially. If $z \in D(z_0, r) \setminus \{z_0\}$, then $(z - z_0)^{-m}$ is independent of k, and so if $\sum_{k=m}^{\infty} a_k (z - z_0)^k$ converges absolutely then so does $\sum_{k=m}^{\infty} [a_k (z - z_0)^k](z - z_0)^{-m} = \sum_{k=m}^{\infty} a_k (z - z_0)^{k-m}$. We define $h_m : D(z_0, r) \to \mathbb{C}$ by $h_m(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^{k-m}$. By Lemma 3.2.10 (Problem 1690), each h_m is holomorphic and therefore continuous in $D(z_0, r)$. Furthermore, if $z \in D(z_0, r)$ then $\lim_{m\to\infty} h_m(z) = 0$.

We have that $f(z) = f_0(z) = (z - z_0)^0 h_0(z)$ for all $z \in D(z_0, r)$ (with the convention $0^0 = 1$ usual in power series).

Suppose that we have established that $f(z) = f_m(z) = (z - z_0)^m h_m(z)$ for some fixed integer m and all $z \in D(z_0, r)$. Then $a_m = h_m(z_0) = \lim_{n\to\infty} h_m(z_n)$ because h_m is continuous. But $h_m(z_n) = f(z_n)/(z_n - z_0)^m = 0$ and so $a_m = 0$. Thus $f(z) = f_m(z) = f_{m+1}(z) = (z - z_0)^{m+1} h_{m+1}(z)$ and $a_m = 0$. By induction we have that $f(z) = f_m(z) = f_{m+1}(z) = (z - z_0)^{m+1} h_{m+1}(z)$ and $a_m = 0$ for all $m \in \mathbb{Z}$, and so as noted above the conclusion follows from Corollary 3.6.2.

(Problem 2040) Give an example of a function f holomorphic in $\mathbb{C} \setminus \{0\}$ and a sequence of points $z_n \in \mathbb{C} \setminus \{0\}$ with $z_n \to 0$ and with $f(z_n) = 0$ but where $f(z) \neq 0$ for some $z \in \mathbb{C} \setminus \{0\}$.

Let $f(z) = \sin(1/z)$ and let $z_n = \frac{1}{n\pi}$. Then $z_n \to 0$ and $f(z_n) = 0$ for all $n \in \mathbb{N}$, but $f(z) \neq 0$ for many values of z.

[Definition: Accumulation point] Let $S \subseteq \mathbb{C}$. Suppose that $P \in \mathbb{C}$ and that, for every $r > 0$, there is a $z \in D(P, r) \cap S$ with $z \neq P$. Then we say that P is an accumulation point for S.

(Problem 2041) Rewrite Theorem 3.6.1 in terms of accumulation points rather than sequences and prove your version.

Theorem. (Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f : \Omega \to \mathbb{C}$ be holomorphic) Let $Z = \{z \in \Omega : f(z) = 0\}$. If Z has an accumulation point that lies in Ω, then $f(z) = 0$ for all $z \in \Omega$.

(Problem 2050) Let Ω be a connected open set and let $f : \Omega \to \mathbb{C}$ be holomorphic and not constant. Let $P \in \Omega$. Show that there is a $r > 0$ with $D(P, r) \subseteq \Omega$ and such that $f \neq 0$ on $D(P, r) \setminus \{P\}$.

Corollary 3.6.3. Suppose that f and g are holomorphic in a connected open set Ω. If $\{z \in \Omega : f(z) = g(z)\}$ has an accumulation point in Ω, then $f(z) = g(z)$ for all $z \in \Omega$.

(Problem 2060) Prove Corollary 3.6.3.

Let $h(z) = f(z) - g(z)$. Then h is holomorphic in Ω and $\{z \in \Omega : f(z) = g(z)\} = \{z \in \Omega : h(z) = 0\}$. Thus $\{z \in \Omega : h(z) = 0\}$ has an accumulation point. By Problem 2041, $h \equiv 0$ in Ω, and so $f \equiv g$ in Ω.

(Problem 2070) Let $\Omega \subseteq \mathbb{C} \setminus \{0\}$ be open and connected and contain a positive real. Show that there is at most one function $f : \Omega \to \mathbb{C}$ such that $f(x) = \ln x$ for all $x \in (0, \infty) \cap \Omega$.

Corollary 3.6.4. Suppose that f and g are holomorphic in a connected open set Ω. If $fg = 0$ everywhere in Ω, then either $f \equiv 0$ or $g \equiv 0$ in Ω.

(Problem 2080) Prove Corollary 3.6.4.

Suppose $f \not\equiv 0$. Then there is a $\zeta \in \Omega$ such that $f(\zeta) \neq 0$. By continuity, there is a $r > 0$ such that $f \neq 0$ in $D(\zeta, r)$, and so we must have that $g \equiv 0$ in $D(\zeta, r)$. The result follows from Corollary 3.6.2.

[Chapter 3, Problem 42] Let f be holomorphic in the connected open set Ω and let $K \subseteq \Omega$ be compact. Show that if f has infinitely many zeroes in K then $f \equiv 0$ in Ω.

4.1. The Behavior of a Holomorphic Function Near an Isolated Singularity

[Definition: Isolated singularity] If $\Omega \subseteq \mathbb{C}$ is open and $P \in \mathbb{C}$, and if f is a function defined and holomorphic in $\Omega \setminus \{P\}$, then we say that f has an isolated singularity at P.

[Definition: Removable singularity] If f has an isolated singularity at P and if f is defined and bounded on $D(P, r) \setminus \{P\}$ for some $r > 0$, then we say that f has a removable singularity at P.

Theorem 4.1.1. [The Riemann removable singularities theorem.] Suppose that f has a removable singularity at P. Then $\lim_{z\to P} f(z)$ exists (and is a finite complex number), and the function

$$\widehat{f}(z) = \begin{cases} f(z), & z \in \Omega \setminus \{P\}, \\ \lim_{z\to P} f(z), & z = P \end{cases}$$

is holomorphic on Ω.

(Observe that if the limit exists, then $\widehat{f}$ is continuous on Ω and holomorphic on $\Omega \setminus \{P\}$, so the fact that $\widehat{f}$ is holomorphic is simply Problem 1490.)

[Chapter 4, Problem 8a] Suppose that $P \in \Omega \subseteq \mathbb{C}$ for some open set Ω. Suppose that $f : \Omega \setminus \{P\} \to \mathbb{C}$ is holomorphic and that $\lim_{z\to P}(z - P)f(z) = 0$. Then $\lim_{z\to P} f(z)$ exists.

(Problem 2090)

(a) Give an example of a C^∞ function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ that is bounded but such that $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ do not exist.
(b) Give an example of a C^∞ function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that f and f' are both bounded but such that $\lim_{x\to 0} f(x)$ does not exist.

(a) Let $f(x) = \sin(1/x)$. Then f is C^∞ on $(0, \infty)$ and $(-\infty, 0)$ but has no limits (even one-sided limits) at 0.
(b) Let $f(x) = \operatorname{sgn}(x)$ (that is, $f(x) = 1$ if $x > 1$ and $f(x) = 0$ if $x < 0$). Then f is bounded and $f' = 0$ is bounded but $\lim_{x\to 0} f(x)$ does not exist.

[Definition: Pole] If f has an isolated singularity at P, and if $\lim_{z\to P} |f(z)| = \infty$, then we say that f has a pole at P.

(Problem 2100) Suppose that Ω is an open set, $P \in \Omega$, $g : \Omega \to \mathbb{C}$ is holomorphic, $g(P) = 0$, and $g \neq 0$ on $\Omega \setminus \{P\}$. Show that $f(z) = 1/g(z)$ is holomorphic on $\Omega \setminus \{P\}$ and that $\lim_{z\to P} |f(z)| = \infty$.

(Problem 2110) Suppose that $P \in \Omega \subseteq \mathbb{C}$ for some open set Ω. Suppose that $f : \Omega \setminus \{P\} \to \mathbb{C}$ is holomorphic and that $\lim_{z\to P} |f(z)| = \infty$. Let $W = \Omega \setminus \{z \in \Omega : f(z) = 0\}$. Observe that $P \in W$. Show that W is open.

[Chapter 4, Problem 15a] Let f be as in the previous problem and let $g : W \setminus \{P\} \to \mathbb{C}$ be given by $g(z) = 1/f(z)$. Then g has a removable singularity at P and $\lim_{z\to P} g(z) = 0$.

[Definition: Essential singularity] If f has an isolated singularity at P, and if f has neither a pole nor a removable singularity at P, then we say that f has an essential singularity at P.

(Problem 2111) State the precise N-δ negation of the statement "$\lim_{z\to P}|f(z)|=\infty$".

If it is false that $\lim_{z\to P}|f(z)|=\infty$ (either the limit does not exists, or it exists but is finite), then there exists a $N\in\mathbb{R}$ such that for every $\delta>0$ there is a z with $0<|z-P|<\delta$ and such that $|f(z)|\leq N$.

(Problem 2120) Let $f:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ be given by $f(z)=\exp(1/z)$. Let $w\in\mathbb{C}$ with $w\neq 0$ and let $r>0$. Show that $w=f(z)$ for some $z\in D(0,r)\setminus\{0\}$.

Recall that there exist real numbers ρ and $theta$ such that $w=\rho e^{i\theta}$. Furthermore $\rho>0$ and we may require $0\leq\theta<2\pi$.

Define z_k by $1/z_k=\ln\rho+i\theta+2k\pi i$. Then $f(z_k)=w$ for all $k\in\mathbb{Z}$. But $\lim_{k\to\infty}|1/z_k|=\infty$, and so we may find a k such that $|z_k|<r$.

(Problem 2130) Show that if $r>0$ then $\sup_{0<|z|<r}|\exp(1/z)|=\infty$ and $\inf_{0<|z|<r}|\exp(1/z)|=0$. Conclude that $\lim_{z\to 0}|\exp(1/z)|$ does not exist (even in the sense of infinite limits).

Theorem 4.1.4. Suppose that f has an essential singularity at P. Let $r>0$ be such that $D(P,r)\subseteq\Omega$. Then $f(D(P,r)\setminus\{P\})$ is dense in $\mathbb{C}$.

(Problem 2140) Prove Theorem 4.1.4.

(Problem 2141) Show that if f has an isolated singularity at P and $\limsup_{z\to P}|f(z)|\neq\liminf_{z\to P}|f(z)|$ then f has an essential singularity at P.

(Problem 2142) Show that if f has an essential singularity at P then $\limsup_{z\to P}|f(z)|=\infty$ and $\liminf_{z\to P}|f(z)|=0$.

(Problem 2150) Give an example of a C^∞ function $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ such that $\limsup_{x\to 0}|f(x)|=\infty$ and $\liminf_{x\to 0}|f(x)|=0$ but such that $f(x)\geq 0$ for all $x\in\mathbb{R}$.

An example of such a function is $f(x)=\frac{1}{x^2}(1+\sin(1/x))$.

4.2. Convergence of Laurent series

[Definition: Laurent series] A Laurent series is a formal expression of the form $\sum_{k=-\infty}^{\infty}a_k(z-P)^k$, where $P\in\mathbb{C}$ and each $a_k\in\mathbb{C}$, with the convention that $0^0=1$ and $0\cdot 0^k=0$ even if $k<0$.

[Definition: Convergence of Laurent series] We say that the Laurent series $\sum_{k=-\infty}^{\infty}a_k(z-P)^k$ converges at z if the two series $\sum_{k=0}^{\infty}a_k(z-P)^k$ and $\sum_{k=1}^{\infty}a_{-k}(z-P)^{-k}$ both converge, and write

$$\sum_{k=-\infty}^{\infty}a_k(z-P)^k=\sum_{k=0}^{\infty}a_k(z-P)^k+\sum_{k=1}^{\infty}a_{-k}(z-P)^{-k}.$$

Lemma 4.2.1. Suppose that the doubly infinite series $\sum_{k=-\infty}^{\infty}a_k(z-P)^k$ converges at $z=w_1$ and at $z=w_2$, where $0<|w_1-P|<|w_2-P|$. Then the series converges absolutely at z for all z such that $|w_1-P|<|z-P|<|w_2-P|$.

(Problem 2160) Prove Lemma 4.2.1.

(Problem 2170) Suppose that the doubly infinite series $\sum_{k=-\infty}^{\infty}a_k(z-P)^k$ converges at $z=w$ and diverges at $z=\zeta$.

- If $0<|w-P|<|\zeta-P|$, show that the series diverges at z for all z such that $|\zeta-P|<|z-P|$.
- If $0<|\zeta-P|<|w-P|$, show that the series diverges at z for all z such that $|z-P|<|\zeta-P|$.

Lemma 4.2.2. Let $\sum_{k=-\infty}^{\infty} a_k(z-P)^k$ be a doubly infinite series that converges at $z = w$ for at least one $w \in \mathbb{C} \setminus \{P\}$. Then there are extended real numbers r and R with $0 \leq r \leq |w-P| \leq R \leq \infty$ such that the series converges absolutely if $r < |z-P| < R$ and diverges if $|z-P| < r$ or $|z-P| > R$.

Furthermore, if $r < \tau < \sigma < R$ then the series converges uniformly on $\overline{D}(P,\sigma) \setminus D(P,\tau)$.

(Problem 2180) Prove the existence of r and R in Lemma 4.2.2.

Let

$$r = \inf\Big\{|z-P| : \sum_{k=-\infty}^{\infty} a_k(z-P)^k \text{ converges}\Big\}$$

and let

$$R = \sup\Big\{|z-P| : \sum_{k=-\infty}^{\infty} a_k(z-P)^k \text{ converges}\Big\}.$$

By the definition and properties of the infimum and supremum, we have that $0 \leq r \leq |w-P| \leq R \leq \infty$ because $|w-P| \in \{|z-P| : \sum_{k=-\infty}^{\infty} a_k(z-P)^k \text{ converges}\}$. Furthermore, we have that the series diverges at z if $|z-P| < r$ or $|z-P| > R$.

It remains only to show that if $r < |z-P| < R$ then the series converges absolutely at z. Choose some such z. By definition of infimum and supremum, there exist w_1 and w_2 such that the series converges at both w_1 and w_2 and such that $|w_1 - P| < |z-P| < |w_2 - P|$. Absolute convergence at z then follows from Lemma 4.2.1.

(Problem 2190) Establish the uniform convergence on $\overline{D}(P,\sigma) \setminus D(P,\tau)$ in Lemma 4.2.2.

Let w_1 and w_2 satisfy $r < |w_1 - P| < \sigma$ and $\tau < |w_2 - P| < R$. Then $\sum_{k=-\infty}^{\infty} a_k(w_1-P)^k$ converges, and so in particular $\sum_{k=0}^{\infty} a_k(w_1-P)^k$ converges. By Proposition 3.2.9 we have that $\sum_{k=0}^{\infty} a_k(z-P)^k$ converges uniformly on $\overline{D}(P,\sigma)$.

Similarly, $\sum_{k=-\infty}^{\infty} a_k(w_2-P)^k$ converges, and so in particular $\sum_{k=-\infty}^{-1} a_k(w_2-P)^k$ converges. Thus the power series $\sum_{n=1}^{\infty} a_{-n}\big(\frac{1}{w_2-P}\big)^n$ converges, and so $\sum_{n=1}^{\infty} a_{-n}\zeta^n$ converges uniformly to some $g(\zeta)$ for $\zeta \in \overline{D}(0, \frac{1}{\tau})$. That is, for every $\varepsilon > 0$ there is a $M > 0$ such that if $m \geq M$ and $|\zeta| \leq 1/\tau$, then $|g(\zeta) - \sum_{n=1}^{m} a_{-n}\zeta^n| < \varepsilon$. Thus, if $m \geq M$ and $|z-P| \geq \tau$, then $\zeta = \frac{1}{z-P}$ satisfies $|\zeta| \leq 1/\tau$ and so $|g(\zeta) - \sum_{k=-m}^{-1} a_k(z-P)^k| < \varepsilon$. Thus the series $\sum_{k=-\infty}^{-1} a_k(z-P)^k$ converges uniformly on $\{z \in \mathbb{C} : |z-P| \geq \tau\} = \mathbb{C} \setminus D(P,\tau)$.

Thus both series converge uniformly on $\overline{D}(P,\sigma) \setminus D(P,\tau)$, and so their sum converges uniformly in this region, as desired.

(Problem 2200) Let $f(z) = \sum_{j=-\infty}^{\infty} a_j(z-P)^j$. Show that f is holomorphic on $D(P,R) \setminus \overline{D}(P,r)$, where r and R are as in Lemma 4.2.2.

If $r = R$ or if the series converges nowhere then there is nothing to prove. Otherwise, let $f_N(z) = \sum_{j=-N}^{N} a_j(z-P)^j$. Then each f_N is holomorphic and $f_N \to f$ uniformly on all compact subsets of $D(P,R) \setminus \overline{D}(P,r)$. By Theorem 3.5.1 f must also be holomorphic.

(Problem 2210) Give examples of Laurent series for which:

- $r = 1$ and $R = 2$.
- $r = 0$, $R = 1$, and $a_j \neq 0$ for infinitely many values of $j < 0$.
- $r = 1$ and $R = \infty$ and $a_j \neq 0$ for infinitely many values of $j > 0$.
- $r = 0$ and $R = \infty$ and $a_j \neq 0$ for infinitely many values of $j > 0$ and also for infinitely many values of $j < 0$.

- One such series is $\sum_{k=-\infty}^{-1} z^k + \sum_{k=0}^{\infty} \frac{1}{2^k} z^k$.
- One such series is $\sum_{k=-\infty}^{-1} \frac{1}{|k|!} z^k + \sum_{k=0}^{\infty} z^k$.
- One such series is $\sum_{k=-\infty}^{-1} z^k + \sum_{k=0}^{\infty} \frac{1}{k!} z^k$.

- One such series is $\sum_{k=-\infty}^{\infty} \frac{1}{|k|!} z^k$.

Proposition 4.2.4. Let $\sum_{j=-\infty}^{\infty} a_j(z-P)^j$ and $\sum_{j=-\infty}^{\infty} b_j(z-P)^j$ be doubly infinite series that both converge to the same value if $r < |z-P| < R$, for some $0 \leq r < R \leq \infty$. Then $a_j = b_j$ for all j.

(Problem 2220) Suppose f is holomorphic in $D(P,R)$ and that $0 < \tau < R$. By Problem 1760 and Proposition 3.2.11, there is a unique sequence of complex numbers $\{a_k\}_{k=0}^{\infty}$ such that $f(z) = \sum_{k=0}^{\infty} a_k(z-P)^k$ for all $z \in D(P,R)$. Combine Theorem 3.3.1 with Theorem 3.1.1 to find a formula for a_k in terms of an integral over $\partial D(P,\tau)$.

By Theorem 3.3.1, we have that if $z \in D(P,R)$ then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(P)}{k!}(z-P)^k.$$

By Theorem 3.1.1, we have that

$$f^{(k)}(P) = \frac{k!}{2\pi i} \oint_{\partial D(P,\tau)} \frac{f(\zeta)}{(\zeta-P)^{k+1}} d\zeta.$$

Thus we must have that

$$a_k = \frac{f^{(k)}(P)}{k!} = \frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \frac{f(\zeta)}{(\zeta-P)^{k+1}} d\zeta.$$

(Problem 2230) Let $\sum_{k=-\infty}^{\infty} a_k(z-P)^k$ be a doubly infinite series that converges to $f(z)$ if $r < |z-P| < R$, for some $0 \leq r < R \leq \infty$. Let $r < \tau < R$. Compute

$$\frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \frac{f(\zeta)}{(\zeta-P)^{n+1}}\, d\zeta$$

for any $n \in \mathbb{Z}$. Then prove Proposition 4.2.4.

Observe that

$$\frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \frac{f(\zeta)}{(\zeta-P)^{n+1}}\, d\zeta = \frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \frac{1}{(\zeta-P)^{n+1}} \sum_{k=-\infty}^{\infty} a_k(\zeta-P)^k\, d\zeta = \frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \sum_{k=-\infty}^{\infty} a_k(\zeta-P)^{k-n-1}\, d\zeta$$

by definition of f. By Lemma 4.2.2, the series converges uniformly on the compact set $\partial D(P,\tau)$. Because $\frac{1}{(\zeta-P)^{n+1}}$ is bounded on $\partial D(P,\tau)$, we have that $\sum_{k=-\infty}^{\infty} a_k(\zeta-P)^{k-n-1}$ also converges uniformly, and so by Memory 1610 we have that

$$\frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \frac{f(\zeta)}{(\zeta-P)^{n+1}}\, d\zeta = \sum_{k=-\infty}^{\infty} a_k \frac{1}{2\pi i} \oint_{\partial D(P,\tau)} (\zeta-P)^{k-n-1}\, d\zeta.$$

Using the parameterization $\gamma(t) = P + \tau e^{it}$, $0 \leq t \leq 2\pi$, of $\partial D(P,\tau)$, we compute

$$\frac{1}{2\pi i} \oint_{\partial D(P,\tau)} (\zeta-P)^{k-n-1}\, d\zeta = \frac{\tau^{k-n}}{2\pi} \int_0^{2\pi} e^{it(k-n)}\, dt$$

which equals one if $k = n$ and equals zero otherwise, so

$$\frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \frac{f(\zeta)}{(\zeta-P)^{n+1}}\, d\zeta = a_n.$$

Similarly, we have that

$$\frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \frac{f(\zeta)}{(\zeta-P)^{n+1}}\, d\zeta = b_n$$

and so $b_n = a_n$ for all n.

Theorem 4.3.2. Let $0 \leq r < R \leq \infty$ and let $\Omega = D(P,R) \setminus \overline{D}(P,r)$ for some $P \in \mathbb{C}$. (We take $D(P,\infty) = \mathbb{C}$.) Suppose that $f : \Omega \to \mathbb{C}$ is holomorphic. Then there exist constants a_k such that the series

$$\sum_{k=-\infty}^{\infty} a_k (z-P)^k$$

converges absolutely to $f(z)$ for all $z \in \Omega$.

Furthermore, if $r < \sigma < \tau < R$ then the series converges uniformly on $\overline{D}(P,\tau) \setminus D(P,\sigma)$.

Theorem 4.3.1. Let f, r, R be as in Theorem 4.3.2. If $r < \sigma < |z-P| < \tau < R$, then

$$f(z) = \frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i} \oint_{\partial D(P,\sigma)} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

(Problem 2240) Let f, r, R, σ, τ, and z be as in Theorem 4.3.1. We will use Theorem 4.3.1 to prove Theorem 4.3.2 (so you may not use Theorem 4.3.2). Begin the proof of Theorem 4.3.1 by computing

$$\frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \frac{f(z)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i} \oint_{\partial D(P,\sigma)} \frac{f(z)}{\zeta - z}\,d\zeta.$$

Because z is outside $D(P,\sigma)$, we have that $\oint_{\partial D(P,\sigma)} \frac{1}{\zeta - z}\,d\zeta = 0$ by Theorem 2.4.3. Because $z \in D(P,\tau)$, we have that $\oint_{\partial D(P,\tau)} \frac{1}{\zeta-z}\,d\zeta = 2\pi i$ by the special case of the Cauchy integral formula (Lemma 2.4.1). Thus

$$\frac{1}{2\pi i} \oint_{\partial D(0,\tau)} \frac{f(z)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i} \oint_{\partial D(0,\sigma)} \frac{f(z)}{\zeta - z}\,d\zeta = f(z).$$

(Problem 2250) Let $r < \sigma < |z-P| < \tau < R$. Complete the proof of Theorem 4.3.1 by computing

$$\frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i} \oint_{\partial D(P,\sigma)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta.$$

Let $g(\zeta) = \frac{f(\zeta)-f(z)}{\zeta - z}$ if $\zeta \neq z$ and let $g(z) = f'(z)$. Then $\lim_{\zeta \to z} g(\zeta) = g(z) \in \mathbb{C}$ and g is holomorphic on $\Omega \setminus \{z\}$, so by Theorem 4.1.1/Problem 1490, g is holomorphic on Ω. Thus

$$\oint_{\partial D(P,\sigma)} g = \oint_{\partial D(P,\tau)} g$$

by Problem 1400, so

$$\frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i} \oint_{\partial D(P,\sigma)} \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta = 0.$$

(Problem 2260) Let f, r, R be as in Theorem 4.3.2. We seek to show that f may be represented by a Laurent series. By Problem 2230, the only possible Laurent series is $\sum_{k=-\infty}^{\infty} a_k (z-P)^k$, where

$$a_k = \frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \frac{f(\zeta)}{(\zeta - P)^{k+1}}\,d\zeta$$

for any $\tau \in (r,R)$. Begin the proof of Theorem 4.3.2 by showing that the sum $\sum_{k=-\infty}^{\infty} a_k (z-P)^k$ converges absolutely for all $z \in \Omega$. (By Problem 2190, this means that the series converges uniformly on compact subsets of Ω.) *Hint*: Let $r < \sigma < |z-P| < \tau < R$ and find upper bounds on a_k in terms of $M_\tau = \sup_{\partial D(P,\tau)} |f|$ and $M_\sigma = \sup_{\partial D(P,\sigma)} |f|$.

(Problem 2270) Continue the proof of Theorem 4.3.2 by showing that, if $r < \sigma < |z-P| < \tau < R$, then

$$\sum_{k=0}^{\infty} a_k (z-P)^k = \frac{1}{2\pi i} \oint_{\partial D(P,\tau)} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Define

$$g(z)=\sum_{k=0}^\infty a_k(z-P)^k, \qquad h(z)=\frac{1}{2\pi i}\oint_{\partial D(P,\tau)}\frac{f(\zeta)}{\zeta-z}\,d\zeta.$$

By the previous problem and Lemma 3.2.10, g is holomorphic in $D(P,\tau)$, while by Theorem 3.1.3 and because holomorphic functions are continuous, we have that h is also holomorphic in $D(P,\tau)$.

Furthermore, by our generalization of Theorem 3.1.3, we have that if $n\geq 0$ is an integer then

$$h^{(n)}(z)=\frac{n!}{2\pi i}\oint_{\partial D(P,\tau)}\frac{f(\zeta)}{(\zeta-z)^{n+1}}\,d\zeta$$

while by Lemma 3.2.10

$$g^{(n)}(z)=\sum_{k=n}^\infty \frac{k!}{(k-n)!}a_k(z-P)^{k-n}.$$

In particular,

$$h^{(n)}(P)=\frac{n!}{2\pi i}\oint_{\partial D(P,\tau)}\frac{f(\zeta)}{(\zeta-P)^{n+1}}\,d\zeta, \quad g^{(n)}(z)=n!a_n.$$

By definition of a_n, we have that $g^{(n)}(P)-h^{(n)}(P)=0$ for all $n\geq 0$, and so by Corollary 3.6.5, $g=h$ everywhere in $D(P,\tau)$.

(Problem 2280) Complete the proof of Theorem 4.3.2.

We need only show that if $r<\sigma<|z-P|<\tau<R$, then

$$\sum_{k=-\infty}^{-1}a_k(z-P)^k=-\frac{1}{2\pi i}\oint_{\partial D(P,\sigma)}\frac{f(\zeta)}{\zeta-z}\,d\zeta.$$

Recall $a_k=\frac{1}{2\pi i}\oint_{\partial D(P,\sigma)}\frac{f(\zeta)}{(\zeta-P)^{k+1}}\,d\zeta$. Then

$$\sum_{k=-\infty}^{-1}a_k(z-P)^k=\sum_{k=-\infty}^{-1}\frac{1}{2\pi i}\oint_{\partial D(P,\sigma)}\frac{f(\zeta)}{(\zeta-P)^{k+1}}\,d\zeta(z-P)^k.$$

Let $M=\sup_{|\zeta-P|=\sigma}|f(\zeta)|$; because f is continuous on Ω and $\partial D(P,\sigma)\subset\Omega$ is compact, M is finite. Then

$$\left|\frac{f(\zeta)}{(\zeta-P)^{k+1}}(z-P)^k\right|\leq M\sigma^{-1}\left(\frac{|z-P|}{\sigma}\right)^k$$

for all $\zeta\in\partial D(P,\sigma)$, and $\sum_{k=-\infty}^{-1}M\sigma^{-1}\left(\frac{|z-P|}{\sigma}\right)^k$ converges because $|z-P|>\sigma$, so $\sum_{k=-\infty}^{-1}\frac{f(\zeta)}{(\zeta-P)^{k+1}}(z-P)^k$ converges uniformly in ζ. Thus

$$\sum_{k=-\infty}^{-1}a_k(z-P)^k=\frac{1}{2\pi i}\oint_{\partial D(P,\sigma)}\sum_{k=-\infty}^{-1}\frac{f(\zeta)}{(\zeta-P)^{k+1}}(z-P)^k\,d\zeta.$$

Applying the formula for the sum of a geometric series, we see that

$$\begin{aligned}\sum_{k=-\infty}^{-1}a_k(z-P)^k&=\frac{1}{2\pi i}\oint_{\partial D(P,\sigma)}f(\zeta)\frac{1}{z-P}\sum_{k=-\infty}^{-1}\left(\frac{z-P}{\zeta-P}\right)^{k+1}d\zeta\\&=\frac{1}{2\pi i}\oint_{\partial D(P,\sigma)}f(\zeta)\frac{1}{z-P}\sum_{n=0}^{\infty}\left(\frac{\zeta-P}{z-P}\right)^{n}d\zeta\\&=\frac{1}{2\pi i}\oint_{\partial D(P,\sigma)}f(\zeta)\frac{1}{z-P}\frac{1}{1-(\zeta-P)/(z-P)}\,d\zeta\\&=\frac{1}{2\pi i}\oint_{\partial D(P,\sigma)}\frac{f(\zeta)}{z-\zeta}\,d\zeta\end{aligned}$$

as desired.

Proposition 4.3.3. Suppose that f is holomorphic in the punctured disc $\Omega = D(P,R)\setminus\{P\}$ for some $P \in \mathbb{C}$ and some $0 < R \leq \infty$. Then f has a unique Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-P)^k$$

which converges absolutely to $f(z)$ for all $z \in D(P,R)\setminus\{P\}$. The convergence is uniform on compact subsets of $D(P,R)\setminus\{P\}$. The coefficients are given by

$$a_k = \frac{1}{2\pi i}\oint_{\partial D(P,\sigma)} \frac{f(\zeta)}{(\zeta-P)^{k+1}}\,d\zeta$$

for any $0 < \sigma < R$.

(Problem 2290) Suppose that f has a removable singularity at P. Show that $a_k = 0$ for all $k < 0$.

By the Riemann removable singularities theorem (Theorem 4.1.1) there is a function $\widehat{f}$ holomorphic in $D(P,R)$ with $\widehat{f}(z) = f(z)$ for all $z \in D(P,R)\setminus\{P\}$. Because $\widehat{f}$ is holomorphic, there are constants b_k such that $\widehat{f}(z) = \sum_{k=0}^{\infty} b_k(z-P)^k$ for all $z \in D(P,R)$. Thus $f(z) = \sum_{k=0}^{\infty} b_k(z-P)^k$ for all $z \in D(P,R)\setminus\{P\}$. By uniqueness of Laurent series (Proposition 4.2.4), $a_k = b_k$ for all $k \geq 0$ and $a_k = 0$ for all $k < 0$.

(Problem 2300) Suppose that $a_k = 0$ for all $k < 0$. Show that f has a removable singularity at P.

We have that $f(z) = \sum_{k=0}^{\infty} a_k(z-P)^k$ for all $z \in D(P,R)\setminus\{P\}$ and the series converges for all such z. By Lemma 3.2.3, the radius of convergence of the power series is at least R. Thus, by Lemma 3.2.10 the series converges to a function $\widehat{f}$ holomorphic in $D(P,R)$. In particular, $\widehat{f}$ is bounded on $\overline{D}(P,R/2)$, and so f must also be bounded on $D(P,R/2)$.

[Definition: Order of a zero] Suppose that f is holomorphic in $D(P,r)$ for some $r > 0$ and $f(P) = 0$. Then $f(z) = \sum_{k=0}^{\infty} a_k(z-P)^k$ for all $z \in D(P,r)$. The order of the zero of f at P is the smallest n such that $a_n \neq 0$; note that the order is at least 1.

[Chapter 4, Problem 15b] Suppose that f is holomorphic in $D(P,R)\setminus\{P\}$ and that f has a pole at P. Then $1/f$ is holomorphic in $D(P,r)\setminus\{P\}$ for some $r > 0$, has a removable singularity at P, and the holomorphic extension g of $1/f$ to $D(P,r)$ satisfies $g(P) = 0$. Let n be the order of the zero of g at P. Then $(z-P)^n f(z)$ has a removable singularity at P.

(Problem 2310) Suppose that f has a pole at P. Show that there is some $N > 0$ such that $a_{-N} \neq 0$ and such that $a_k = 0$ for all $k < -N$.

By Proposition 4.3.3, f has a Laurent series in $D(P,r)\setminus\{P\}$, so

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-P)^k$$

for all $z \in D(P,r)\setminus\{P\}$.

Let n be as in Problem 4.15b. Then

$$(z-P)^n f(z) = \sum_{k=-\infty}^{\infty} a_k(z-P)^{k+n} = \sum_{\ell=-\infty}^{\infty} a_{\ell-n}(z-P)^{\ell}$$

has a removable singularity at P, and so by Problem 2290 we must have that $a_{\ell-n} = 0$ for all $\ell < 0$, that is, $a_k = 0$ for all $k < -n$.

Conversely, f does not have a removable singularity at P, and so by the contrapositive to Problem 2300, there is at least one $m < 0$ such that $a_m \neq 0$. Thus $\{m \in \mathbb{Z} : a_m \neq 0,\ m < 0\}$ is nonempty and bounded below. Letting $N = \min\{m \in \mathbb{Z} : a_m \neq 0,\ m < 0\}$ completes the proof.

[Chapter 4, Problem 15c] Suppose that f is holomorphic in $D(P,R)\setminus\{P\}$, that f is not bounded in $D(P,R/2)\setminus\{P\}$, and that there is a $m\in\mathbb{N}$ such that the function g given by $g(z)=(z-P)^m f(z)$ is bounded in $D(P,R/2)$. Show that f has a pole at P.

(Problem 2320) Suppose that there is some $N>0$ such that $a_{-N}\neq 0$ and such that $a_k=0$ for all $k<-N$. Show that f has a pole at P.

We may write

$$f(z)=\sum_{k=-N}^{\infty} a_k(z-P)^k$$

for all $z\in D(P,R)\setminus\{P\}$. Because $a_{-N}\neq 0$, f does not have a removable singularity, and so f cannot be bounded in $D(P,R/2)\setminus\{P\}$.

Then

$$(z-P)^N f(z)=\sum_{k=-N}^{\infty} a_k(z-P)^{k+N}=\sum_{\ell=0} a_{\ell-N}(z-P)^\ell$$

for all $z\in D(P,R)\setminus\{P\}$ and the series converges in that region. Thus the power series $\sum_{k=-N}^{\infty} a_k(z-P)^{k+N}=\sum_{\ell=0} a_{\ell-N}(z-P)^\ell$ must yield a function holomorphic in $D(P,R)$, and in particular bounded in the compact set $\overline{D}(P,R/2)$. The result follows from Problem 4.15c.

[Definition: Order of a pole] Suppose that f has a pole at P. Then there is some $N>0$ such that $a_{-N}\neq 0$ and such that $f(z)=\sum_{k=-N}^{\infty} a_k(z-P)^k$ for all z in a punctured neighborhood of P (that is, for all z in in $D(P,r)\setminus\{P\}$ for some $r>0$). We call N the order of the pole at P. If $N=1$ we say that f has a simple pole at P.

[Definition: Pole/zero of nonpositive order] If you write "pole of order 0", I will assume that you mean "removable singularity". If you write "zero of order 0 at P", I will assume that you mean "holomorphic near P and nonzero at P." If you write "zero/pole of order $-n$", for $n\in\mathbb{N}$, I will assume that you mean a pole/zero of order n (possibly after the additional step of applying the Riemann removable singularities theorem).

(Problem 2330) Show that f has an essential singularity at P if and only if, for all $N>0$, there is a $k\in\mathbb{Z}$ with $k<-N$ such that $a_k\neq 0$.

Suppose that f has an isolated singularity at P. Then f has an essential singularity if and only if it does not have either a pole or removable singularity. By the previous four problems, f has an essential singularity if and only if both of the following conditions are false:

- $a_k=0$ for all $k<0$,
- The previous statement is false, but there is a $N\in\mathbb{N}$ such that if $k<-N$ then $a_k=0$.

Thus, f has a pole or removable singularity if and only if there is a $N\in\mathbb{N}$ such that if $k<-N$ then $a_k=0$ (with a pole if $a_k\neq 0$ for some $-N\le k<0$ and a removable singularity otherwise). The negation of this statement is precisely the condition given in the problem statement.

4.4. Examples of Laurent Expansions

(Problem 2340) Suppose that f has a zero of order k at P. Show that $\frac{1}{(z-P)^k}f(z)$ has a removable singularity at P and that its limit at P is not zero.

(Problem 2350) Suppose that f has a pole of order n at P. Show that $(z-P)^n f(z)$ has a removable singularity at P and that its limit at P is not zero.

By Proposition 4.3.3 and definition of order, we have that if f is holomorphic in $D(P,R)\setminus\{P\}$ then there are coefficients a_k such that if $z\in D(P,R)\setminus\{P\}$ then

$$f(z)=\sum_{k=-n}^{\infty} a_k(z-P)^k$$

and that $a_{-n} \neq 0$. Then $g(z) = (z-P)^n f(z)$ has an isolated singularity at P because $f(z)$, $(z-P)^n$ are holomorphic in $D(P,R) \setminus \{P\}$, and so g must also have a Laurent series. It is

$$g(z) = \sum_{k=-n}^{\infty} a_k (z-P)^{k+n} = \sum_{\ell=0}^{\infty} a_{\ell-n}(z-P)^{\ell}$$

which converges in $D(P,R) \setminus \{P\}$. Thus by Problem 2300 g has a removable singularity at P. By Lemma 3.2.10, the power series converges to a function holomorphic (thus continuous) on $D(P,R)$, and in particular satisfies

$$\lim_{z\to P} g(z) = a_{-n} \neq 0$$

as desired.

We observe that the same argument works if $n=0$ or $n<0$ under the convention that a zero is a pole of negative order and a removable singularity with nonzero limit is a pole of order 0.

(Problem 2360) Suppose that f is holomorphic in $D(P,r)\setminus\{P\}$ and that $(z-P)^k f(z)$ has a removable singularity at P for $k \in \mathbb{N}$. Show that either f has a removable singularity at P or f has a pole of order at most k at P. If in addition $\lim_{z\to P}(z-P)^k f(z) \neq 0$ then f has a pole of order exactly k.

We may write $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-P)^n$ in $D(P,r)\setminus\{P\}$. Then $(z-P)^k f(z) = \sum_{m=-\infty}^{\infty} a_{m-k}(z-P)^m$. Because this function has a removable singularity, we have by Problem 2290 that $a_{m-k} = 0$ for all $m<0$; thus, $a_n = 0$ if $n < -k$.

Thus $f(z) = \sum_{n=-k}^{\infty} a_n(z-P)^n$ in $D(P,r) \setminus \{P\}$, and so either $a_n = 0$ for all $n<0$ and so f has a removable singularity at P by Problem 2300, or there is an m with $1 \leq m \leq k$ such that $a_{-m} \neq 0$ and $a_n = 0$ if $n < -m$, and so by Problem 2320 and the definition of order f has a pole of order $m \leq k$.

Furthermore, $\lim_{z\to P}(z-P)^k f(z) = a_{-k}$, and so if $\lim_{z\to P}(z-P)^k f(z) \neq 0$, then $a_{-k} \neq 0$ and so f has a pole of order exactly k.

[Chapter 4, Problem 29] Suppose that f has a pole of order $n>0$ at P. Let $k \geq -n$ be an integer. Let the Laurent series for f in a punctured neighborhood of P be $\sum_{k=-n}^{\infty} a_k (z-P)^k$. Show that

$$a_k = \lim_{z\to P} \frac{1}{(n+k)!}\left(\frac{\partial}{\partial z}\right)^{n+k}((z-P)^n f(z)).$$

(Bonus Problem 2361) Suppose that f is holomorphic in $D(P,r) \setminus \{P\}$ for some $P \in \mathbb{C}$, $r>0$, and that

$$\frac{1}{(\ell+k)!}\left(\frac{\partial}{\partial z}\right)^{\ell+k}((z-P)^{\ell} f(z))$$

has a removable singularity at $z=P$ for some integers k and ℓ with $k+\ell \geq 0$. Show that f has a removable singularity or pole of order at most ℓ at P and that

$$a_k = \lim_{z\to P} \frac{1}{(\ell+k)!}\left(\frac{\partial}{\partial z}\right)^{\ell+k}((z-P)^{\ell} f(z)).$$

(Problem 2370) Suppose that f is holomorphic in $D(P,r)\setminus\{P\}$ and that $(z-P)^{\ell} f(z)$ has a removable singularity at P for some $\ell \geq 0$. Let $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-P)^k$ in $D(P,r) \setminus \{P\}$. Show that

$$a_k = \lim_{z\to P}\left((z-P)^{-k} f(z) - \sum_{n=-\ell}^{k-1} a_n (z-P)^{n-k}\right).$$

If $z \in D(P,r) \setminus \{P\}$ then

$$(z-P)^{\ell} f(z) = \sum_{n=-\infty}^{\infty} a_n (z-P)^{n+\ell} = \sum_{m=-\infty}^{\infty} a_{m-\ell}(z-P)^n.$$

By Problem 2290, we have that $a_{m-\ell} = 0$ for all $m < 0$, that is, $a_n = 0$ for all $n < -\ell$. Thus

$$f(z) = \sum_{n=-\ell}^{\infty} a_n (z-P)^n.$$

We compute

$$(z-P)^{-k} f(z) - \sum_{n=-\ell}^{k-1} a_n (z-P)^{n-k} = \sum_{n=k}^{\infty} a_n (z-P)^{n-k} = \sum_{m=0}^{\infty} a_{m+k} (z-P)^m.$$

The series converges for such z and so by Lemma 3.2.3 the radius of convergence is at least r; thus, the series converges to a holomorphic (thus continuous) function on $D(P, r)$. In particular,

$$\lim_{z \to P} \left((z-P)^{-k} f(z) - \sum_{n=-\ell}^{k-1} a_n (z-P)^{n-k} \right) = \lim_{z \to P} \sum_{m=0}^{\infty} a_{m+k} (z-P)^m = a_k$$

as desired.

[Definition: Principal part] The principal part of the Laurent series $\sum_{k=-\infty}^{\infty} a_k (z-P)^k$ is $\sum_{k=-\infty}^{-1} a_k (z-P)^k$.

(Problem 2380) Let $f(z) = z/(z-1)$. Find the Laurent series for f about $z = 1$ by direct computation.

(Problem 2390) Find the Laurent series for $f(z) = z/(z-1)$ at $P = 1$ by using Problem 4.29.

$\lim_{z \to 1} (z-1) f(z) = 1$, and so f has a pole at 1 of order 1. Thus

$$a_k = \lim_{z \to 1} \frac{1}{(1+k)!} \left(\frac{\partial}{\partial z} \right)^{1+k} (z).$$

Thus $a_{-1} = \frac{1}{0!} z|_{z=1} = 1$, $a_0 = \frac{1}{1!} \frac{\partial}{\partial z} z = 1$, and $a_k = 0$ for any $k > 0$ because $\left(\frac{\partial}{\partial z}\right)^{1+k} (z) = 0$. Thus

$$f(z) = 1 + \frac{1}{z-1}.$$

(Problem 2400) Find the Laurent series for $f(z) = \frac{e^z}{(z-2)^2}$ at $P = 2$.

$\lim_{z \to 2} (z-2)^2 f(z) = e^2$ exists and is not zero, so f must have a pole of order 2 at 2. Thus

$$a_k = \lim_{z \to 2} \frac{1}{(2+k)!} \left(\frac{\partial}{\partial z} \right)^{2+k} (e^z) = \frac{e^2}{(2+k)!}.$$

Thus

$$f(z) = \sum_{n=-2}^{\infty} \frac{e^2}{(n+2)!} (z-2)^n.$$

(Problem 2410) Find the principal part of the Laurent series for $f(z) = \frac{e^z}{(z-1)(z-3)^2}$ at $P = 3$.

(Problem 2420) Find the principal part of the Laurent series for $f(z) = \frac{e^z}{\sin z}$ at $z = 0$.

$\sin z$ has an isolated zero at 0 and e^z is continuous and nonzero at 0, and so $\lim_{z \to 0} \left| \frac{e^z}{\sin z} \right| = \infty$. Thus f has a pole at 0. Therefore, by Problems 2310, 2300 and 2320, $z f(z)$ has either a pole or a removable singularity at 0.

$$\lim_{\substack{x \to 0 \\ x \in \mathbb{R}}} x \frac{e^x}{\sin x} = 1$$

by l'Hôpital's rule in the real numbers. Therefore $\lim_{z \to 0} |z f(z)| \neq \infty$ and so $z f(z)$ does not have a pole at 0. It must have a removable singularity and satisfy $\lim_{z \to 0} z f(z) = \lim_{\substack{x \to 0 \\ x \in \mathbb{R}}} x f(x) = 1$. So by Problem 2360 f has a pole of order 1 at 0, and by Problem 4.29 $a_{-1} = 1$. Thus the principal part of the Laurent series is $\frac{1}{z}$.

(Memory 2430) Let (X, d) be a metric space and let $Y \subset X$ be a closed subset. Suppose that $F \subseteq Y$ is relatively closed, that is, closed in (Y, d). Show that F is also closed in (X, d).

(Memory 2440) Let (X, d) be a metric space and let $Y \subset X$ be an open subset. Suppose that $G \subseteq Y$ is relatively open, that is, open in (Y, d). Show that G is also open in (X, d).

4.5. The index of a curve around a point

(Problem 2450) Let $r : [0, 1] \to (0, \infty)$ and $\theta : [0, 1] \to \mathbb{R}$ be two C^1 functions. Let $\psi(t) = r(t)e^{i\theta(t)}$. Show that $\theta'(t) = \mathrm{Im}(\psi'(t)/\psi(t))$ and $r'(t)/r(t) = \mathrm{Re}(\psi'(t)/\psi(t))$.

(Problem 2460) Suppose that ψ is a closed curve. What can you say about $r(0)$, $r(1)$, $\theta(0)$, and $\theta(1)$? What is the geometrical significance of the number $\frac{1}{2\pi}(\theta(1) - \theta(0))$?

We have that $\psi(0) = \psi(1)$, so $r(0) = |\psi(0)| = |\psi(1)| = r(1)$ and $\theta(0) = \theta(1) + 2n\pi$ for some $n \in \mathbb{Z}$. The number n denotes the number of times the curve wraps around the origin counterclockwise.

(Problem 2470) Suppose that ψ is a closed curve. Show that

$$\frac{1}{2\pi}(\theta(1) - \theta(0)) = \frac{1}{2\pi i}\oint_\psi \frac{d\zeta}{\zeta}.$$

We call this number the index of ψ with respect to 0, or the winding number of ψ about 0.

We compute that

$$\begin{aligned}\frac{1}{2\pi}(\theta(1) - \theta(0)) &= \frac{1}{2\pi}\int_0^1 \theta'(t)\,dt \\ &= \frac{1}{2\pi}\int_0^1 \mathrm{Im}\,\frac{\psi'(t)}{\psi(t)}\,dt\end{aligned}$$

and

$$\begin{aligned}0 = \frac{1}{2\pi}(\ln r(1) - \ln r(0)) &= \frac{1}{2\pi}\int_0^1 \frac{r'(t)}{r(t)}\,dt \\ &= \frac{1}{2\pi}\int_0^1 \mathrm{Re}\,\frac{\psi'(t)}{\psi(t)}\,dt.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{2\pi}(\theta(1) - \theta(0)) &= \frac{1}{2\pi i}\int_0^1 \frac{\psi'(t)}{\psi(t)}\,dt \\ &= \frac{1}{2\pi i}\oint_\psi \frac{1}{\zeta}\,d\zeta.\end{aligned}$$

(Lemma 2471) Suppose $\gamma : [0, 1] \to \mathbb{C} \setminus \{0\}$ is a C^1 curve. Then there exist two C^1 functions $r : [0, 1] \to (0, \infty)$ and $\theta : [0, 1] \to \mathbb{R}$ such that $\gamma(t) = r(t)e^{i\theta(t)}$ for all $t \in [0, 1]$.

(Problem 2480) Begin the proof of Lemma 2471 by letting $r(t) = |\gamma(t)|$ and showing that r is a C^1 function. (In particular, show that $\frac{r'(t)}{r(t)} = \mathrm{Re}\,\frac{\gamma'(t)}{\gamma(t)}$.)

We have that

$$r(t) = \sqrt{(\mathrm{Re}\,\gamma(t))^2 + (\mathrm{Im}\,\gamma(t))^2}.$$

Furthermore, $\mathrm{Re}\,\gamma$ and $\mathrm{Im}\,\gamma$ are real-valued C^1 functions. Because $\gamma(t) \neq 0$ for all t, we have that the function $R(t) = (\mathrm{Re}\,\gamma(t))^2 + (\mathrm{Im}\,\gamma(t))^2$ maps $[0, 1]$ to $(0, \infty)$ and is C^1.

The function $h(t) = \sqrt{t}$ is C^1 (in fact, C^∞) on $(0,\infty)$, and so by the chain rule, r is C^1. Furthermore, again by the chain rule

$$r'(t) = \frac{(\operatorname{Re}\gamma(t))(\operatorname{Re}\gamma'(t)) + (\operatorname{Im}\gamma(t))(\operatorname{Im}\gamma'(t))}{\sqrt{(\operatorname{Re}\gamma(t))^2 + (\operatorname{Im}\gamma(t))^2}}.$$

Conversely,

$$\begin{aligned}\frac{\gamma'(t)}{\gamma(t)} &= \frac{(\operatorname{Re}\gamma'(t)) + i(\operatorname{Im}\gamma'(t))}{(\operatorname{Re}\gamma(t)) + i(\operatorname{Im}\gamma(t))} = \frac{(\operatorname{Re}\gamma'(t) + i\operatorname{Im}\gamma'(t))(\operatorname{Re}\gamma(t) - i\operatorname{Im}\gamma(t))}{|\gamma(t)|^2} \\ &= \frac{\operatorname{Re}\gamma'(t)\operatorname{Re}\gamma(t) + \operatorname{Im}\gamma'(t)\operatorname{Im}\gamma(t)}{|\gamma(t)|^2} + i\frac{\operatorname{Im}\gamma'(t)\operatorname{Re}\gamma(t) - \operatorname{Re}\gamma'(t)\operatorname{Im}\gamma(t)}{|\gamma(t)|^2}\end{aligned}$$

and so $\frac{r'(t)}{r(t)} = \frac{r'(t)}{|\gamma(t)|} = \operatorname{Re}\frac{\gamma'(t)}{\gamma(t)}$, as desired.

(Problem 2490) Show that there exists a C^1 function $\theta : [0,1] \to \mathbb{R}$ such that $\gamma(t) = r(t)e^{i\theta(t)}$ for all t, where $r(t) = |\gamma(t)|$ as in the previous problem. *Hint*: There is a $\theta_0 \in \mathbb{R}$ such that $\gamma(0) = r(0)e^{i\theta_0}$. What do you think $\theta'(t)$ ought to equal?

Lemma 4.5.5. Let γ be a C^1 closed curve and let $P \in \mathbb{C} \setminus \widetilde{\gamma}$. Then

$$\frac{1}{2\pi i}\oint_\gamma \frac{1}{\zeta - P}\,d\zeta$$

is an integer.

(Problem 2500) Prove Lemma 4.5.5.

Definition 4.5.4. We define $\operatorname{Ind}_\gamma(P) = \frac{1}{2\pi i}\oint_\gamma \frac{1}{\zeta - P}\,d\zeta$; this is the index of γ with respect to P, or the winding number of γ about P.

[Definition: Simply connected] A connected open set $\Omega \subseteq \mathbb{C}$ is simply connected if, whenever $\gamma : [0,1] \to \Omega$ is a closed curve, we have that γ is homotopic to a point (that is, to some constant function $\gamma_0 : [0,1] \to \mathbb{C}$).

(Problem 2510) Suppose that Ω is simply connected, $\gamma : [0,1] \to \Omega$ is a C^1 closed curve, and $P \in \mathbb{C}\setminus\widetilde{\gamma}$ satisfies $\operatorname{Ind}_\gamma(P) \neq 0$. Show that $P \in \Omega$.

Suppose that $P \notin \Omega$. Then $f(\zeta) = 1/(\zeta - P)$ is holomorphic in Ω, and so $\oint \frac{1}{\zeta - P}\,d\zeta = 0$ by Problem 1450. Taking the contrapositive, if $\operatorname{Ind}_\gamma(P) \neq 0$ then $P \in \Omega$.

(Problem 2520) Let $\gamma : [0,1] \to \mathbb{C}$ be a C^1 closed curve. Show that $\operatorname{Ind}_\gamma$ is a continuous function on $\mathbb{C} \setminus \widetilde{\gamma}$.

Pick $P \in \mathbb{C} \setminus \widetilde{\gamma}$ and $\varepsilon > 0$. Because $\widetilde{\gamma}$ is closed, there is an $r > 0$ such that $D(P,r) \subset \mathbb{C} \setminus \widetilde{\gamma}$. Let $\delta = \min\left(r/2, \frac{r^2\pi\varepsilon}{\ell(\gamma)+1}\right)$. If $w \in D(P,\delta)$, then $w \in D(P,r/2) \subset D(P,r) \subset \mathbb{C}\setminus\widetilde{\gamma}$ and

$$\begin{aligned}|\operatorname{Ind}_\gamma(w) - \operatorname{Ind}_\gamma(P)| &= \left|\frac{1}{2\pi i}\oint_\gamma \frac{1}{z - P} - \frac{1}{z - w}\,dz\right| \\ &= \frac{1}{2\pi}\left|\oint_\gamma \frac{P - w}{(z - P)(z - w)}\,dz\right|.\end{aligned}$$

By Proposition 2.1.8

$$|\operatorname{Ind}_\gamma(w) - \operatorname{Ind}_\gamma(P)| \leq \frac{1}{2\pi}\ell(\gamma)\sup_{z\in\widetilde{\gamma}} \frac{|w - P|}{|z - P||z - w|}.$$

By assumption $|w - P| < \delta \leq$. If $z \in \widetilde{\gamma}$ then $z \notin D(P,r)$ and so $|z - P| \geq r$. By the triangle inequality $|w - z| \geq |w - P| - r > r/2$. Thus

$$|\operatorname{Ind}_\gamma(w) - \operatorname{Ind}_\gamma(P)| < \frac{1}{2\pi}\ell(\gamma)\frac{2}{r^2}\delta \leq \varepsilon$$

as desired.

(Problem 2530) Show that $\operatorname{Ind}_\gamma$ is constant on every connected component of $\mathbb{C} \setminus \widetilde{\gamma}$.

Let Ω be such a connected component. Let $z \in \Omega$, $n = \operatorname{Ind}_\gamma(z)$. Let $f = \operatorname{Ind}_\gamma\big|_\Omega$.

If $w \in \Omega$ and $f(w) \in D(n, 1/2)$, then because $f(w)$ is a (real) integer we must have that $f(w) = n$. Thus

$$f^{-1}(\{n\}) = f^{-1}(D(n, 1/2)).$$

But $\{n\}$ is a closed set and $D(n, 1/2)$ is an open set, and so $f^{-1}(\{n\}) = f^{-1}(D(n, 1/2))$ must be both relatively open and relatively closed in Ω. By definition of connected set, this implies that $\Omega = f^{-1}(\{n\})$ and so $f \equiv n$ (and in particular is constant) on all of Ω.

(Problem 2540) Suppose that γ_1 and γ_2 are homotopic closed curves in $\mathbb{C}\setminus\{P\}$. Show that $\operatorname{Ind}_{\gamma_1}(P) = \operatorname{Ind}_{\gamma_2}(P)$.

The function $f(\zeta) = \frac{1}{\zeta - P}$ is holomorphic in $\mathbb{C} \setminus \{P\}$. Thus $\operatorname{Ind}_{\gamma_1}(P) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{d\zeta}{\zeta - P} = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{d\zeta}{\zeta - P} = \operatorname{Ind}_{\gamma_2}(P)$ by Problem 1440.

(Problem 2550) Show that if $\widetilde{\gamma} = \partial D(P, r)$ is a circle traversed once counterclockwise, then $\operatorname{Ind}_\gamma(z) = 1$ if $z \in D(P, r)$ and $\operatorname{Ind}_\gamma(z) = 0$ if $z \notin \overline{D}(P, r)$.

The case $z \in D(P, r)$ follows immediately from the Cauchy integral formula (Theorem 2.4.2).

If $z \notin \overline{D}(P, r)$, then $f(\zeta) = \frac{1}{\zeta - z}$ is holomorphic in $D(P, |\zeta - P|) \supset \overline{D}(P, r)$, and so the result follows from the Cauchy integral theorem (Theorem 2.4.3).

(Bonus Problem 2560) Let $t \in (0, 1)$ be such that $\gamma'(t) \neq 0$. Let $r > 0$ be such that if $z \in \overline{D}(\gamma(t), r)$ then $z = \gamma(s)$ for at most one $s \in [0, 1]$. Show that there is some $s > 0$ such that, if $\tau \in \mathbb{R}$ and $|\tau| < s$, then $\gamma(t) + i\tau\gamma'(t) \notin \widetilde{\gamma}$.

(Bonus Problem 2570) Show that for such τ we have that $\operatorname{Ind}_\gamma(\gamma(t) + i\tau\gamma'(t)) = \operatorname{Ind}_\gamma(\gamma(t) - i\tau\gamma'(t)) + 1$.

4.5. The Calculus of Residues

[Definition: Simply connected open set] An open set $\Omega \subseteq \mathbb{C}$ is simply connected if every closed curve $\gamma : [0, 1] \to \Omega$ is homotopic to a point (that is, to a constant curve).

[Definition: Residue] If $\Omega \subseteq \mathbb{C}$ is open, $P \in \Omega$, and $f : \Omega \setminus \{P\} \to \mathbb{C}$ is holomorphic, then $\operatorname{Res}_f(P)$ is defined to be the coefficient of $(z - P)^{-1}$ in the Laurent expansion of f about P.

Theorem 4.5.3. Suppose that $\Omega \subseteq \mathbb{C}$ is open and simply connected, $\{P_1, P_2, \dots, P_n\} \subset \Omega$ is a set of n distinct points, $\gamma : [0, 1] \to \Omega \setminus \{P_1, P_2, \dots, P_n\}$ is a C^1 closed curve, and $f : \Omega \setminus \{P_1, P_2, \dots, P_n\}$ is holomorphic. Then

$$\oint_\gamma f = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_f(P_k) \cdot \operatorname{Ind}_\gamma(P_k).$$

(Memory 2580) By Problem 1450, Theorem 4.5.3 is true in the special case where $n = 0$, that is, where f is holomorphic in all of Ω.

(Problem 2590) Prove Theorem 4.5.3 in the special case where $n = 1$ and where the Laurent series for f about $P = P_1$ converges uniformly on $\widetilde{\gamma}$.

Let $r > 0$ be such that $D(P, r) \subseteq \Omega$ and let $f(z) = \sum_{k=-\infty}^{\infty} a_k(z - P)^k$ for all $z \in D(P, r)$.

By Problems 550 and 630, if $k \neq -1$ then the function $(z - P)^k$ has a holomorphic antiderivative in $\mathbb{C} \setminus \{P\}$, and so by Proposition 2.1.6, $\oint_\gamma (z - P)^k \, dz = 0$.

By Problem 1610,

$$\oint_\gamma f(z)\, dz = \oint_\gamma \sum_{k=-\infty}^{\infty} a_k(z - P)^k \, dz = \sum_{k=-\infty}^{\infty} \oint_\gamma a_k(z - P)^k \, dz$$
$$= a_{-1} \oint_\gamma \frac{1}{z - P} \, dz$$

and the result follows by definition of $\mathrm{Ind}_\gamma(P)$.

(Problem 2600) Suppose that $\Omega \subseteq \mathbb{C}$ is open, $P \in \Omega$, and $f : \Omega \setminus \{P\} \to \mathbb{C}$ is holomorphic. Show that the principal part of the Laurent series for f at P converges absolutely for all $z \in \mathbb{C} \setminus \{P\}$ to a function holomorphic in $\mathbb{C} \setminus \{P\}$.

Because Ω is open, there is a $r > 0$ with $D(P, r) \subseteq \Omega$. By Theorem 4.3.2, there is a Laurent series $\sum_{k=-\infty}^{\infty} a_k(z-P)^k$ that converges absolutely to $f(z)$ for $z \in D(P, r) \setminus \{P\}$. The principal part is $\sum_{k=-\infty}^{-1} a_k(z-P)^k$; by definition of convergence of a doubly infinite series, it converges absolutely for $z \in D(P, r) \setminus \{P\}$.

If $|w| > 1/r$, then the series converges at $z = P + 1/w$, and by reindexing we see that $\sum_{\ell=1}^{\infty} a_{-\ell} w^\ell$ converges. Thus by Lemma 3.2.3, $\sum_{\ell=1}^{\infty} a_{-\ell}\zeta^\ell$ converges for all $\zeta \in \mathbb{C}$. Reindexing again we see that $\sum_{k=-\infty}^{-1} a_k(z-P)^k$ converges for all $z \in \mathbb{C} \setminus \{P\}$.

By Problem 2200, $\sum_{k=-\infty}^{-1} a_k(z-P)^k$ is holomorphic in the interior of the annulus of convergence, that is, in $\mathbb{C} \setminus \{P\}$.

(Problem 2601) Suppose that h is holomorphic in $D(P, r)$ and that g is holomorphic in $D(P, r) \setminus \{P\}$. If $f = g + h$ in $D(P, r) \setminus \{P\}$, show that $\mathrm{Res}_f(P) = \mathrm{Res}_g(P)$.

f is holomorphic in $D(P, r) \setminus \{P\}$. By Theorem 3.3.1 and Theorem 4.3.2, there are constants a_k, b_k, and c_k such that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-P)^k, \quad g(z) = \sum_{k=0}^{\infty} b_k(z-P)^k, \quad h(z) = \sum_{k=-\infty}^{\infty} c_k(z-P)^k$$

for all $z \in D(P, r) \setminus \{P\}$. Define $b_k = 0$ for all $k < 0$; then $g(z) = \sum_{k=-\infty}^{\infty} b_k(z-P)^k$ and so

$$\sum_{k=-\infty}^{\infty} a_k(z-P)^k = f(z) = g(z) + h(z) = \sum_{k=-\infty}^{\infty} (b_k + c_k)(z-P)^k$$

for all $z \in D(P, r) \setminus \{P\}$. By Proposition 4.2.4, we have that $a_k = b_k + c_k$ for all k; in particular, $\mathrm{Res}_f(P) = a_{-1} = b_{-1} + c_{-1} = c_{-1} = \mathrm{Res}_h(P)$.

(Problem 2610) Prove Theorem 4.3.2.

By Memory 2580 the theorem is true if $n = 0$.

Because $\Omega \setminus (\{P_1, P_2, \dots, P_n\} \setminus \{P_k\})$ is open, for each $1 \leq k \leq n$ there is a $r_k > 0$ such that $D(P_k, r_k) \subseteq \Omega \setminus (\{P_1, P_2, \dots, P_n\} \setminus \{P_k\})$, that is, such that $D(P_k, r_k) \setminus \{P_k\} \subseteq \Omega \setminus \{P_1, P_2, \dots, P_n\}$. Observe that f is holomorphic in $D(P_k, r_k) \setminus \{P_k\}$ for each k.

Suppose $n \geq 1$ and the theorem is true for $n-1$. Then there are coefficients a_k such that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k(z-P_n)^k \text{ for all } z \in D(P_n, r_n) \setminus \{P\}.$$

Let $g(z) = \sum_{k=-\infty}^{-1} a_k(z-P_n)^k$; by Problem 2600, g is holomorphic on $\mathbb{C} \setminus \{P_n\}$, and by Problem 2590,

$$\oint_\gamma g = 2\pi i \, \mathrm{Ind}_\gamma(P_n) \, \mathrm{Res}_g(P_n).$$

Observe that $\mathrm{Res}_g(P_n) = a_{-1} = \mathrm{Res}_f(P_n)$.

Define $h : \Omega \setminus \{P_1, \dots, P_{n-1}\} \to \mathbb{C}$ by

$$h(z) = \begin{cases} f(z) - g(z), & z \neq P_n, \\ a_0, & z = P_n. \end{cases}$$

We then have that $h(z) = \sum_{k=0}^{\infty} a_k(z-P)^k$ and the series converges for all $z \in D(P_n, r_n)$, and so by Lemma 3.2.10, h is holomorphic in $D(P_n, r_n)$, and thus in $\Omega \setminus \{P_1, \dots, P_{n-1}\}$. Thus by our induction hypothesis

$$\oint_\gamma h = 2\pi i \sum_{k=1}^{n-1} \mathrm{Res}_h(P_k)\,\mathrm{Ind}_\gamma(P_k).$$

Because $P_n \notin \widetilde{\gamma}$ we have that $f = g + h$ on $\widetilde{\gamma}$, and so

$$\oint_\gamma f = 2\pi i \Big(\sum_{k=1}^{n-1} \mathrm{Res}_h(P_k)\,\mathrm{Ind}_\gamma(P_k) + \mathrm{Res}_f(P_n)\,\mathrm{Ind}_\gamma(P_k)\Big).$$

The proof follows by Problem 2601.

4.6. Applications of the Calculus of Residues to the Calculation of Definite Integrals and Sums

(Memory 2620) By Problem 4.29, if f has a pole of order k at P, then

$$\mathrm{Res}_f(P) = \frac{1}{(k-1)!}\left(\frac{\partial}{\partial z}\right)^{k-1}((z-P)^k f(z))\Big|_{z=P}.$$

In particular, if f has a simple pole at P, then

$$\mathrm{Res}_f(P) = \lim_{z\to P}(z-P)f(z).$$

[Chapter 3, Problem 22] L'Hôpital's rule is valid for quotients of meromorphic functions.

That is, let $r > 0$ and $P \in \mathbb{C}$. Suppose that f and g are both holomorphic in $D(P,r) \setminus \{P\}$, and that neither f nor g has an essential singularity at P. Finally suppose that g is not a constant in $D(P,r) \setminus \{P\}$.

Then there is a ϱ with $0 < \varrho \leq r$ such that $g(z) \neq 0 \neq g'(z)$ for all $z \in D(P,\varrho) \setminus \{P\}$. Furthermore, the functions f/g and f'/g', which are holomorphic in $D(P,\varrho) \setminus \{P\}$, do not have essential singularities at P.

Finally, if f and g both have poles at P or if $\lim_{z\to P} f(z) = 0 = \lim_{z\to P} g(z)$, then f/g has a pole at P if and only if f'/g' has a pole at P, and if f'/g' has a removable singularity then $\lim_{z\to P} f(z)/g(z) = \lim_{z\to P} f'(z)/g'(z)$.

(Problem 2630) Consider the integral $\int_{-\infty}^{\infty} \frac{x^2}{x^4+4}\,dx$. Write down a family of closed C^1 curves γ_R and a function f holomorphic in a neighborhood of $\widetilde{\gamma}_R$ such that $\lim_{R\to\infty} \oint_{\gamma_R} f = \int_{-\infty}^{\infty} \frac{x^2}{x^4+4}\,dx$.

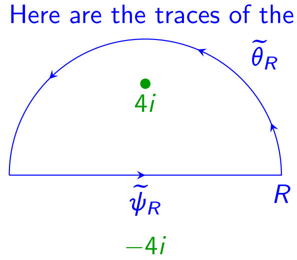
Then find $\int_{-\infty}^{\infty} \frac{x^2}{x^2+4} dx$ by finding $\oint_{\psi_R} f$ for all R large enough.

(Problem 2640) Compute $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4}\,dx$. *Hint*: Recall that $\mathrm{Re} \int_a^b f(x)\,dx = \int_a^b \mathrm{Re}\, f(x)\,dx$.

Let $f(z) = \frac{e^{iz}}{z^2+4}$. Then f has two poles, at $\pm 4i$, and both are of order one.

Let $\psi_R : [-R, R] \to \mathbb{C}$ be given by $\psi_R(t) = t$. Let $\theta_R : [0,\pi] \to \mathbb{C}$ be given by $\theta_R(t) = Re^{it}$. Then ψ_R and θ_R are two C^1 curves with $\psi_R(R) = \theta_R(0)$, and so we can combine them using Problem 1050 to get a C^1 curve $\gamma_R : [-1,1] \to \mathbb{C}$. Because $\gamma_R(-1) = \psi_R(-R) = -R = Re^{i\pi} = \theta_R(\pi) = \gamma_R(1)$, γ_R is a closed curve.

Here are the traces of the curves ψ_R and θ_R together with the singularities of f:



If you would like to use the above picture on your homework, you may generate it with the following code:

```
\usepackage{tikz}
...
\begin{tikzpicture} [scale=0.3]
\begin{scope}[green!60!black]
\node at (0,4) {$\bullet$};
\node [below] at (0,4) {$4i$};
\node at (0,-4) {$\bullet$};
\node [above] at (0,-4) {$-4i$};
\end{scope}
\draw
   (-3,0)-- node [below] {$\widetilde\psi_R$}
   (3,0) node [below] {$R$}
   arc (0:45:3) node [above right] {$\widetilde\theta_R$}
   arc (45:90:3)
   arc (90:180:3);
\end{tikzpicture}
```

Then

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos x}{x^2+4}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{\operatorname{Re} e^{ix}}{x^2+4}\,dx$$
$$= \lim_{R\to\infty} \operatorname{Re} \int_{-R}^{R} \frac{e^{ix}}{x^2+4}\,dx = \lim_{R\to\infty} \operatorname{Re} \oint_{\psi_R} f.$$

As before

$$\left|\oint_{\theta_R} f\right| \le \ell(\theta_R) \sup_{\widetilde\theta_R} |f| \le \pi R \frac{1}{R^2-4}$$

and so $\lim_{R\to\infty} \oint_{\theta_R} f = 0$.

By Problem 1050 $\oint_{\gamma_R} f = \oint_{\theta_R} f + \oint_{\psi_R} f$, and so

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4}\,dx = \lim_{R\to\infty} \operatorname{Re} \oint_{\gamma_R} f.$$

But by Theorem 4.5.3, if $R > 2$ then

$$\oint_{\gamma_R} f = 2\pi i \operatorname{Res}_f(2i) \operatorname{Ind}_{\gamma_R}(2i) = 2\pi i \operatorname{Res}_f(2i).$$

We find

$$\operatorname{Res}_f(2i) = \lim_{z\to 2i} (z-2i)f(z) = \lim_{z\to 2i} \frac{(z-2i)e^{iz}}{z^2+4} = \lim_{z\to 2i} \frac{e^{iz}}{z+2i} = \frac{e^{-2}}{4i}.$$

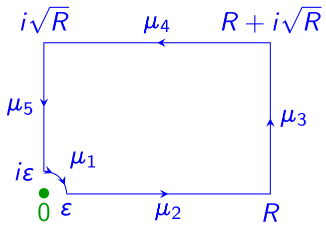
Thus

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4}\,dx = \operatorname{Re} 2\pi i \frac{e^{-2}}{4i} = \frac{e^{-2}\pi}{2}.$$

(Problem 2650) Compute $\int_0^\infty \frac{\sin x}{x}\,dx$ by finding an appropriate function f and closed contour $\gamma_{R,S}$ that satisfies $\oint_{\gamma_{R,S}} f = 0$. (This is the exercise immediately after Example 4.6.3 in your book; a verbatim presentation of Example 4.6.3 will not be accepted.)

Let $f(z) = \frac{e^{iz}}{z}$.

Define the curves μ_j to have the following traces, with the curve μ given by Problem 1050 oriented counterclockwise.




```
\usepackage{tikz}
...
\begin{tikzpicture}
\begin{scope}[green!60!black]
\node at (0,0) {$\bullet$};
\node [below] at (0,0) {$0$};
\end{scope}
\draw
(0.3,0) node [below] {$\varepsilon$}
-- node [below] {$\mu_2$}
(3,0) node [below] {$R$}
-- node [right]{ $\mu_3$}
(3,2) node [above] {$R+i\sqrt{R}$}
-- node [above] {$\mu_4$}
(0,2) node [above] {$i\sqrt{R}$}
-- node [left] {$\mu_5$}
(0,0.3) node [left] {$i\varepsilon$}
arc (90:45:.3) node [above right] {$\mu_1$}
arc (45:1:0.3)
;
\end{tikzpicture}
```

Then $\oint_\mu f = 0$ because f is holomorphic on $\mathbb{C}\setminus\{0\}$.

By definition of line integral and Problem 1010, and parameterizing the reverse of μ_1 by $\hat\mu_1(t) = \varepsilon e^{it}$, we see that

$$\oint_{\mu_1} f = -\int_0^{\pi/2} f(\varepsilon e^{it}) i\varepsilon e^{it}\,dt = -i\int_0^{\pi/2} e^{i\varepsilon e^{it}}\,dt.$$

Because $e^{i\varepsilon e^{it}} \to 1$, uniformly in t, as $\varepsilon \to 0^+$, we have that

$$\lim_{\varepsilon\to 0^+} \oint_{\mu_1} f = -i\frac{\pi}{2}.$$

By the definition of line integral and of integral of complex function, using the natural parameterization of μ_2, we have that

$$\int_0^\infty \frac{\sin x}{x}\,dx = \operatorname{Im} \lim_{R\to\infty} \lim_{\varepsilon\to 0^+} \oint_{\mu_2} f.$$

We compute

$$\left|\oint_{\mu_3} f\right| \leq \ell(\mu_3)\sup_{\widetilde{\mu}_3}|f| = \sqrt{R}\cdot\frac{1}{R} = \frac{1}{\sqrt{R}},$$

$$\left|\oint_{\mu_4} f\right| \leq \ell(\mu_4)\sup_{\widetilde{\mu}_4}|f| = R\cdot\frac{e^{-\sqrt{R}}}{\sqrt{R}}$$

and so $\lim_{R\to\infty}\oint_{\mu_3} f = \lim_{R\to\infty}\oint_{\mu_4} f = 0$.

Finally, parameterizing the reverse of μ_5 by $\hat{\mu}_5(t) = it$,

$$\oint_{\mu_5} f = -\int_\varepsilon^R f(it)\,i\,dt = -\int_\varepsilon^R \frac{e^{-t}}{it} i\,dt$$

is an entirely real integral, and so $\operatorname{Im}\oint_{\mu_5} f = 0$.

Combining the above results, we have that

$$\begin{aligned}
0 &= \lim_{R\to\infty}\lim_{\varepsilon\to 0^+}\operatorname{Im}\oint_\mu f \\
&= \lim_{\varepsilon\to 0^+}\operatorname{Im}\oint_{\mu_1} f + \lim_{R\to\infty}\lim_{\varepsilon\to 0^+}\operatorname{Im}\oint_{\mu_2} f \\
&\qquad + \lim_{R\to\infty}\operatorname{Im}\oint_{\mu_3} f + \lim_{R\to\infty}\operatorname{Im}\oint_{\mu_4} f + 0 \\
&= -\frac{\pi}{2} + \int_0^\infty \frac{\sin x}{x}\,dx.
\end{aligned}$$

Thus

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

(Problem 2660) Let $n \in \mathbb{N}$ and let $\theta_0 \in \mathbb{R}$. Define $z^{1/n}$ with branch cut at angle θ_0 by $(re^{i\theta})^{1/n} = r^{1/n}e^{i\theta/n}$ for all $r > 0$ and all $\theta_0 \leq \theta < \theta_0 + 2\pi$. Show that $z^{1/n}$ is well defined and holomorphic on $\mathbb{C}\setminus\{te^{i\theta_0} : t \in \mathbb{R},\ t \geq 0\}$ and that $\frac{\partial}{\partial z} z^{1/n} = \frac{1}{n(z^{1/n})^{n-1}}$.

That $z^{1/n}$ is well defined follows from Problem 1.25 in your book and from Problem 360.

Recall from Homework 5 that the function ln given by

$$\ln(re^{i\theta}) = \ln r + i\theta \quad \text{if } r > 0,\ -\pi < \theta < \pi$$

is holomorphic on $\mathbb{C}\setminus(-\infty, 0]$ and satisfies $\frac{\partial}{\partial z}\ln z = \frac{1}{z}$.

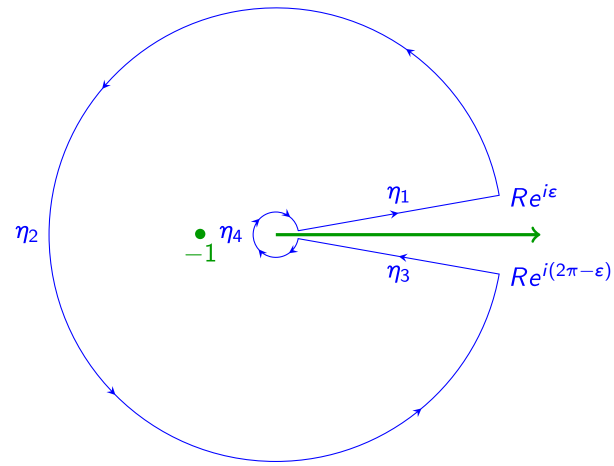
Observe that

$$z^{1/n} = \exp((\ln(ze^{i(-\pi-\theta_0)}) + i\theta_0 + i\pi)/n)$$

and so $z^{1/n}$ is holomorphic, as it is a composition of holomorphic functions.

(Problem 2670) Find $\int_0^\infty \frac{x^{1/2}}{x^2+2x+1}\,dx$.

If $0<\varepsilon<\pi/2<R<\infty$, define the contours η_k as shown below. η_2 is oriented counterclockwise, and $\widetilde{\eta}_4$



is a subset of the circle of radius $1/R$ centered at 0.

```
\usepackage{tikz}
...
\begin{tikzpicture}
\draw
(10:0.3)
-- node [above] {$\eta_1$}
(10:3) node [right]{$Re^{i\varepsilon}$}
arc (10:180:3)
node[left] {$\eta_2$}
arc (180:350:3)
 node [right]{$Re^{i(2\pi-\varepsilon)}$}
-- node [below] {$\eta_3$}
(-10:0.3)
arc (-10:-180:0.3)
node [left] {$\eta_4$}
arc (-180:-350:0.3);
\end{tikzpicture}
```

Let $f(z)=\frac{z^{1/2}}{(z+1)^2}$ where the branch cut of $z^{1/2}$ is taken to be at $\theta_0=0$, that is, the positive real axis.

We may then compute that f is holomorphic in a neighborhood of $\widetilde{\eta}$ for any such ε and R and has only one singularity in the interior of η, namely $z=-1$. Because f has a pole of order 2 at $z=-1$, we compute using Problem 4.29 that

$$\mathrm{Res}_f(-1)=\lim_{z\to-1}\frac{\partial}{\partial z}z^{1/2}=\frac{1}{2i}$$

and so

$$\pi=\oint_\eta f=\oint_{\eta_1} f+\oint_{\eta_2} f+\oint_{\eta_3} f+\oint_{\eta_4} f.$$

By Proposition 2.1.8, if $R>1$ then

$$\left|\oint_{\eta_2} f\right|\leq 2\pi R\frac{\sqrt{R}}{(R-1)^2},\qquad \left|\oint_{\eta_4} f\right|\leq 2\pi\frac{1}{R}\frac{\sqrt{1/R}}{(1-1/R)^2}$$

and so $\lim_{R\to\infty}\oint_{\eta_2} f+\oint_{\eta_4} f=0$. Thus

$$\pi=\lim_{R\to\infty}\oint_{\eta_1} f+\oint_{\eta_3} f.$$

By definition of line integral,

$$\oint_{\eta_1} f = \int_{1/R}^{R} \frac{\sqrt{t}e^{i\varepsilon/2}}{(te^{i\varepsilon}+1)^2} e^{i\varepsilon}\,dt$$

and so

$$\int_0^\infty \frac{x^{1/2}}{x^2+2x+1}\,dx = \lim_{R\to\infty}\int_{1/R}^{R} \frac{x^{1/2}}{x^2+2x+1}\,dx = \lim_{R\to\infty}\lim_{\varepsilon\to0^+}\oint_{\eta_1} f.$$

By Problem 1010

$$\oint_{\eta_3} f = -\int_{1/R}^{R} \frac{\sqrt{t}e^{i(\pi-\varepsilon/2)}}{(te^{i\pi-i\varepsilon}+1)^2} e^{i(2\pi-\varepsilon)}\,dt$$

and so

$$\int_0^\infty \frac{x^{1/2}}{x^2+2x+1}\,dx = \lim_{R\to\infty}\int_{1/R}^{R} \frac{x^{1/2}}{x^2+2x+1}\,dx = \lim_{R\to\infty}\lim_{\varepsilon\to0^+}\oint_{\eta_3} f.$$

Thus

$$\int_0^\infty \frac{x^{1/2}}{x^2+2x+1}\,dx = \frac{1}{2}\lim_{R\to\infty}\lim_{\varepsilon\to0^+}\oint_{\eta_1} f + \oint_{\eta_3} f = \frac{\pi}{2}$$

as desired.

(Problem 2680) Use the calculus of residues to compute $\int_0^\infty \frac{dx}{x^2+5x+6}$.

Recall from Homework 5 that the function $\ln z$ is holomorphic on $\mathbb{C}\setminus(-\infty,0]$. Let $\log(z) = i\pi + \ln(-z)$; then log is holomorphic as a composition of holomorphic functions and satisfies $\log(re^{i\theta}) = \log r + i\theta$ if $0 < r < \infty$ and $0 < \theta < 2\pi$. Define $g(z) = \frac{\log z}{z^2+5z+6}$. Let η_k be as in the previous problem.

We can show as usual that $\lim_{R\to\infty}\oint_{\eta_2} g = \lim_{R\to\infty}\oint_{\eta_4} g = 0$.

Observe that

$$\oint_{\eta_1} g = \int_{1/R}^{R} \frac{\log t + i\varepsilon}{t^2e^{2i\varepsilon}+5te^{i\varepsilon}+6}\,dt$$

and

$$\oint_{\eta_3} g = -\int_{1/R}^{R} \frac{\log t + 2\pi i - i\varepsilon}{t^2e^{-2i\varepsilon}+5te^{-i\varepsilon}+6}\,dt.$$

Fix R. The integrands converge uniformly as $\varepsilon\to0^+$, and so

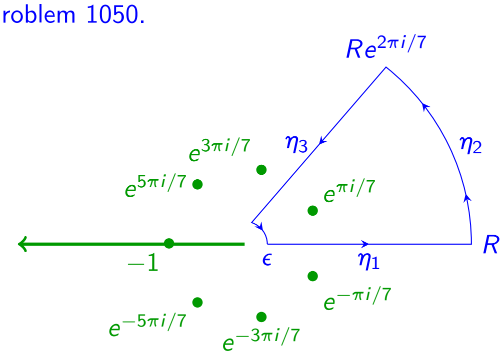
$$\lim_{\varepsilon\to0^+}\left(\oint_{\eta_1} g + \oint_{\eta_3} g\right) = \int_{1/R}^{R} \frac{-2\pi i}{t^2+5t+6}\,dt.$$

Thus

$$\begin{aligned}\int_0^\infty \frac{1}{x^2+5x+6}\,dx &= -\frac{1}{2\pi i}\lim_{R\to\infty}\lim_{\varepsilon\to0^+}\oint_\eta g\\ &= -\operatorname{Res}_g(-2) - \operatorname{Res}_g(-3)\\ &= -\frac{\log 2 + i\pi}{-2+3} - \frac{\log 3 + i\pi}{-3+2}\\ &= -\log 2 + \log 3.\end{aligned}$$

(Problem 2690) Find $\int_0^\infty \frac{\sqrt[5]{x}}{x^7+1}\,dx$.

Let η_1, η_2, η_3, and η_4 (the small unlabeled contour) be as shown, and let η be the contour obtained by Problem 1050.



Let $f(z) = \frac{z^{1/5}}{z^7+1}$, where $z^{1/5} = r^{1/5}e^{i\theta/5}$ where r and θ are the unique real numbers with $r \geq 0$, $-\pi < \theta < \pi$ and $z = re^{i\theta}$. A standard argument shows that

$$\lim_{R\to\infty} \oint_{\eta_2} f = 0 = \lim_{\varepsilon\to 0^+} \oint_{\eta_4} f$$

and

$$\oint_\eta f = 2\pi i \operatorname{Res}_f(e^{\pi i/7}).$$

Thus

$$2\pi i \operatorname{Res}_f(e^{\pi i/7}) = \lim_{R\to\infty,\varepsilon\to 0^+} \oint_{\eta_1} f + \oint_{\eta_2} f.$$

We compute that

$$\begin{aligned}\oint_{\eta_1} f &= \int_\varepsilon^R \frac{x^{1/5}}{x^7+1}dx,\\ \oint_{\eta_2} f &= -\int_\varepsilon^R \frac{(xe^{2\pi i/7})^{1/5}}{x^7+1}e^{2\pi i/7}dx\\ &= -e^{12\pi i/35}\int_\varepsilon^R \frac{x^{1/5}}{x^7+1}dx\end{aligned}$$

so

$$(1-e^{12\pi i/35})\int_0^\infty \frac{x^{1/5}}{x^7+1}\,dx = 2\pi i \operatorname{Res}_f(e^{\pi i/7}).$$

We now consider

$$\lim_{z\to e^{i\pi/7}} (z-e^{i\pi/7})f(z) = \lim_{z\to e^{i\pi/7}} \frac{(z-e^{i\pi/7})z^{1/5}}{z^7+1}.$$

The numerator and denominator are both holomorphic in a neighborhood of $e^{i\pi/7}$ and equal zero at $e^{i\pi/7}$. Thus we may apply l'Hôpital's rule. We observe that

$$\frac{\partial}{\partial z}z^{1/n} = \frac{\partial}{\partial x}z^{1/n}$$

by Proposition 1.4.3, and so $\frac{\partial}{\partial z}z^{1/n} = \frac{1}{n}z^{1/n-1}$ if z is a positive real number. By Corollary 3.6.3 this implies that $\frac{\partial}{\partial z}z^{1/n} = \frac{1}{n}z^{1/n-1}$ for all z in the domain of $z^{1/n}$. Thus

$$\lim_{z\to e^{i\pi/7}} (z-e^{i\pi/7})f(z) = \lim_{z\to e^{i\pi/7}} \frac{z^{1/5}+(1/5)z^{-4/5}(z-e^{i\pi/7})}{7z^6} = \frac{e^{i\pi/35}}{7e^{6i\pi/7}}.$$

Thus f has a simple pole at $e^{i\pi/7}$ (this may also be seen by factoring the denominator) and the residue is given by the above limit, and so

$$\int_0^\infty \frac{x^{1/5}}{x^7+1}\,dx = \frac{2\pi i}{1-e^{12\pi i/35}}\frac{e^{i\pi/35}}{7e^{6i\pi/7}}$$
$$= \frac{2i}{e^{29i\pi/35}-e^{41\pi i/35}}\frac{\pi}{7}.$$

But $e^{41\pi i/35} = e^{41\pi i/35-2\pi i} = e^{-29\pi i/35}$, so

$$\int_0^\infty \frac{x^{1/5}}{x^7+1}\,dx = \frac{2i}{e^{29i\pi/35}-e^{-29\pi i/35}}\frac{\pi}{7} = \frac{\pi}{7}\csc(29\pi/35).$$

Because $\sin(x+\pi) = -\sin(x)$ for all $x\in\mathbb{R}$, we compute that $\csc(29\pi/35) = -\csc(29\pi/35-\pi) = -\csc(-6\pi/35) = \csc(6\pi/35)$. Thus

$$\int_0^\infty \frac{x^{1/5}}{x^7+1}\,dx = \frac{\pi}{7}\csc(6\pi/35).$$

(Problem 2700) Compute $\int_0^{2\pi} \frac{d\theta}{3+\sin\theta}$.

Recall that $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. Thus,

$$\int_0^{2\pi} \frac{d\theta}{3+\sin\theta} = \int_0^{2\pi}\frac{2i\,d\theta}{6i+2i\sin\theta}$$
$$= \int_0^{2\pi}\frac{2i\,d\theta}{6i+e^{i\theta}-e^{-i\theta}}$$
$$= \int_0^{2\pi}\frac{2ie^{i\theta}\,d\theta}{6ie^{i\theta}+e^{2i\theta}-1}$$
$$= \int_0^{2\pi}\frac{2ie^{i\theta}\,d\theta}{6ie^{i\theta}+(e^{i\theta})^2-1}.$$

Let $\gamma(t) = e^{it}$, $0\le t\le 2\pi$. Let $f(z) = \frac{2}{6iz+z^2-1}$. Then

$$\oint_\gamma f = \int_0^{2\pi} f(\gamma(t))\gamma'(t)\,dt = \int_0^{2\pi}\frac{2}{6ie^{it}+e^{2it}-1}ie^{it}\,dt.$$

Thus

$$\int_0^{2\pi}\frac{d\theta}{3+\sin\theta} = \oint_\gamma f.$$

The function f has poles at $(-3\pm\sqrt{8})i$. The point $(-3-\sqrt{8})i$ lies outside the unit disc, while the point $(-3+\sqrt{8})i$ lies inside the unit disc. Thus

$$\int_0^{2\pi}\frac{d\theta}{3+\sin\theta} = \oint_\gamma f = 2\pi i\,\mathrm{Res}_f(i(\sqrt{8}-3))$$
$$= 2\pi i \lim_{z\to i(\sqrt{8}-3)} (z-i(\sqrt{8}-3))f(z)$$
$$= 2\pi i \lim_{z\to i(\sqrt{8}-3)} (z-i(\sqrt{8}-3))\frac{2}{z^2+6iz-1}$$
$$= 2\pi i \lim_{z\to i(\sqrt{8}-3)} \frac{2(z-i(\sqrt{8}-3))}{(z-i(\sqrt{8}-3))(z-(i(-\sqrt{8}-3)))}$$
$$= 2\pi i\frac{2}{(i(\sqrt{8}-3)-(i(-\sqrt{8}-3)))}$$
$$= \frac{2\pi}{\sqrt{8}} = \frac{\pi}{\sqrt{2}}.$$

(Problem 2710) Let $\cot z = \frac{\cos z}{\sin z}$. Show that cot is holomorphic on $\mathbb{C} \setminus \{n\pi : n \in \mathbb{Z}\}$ and that $\mathrm{Res}_{\cot}(n\pi) = 1$ for all $n \in \mathbb{Z}$.

Recall that $\sin z = \frac{e^{iz}+e^{-iz}}{2i}$, while $\cos z = \frac{e^{iz}+e^{-iz}}{2}$. Thus, $\cot z = i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}$.
If x, $y \in \mathbb{R}$, then

$$\cot(x+iy) = i\frac{e^{i(x+iy)} + e^{-i(x+iy)}}{e^{i(x+iy)} - e^{-i(x+iy)}} = i\frac{e^{ix-y} + e^{-ix+y}}{e^{ix-y} - e^{-ix+y}}.$$

The singularities occur when the denominator equals zero, that is, when

$$e^{ix-y} = e^{-ix+y}.$$

Recall that $e^{ix-y} = e^{ix}e^{-y} = (\cos x + i\sin x)e^{-y}$. Thus, singularities occur when

$$(\cos x + i\sin x)e^{-y} = (\cos(-x) + i\sin(-x))e^{y}.$$

Because cos is an even function and sin is an odd function, singularities occur precisely when

$$(\cos x + i\sin x)e^{-y} = (\cos x - i\sin x)e^{y}.$$

Taking the real and imaginary parts of this equation, we see that cot has a pole at $x+iy$ if and only if

$$e^{-y}\cos x = e^{y}\cos x \quad \text{and} \quad e^{-y}\sin x = -e^{y}\sin x.$$

Considering the second equation, we can solve to see

$$(e^{y} + e^{-y})\sin x = 0.$$

Since $e^{y} + e^{-y} > 0$ for all real numbers y, we see that $\sin x = 0$, so $x = n\pi$ for some $n \in \mathbb{Z}$.
Considering the second equation, we see that $\cos x = \pm 1$. In particular, $\cos x \neq 0$, so $e^{-y} = e^{y}$. This is true only for $y = 0$.
Thus, $\cot z$ has singularities at $z = n\pi$ for $n \in \mathbb{Z}$.
Now, choose some $n \in \mathbb{Z}$. By l'Hôpital's rule,

$$\lim_{z\to n\pi}(z-n\pi)\cot z = \lim_{z\to n\pi}\frac{(z-n\pi)\cos z}{\sin z} = \lim_{z\to n\pi}\frac{\cos z - (z-n\pi)\sin z}{\cos z} = 1$$

because $\sin(n\pi) = 0$ and $\cos(n\pi) = \pm 1 \neq 0$ for all $n \in \mathbb{Z}$. Thus, $\mathrm{Res}_{\cot}(n\pi) = 1$ for all $n \in \mathbb{Z}$.

(Problem 2720) Show that $\cot(x+iy)$ converges to $-i$ as $y \to \infty$ and to i as $y \to -\infty$, uniformly in $x \in \mathbb{R}$.

If x, $y \in \mathbb{R}$, then

$$\cot(x+iy) + i = i\frac{e^{ix-y} + e^{-ix+y}}{e^{ix-y} - e^{-ix+y}} + i = i\frac{e^{ix}e^{-y} + e^{-ix}e^{y}}{e^{ix}e^{-y} - e^{-ix}e^{y}} + i = i\frac{2e^{ix}e^{-y}}{e^{ix}e^{-y} - e^{-ix}e^{y}}.$$

By the reverse triangle inequality $|e^{ix}e^{-y} - e^{-ix}e^{y}| \geq |e^{-ix}e^{y}| - |e^{ix}e^{-y}| = e^{y} - e^{-y}$. Thus, for all $x \in \mathbb{R}$ and all $y > 0$ we have that

$$|\cot(x+iy) + i| \leq \frac{2e^{-y}}{e^{y} - e^{-y}} = \frac{2}{e^{2y} - 1}.$$

We know from real analysis that $\lim_{y\to\infty}\frac{2}{e^{2y}-1} = 0$, that is, for every $\varepsilon > 0$ there is a $N \in \mathbb{R}$ such that if $y > 0$ then $\frac{2}{e^{2y}-1} < \varepsilon$. Thus, if $y > N$, then for all $x \in \mathbb{R}$ we have that $|\cot(x+iy) + i| < \varepsilon$, and so $\cot(x+iy) \to -i$ as $y \to \infty$ uniformly in x.

(Problem 2730) Show that if $y \in \mathbb{R}$ and $x = (n+1/2)\pi$ for some $n \in \mathbb{Z}$, then $|\cot(x+iy)| \leq 1$ for all $y \in \mathbb{R}$.

Recall

$$\cot(x+iy) = i\frac{e^{ix-y}+e^{-ix+y}}{e^{ix-y}-e^{-ix+y}} = i\frac{e^{ix}e^{-y}+e^{-ix}e^{y}}{e^{ix}e^{-y}-e^{-ix}e^{y}}.$$

If $x = (n\pi + \pi/2)$, then $e^{ix} = \cos x + i\sin x = i\sin(n\pi+\pi/2) = i(-1)^n$ because $\cos(n\pi+\pi/2) = 0$ and $\sin(n\pi) = (-1)^n$ for all $n \in \mathbb{Z}$. In particular, $e^{-i(n+1/2)\pi} = e^{i(-n-1/2)\pi} = i(-1)^{-n-1} = -e^{i(n+1/2)\pi}$.

Thus, in either case, $e^{ix} = -e^{-ix} \neq 0$, so

$$\cot((n+1/2)\pi + iy) = i\frac{e^{-y}-e^{y}}{e^{-y}+e^{y}}.$$

Note that $e^{-y}-e^{y}$ is a real number with $|e^{-y}-e^{y}| \leq e^{-y}+e^{y}$, and so

$$|\cot((n+1/2)\pi+iy)| = \frac{|e^{-y}-e^{y}|}{e^{-y}+e^{y}} \leq 1.$$

(Problem 2740) We would like to compute $\sum_{n=1}^{\infty}\frac{1}{n^2}$. In this problem we begin the proof. Let $\gamma_n : [0,1] \to \mathbb{C}$ be a C^1 parameterization of the rectangle with corners at $\pm(n\pi+\frac{\pi}{2}) \pm in$, for $n \in \mathbb{N}$. Let $f(z) = \frac{\cot z}{z^2}$. Show that $\lim_{n\to\infty}\oint_{\gamma_n} f = 0$.

(Problem 2750) Find the residue of f at zero.

First observe that

$$\begin{aligned}\lim_{z\to 0} z^3 f(z) &= \lim_{z\to 0} z\cot z = \lim_{z\to 0}\frac{z\cos z}{\sin z}\\ &= \lim_{z\to 0}\frac{\cos z - z\sin z}{\cos z} = 1\end{aligned}$$

by l'Hôpital's rule. This is a nonzero finite complex number, so f must have a pole of order 3 at 0.

We write the Laurent series of f about $z=0$ as

$$f(z) = \sum_{k=-3}^{\infty} a_k z^k.$$

Note that $a_{-3} = 1$. Thus

$$f(z) - \frac{1}{z^3} = \sum_{k=-2}^{\infty} a_k z^k$$

and

$$\begin{aligned}a_{-2} &= \lim_{z\to 0}\sum_{k=-2}^{\infty} a_k z^{k+2} = \lim_{z\to 0} z^2 \sum_{k=-2}^{\infty} a_k z^k\\ &= \lim_{z\to 0} z^2\left(f(z) - \frac{1}{z^3}\right) = \lim_{z\to 0}\cot z - \frac{1}{z}\\ &= \lim_{z\to 0}\frac{z\cos z - \sin z}{z\sin z} = \lim_{z\to 0}\frac{-z\sin z}{\sin z + z\cos z}\\ &= \lim_{z\to 0}\frac{-\sin x - z\cos z}{2\cos z - z\sin z} = 0.\end{aligned}$$

Thus,

$$\begin{aligned}\operatorname{Res}_f(0) = a_{-1} &= \lim_{z\to 0} z\left(f(z) - \frac{1}{z^3} - \frac{0}{z^2}\right) = \lim_{z\to 0} \frac{\cot z}{z} - \frac{1}{z^2} \\ &= \lim_{z\to 0} \frac{z\cot z - 1}{z^2} = \lim_{z\to 0} \frac{z\cos z - \sin z}{z^2 \sin z} \\ &= \lim_{z\to 0} \frac{-z\sin z}{2z\sin z + z^2\cos z} \\ &= \lim_{z\to 0} \frac{-\sin z - z\cos z}{2\sin z + 4\,z\cos z - z^2\sin z} \\ &= \lim_{z\to 0} \frac{-2\cos z + z\sin z}{6\cos z - 6\,z\sin z - z^2\cos z} = -\frac{1}{3}.\end{aligned}$$

(Problem 2760) Find all the singularities of f and then find the residues of f at each singularity.

(Problem 2770) Use the above results to compute $\sum_{j=1}^{\infty} \frac{1}{j^2}$.

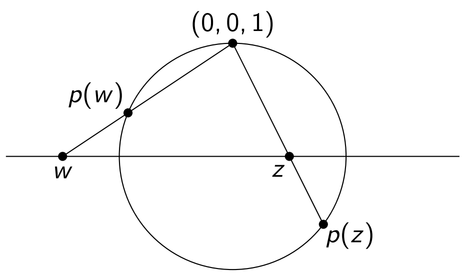
4.7. Real analysis

(Problem 2780) Let $X = \mathbb{C} \cup \{\infty\}$ (where ∞ is a single point not in $\mathbb{C}$) and define $d : X \times X \to [0, \infty)$ by

$$d(z,w) = \begin{cases} 0, & z = \infty = w, \\ \frac{2}{\sqrt{1+|z|^2}}, & z \in \mathbb{C},\ w = \infty, \\ \frac{2}{\sqrt{1+|w|^2}}, & w \in \mathbb{C},\ z = \infty, \\ \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, & z \in \mathbb{C},\ w \in \mathbb{C}. \end{cases}$$

This is called the *spherical metric* on X. Then (X, d) is a metric space. (See Problem 4.32 in your book.)

This metric arises as follows. If $z = \xi + i\eta$, where $\xi, \eta \in \mathbb{R}$, then we let the stereographic projection $p(z)$ be the point in $\mathbb{R}^3$ that lies on the unit sphere $\{(x, y, t) : x^2 + y^2 + t^2 = 1\}$ and also lies on the line through $(0, 0, 1)$ (the north pole) and the point $(x, y, 0)$. See the following figure. Then $d(z, w) = \|p(z) - p(w)\|$ (if z, $w \in \mathbb{C}$ and $\|\cdot\|$ denotes the standard Euclidean metric in $\mathbb{R}^3$) and $d(z, \infty) = \|(0, 0, 1) - p(z)\|$.



(Problem 2790) The subspace $(\mathbb{C}, d)$ is equivalent to $(\mathbb{C}, |\cdot - \cdot|)$ (that is, $\mathbb{C}$ equipped with the standard metric) in the sense that, if x, $x_n \in \mathbb{C}$, then $x_n \to x$ in $(\mathbb{C}, d)$ if and only if $x_n \to x$ in $(\mathbb{C}, |\cdot - \cdot|)$.

(Problem 2800) The subspace $(\mathbb{C}, d)$ is equivalent to $(\mathbb{C}, |\cdot - \cdot|)$ in the sense that if $z \in \Omega \subseteq \mathbb{C}$ and f is a function defined on Ω, then f is continuous at z as a function on (Ω, d) if and only if f is continuous at z as a function on $(\Omega, |\cdot - \cdot|)$.

(Problem 2810) The subspace $(\mathbb{C}, d)$ is equivalent to $(\mathbb{C}, |\cdot - \cdot|)$ in the sense that if $f : Y \to \mathbb{C}$ is a function defined on a metric space Y and $a \in Y$, then f is continuous at a as a function mapping into (X, d) (or $(\mathbb{C}, d)$) if and only if f is continuous at a as a function mapping into $(\mathbb{C}, |\cdot - \cdot|)$.

(Problem 2820) If $\{x_n\}_{n=1}^{\infty} \subset \mathbb{C}$, then $x_n \to \infty$ in (X, d) if and only if $\lim_{n\to\infty} |x_n| = \infty$ in the sense of real analysis.

(Problem 2830) If $\Omega \subseteq \mathbb{C}$ is unbounded and $f : \Omega \to Y$ for some metric space (Y, ρ), we may define $\lim_{z\to\infty} f(z)$ in $(\mathbb{C}, |\cdot - \cdot|)$ as follows: $\lim_{z\to\infty} f(z) = L$ if, for every $\varepsilon > 0$, there is a $N > 0$ such that if $z \in \Omega$ and $|z| > N$, then $\rho(f(z), L) < \varepsilon$. Show that $\lim_{z\to\infty} f(z) = L$ in $(\mathbb{C}, |\cdot - \cdot|)$ if and only if $\lim_{z\to\infty} f(z) = L$ in (X, d).

(Problem 2840) If (Y, ρ) is a metric space, $a \in Y$, and $f : Y \setminus \{a\} \to C$, then we say that $\lim_{y\to a} f(y) = \infty$ in $(\mathbb{C}, |\cdot - \cdot|)$ if for every $R > 0$ there is a $\delta > 0$ such that if $0 < \rho(a, y) < \delta$ then $|f(y)| > R$. Show that $\lim_{y\to a} f(y) = \infty$ in $(\mathbb{C}, |\cdot - \cdot|)$ if and only if $\lim_{y\to a} f(y) = \infty$ in (X, d).

4.7. Meromorphic Functions

Definition 4.7.2. Let $\Omega \subseteq \mathbb{C}$ be open. A function f is said to be *meromorphic* on Ω if there is a set S such that

- $S \subset \Omega$,
- S has no accumulation points in Ω,
- $f : \Omega \setminus S \to \mathbb{C}$,
- f is holomorphic on $\Omega \setminus S$,
- If $P \in S$, then f has a pole at P.[2]

We call S the singular set for f.

Recall [Problem 2050]: Let Ω be a connected open set and let $f : \Omega \to \mathbb{C}$ be holomorphic and not constant. Then the set $S = \{z \in \Omega : f(z) = 0\} \subset \Omega$ has no accumulation points in Ω.

(Problem 2850) Let $\Omega \subseteq \mathbb{C}$ be open and let $S \subset \Omega$ have no accumulation points in Ω. Show that $\Omega \setminus S$ is both open and dense in Ω.

Let $z \in \Omega$. Then there is an $r > 0$ such that $D(z, r) \subseteq \Omega$.

z is not an accumulation point for S, and so there is an $\varepsilon > 0$ such that $D(z, \varepsilon) \setminus \{z\}$ contains no points of S.

Let $\varrho = \min(r, \varepsilon)$; since $r > 0$ and $\varepsilon > 0$ we have that $\varrho > 0$ as well.

If $z \in \Omega \setminus S$ then $D(z, \varrho) \subset \Omega \setminus S$ and $\varrho > 0$. This is true for all $z \in \Omega \setminus S$, and so $\Omega \setminus S$ is open.

If $z \in \Omega$, let $\delta > 0$. Then $D(z, \min(\delta, \varrho)) \setminus \{z\}$ is not empty because $\min(\delta, \varrho) > 0$ and it lies entirely within $\Omega \setminus S$; thus, in particular $D(z, \delta)$ contains at least one point of $\Omega \setminus S$ (all of the points in $D(z, \min(\delta, \varrho)) \setminus \{z\}$), and so $\Omega \setminus S$ is dense in Ω.

[Chapter 3, Problem 42 (generalized)] Let $K \subset \Omega$ be compact. Then $K \cap S$ is finite.

(Problem 2860) Suppose in addition that Ω is connected. Show that $\Omega \setminus S$ is connected.

(Problem 2870) If $\Omega \subseteq \mathbb{C}$ is open, and if S and Z are two subsets of Ω with no accumulation points in Ω, show that $S \cup Z$ has no accumulation points in Ω.

In particular, let f be meromorphic and non-constant in Ω for some $\Omega \subseteq \mathbb{C}$ open and connected. Let S be the singular set for f and let $Z = \{z \in \Omega \setminus S : f(z) = 0\}$ be the zero set. Show that $S \cup Z$ has no accumulation points in Ω.

Let $z \in \Omega$. z is not an accumulation point for S, and so there is a $r_S > 0$ such that $S \cap D(z, r_S) \setminus \{z\} = \emptyset$. Similarly, there is a $r_Z > 0$ such that $Z \cap D(z, r_Z) \setminus \{z\} = \emptyset$. Letting $r = \min(r_S, r_Z)$, we see that $r > 0$ and that $(S \cup Z) \cap D(z, r) \setminus \{z\} = \emptyset$, as desired.

[Chapter 4, Problem 63] The sum of two meromorphic functions is meromorphic.

[Chapter 4, Problem 64a] The product of two meromorphic functions is meromorphic.

(Problem 2880) Show that the derivative of a meromorphic function is meromorphic.

Let S be the singular set for Ω; by definition of meromorphic $S \subset \Omega$ and S has no accumulation points in Ω. Then f is holomorphic in $\Omega \setminus S$. $\Omega \setminus S$ is open by Problem 2850, and so by Corollary 3.1.2 f' is also holomorphic in $\Omega \setminus S$.

[2] Occasionally it is convenient to allow f to also have removable singularities at points of S; we can however take the convention that all removable singularities should be extended as much as possible using the Riemann removable singularities theorem.

We need only show that, if $s \in S$, then f' has a pole at s. Because Ω is open and the points of S are isolated, there is a $r > 0$ such that $D(s,r) \subset \Omega$ and $D(s,r) \cap S = \{s\}$. Therefore f has a Laurent series in $D(s,r) \setminus \{s\}$. By assumption f has a pole at S, and so by Problem 2310 there is a $n > 0$ such that $a_{-n} \neq 0$ and such that

$$f(z) = \sum_{k=-n}^{\infty} a_k (z-s)^k$$

for all $z \in D(s,r) \setminus \{s\}$.

By Proposition 4.3.3 and Corollary 3.5.2, if $z \in D(s,r) \setminus \{s\}$ then

$$f'(z) = \sum_{k=-n}^{\infty} k a_k (z-s)^{k-1} = \sum_{j=-n-1}^{\infty} (j+1) a_{j+1} (z-s)^j$$

and so by Problem 2320 f' has a pole at s.

[Chapter 4, Problem 64b] Suppose that f is meromorphic in Ω. Then the function obtained by extending $1/f$ as much as possible using the Riemann removable singularities theorem is meromorphic in Ω.

(Problem 2890) Suppose that Ω, $W \subseteq \mathbb{C}$ are open and connected, that $f : \Omega \to W$ is holomorphic and not constant, and that $g : W \to \mathbb{C}$ is meromorphic in W. Show that $g \circ f$ is meromorphic in Ω.

Let $S_g \subset W$ be the singular set for g. Let $S = f^{-1}(S_g) = \{z \in \Omega : f(z) \in S_g\}$.

First, we claim that S has no accumulation points in Ω.

Suppose not. Let $\zeta \in \Omega$. Then $f(\zeta) \in W$ and so $f(\zeta)$ is not an accumulation point of S_g. There is thus a $\varepsilon > 0$ such that $D(f(\zeta),\varepsilon) \cap S_g \subseteq \{f(\zeta)\}$ (that is, the intersection is either empty or the single point $f(z)$).

Suppose that ζ is an accumulation point for S. Let $\{s_k\}_{k=1}^{\infty}$ be a sequence of distinct points in S with $s_k \to \zeta$. Then by continuity of f we have that $f(s_k) \to f(\zeta) \in W$. Thus, there is a $N \in \mathbb{N}$ such that if $k > N$ then $f(s_k) \in D(f(\zeta),\varepsilon)$.

By definition of S, $f(s_k) \in S_g$ for all k. Thus if $k > N$ then $f(s_k) \in D(f(\zeta),\varepsilon) \cap S_k \subseteq \{f(\zeta)\}$, and so we must have $f(s_k) = f(\zeta)$ for all $k > N$. Thus $\{z \in \Omega : f(z) = f(\zeta)\}$ has an accumulation point, namely ζ; by Theorem 3.6.1 f is a constant, contradicting our assumption.

Thus $S \subset \Omega$ has no accumulation points. By Problem 1.49 in your book, $g \circ f$ is holomorphic in $\Omega \setminus S$.

Now, let $s \in S$. We need only show that $g \circ f$ has a pole at s.

Because s is not an accumulation point of S there is a $r > 0$ such that $D(s,r) \cap S = \{s\}$.

We know that $f(s) \in S_g$, and so g has a pole at $f(s)$. Thus $\lim_{w \to f(s)} |g(w)| = \infty$: for any $N \in \mathbb{R}$ there is a $\varepsilon > 0$ such that if $0 < |w - f(s)| < \varepsilon$ then $|g(w)| > N$. But f is continuous at s, and so there is a $\delta > 0$ such that if $|z - s| < \delta$ then $|f(z) - f(s)| < \varepsilon$.

Thus if $0 < |z - s| < \min(\delta, r)$ then $z \notin S$ and so $f(z) \notin S_g$, and so $f(z) \neq f(s) \in S_g$. Thus $0 < |f(z) - f(s)| < \varepsilon$ and so $|g(f(z))| > N$, as desired.

(Problem 2891) Suppose that $\Omega \subseteq \mathbb{C}$ is open and that $S \subset \Omega$ has no accumulation points in Ω. Show that f is meromorphic in Ω with singular set S if and only if f is holomorphic in $\Omega \setminus S$ and if the function $\hat{f}$ given by

$$\hat{f}(z) = \begin{cases} f(z), & z \in \Omega \setminus S, \\ \infty, & z \in S \end{cases}$$

is continuous as a function from Ω to (X,d), where (X,d) is the metric space in Problem 2780.

4.7. Singularities at Infinity

Definition 4.7.4. Let $\Omega \subseteq \mathbb{C}$ be open. Suppose that there is some $R > 0$ such that

$$\mathbb{C} \setminus \overline{D}(0,R) = \{z \in \mathbb{C} : |z| > R\} \subseteq \Omega.$$

If f is holomorphic on Ω, then we say that f has an isolated singularity at ∞.

Let $W = \{z \in \mathbb{C} \setminus \{0\} : 1/z \in \Omega\}$ and define $g : W \to \mathbb{C}$ by $g(z) = f(1/z)$.

(i) If g has a removable singularity at 0, we say that f has a removable singularity at ∞.

(ii) If g has a pole at 0, we say that f has a pole at ∞.

(iii) If g has an essential singularity at 0, we say that f has an essential singularity at ∞.

If f has a removable singularity or pole at ∞, we say that f is meromorphic at ∞.

Recall [Theorem 4.3.2]: If f is holomorphic on an open set Ω, and $\Omega \supseteq D(0,\infty)\setminus\overline{D}(0,R) = \{z \in \mathbb{C} : |z| > R\}$, then there is a unique Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$ that converges to f on $\{z \in \mathbb{C} : |z| > R\}$.

[Definition: Laurent expansion around infinity] If there is an $R > 0$ such that $\sum_{n=-\infty}^{\infty} a_n z^n$ that converges to f on $\{z \in \mathbb{C} : |z| > R\}$, then we call $\sum_{n=-\infty}^{\infty} a_n z^n$ the Laurent expansion of f around ∞.

(Problem 2900) Let f have an isolated singularity at ∞.

(a) State equivalent conditions for f to have a removable singularity, pole, or essential singularity at ∞ in terms of the limits of f and $|f|$ as $|z| \to \infty$.

(b) State equivalent conditions for f to have a removable singularity, pole, or essential singularity at ∞ in terms of the Laurent expansion of f about ∞.

If f has an isolated singularity at ∞, and the Laurent expansion of f about ∞ is $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ for all $|z| > R$, then the rows in this table all contain three equivalent statements.

f has a removable singularity at ∞	$\limsup_{z\to\infty} |f(z)| < \infty$ (this means that there is a $r > 0$ such that $\sup_{|z|>r} |f(z)| < \infty$). Alternatively, $\lim_{z\to\infty} f(z) = L$ for some $L \in \mathbb{C}$.	$a_n = 0$ for all $n > 0$
f has a pole at ∞	$\lim_{z\to\infty} |f(z)| = \infty$	There is a $n \in \mathbb{N}$ (with $n > 0$) such that $a_n \neq 0$ and $a_k = 0$ for all $k > n$
f has an essential singularity at ∞	$\limsup_{z\to\infty} |f(z)| = \infty$ and $\liminf_{z\to\infty} |f(z)| = 0$ (or $\limsup_{z\to\infty} |f(z)| \neq \liminf_{z\to\infty} |f(z)|$)	If $n \in \mathbb{N}$ then there is a $k \in \mathbb{N}$ with $k > n$ and $a_k \neq 0$

(Problem 2910) Let Ω, $W \subset \mathbb{C}$ be open. Suppose that f is meromorphic and not constant in Ω and that g is meromorphic in W. If f has any poles, we require that g be meromorphic at ∞; that is, there is an $R > 0$ such that$\{z \in \mathbb{C} : |z| > R\} \subseteq W$ and g has no poles in $\{z \in \mathbb{C} : |z| > R\}$, and furthermore that g has either a removable singularity or a pole at ∞. Suppose furthermore that $f(\Omega \setminus S_f) \subseteq W$ where S_f is the singular set for f. Show that $g \circ f$ is meromorphic on Ω (possibly after "filling in" removable singularities).

We claim that $g \circ f$ is meromorphic with singular set a subset of $S_f \cup f^{-1}(S_g)$.

By Problem 2890, we have that $g \circ f$ is meromorphic in $\Omega \setminus S_f$ with singular set $f^{-1}(S_g)$; that is, if $\zeta \in \Omega \setminus S_f$ then ζ is not an accumulation point of $f^{-1}(S_g)$, and either $g \circ f$ is holomorphic in a neighborhood of ζ or $g \circ f$ has a pole at ζ.

We must show that if $\zeta \in S_f$ then ζ is not an accumulation point of $f^{-1}(S_g)$, and that $g \circ f$ has a pole or removable singularity at ζ. By Problem 2870, this will imply that the set $S_f \cup f^{-1}(S_g)$ has no accumulation points in Ω.

Recall that if $\zeta \in S_f$ then f has a pole at ζ. Thus there is a $\delta > 0$ such that if $0 < |z - \zeta| < \delta$ then $|f(z)| > R$. But by assumption g has no poles in $\{z \in \mathbb{C} : |z| > R\}$, and so $f(z)$ cannot be in S_g; thus, $D(\zeta, \delta) \setminus \{\zeta\}$ contains no points of $f^{-1}(S_g)$ and so ζ is not an accumultation point of $f^{-1}(S_g)$.

Finally, it follows from the properties of limits that $\lim_{z\to\zeta} |g \circ f(z)| = \lim_{w\to\infty} |g(w)|$, which is either finite or infinite but in either case exists, and so $g \circ f$ has either a removable singularity or a pole at ζ.

(Problem 2920) Suppose that $\Omega \subseteq \mathbb{C}$ is open and that $S \subset \Omega$ has no accumulation points in Ω. Further suppose that there is some $R > 0$ such that $\{z \in \mathbb{C} : |z| > R\} \subseteq \Omega \setminus S$. Show that f is meromorphic in Ω and at ∞ with singular set S if and only if f is holomorphic in $\Omega \setminus S$ and if the function $\hat{f}$ given by

$$\hat{f}(z) = \begin{cases} f(z), & z \in \Omega \setminus S, \\ \infty, & z \in S, \\ \lim_{\zeta \to \infty} f(\zeta), & z = \infty \end{cases}$$

is continuous as a function from $\Omega \cup \{\infty\} \subseteq X$ (with the metric d) to (X, d), where (X, d) is the metric space in Problem 2780.

Theorem 4.7.5. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire and is also meromorphic at ∞. Then f is a polynomial.

(Problem 2930) Prove Theorem 4.7.5.

Because f is entire, f has a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for all $z \in \mathbb{C}$. This is then the Laurent expansion about ∞ of f, and so by Problem 2900 we have that there is a $N \in \mathbb{N}_0$ such that $a_n = 0$ for all $n > N$. Thus

$$f(z) = \sum_{n=0}^{N} a_n z^n$$

and so f is a polynomial.

Theorem 4.7.7. Suppose that f is meromorphic in $\mathbb{C}$ and also is meromorphic at ∞. Then there are two polynomials p and q such that the singular set of f is equal to the zero set of q and such that $f(z) = \frac{p(z)}{q(z)}$ for all z in $\mathbb{C}$ outside of the singular set.

(Problem 2940) Prove Theorem 4.7.7.

Because f is meromorphic at ∞, we have that f has an isolated singularity at ∞; that is, there is a $R > 0$ such that if S is the singular set for f then $S \cap \{z \in \mathbb{C} : |z| > R\} = \emptyset$.

Thus $S \subset \overline{D}(0, R)$. $\overline{D}(0, R)$ is compact and S has no accumulation points, so by Problem 3.42 $\overline{D}(0, R)$ contains only finitely many points of S. But $\overline{D}(0, R)$ contains all points of S, so S is finite. Let $S = \{s_1, s_2, \ldots, s_m\}$.

Let n_k be the order of the pole of f at s_k. Then $q(z) = \prod_{k=1}^{m} (z - s_k)^{n_k}$ is a polynomial. By Problems 2340 and 2350, $q(z)f(z)$ has removable singularities at each s_k. We may extend qf using the Riemann removable singularities theorem to an entire function p.

The function f has a Laurent expansion $\sum_{k=-\infty}^{n} a_k z^k$ about ∞; because f is meromorphic at ∞, we have that $n < \infty$. Multiplying by a polynomial yields that the Laurent expansion of p about ∞ is $\sum_{k=-\infty}^{n+m} b_k z^k$ where m is the degree of the polynomial $q(z)$; thus p is also meromorphic at ∞.

Because p is meromorphic at ∞ and also entire, p is a polynomial by Theorem 4.7.5. Then

$$f(z) = \frac{p(z)}{q(z)}$$

where p and q are polynomials, as desired.

Theorem 5.1.4. (Argument principle for meromorphic functions.) Let $\Omega \subseteq \mathbb{C}$ be open and let f be meromorphic in Ω. Let $\overline{D}(P,r) \subset \Omega$. Suppose that f has no poles on $\partial D(P,r)$ and that $f(z) \neq 0$ for all $z \in \partial D(P,r)$. Then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f'}{f} = \sum_{j=1}^{p} n_j - \sum_{k=1}^{q} m_k,$$

where $n_1, \dots n_p$ are the multiplicities of the zeroes $z_1, \dots, z_p$ of f in $D(P,r)$ and where $m_1, \dots m_q$ are the orders of the poles $w_1, \dots, w_q$ of f in $D(P,r)$.

[Chapter 5, Problem 1] Suppose that $\Omega \subseteq \mathbb{C}$ is open, f is meromorphic in Ω, and $\overline{D}(P,r) \subset \Omega$. Then f has at most finitely many poles and at most finitely many zeroes in $\overline{D}(P,r)$.

(Problem 2950) In this problem we will prove Theorem 5.1.4. Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f : \Omega \to \mathbb{C}$ be meromorphic and not uniformly zero. Let $g(z) = \frac{f'(z)}{f(z)}$; then g is meromorphic in Ω.

- Let $z_j \in \Omega$ be a zero of f and let n_j be the order of the zero at z_j. Show that $\mathrm{Res}_g(z_j) = n_j$.
- Let $w_k \in \Omega$ be a pole of f and let m_k be the order of the pole of f at w_k. Show that $\mathrm{Res}_g(z_k) = -m_k$.
- Prove Theorem 5.1.4.

Suppose that f has a zero or pole at ζ. Let n be the order of the zero or the negative of the order of the pole; then $(z-\zeta)^{-n} f(z)$ has a removable singularity at ζ and $\lim_{z\to\zeta} f(z) \neq 0$. Let $h(z)$ be the extension of $(z-\zeta)^{-n} f(z)$ given by the Riemann removable singularities theorem.

Then $f(z) = (z-\zeta)^n h(z)$ in a punctured neighborhood of ζ, and so

$$\frac{f'(z)}{f(z)} = \frac{n(z-\zeta)^{n-1} h(z) + (z-\zeta)^n h'(z)}{(z-\zeta)^n h(z)} = \frac{n}{z-\zeta} + \frac{h'(z)}{h(z)}.$$

But $h(\zeta) \neq 0$ and h is continuous, so $h \neq 0$ in a neighborhood of ζ and so h'/h is holomorphic in that neighborhood. Thus $\mathrm{Res}_{f'/f}(\zeta) = n + \mathrm{Res}_{h'/h} = n$.

The conclusion follows from the residue theorem (Theorem 4.5.3).

(Problem 2960) Write down a version of the argument principle that allows us to count the number of solutions to $f(z) = 3$ in $D(P,r)$.

5.2. The Local Geometry of Holomorphic Functions

Recall [Problem 1140]: If γ is a C^1 curve and f is holomorphic in a neighborhood of $\widetilde{\gamma}$ then $f \circ \gamma$ is also a C^1 curve. Clearly, if γ is closed then so is $f \circ \gamma$.

(Problem 2970) Suppose that γ is the standard counterclockwise parameterization of $\partial D(P,r)$, that f is holomorphic in $D(P,R)$ for some $R > r$, and that $f(\partial D(P,r)) \subset \mathbb{C} \setminus \{0\}$. Show that $\mathrm{Ind}_{f\circ\gamma}(0)$ is equal to the number of solutions to $f(z) = 0$ in $D(P,r)$ (counted with multiplicity). *Hint*: Use the argument principle.

Recall that, if η is a closed path and $w \in \mathbb{C} \setminus \widetilde{\eta}$, then $\mathrm{Ind}_\eta(w)$ is defined to be

$$\mathrm{Ind}_\eta(w) = \frac{1}{2\pi i} \oint_\eta \frac{1}{\zeta - w}\, d\zeta.$$

Thus

$$\mathrm{Ind}_{f\circ\gamma}(0) = \frac{1}{2\pi i} \oint_{f\circ\gamma} \frac{1}{\zeta}\, d\zeta = \frac{1}{2\pi i} \oint_{f\circ\gamma} g$$

where $g(z) = \frac{1}{z}$.

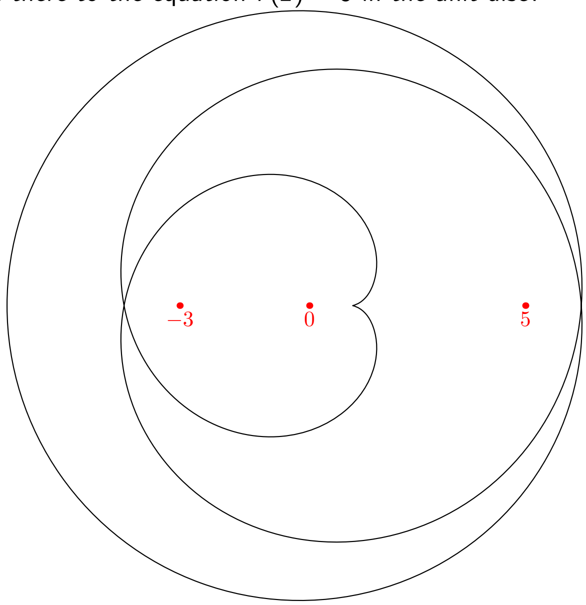
Because f is continuous and $\partial D(P,r)$ is compact, we have that there exist r_1, R_1 with $0 < r_1 < r < R_1 < R$ such that f is nonzero on $\Omega = D(P,R_1) \setminus \overline{D}(P,r_1)$, that is, $f : \Omega \to W = \mathbb{C} \setminus \{0\}$. Observe that g is continuous on W.

By our change of variable result 1380 with u replaced by f and f replaced by g, we have that

$$\mathrm{Ind}_{f\circ\gamma}(0) = \frac{1}{2\pi i} \oint_{f\circ\gamma} g = \frac{1}{2\pi i} \oint_\gamma g \circ f\, f' = \frac{1}{2\pi i} \oint_\gamma \frac{f'}{f}.$$

The argument principle completes the proof.

(Problem 2980) You are given that f is holomorphic in $D(0,2)$ and that there are at most finitely many points w such that $f(z) = w$ for more than one $z \in \partial D(0,1)$. Illustrated are the three points 0, -3, and 5 and the set $f(\partial D(0,1))$. Counted with multiplicity, how many solutions are there to the equation $f(z) = -3$ in the unit disc? How many solutions are there to the equation $f(z) = 5$ in the unit disc?

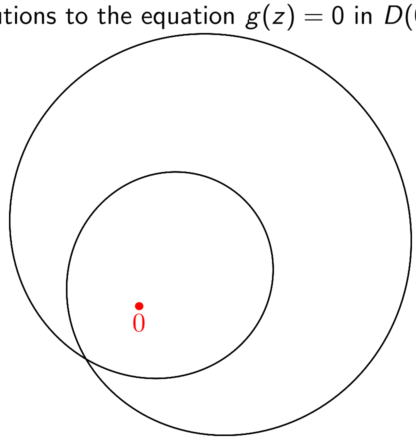


We must have that $f(z) = -3$ has three solutions and $f(z) = 5$ has two solutions (counted with multiplicity).

(Problem 2990) For the function f illustrated above, the only solution to $f(z) = 0$ in $D(0,1)$ is $z = 0$. How is this possible?

The multiplicity of the zero at 0 must be 3.

(Problem 3000) You are given that g is holomorphic in $D(0,2)$. Illustrated is the point 0 and the set $g(\partial D(0,1))$. You are given that there are four solutions to the equation $g(z) = 0$ in $D(0,1)$. How is this possible?



The curve $g \circ \gamma$, where $\gamma : [0, 2\pi] \to \mathbb{C}$ and $\gamma(t) = e^{it}$, must be 2-to-1 at most points; that is, most points in $g(\partial D(0,1))$ must have two preimages instead of one.

Recall [Problem 850]: Let X be a compact metric space and let $f : X \to Z$ be a continuous function. Then $f(X)$ is compact.

(Memory 3010) Let $K \subseteq \mathbb{C}$ be compact and let $\Omega \subseteq \mathbb{C}$ be a connected component of $\mathbb{C} \setminus K$. Then Ω is open.

5.2. The Open Mapping Theorem

Theorem 5.2.1. (The open mapping theorem.) Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f : \Omega \to \mathbb{C}$ be holomorphic. Then either $f(\Omega) = \{Q\}$ for some $Q \in \mathbb{C}$ or $f(\Omega)$ is an open subset of $\mathbb{C}$.

(Problem 3020) In this problem we begin the proof of Theorem 5.2.1. Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f : \Omega \to \mathbb{C}$ be holomorphic and not constant. Let $P \in \Omega$ and let $Q = f(P)$. Show that if $r > 0$ and $\overline{D}(P, r) \subset \Omega$, and if $\gamma(t) = P + re^{it}$, $0 \leq t \leq 2\pi$, then $\mathrm{Ind}_{f\circ\gamma}(Q) > 0$.

There is (counted with multiplicity) at least one solution to $f(z) = Q$ in $D(P, r)$, namely $z = P$. Thus by Problem 2970 the index must be positive.

(Problem 3030) [Redacted]

(Problem 3040) Let Ω be the connected component of $\mathbb{C} \setminus f(\partial D(P, r))$ containing Q. Show that $U \subseteq f(\Omega)$.

(Problem 3050) [Prove the open mapping theorem.] Show that $f(\Omega)$ is open.

(Problem 3060) Give an example of an open set $\Omega \subseteq \mathbb{R}^2$ and a non-constant C^∞ function $f : \Omega \to \mathbb{R}^2$ such that $f(\Omega)$ is not open.

Let $f(x, y) = (x^2 + y^2, 2xy)$ and let $\Omega = \mathbb{R}^2$. Then $(0, 0) = f(0, 0) \in f(\mathbb{R}^2)$ but $f(\mathbb{R}^2) \subseteq [0, \infty) \times \mathbb{R}$, so $(0, 0) \in f(\mathbb{R}^2)$ is a boundary point of $f(\mathbb{R}^2)$ and so $f(\mathbb{R}^2)$ cannot be open.

(Problem 3061) Let $\Omega \subset \mathbb{C}$ be open, connected, and bounded, and let $f : \overline{\Omega} \to \mathbb{C}$ be continuous on $\overline{\Omega}$ and holomorphic in Ω. Show that $\partial(f(\Omega)) \subseteq f(\partial\Omega)$.

(Problem 3062) What can you say if Ω is unbounded?

5.2. Simple points, multiple points, and inverses

[Definition: Multiple point] Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f : \Omega \to \mathbb{C}$ be holomorphic and not constant. Then $P \in \Omega$ is a multiple point of f if the function $g(z) = f(z) - f(P)$ has a zero of multiplicity at least 2 at P. If $f(P) = Q$ and g has a zero of multiplicity k, we say that $f(P) = Q$ with order k.

[Definition: Simple point] Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f : \Omega \to \mathbb{C}$ be holomorphic and not constant. Then $P \in \Omega$ is a simple point of f if it is not a multiple point of f, that is, if the function $g(z) = f(z) - f(P)$ has a zero of multiplicity 1 at P. If $f(P) = Q$, we say that $f(P) = Q$ with order 1.

Theorem 5.2.2. Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $f : \Omega \to \mathbb{C}$ be holomorphic and not constant. Let $P \in \Omega$ and let $Q = f(P)$ with order k.

Then there exists a $\delta > 0$ and a $\varepsilon > 0$ such that if $w \in D(Q, \varepsilon) \setminus \{Q\}$ then there are exactly k points $z_1, \dots, z_k \in D(P, \delta)$ such that $f(z_j) = w$. Furthermore, each z_j is a simple point.

(Problem 3070) In this problem we begin the proof of Theorem 5.2.2. Show that if $P \in \Omega$, then there is a $\rho > 0$ such that there are no multiple points in $D(P, \rho) \setminus \{P\} \subset \Omega$. *Hint:* Start by showing that P is a multiple point of f if and only if $f'(P) = 0$.

(Problem 3080) [Redacted]

(Problem 3090) Let $R > 0$ be such that $z = P$ is the only solution in $D(P, R)$ to $f(z) = f(P)$. Such an R must exist by Problem 2050. Let ρ be as in Problem 3070, and let $0 < \delta < \min(R, \rho)$. Let U be the connected component of $\mathbb{C} \setminus f(\partial D(P, \delta))$ containing Q; recall from Problem 3010 that U is open. Suppose that $w \in U \setminus \{Q\}$. Show that there are exactly k points $z_1, \dots, z_k \in D(P, \delta) \setminus \{P\}$ with $f(z_j) = w$.

(Problem 3100) Prove Theorem 5.2.2.

(Problem 3110) Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be holomorphic. Let $P \in \Omega$ and suppose that $f'(P) \neq 0$. Show that there is a $r > 0$ with $D(P, r) \subseteq \Omega$ and such that f is one-to-one on $D(P, r)$.

Let $Q = f(P)$. By Problem 3070, P is a simple point of f. Thus, by Theorem 5.2.2, there is a $\varepsilon > 0$ and a $\delta > 0$ such that if $w \in D(Q, \varepsilon)$ then there is exactly one solution $z \in D(P, \delta)$ to the equation $f(z) = w$. Let $U = f^{-1}(D(Q, \varepsilon)) \cap D(P, \delta)$; observe that f is continuous and so Ω is open. Thus, there is a $r > 0$ such that $D(P, r) \subseteq U$.

If $\zeta \in D(P, r) \subseteq D(P, \delta)$, then there is precisely one solution z in $D(P, \delta)$ to $f(z) = f(\zeta)$; it is necessarily $z = \zeta$. Thus, f must be one-to-one on $D(P, r)$.

This may also be established using the Inverse Function Theorem of real analysis, but this proof is shorter and involves techniques that you have probably seen more recently.

(Problem 3111) Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be holomorphic. Let $P \in \Omega$ and suppose that $f(P) = 0$. Show that there does not exist a $r > 0$ with $D(P, r) \subseteq \Omega$ and such that f is one-to-one on $D(P, r)$.

(Problem 3120) Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be holomorphic and one-to-one. Let $W = f(\Omega)$. Show that $f^{-1} : W \to \Omega$ is continuous.

The statement is vacuous if $\Omega = \emptyset$ so we assume Ω is not empty; because it is open each connected component must contain infinitely many points. Because f is one-to-one, f cannot be constant on any connected component.

By the Open Mapping Theorem, if $U \subseteq \Omega$ is open, then $(f^{-1})^{-1}(U) = f(U)$ is open. Thus, the preimages of all open sets in $(f^{-1})(W) = \Omega$ under f^{-1} are open. Thus, f^{-1} is continuous.

[Chapter 5, Problem 7] Let Ω be open and connected and let $f : \Omega \to \mathbb{C}$ be holomorphic and injective. Then $f^{-1} : f(\Omega) \to \Omega$ is also holomorphic.

5.3. Further Results on the Zeros of Holomorphic Functions: Rouché's theorem

Theorem 5.3.1. [Rouché's theorem.] Let $\Omega \subseteq \mathbb{C}$ be open. Let $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ be holomorphic. Suppose that $\overline{D}(P, r) \subseteq \Omega$ and that

$$|f(z) - g(z)| \neq |f(z)| + |g(z)|$$

for all $z \in \partial D(P, r)$. Then f and g have the same number of zeroes (counted with multiplicities) in $D(P, r)$.

(Problem 3130) In this problem we show an example of an application of Rouché's theorem. The polynomial $p(z) = z^7 + 4z^2 + 2$ has seven zeroes (with multiplicity). Use Rouché's theorem to determine how many of these zeroes are in the disc $D(0, 1)$.

(Problem 3140) The polynomial $p(z) = z^7 + 3z^2 + 2$ has seven zeroes (with multiplicity). Use Rouché's theorem to determine how many of these zeroes are in the disc $D(0, 1)$.

(Problem 3150) In this problem we begin the proof of Rouché's theorem. Let η, $\phi : [0, 1] \to \mathbb{C}$ be two C^1 closed curves. Suppose that

$$|\eta(t) - \phi(t)| \neq |\eta(t)| + |\phi(t)|$$

for all $0 \leq t \leq 1$. Show that η and ϕ are homotopic in $\mathbb{C} \setminus \{0\}$.

Let $\Psi(t, s) = s\eta(t) + (1 - s)\phi(t)$. Then Ψ is clearly C^1 in both t and s and satisfies $\Psi(t, 0) = \phi(t)$, $\Psi(t, 1) = \eta(t)$, so Ψ is a C^1 homotopy between ϕ and η.

It remains to show that $\Psi : [0, 1] \times [0, 1] \to \mathbb{C} \setminus \{0\}$, that is, that 0 is not in the image of Ψ. Suppose for the sake of contradiction that $\Psi(t, s) = 0$. Then

$$0 = s\eta(t) + (1 - s)\phi(t).$$

If $s = 0$ then $\phi(t) = 0$ and so $|\eta(t) - \phi(t)| = |\eta(t)| = |\eta(t)| + |\phi(t)|$, which is a contradiction.

If $s \neq 0$ then $\eta(t) = \frac{s-1}{s}\phi(t)$. Then

$$|\eta(t) - \phi(t)| = |(\frac{s - 1}{s} - 1)\phi(t)| = |\frac{1}{s}||\phi(t)|.$$

But $s > 0$, so $|\frac{1}{s}| = \frac{1}{s}$ and so
$$|\eta(t) - \phi(t)| = \frac{1}{s}|\phi(t)| = (1 + \frac{1-s}{s})|\phi(t)| = |\phi(t)| + \frac{1-s}{s}|\phi(t)|.$$
But $\frac{1-s}{s} \geq 0$ and so $\frac{1-s}{s} = |\frac{1-s}{s}|$. Thus
$$|\eta(t) - \phi(t)| = |\phi(t)| + |\eta(t)|.$$
This is again a contradiction.

Recall [Problem 2540]: Suppose that ϕ and η are homotopic closed curves in $\Omega\setminus\{P\}$. Then $\mathrm{Ind}_\phi(P) = \mathrm{Ind}_\eta(P)$.

Recall [Problem 2970]: Suppose that γ is the standard counterclockwise parameterization of $\partial D(P, r)$, that f is holomorphic in $D(P, R)$ for some $R > r$, and that $f(\partial D(P, r)) \subset \mathbb{C} \setminus \{0\}$. Then $\mathrm{Ind}_{f\circ\gamma}(0)$ is equal to the number of solutions to $f(z) = 0$ in $D(P, r)$ (counted with multiplicity).

(Problem 3160) Prove Rouché's theorem.

Let $\gamma(t) = P + re^{it}$, $0 \leq t \leq 2\pi$. Then γ is a C^1 closed curve. By Problem 1140, if $\eta = f \circ \gamma$ and $\phi = g \circ \gamma$, then η and ϕ are also C^1 closed curves.

But
$$|\phi(t) - \eta(t)| = |f(\gamma(t)) - g(\gamma(t))| \neq |f(\gamma(t))| + |g(\gamma(t))| = |\phi(t)| + |\eta(t)|$$
and so by the previous problem η and ϕ are homotopic in $\mathbb{C} \setminus \{0\}$.

Thus by Problem 2540 $\mathrm{Ind}_\eta(0) = \mathrm{Ind}_\phi(0)$.

But by Problem 2970 $\mathrm{Ind}_\eta(0)$ is the number of zeroes of f in $D(P, r)$ with multiplicity and $\mathrm{Ind}_\phi(0)$ is the number of zeroes of g in $D(P, r)$, so they must be equal, as desired.

5.4. The Maximum Modulus Principle

Theorem 5.4.2. (The maximum modulus principle.) Let $\Omega \subseteq \mathbb{C}$ be a connected open set and let $f : \Omega \to \mathbb{C}$ be holomorphic. Suppose that there is a $w \in \Omega$ such that $|f(w)| \geq |f(z)|$ for all $z \in \Omega$. Then f is constant.

(Problem 3170) Prove the maximum modulus principle. *Hint*: Show that $f(\Omega)$ is not open.

We will prove the contrapositive. Suppose that f is not constant and $w \in \Omega$; we need only show that $|f(z)| > |f(w)|$ for some $z \in \Omega$.

By Theorem 5.2.1, $f(\Omega)$ is open and thus there is a $r > 0$ such that $D(f(w), r) \subseteq f(\Omega)$. In particular, $\zeta = f(w) + r\frac{f(w)}{|f(w)|} \in f(\Omega)$ if $f(w) \neq 0$ and $\zeta = r/2 \in f(\Omega)$ if $f(w) = 0$.

But then either $|\zeta| = |f(w) + r\frac{f(w)}{|f(w)|}| = |f(w)|(1 + r/2|f(w)|) > |f(w)|$ or $|\zeta| = |r/2| = r/2 > 0 = |f(w)|$, and so in either case $|\zeta| > |f(w)|$.

But by definition of $f(\Omega)$ we have that $\zeta = f(z)$ for some $z \in \Omega$, and so $|f(z)| > |f(w)|$ for some $z \in \Omega$, as desired.

Theorem 5.4.4. (The maximum modulus principle, sharpened.) Let $\Omega \subseteq \mathbb{C}$ be a connected open set and let $f : \Omega \to \mathbb{C}$ be holomorphic. Suppose that there is a $r > 0$ and a $w \in \Omega$ such that $D(w, r) \subseteq \Omega$ and such that $|f(w)| \geq |f(z)|$ for all $z \in D(w, r)$. Then f is constant.

(Problem 3180) Prove Theorem 5.4.4.

Corollary 5.4.3. (The maximum modulus theorem.) Let $\Omega \subseteq \mathbb{C}$ be a bounded open set. Let $f : \overline{\Omega} \to \mathbb{C}$ be continuous on $\overline{\Omega}$ and holomorphic on Ω. Then there is a $w \in \partial\Omega$ such that $|f(w)| \geq |f(z)|$ for all $z \in \overline{\Omega}$.

(Problem 3190) Prove Corollary 5.4.3.

$\overline{\Omega}$ is a closed bounded subset of $\mathbb{C} = \mathbb{R}^2$, and so by the Heine-Borel theorem $\overline{\Omega}$ is compact.

Since f is continuous on $\overline{\Omega}$, so is $|f|$. Because continuous functions on compact sets attain their maxima, there is a $w \in \overline{\Omega}$ such that $|f(w)| \geq |f(z)|$ for all $z \in \Omega$.

If $w \in \partial\Omega$ then we are done. Otherwise, by the maximum modulus principle f is constant on the connected component U of Ω containing w. Because Ω is bounded, there is at least one point $\omega \in \partial U$. It is an elementary argument in real analysis to show that $\partial U \subseteq \partial\Omega$. Because f is continuous on $\overline{\Omega} \supseteq \overline{U}$, we have that $|f(\omega)| = |f(w)| \geq |f(z)|$ for all $z \in \overline{\Omega}$.

(Problem 3200) Give an example of an unbounded connected open set Ω with nonempty boundary and a continuous function $f : \overline{\Omega} \to \mathbb{C}$ such that f is holomorphic in Ω and such that $|f(w)| < \sup_{z\in\overline{\Omega}} |f(z)|$ for all $w \in \overline{\Omega}$. *Bonus*: Can you give an example in which f is bounded in Ω and another example in which f is bounded on $\partial\Omega$ but unbounded in Ω?

Let $\Omega = \{x + iy : x > 0\}$ be the right half plane and let $f(z) = \exp(z)$. Then $|f(z)| = 1$ for all $z \in \partial\Omega$ but $|f|$ is unbounded in Ω.

Now let $g(z) = \frac{z}{z+1}$. Then $\lim_{z\to\infty} |g(z)| = 1$ and so we may find z such that $|g(z)|$ is arbitrarily close to 1, but if $|g(z)| \geq 1$ then $\operatorname{Re} z \leq -1/2$ and so $|g|$ cannot achieve its maximum in Ω.

Proposition 5.4.5. (The minimum modulus principle.) Let $\Omega \subseteq \mathbb{C}$ be a connected open set and let $f : \Omega \to \mathbb{C}$ be holomorphic. Suppose that there is a $w \in \Omega$ such that $|f(w)| \leq |f(z)|$ for all $z \in \Omega$. Then either f is constant or...

(Problem 3210) Finish the statement of Proposition 5.4.5 and prove that your claim is correct.

Either f is constant or $f(w) = 0$.

The proof is as follows. If $f(w) = 0$ then we are done, so suppose that $f(w) \neq 0$. Then $0 < |f(w)| \leq |f(z)|$ for all $z \in \Omega$, and so f is never zero.

Let $g(z) = 1/f(z)$; then g is also holomorphic in Ω. Furthermore, $|g(z)| = 1/|f(z)| \leq 1/|f(w)| = |g(w)|$ for all $z \in \Omega$, and so by the maximum modulus principle (Theorem 5.4.2) g is constant. Thus f is constant, as desired.

5.3. Further Results on the Zeros of Holomorphic Functions: Hurwitz's theorem

Theorem 5.3.3. [Hurwitz's theorem.] Let $\Omega \subseteq \mathbb{C}$ be a connected open set. If $k \in \mathbb{N}$, let $f_k : \Omega \to \mathbb{C} \setminus \{0\}$ be a nowhere zero holomorphic function. Suppose that $f_k \to f$, uniformly on compact subsets of Ω. Suppose that $f(w) \neq 0$ for at least one $w \in \Omega$. Then $f(z) \neq 0$ for all $z \in \Omega$.

(Problem 3220) Give an example of a connected open set $\Omega \subseteq \mathbb{C}$ and a sequence of holomorphic functions $f_k : \Omega \to \mathbb{C} \setminus \{0\}$ such that $f_k(z) \neq 0$ for all $z \in \Omega$ and all $k \in \mathbb{N}$ but such that $f_k \to 0$ uniformly on all compact subsets of Ω.

Let $\Omega = \mathbb{C}$ and let $f_k(z) = \frac{1}{k} e^z$. Then $f_k \to 0$ uniformly on all compact sets, but $f_k(z) \neq 0$ for all k and z.

(Problem 3230) Prove Hurwitz's theorem.

By Theorem 3.5.1, f is holomorphic. Suppose that f is not identically zero, and for the sake of contradiction suppose that $f(P) = 0$ for some $P \in \Omega$.

Then by Problem 2050 (a corollary of Theorem 3.6.1), the zeroes of f are isolated. Thus, there is a $R > 0$ such that $f \neq 0$ in $D(P, R) \setminus \{P\}$.

Let $0 < r < R$. Then $f \neq 0$ on $\partial D(P, r)$. $|f|$ is continuous, so $|f|$ attains its minimum on the compact set $\partial D(P, r)$. Let $m = \min_{z\in\partial D(P,r)} |f(z)| > 0$ and let $\varepsilon = m/3$.

The set $\overline{D}(P, r)$ is compact and so $f_k \to f$ uniformly on $\overline{D}(P, r)$. In particular, there is a $k \in \mathbb{N}$ such that $|f_k(z) - f(z)| < \varepsilon$ for all $z \in \overline{D}(P, r)$.

By the triangle inequality, $|f_k(z)| \geq |f(z)| - \varepsilon \geq 2m/3$ for all $z \in \partial D(P, r)$. Because $f_k \neq 0$, the minimum modulus principle applies in $D(P, r)$ and so $|f_k| \geq 2m/3$ in $D(P, r)$. In particular $|f_k(P)| \geq 2m/3$. Again by the triangle inequality $|f(P)| \geq |f_k(P)| - |f_k(P) - f(P)| > m/3 > 0$ and so $f(P) \neq 0$, contradicting our assumption. This completes the proof.

(Problem 3231) Let $\Omega \subseteq \mathbb{C}$ be a connected open set. If $k \in \mathbb{N}$, let $f_k : \Omega \to \mathbb{C} \setminus \{0\}$ be an injective holomorphic function. Suppose that $f_k \to f$, uniformly on compact subsets of Ω. Show that f is either constant or injective.

[Definition: The unit disc] We will let $\mathbb{D} = D(0, 1)$.

Theorem 5.5.1. (Schwarz's lemma.) Let $f : \mathbb{D} \to \mathbb{C}$ be a function such that

- $f(0) = 0$,
- f is holomorphic in $\mathbb{D}$,
- $f(\mathbb{D}) \subseteq \mathbb{D}$, that is, $|f(z)| < 1$ for all $z \in \mathbb{D}$.

Then we have that both of the following statements are true:

- $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$,
- $|f'(0)| \leq 1$.

If in addition either

- $|f'(0)| = 1$, or
- $|f(w)| = |w|$ for at least one $w \in \mathbb{D} \setminus \{0\}$,

then there is a $\theta \in \mathbb{R}$ such that $f(z) = ze^{i\theta}$ for all $z \in \mathbb{D}$.

(Problem 3240) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Begin the proof of Schwarz's lemma by proving that $|f'(0)| \leq 1$ and that $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.

Define

$$g(z) = \begin{cases} f(z)/z, & z \in \mathbb{D} \setminus \{0\}, \\ f'(0), & z = 0. \end{cases}$$

g is clearly holomorphic on $\mathbb{D} \setminus \{0\}$. By definition of the complex derivative and because $f(0) = 0$,

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{f(z)}{z}$$

and in particular the limit exists. Thus by Theorem 4.1.1 (the Riemann removable singularities theorem) we have that g is holomorphic at 0 and thus on $\mathbb{D}$.

If $0 < \varepsilon < 1$ and $|z| = 1 - \varepsilon$, then $|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{1-\varepsilon}$. By the maximum modulus principle, $|g(z)| \leq \frac{1}{1-\varepsilon}$ for all $z \in D(0, 1 - \varepsilon)$. Taking the limit as $\varepsilon \to 0^+$ we have that $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. Recalling the definition of g completes the proof.

(Problem 3250) Complete the proof of Schwarz's lemma. That is, suppose that in addition either $|f(z)| = |z|$ for some $z \in \mathbb{D}$ or $|f'(0)| = 1$. Show that there is a $\theta \in \mathbb{R}$ such that $f(z) = ze^{i\theta}$ for all $z \in \mathbb{D}$.

Again let

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \in \mathbb{D} \setminus \{0\}, \\ f'(0), & z = 0. \end{cases}$$

By the limit definition of f', g is continuous at 0, so by the Riemann removable singularities theorem, g is holomorphic on $\mathbb{D}$.

By the previous problem, $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. If $|f'(0)| = 1$ or $|f(w)| = |w|$ for some $w \in \mathbb{D} \setminus \{0\}$, then $|g(w)| = 1$ for some $w \in \mathbb{D}$. By the maximum modulus principle g is constant in $\mathbb{D}$, so there is an $\alpha \in \mathbb{C}$ such that $g(z) = \alpha$ or $f(z) = \alpha z$ for all $z \in \mathbb{D}$. Since $\alpha = g(w)$, we have $|\alpha| = |g(w)| = 1$ and so $\alpha = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

(Lemma 3260) Let $c \in \mathbb{D}$ and define

$$\phi_c(z) = \frac{z-c}{1-\bar{c}z}.$$

Then ϕ_c is a holomorphic bijection from $\mathbb{D}$ to itself, a continuous bijection from $\partial\mathbb{D}$ to itself, and a continuous bijection from $\overline{\mathbb{D}}$ to itself.

(Problem 3270) In this problem we begin the proof of Lemma 3260. Show that ϕ_c is holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.

(Problem 3280) Show that $\phi_c(\mathbb{D}) \subseteq \mathbb{D}$ and that $\phi_c(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$.

Let $z \in \partial\mathbb{D}$; then $z = e^{i\theta}$ for some $\theta \in \mathbb{R}$. We compute

$$\phi_c(e^{i\theta}) = e^{i\theta}\frac{1-ce^{-i\theta}}{1-\bar{c}e^{i\theta}}.$$

Let $w = 1 - ce^{-i\theta}$; because $|c| < 1$ we have that $w \neq 0$ and so $|w| = |\bar{w}| \neq 0$. Thus

$$|\phi_c(e^{i\theta})| = \left|e^{i\theta}\frac{w}{\bar{w}}\right| = 1.$$

Since $z = e^{i\theta}$ was an arbitrary point of $\partial\mathbb{D}$, we have that $\phi_c(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$.

We now observe that ϕ_c is not a constant function. In particular, $\phi_0(z) = z$, and if $c \neq 0$ then $\phi_c(c) = 0$ and $\phi_c(0) = -c$, so in either case ϕ_c is not a constant.

Because ϕ_c is holomorphic and not constant in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, by the maximum modulus principle we have that if $z \in \mathbb{D}$ then $|\phi_c(z)| < \sup_{\partial\mathbb{D}} |\phi_c| = 1$. So $\phi_c(\mathbb{D}) \subseteq \mathbb{D}$.

(Problem 3290) Show that $\phi_c(\phi_{-c}(z)) = z$ for all $z \in \overline{\mathbb{D}}$.

Because $\phi_{-c}(\overline{\mathbb{D}}) \subseteq \overline{\mathbb{D}}$ and ϕ_c is defined on $\overline{\mathbb{D}}$, we have that $\phi_{-c} \circ \phi_c$ is well defined on $\overline{\mathbb{D}}$. Then

$$\begin{aligned}
\phi_c(\phi_{-c}(z)) &= \frac{\phi_{-c}(z) - c}{1 - \bar{c}\phi_{-c}(z)} \\
&= \frac{(1+\bar{c}z)\phi_{-c}(z) - c(1+\bar{c}z)}{(1+\bar{c}z) - (1+\bar{c}z)\bar{c}\phi_{-c}(z)} \\
&= \frac{z + c - c(1+\bar{c}z)}{(1+\bar{c}z) - (c+z)\bar{c}} \\
&= \frac{z(1-c\bar{c})}{1-c\bar{c}} = z.
\end{aligned}$$

(Problem 3300) Show that $\phi_c(\mathbb{D}) = \mathbb{D}$ and that $\phi_c(\partial\mathbb{D}) = \partial\mathbb{D}$. This completes the proof of Lemma 3260.

We have shown that $\phi_c(\mathbb{D}) \subseteq \mathbb{D}$. Conversely, let $w \in \mathbb{D}$. Then $\phi_{-c}(w) \in \mathbb{D}$ so $w = \phi_c(\phi_{-c}(w)) \in \phi_c(\mathbb{D})$. Thus $\mathbb{D} \subseteq \phi_c(\mathbb{D})$ and so $\mathbb{D} = \phi_c(\mathbb{D})$.

Similarly $\phi_c(\partial\mathbb{D}) = \partial\mathbb{D}$.

(Problem 3310) We will now establish some further properties of ϕ_c. Show that

$$\phi_c'(z) = \frac{1-|c|^2}{(1-\bar{c}z)^2}.$$

(Problem 3320) If $\theta \in \mathbb{R}$, show that $\phi_c(e^{i\theta}z) = e^{i\theta}\phi_{ce^{-i\theta}}(z)$.

This is a straightforward computation.

(Problem 3330) If c, $w \in \mathbb{D}$, show that $\phi_c \circ \phi_w = e^{i\theta}\phi_b$ for some $b \in \mathbb{D}$ and some $\theta \in \mathbb{R}$.

We compute

$$\begin{aligned}\phi_c(\phi_w(z)) &= \frac{\phi_w(z) - c}{1 - \overline{c}\phi_w(z)}\\ &= \frac{(1-\overline{w}z)\phi_w(z) - c(1-\overline{w}z)}{(1-\overline{w}z) - (1-\overline{w}z)\overline{c}\phi_w(z)}\\ &= \frac{(z-w) - c(1-\overline{w}z)}{(1-\overline{w}z) - (z-w)\overline{c}}\\ &= \frac{z(1+c\overline{w}) - (w+c)}{1 + w\overline{c} - (\overline{w}+\overline{c})z}\\ &= \frac{1+c\overline{w}}{1+w\overline{c}}\frac{z-b}{1-\overline{b}z}\end{aligned}$$

where $b = \frac{c+w}{1+c\overline{w}} = \phi_{-w}(c) \in \mathbb{D}$. Because $\overline{1+c\overline{w}} = 1 + w\overline{c}$, we have that $|\frac{1+c\overline{w}}{1+w\overline{c}}| = 1$ and so $\frac{1+c\overline{w}}{1+w\overline{c}} = e^{i\theta}$ for some $\theta \in \mathbb{R}$.

(Problem 3331) Let $G = \{\phi_c : c \in \mathbb{D}\} \cup \{f : f(z) = ze^{i\theta} \text{ for some } \theta \in \mathbb{R}\}$. Show that G is a group (with function composition as the group action) and is a subgroup of the group of all holomorphic bijections from $\mathbb{D}$ to itself.

5.5. The Schwarz-Pick lemma

(Problem 3340) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. Let $a \in \mathbb{D}$ and let $b = f(a)$. Let $g = \phi_b \circ f \circ \phi_{-a}$. Show that g satisfies the conditions of Schwarz's lemma (Theorem 5.5.1).

By Lemma 3260 we have that $\phi_{-a} : \mathbb{D} \to \mathbb{D}$ and is holomorphic. By assumption $f : \mathbb{D} \to \mathbb{D}$ and is holomorphic. Again by Lemma 3260 we have that $\phi_b : \mathbb{D} \to \mathbb{D}$ and is holomorphic. Thus $g : \mathbb{D} \to \mathbb{D}$, and by Problem 1.49 in your book g is holomorphic.

Finally, $g(0) = \phi_b(f(\phi_{-a}(0))) = \phi_b(f(a)) = \phi_b(b) = 0$.

(Problem 3350) Apply Schwarz's lemma to g to derive an upper bound on $|f'(a)|$.

By Schwarz's lemma $|g'(0)| \leq 1$, and if $|g'(0)| = 1$, then g is a rotation.

By the chain rule (Problem 1.49 in your book), $g'(0) = \phi_b'(f(\phi_{-a}(0))) \cdot f'(\phi_{-a}(0)) \cdot \phi_{-a}'(0)$. So

$$\begin{aligned}1 \geq |g'(0)| &= |\phi_b'(b)||f'(a)||\phi_{-a}'(0)|\\ &= \frac{1}{1-|b|^2}|f'(a)|\frac{1-|a|^2}{1}.\end{aligned}$$

This simplifies to

$$|f'(a)| \leq \frac{1-|f(a)|^2}{1-|a|^2}.$$

(Problem 3351) What can you say about f if $|g'(0)| = 1$?

We observe that $|g'(0)| = 1$ if and only if $|f'(a)| = \frac{1-|f(a)|^2}{1-|a|^2}$.

Thus, if $|f'(a)| = \frac{1-|b|^2}{1-|a|^2}$ then there is a $\theta \in \mathbb{R}$ such that $g(z) = e^{i\theta}z$.

We observe that $f(z) = \phi_{-b}(g(\phi_a(z))) = \phi_{-b}(e^{i\theta}(\phi_a(z)))$. By Problems 3320 and 3330, we have that $f(z) = e^{i\alpha}\phi_c(z)$ for some $c \in \mathbb{D}$ and some $\alpha \in \mathbb{R}$, and so f is a holomorphic bijection from $\mathbb{D}$ to itself.

(Problem 3360) Let $w \in \mathbb{D} \setminus \{a\}$. What does Schwarz's lemma tell you about $f(a)$ and $f(w)$?

Recall $\phi_a(w) \in \mathbb{D}$. By Schwarz's lemma, $|g(\phi_a(w))| \leq |\phi_a(w)|$ and so

$$\left|\frac{f(w)-f(a)}{1-\overline{f(a)}\,f(w)}\right| = \left|\frac{f(w)-b}{1-\overline{b}\,f(w)}\right| = |\phi_b(f(w))| = |\phi_b(f(\phi_{-a}(\phi_a(w))))| = |g(\phi_a(w))|$$
$$\leq |\phi_a(w)| = \left|\frac{w-a}{1-\bar{a}w}\right|.$$

Furtermore, if we have equality then g must be a rotation, and so as in the previous problem there is an $\alpha \in \mathbb{R}$ and a $c \in \mathbb{D}$ such that $f(z) = e^{i\alpha}\phi_c(z)$ for all $z \in \mathbb{D}$.

Theorem 5.5.2. [The Schwarz-Pick lemma.] Let $f : \mathbb{D} \to \mathbb{C}$ be a function such that

- f is holomorphic in $\mathbb{D}$,
- $f(\mathbb{D}) \subset \mathbb{D}$, that is, $|f(z)| < 1$ for all $z \in \mathbb{D}$.

Then we have that

- If $a \in \mathbb{D}$ then $|f'(a)| \leq \frac{1-|f(a)|^2}{1-|a|^2}$.
- If a, $w \in \mathbb{D}$ then $\left|\frac{f(a)-f(w)}{1-\overline{f(a)}f(w)}\right| \leq \left|\frac{a-w}{1-\bar{a}w}\right|$.

If a nontrivial equality holds (either $|f'(a)| = \frac{1-|f(a)|^2}{1-|z|^2}$ for some $a \in \mathbb{D}$, or $\left|\frac{f(a)-f(w)}{1-\overline{f(a)}f(w)}\right| = \left|\frac{a-w}{1-\bar{a}w}\right|$ for some a, $w \in \mathbb{D}$ with $a \neq w$), then there is a $\theta \in \mathbb{R}$ and a $c \in \mathbb{D}$ such that

$$f(z) = e^{i\theta}\frac{z-c}{1-\bar{c}z}$$

for all $z \in \mathbb{D}$.

6.3. Linear Fractional Transformations

Definition 6.3.1. We say that a function $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is a *fractional linear transformation* if there exist numbers a, b, c, $d \in \mathbb{C}$ such that $ad - bc \neq 0$ and such that

$$f(z) = \begin{cases} \frac{az+b}{cz+d}, & z \in \mathbb{C} \text{ and either } c = 0 \text{ or } z \neq -d/c, \\ \infty, & z = -d/c,\ c \neq 0, \\ \infty, & z = \infty,\ c = 0, \\ \frac{a}{c}, & z = \infty,\ c \neq 0. \end{cases}$$

Theorem 6.3.4. Let a, b, c, and $d \in \mathbb{C}$ with $ad - bc \neq 0$. Let f be the fractional linear transformation given in Definition 6.3.1.

Then f is continuous as a function from $\mathbb{C} \cup \{\infty\}$ to itself if we use the metric in Problem 2780.

Furthermore, f is a bijection from $\mathbb{C} \cup \{\infty\}$ to itself and its inverse is a fractional linear transformation.

Finally, if g is another fractional linear transformation then so is $f \circ g$.

(Problem 3361) Begin the proof of Theorem 6.3.4 by showing that f is continuous as a function from $\mathbb{C} \cup \{\infty\}$ to itself if we use the metric in Problem 2780. Furthermore, show that f is the *only* continuous function from $\mathbb{C} \cup \{\infty\}$ to itself that also satisfies $f(z) = \frac{az+b}{cz+d}$ for all $z \in \mathbb{C}$ such that $cz + d \neq 0$.

(Problem 3370) Continue the proof of Theorem 6.3.4 by showing that f is a bijection from $\mathbb{C} \cup \{\infty\}$ to itself and that its inverse is a fractional linear transformation.

Define g by

$$g(z) = \begin{cases} \frac{dz-b}{-cz+a}, & z \in \mathbb{C} \text{ and either } c = 0 \text{ or } z \neq a/c, \\ \infty, & z = a/c,\ c \neq 0, \\ \infty, & z = \infty,\ c = 0, \\ -\frac{d}{c}, & z = \infty,\ c \neq 0. \end{cases}$$

g is a fractional linear transformation because $da - (-b)(-c) = ad - bc \neq 0$ and d, $-b$, $-c$, a are complex numbers.

We claim that $f(g(z)) = z$ and $g(f(z)) = z$ for all $z \in \mathbb{C} \cup \{\infty\}$. This suffices to show that f has an inverse and thus is a bijection from $\mathbb{C} \cup \{\infty\}$ to itself.

We further observe that interchanging a and d and negating b and c interchanges f and g and does not alter the value of $ad - bc$, and so we need only show that $f(g(z)) = z$ for all $z \in \mathbb{C} \cup \{\infty\}$.

We see immediately that if $c = 0$ then $f(g(\infty)) = f(\infty) = \infty$, while if $c \neq 0$ then $f(g(\infty)) = f(-d/c) = \infty$. In either case $f(g(\infty)) = \infty$.

It remains only to show that $f(g(z)) = z$ for all $z \in \mathbb{C}$. Suppose that $z \in \mathbb{C}$.

If $c = 0$ then $a \neq 0 \neq d$. Furthermore $f(z) = (a/d)z + (b/d)$ and $g(z) = (d/a)z - (b/a)$ for all $z \in \mathbb{C}$, and it is straightforward to compute that $f(g(z)) = z$ for all $z \in \mathbb{C}$.

Suppose that $c \neq 0$. Then $f(g(a/c)) = f(\infty) = a/c$.

Now, we claim that if $z \in \mathbb{C} \setminus \{a/c\}$ then $g(z) \in \mathbb{C} \setminus \{-d/c\}$. We have $g(z) \neq \infty$ by definition of g. If $g(z) = -d/c$ then $\frac{dz-b}{-cz+a} = -d/c$ and so $cdz - cb = cdz - ad$ and $ad - bc = 0$, contradicting the definition of fractional linear transformation.

Thus $g(z) \in \mathbb{C} \setminus \{-d/c\}$ and so $f(g(z)) \in \mathbb{C}$. A straightforward computation establishes that $f(g(z)) = z$ for all $z \in \mathbb{C} \setminus \{-d/c\}$, and so $f(g(z)) = z$ for all $z \in \mathbb{C} \cup \{\infty\}$.

(Problem 3380) Complete the proof of Theorem 6.3.4 by showing that the composition of two fractional linear transformations is a fractional linear transformation.

Let f and g be two fractional linear transformations and let a, b, c, d, α, β, γ, δ be the complex numbers such that
$$f(z) = \frac{az+b}{cz+d} \text{ if } z \in \mathbb{C},\ cz+d \neq 0, \quad g(z) = \frac{\alpha z+\beta}{\gamma z+\delta} \text{ if } z \in \mathbb{C},\ \gamma z+\delta \neq 0.$$
Let ζ, η, θ, κ be the complex numbers that satisfy
$$\begin{pmatrix} \zeta & \eta \\ \theta & \kappa \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$
Then
$$\zeta\kappa - \eta\theta = \det \begin{pmatrix} \zeta & \eta \\ \theta & \kappa \end{pmatrix}$$
which we know from linear algebra to be equal to
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$
Because f and g are fractional linear transformations, we know that
$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \neq \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$
and so $\zeta\kappa - \eta\theta \neq 0$. There is thus a fractional linear transformation h such that
$$h(z) = \frac{\zeta z+\eta}{\theta z+\kappa} \text{ if } z \in \mathbb{C},\ \theta z+\kappa \neq 0.$$
We claim that $h = f \circ g$.

To prove this, let $\Omega = \{z \in \mathbb{C} : g(z) \in \mathbb{C} \text{ and } f(g(z)) \in \mathbb{C}\}$. Because f and g are bijections from $\mathbb{C} \cup \{\infty\}$ to itself, we have that $\mathbb{C} \setminus \Omega$ contains at most two points, $g^{-1}(\infty)$ and $g^{-1}(f^{-1}(\infty))$ (and might contain one or zero points if these two points coincide with each other or with ∞). If $z \in \Omega$, then $g(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$ (because $g(z) \neq \infty$) and also $f(g(z)) = \frac{az+b}{cz+d}$ (because $f(g(z)) \neq \infty$). It is then straightforward to compute that $f(g(z)) = h(z)$ for such z.

We are left with the points ∞, $g^{-1}(\infty)$, and $g^{-1}(f^{-1}(\infty)$. But h and $f \circ g$ are continuous in the metric space of Problem 2780, and so if they are equal on a dense set then they must be equal everywhere.

[Definition: Line or circle] Let $S \subset \mathbb{C} \cup \{\infty\}$.

If there are real numbers a, b, and $r > 0$ such that $S = \{x + iy : x \in \mathbb{R},\ y \in \mathbb{R},\ (x-a)^2 + (y-b)^2 = r^2\}$, then we say that S is a circle. (Observe that circles by definition have positive radius.)

If there are real numbers a, b, and c, with a and b not both zero, such that $S = \{\infty\} \cup \{x + iy : x \in \mathbb{R},\ y \in \mathbb{R},\ ax + by = c\}$, then we say that S is a line. (Observe that lines include the point at ∞ and circles do not.)

Theorem 6.3.7. Let f be a fractional linear transformation and let $S \subset \mathbb{C} \cup \{\infty\}$. If S is a circle, then $f(S)$ is either a line or a circle, and if S is a line, then $f(S)$ is either a line or a circle.

(Problem 3390) Suppose that $ad - bc \neq 0$. Show that if $az + b = 0$ then $cz + d \neq 0$.

(Problem 3400) In this problem we begin the proof of Theorem 6.3.7. Specifically, we begin by examining the preimages of the particular circle $\partial\mathbb{D}$ under fractional linear transformations. Let f be a fractional linear transformation. Let $S = \{z \in \mathbb{C} \cup \{\infty\} : |f(z)| = 1\}$. Show that $S \cap \mathbb{C}$ and $\mathbb{C} \setminus S$ both contain infinitely many points.

(Alex, Problem 3410) Let f be a fractional linear transformation and let a, b, c, d be such that $f(z) = \frac{az+b}{cz+d}$ whenever $cz + d \neq 0$. Let $S = \{z \in \mathbb{C} \cup \{\infty\} : |f(z)| = 1\}$. Show that there exist real numbers α and β such that

$$S \cap \mathbb{C} = \{x + iy : x \in \mathbb{R},\ y \in \mathbb{R},\ (|a|^2 - |c|^2)(x^2 + y^2) + \alpha x + \beta y = |d|^2 - |b|^2\}.$$

(Clayton, Problem 3420) If $|a| = |c|$, show that S is a line (recall this means that $\infty \in S$).

(David, Problem 3430) If $|a| \neq |c|$, show that S is a circle (of positive radius).

(Problem 3440) Let $S \subset \mathbb{C} \cup \{\infty\}$ be either a straight line or a circle of positive radius. Show that there is a fractional linear transformation such that $S = \{z \in \mathbb{C} \cup \{\infty\} : |f(z)| = 1\} = f^{-1}(\partial\mathbb{D})$.

(Emily, Problem 3450) Complete the proof of Theorem 6.3.7 by showing that if f is any fractional linear transformation and S is either a line or a circle, then $f(S)$ is also either a line or a circle.

(Irina, Problem 3460) Let $f(z) = \frac{z+1}{iz-i}$. Show that $f(\mathbb{D}) = \mathbb{H}$, where $\mathbb{H} = \{x + iy : x \in \mathbb{R}, y \in (0, \infty)\}$.

(Michael, Problem 3470) Let $f(z) = \frac{z-1}{iz+i}$. Show that $f(\{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}) = \{x+iy : x, y \in (0, \infty)\}$.

6. Holomorphic functions as Geometric Mappings

(Timmy, Problem 3480) Find a holomorphic bijection f from the quarter-plane $\{x + iy : x, y \in (0, \infty)\}$ to the upper half-plane $\mathbb{H}$.

(Zach, Problem 3490) Find a holomorphic bijection from the strip $\{x + iy : x \in \mathbb{R},\ y \in (0, \pi)\}$ to the upper half plane.

(Alex, Problem 3500) Find a holomorphic bijection from $\mathbb{D} \setminus \{0\}$ to $\mathbb{C} \setminus \overline{\mathbb{D}}$.

(Clayton, Problem 3510) Find a holomorphic bijection from $\mathbb{D}$ to the strip $\{x + iy : x \in \mathbb{R},\ y \in (0, \pi)\}$.

(David, Problem 3520) Find a holomorphic bijection from the quarter-circle $\{z \in \mathbb{C} : |z| < 1, \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ to the upper half plane.

6.1. Biholomorphic self-maps of $\mathbb{C}$

[Definition: Biholomorphic self-map] Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \Omega$. We say that f is a biholomorphic self-map if it is holomorphic in Ω and is a bijection from Ω to Ω.

Theorem 6.1.1. Let $f : \mathbb{C} \to \mathbb{C}$. Then f is a biholomorphic self-map if and only if there are complex numbers a, $b \in \mathbb{C}$ with $a \neq 0$ such that that $f(z) = az + b$ for all $z \in \mathbb{C}$.

(Problem 3560) In this problem we prove the more straightforward direction of Theorem 6.1.1. Let a, $b \in \mathbb{C}$ with $a \neq 0$. Show that $f(z) = az + b$ is a biholomorphic self-map of $\mathbb{C}$.

(Emily, Problem 3570) Let $f : \mathbb{C} \to \mathbb{C}$ be a biholomorphic self-map. Show that $\lim_{z\to\infty} |f(z)| = \infty$.

(Irina, Problem 3580) Let $f : \mathbb{C} \to \mathbb{C}$ be a biholomorphic self-map. Given that f has a pole at ∞ and no singularity at 0, what can you say about the Laurent series for f in $\mathbb{C} \setminus \{0\}$?

(Problem 3590) Let $f : \mathbb{C} \to \mathbb{C}$ be a biholomorphic self-map. Show that $f(z) = az + b$ for some $b \in \mathbb{C}$ and some $a \in \mathbb{C} \setminus \{0\}$.

[Chapter 6, Problem 2] Suppose that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and one-to-one. Then f is linear (and, in particular, is also surjective).

(Michael, Problem 3600) Let X and Z be two topological spaces and let $f : X \to Z$ be a continuous bijection with continuous inverse. Let $Y \subseteq X$. Show that $f(\partial Y) = \partial f(Y)$.

Recall that

$$\partial f(Y) = \operatorname{cl} f(Y) \setminus \operatorname{int} f(Y) = \Big(\bigcap_{\substack{f(Y)\subseteq F \\ F \text{ closed}}} F \Big) \setminus \Big(\bigcup_{\substack{f(Y)\supseteq G \\ G \text{ open}}} G \Big).$$

Because f and f^{-1} are continuous, we have that $E \subseteq X$ is open (respectively closed) if and only if $f(E)$ is open (respectively closed). Thus

$$\partial f(Y) = \Big(\bigcap_{\substack{f(Y)\subseteq f(F) \\ F \text{ closed}}} f(F) \Big) \setminus \Big(\bigcup_{\substack{f(Y)\supseteq f(G) \\ G \text{ open}}} f(G) \Big).$$

But $A \subseteq B$ if and only if $f(A) \subseteq f(B)$, and so

$$\partial f(Y) = \Big(\bigcap_{\substack{Y\subseteq F \\ F \text{ closed}}} f(F) \Big) \setminus \Big(\bigcup_{\substack{Y\supseteq G \\ G \text{ open}}} f(G) \Big).$$

Because f is a bijection it commutes with set theoretic operations, and so

$$\partial f(Y) = f\bigg(\Big(\bigcap_{\substack{Y\subseteq F \\ F \text{ closed}}} F \Big) \setminus \Big(\bigcup_{\substack{Y\supseteq G \\ G \text{ open}}} G \Big) \bigg) = f(\partial Y)$$

as desired.

6.2. BIHOLOMORPHIC SELF-MAPS OF $\mathbb{D}$

(Timmy, Problem 3610) Let $f : \mathbb{D} \to \mathbb{D}$ be a biholomorphic self-map. Show that $f(z) = e^{i\theta} \frac{z-c}{1-\bar{c}z}$ for some $c \in \mathbb{D}$ and some $\theta \in \mathbb{R}$. *Hint*: Let $g = f^{-1}$. Compute $|g'(f(0))\, f'(0)|$ and use the Schwarz-Pick lemma.

[Chapter 1, Problem 10] A fractional linear transformation $f(z) = \frac{az+b}{cz+d}$ is a bijection from $\mathbb{H}$ to itself if and only if a, b, c, and d are real numbers and $ad - bc > 0$.

(Zach, Problem 3620) Let $f : \mathbb{H} \to \mathbb{H}$ be a function. Show that f is a biholomorphic self-map if and only if $f(z) = \frac{az+b}{cz+d}$ for some real numbers a, b, c, and d that satisfy $ad - bc > 0$. *Hint*: Most of the work consists in showing that f is a fractional linear transformation.

6.3. BIMEROMORPHIC SELF-MAPS OF $\mathbb{C} \cup \{\infty\}$

(Alex, Problem 3630) We have seen that fractional linear transformations are bimeromorphic self-maps of $\mathbb{C} \cup \{\infty\}$. Conversely, let $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be a meromorphic bijection (with $f(z) = \infty$ if f has a pole at z; because f is a bijection it has a single pole). Show that f is a fractional linear transformation. *Hint*: Does it suffice to prove this in the case that $f(\infty) = \infty$?

First, suppose that $f(\infty) = \infty$. Then $f : \mathbb{C} \to \mathbb{C}$ must be a bijection and so by Theorem 6.1.1 we have that $f(z) = az + b$ is a fractional linear transformation.

Now, suppose that $f(w) = \infty$ for some $w \in \mathbb{C}$. Because f is a bijection there can be at most one such w. Let $g(z) = f(w + \frac{1}{z}) = f(\frac{wz+1}{z})$. Because $h(z) = \frac{wz+1}{z}$) is a fractional linear transformation it is a bimeromorphic self-map of $\mathbb{C} \cup \{\infty\}$. Thus $g = f \circ h$ is as well and $g(\infty) = f(w) = \infty$, and so by the above analysis $g(z) = az + b$ for some $a, b \in \mathbb{C}$. Then $f(z) = g(\frac{1}{z-w})$ is a composition of fractional linear transformations, and so is a fractional linear transformation by Theorem 6.3.4.

(Clayton, Problem 3640) Let $\Omega \subseteq \mathbb{C}$ be open and simply connected. Let $f : \Omega \to \mathbb{C}$ be holomorphic. Show that there exists a holomorphic function $F : \Omega \to \mathbb{C}$ such that $F' = f$.

[Definition: Holomorphically simply connected] A connected open set $\Omega \subset \mathbb{C}$ is holomorphically simply connected if, whenever $f : \Omega \to \mathbb{C}$ is holomorphic, there exists a holomorphic function $F : \Omega \to \mathbb{C}$ with $F' = f$.

(Bonus Problem 3650) Let $\Omega \subseteq \mathbb{C}$ be open and holomorphically simply connected. Show that Ω is simply connected.

Theorem 6.4.2. [The Riemann mapping theorem.] Suppose that $\Omega \subsetneq \mathbb{C}$ is a holomorphically simply connected open set; we emphasize $\Omega \neq \mathbb{C}$. Then there exists a conformal mapping (holomorphic bijection) $f : \Omega \to \mathbb{D}$.

Furthermore, for any $a \in \Omega$, there exists a unique conformal mapping $f : \Omega \to \mathbb{D}$ such that $f(a) = 0$ and such that $f'(a)$ is a positive real number.

(David, Problem 3660) Assume the Riemann mapping theorem is true. Prove that every holomorphically simply connected region is simply connected.

By Problem 3630, if Ω is simply connected then it is holomorphically simply connected. Conversely, let $\Omega \subseteq \mathbb{C}$ be holomorphically simply connected. If $\Omega = \emptyset$ or $\Omega = \mathbb{C}$ then Ω is simply connected, so suppose not.

Then by the Riemann mapping theorem there is a holomorphic (thus continuous) bijection from Ω to $\mathbb{D}$. Because $\mathbb{D}$ is simply connected, we must have that Ω is as well.

(Emily, Problem 3670) We now begin the proof of Theorem 6.4.2. Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is a conformal mapping, that $\varphi(0) = 0$, and that $\varphi'(0) > 0$ (that is, $\varphi'(0)$ is a positive real number). Prove that φ is the identity.

By Problem 3610, there is a $c \in \mathbb{D}$ and a $\theta \in \mathbb{R}$ such that $\varphi(z) = e^{i\theta} \frac{z-c}{1-\bar{c}z}$.

But then $0 = \varphi(0) = e^{i\theta}c$, and so we must have that $c = 0$ and so $\varphi(z) = e^{i\theta}z$. But then $\varphi'(0) = e^{i\theta}$ is a positive real number, and so we must have that $e^{i\theta} = 1$ and so $\varphi(z) = z$ for all $z \in \mathbb{D}$.

(Irina, Problem 3680) Let $\Omega \subset \mathbb{C}$ be a connected open set and let $a \in \Omega$. Prove that there is at most one function $f : \Omega \to \mathbb{D}$ that satisfies the conditions of the Riemman mapping theorem.

Suppose that $f : \Omega \to \mathbb{D}$ and $g : \Omega \to \mathbb{D}$ both satisfy the conditions of the Riemman mapping theorem. Then g^{-1} is a bijection from $\mathbb{D}$ to Ω and is holomorphic by Problem 5.7. Thus $h = f \circ g^{-1}$ is a holomorphic bijection from $\mathbb{D}$ to $\mathbb{D}$.

We observe that $h(0) = f(g^{-1}(0)) = f(a) = 0$ and

$$h'(0) = f'(g^{-1}(0)) \cdot (g^{-1})'(0) = f'(a) \cdot (g^{-1})'(0).$$

Let $i(z) = g(g^{-1}(z))$. Then $i(z) = z$ for all $z \in \mathbb{C}$. By the chain rule,

$$1 = i'(0) = g'(g^{-1}(0)) \cdot (g^{-1})'(0) = g'(a) \cdot (g^{-1})'(0)$$

and so

$$h'(0) = \frac{f'(a)}{g'(a)}$$

is a positive real number.

By the previous problem, we must have that h is the identity and so $f = (g^{-1})^{-1} = g$.

Lemma 6.6.4. If Ω is holomorphically simply connected, f is holomorphic on Ω, and $f \neq 0$ on Ω, then there is some holomorphic function h on Ω such that $e^h = f$.

(Michael, Problem 3690) Prove Lemma 6.6.4.

Because $f(z) \neq 0$ for all $z \in \Omega$, the function $g(z) = \frac{f'(z)}{f(z)}$ is holomorphic in Ω. By definition of holomorphically simply connected, there is a function $H : \Omega \to \mathbb{C}$ such that $H'(z) = g(z) = \frac{f'(z)}{f(z)}$.

Choose some $z_0 \in \Omega$ and let $C \in \mathbb{C}$ be such that $e^{H(z)+C} = f(z_0)$. Because $f(z_0) \neq 0$, some such C must exist. Let $h(z) = H(z) + C$; we still have that $h'(z) = \frac{f'(z)}{f(z)}$.

Now, let $g(z) = f(z)e^{-h(z)}$. Then $g(z_0) = 1$. We compute $g'(z) = f'(z)e^{h(z)} - f(z)e^{h(z)}h'(z) = 0$ because $f(z)\,h'(z) = f'(z)$. Because Ω is connected, we must have that g is a constant, so $g(z) = 1$ for all $z \in \Omega$; thus $f(z) = e^{h(z)}$ for all $z \in \Omega$.

(Timmy, Problem 3700) Let $\Omega = \mathbb{C} \setminus \{x + 0i : x \leq 0\}$ be the complex plane minus a slit. Let $f(z) = z^2$ be holomorphic on Ω.

- Find an explicit formula for a function $h(z)$ such that $e^h = f$ on Ω.
- Compute $f(i)$ and $f(-i)$. Are they equal?
- Compute $h(i)$ and $h(-i)$. Are they equal?
- Is there some branch of log such that $h(z) = \log f(z)$?

If $z \in \Omega$, then there is a unique positive real number r and a unique real number θ with $-\pi < \theta < \pi$ such that $z = re^{i\theta}$. Define $h(z) = h(re^{i\theta}) = 2\ln r + 2i\theta$. It is elementary to check that $e^h = f$.

Then $f(i) = -1 = f(-i)$, but $h(i) = i\pi \neq -i\pi = h(-i)$. For any possible branch of log, $\log f(i) log f(-i)$ because $\log f(i) = \log -1 = \log f(-i)$; thus, we cannot find a branch of log such that $h = \log \circ f$.

Corollary 6.6.5. If Ω is holomorphically simply connected, f is holomorphic on Ω, and $f \neq 0$ on Ω, then there is some holomorphic function k on Ω such that $k^2 = f$.

(Problem 3710) Prove Corollary 6.6.5.

(Zach, Problem 3720) Suppose that $\Omega \subset \mathbb{C}$ is open and that for some $Q \in \mathbb{C}$ and $r > 0$, we have that $\Omega \cap D(Q, r) = \emptyset$. Find a one-to-one holomorphic function $g : \Omega \to \mathbb{D}$. ($g$ need not be a bijection.)

Let $g(z) = \frac{r/2}{z-Q}$.

Since $Q \notin \Omega$ we have that g is well defined on Ω, and by inspection g is one-to-one and holomorphic on $\mathbb{C} \setminus \{Q\}$.

Now, if $w \in \Omega$ then $w \notin D(Q, r)$, and so $|w - Q| \geq r$. Thus $|g(w)| \leq 1/2$ and so $g(\Omega) \subset \mathbb{D}$, as desired.

(Problem 3730) Suppose that Ω is holomorphically simply connected and that $P \notin \Omega$ for some $P \in \mathbb{C}$. Using the function h of Problem 3690, show that there exists a one-to-one holomorphic function $g : \Omega \to \mathbb{D}$. [This problem will be assigned as homework. Note that your book does this exercise using the function k of Problem 3710.]

(Problem 3740) Let $\Omega \subsetneq \mathbb{C}$ be holomorphically simply connected with $a \in \Omega$. Let $\mathcal{F}$ be the set of all functions f such that

- f is holomorphic on Ω,
- f is one-to-one,
- $|f(z)| < 1$ for all $z \in \Omega$, so $f(\Omega) \subseteq \mathbb{D}$,
- $f(a) = 0$,
- $f'(a) > 0$.

To prove the Riemann mapping theorem, what do we need to prove about $\mathcal{F}$?

(Problem 3750) Show that $\mathcal{F}$ is nonempty.

(Alex, Problem 3760) Let $R = \sup\{f'(a) : f \in \mathcal{F}\}$. Show that $0 < R < \infty$.

By the previous problem there is a $g \in \mathcal{F}$. By definition of $\mathcal{F}$ we have that $g'(a) > 0$, and so $R \geq g'(a) > 0$.

We now turn to the proof that R is finite. Because Ω is open, there is a $r > 0$ such that $D(a, r) \subseteq \Omega$.

By the Cauchy estimates (Theorem 3.4.1), if $f \in \mathcal{F}$, then

$$|f'(a)| \leq \frac{2}{r} \sup_{\partial D(a,r/2)} |f|.$$

But $f(\Omega) \subseteq \mathbb{D}$ and so $|f(z)| < 1$ for all $z \in \Omega \supseteq D(a, r)$; thus $|f'(a)| \leq 2/r$ for all $f \in \mathcal{F}$ and so $R \leq 2/r$.

Claim 6.7.3. If $f \in \mathcal{F}$ and $f'(a) = R$, then $f : \Omega \to \mathbb{D}$ is surjective.

(Clayton, Problem 3770) Let $0 < r < 1$. Let $W = \mathbb{D} \setminus (-1, -r] = \{x + iy : x^2 + y^2 < 1 \text{ and either } y \neq 0 \text{ or } x > -r\}$ be the unit disc with a slit removed; then $0 \in W \subsetneq \mathbb{D}$. Let

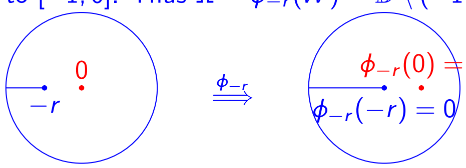
$$\psi(z) = \phi_{\sqrt{r}}(\sqrt{\phi_{-r}(z)})$$

where ϕ_c is as in Lemma 3260 and where $\sqrt{re^{i\theta}} = \sqrt{r}e^{i\theta/2}$ if $0 < r < \infty$ and $-\pi < \theta < \pi$. Show that

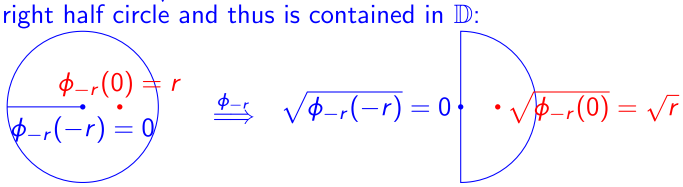
- $\psi(W) \subseteq \mathbb{D}$,
- ψ is holomorphic on W,
- ψ is one-to-one on W,
- $\psi(0) = 0$,
- $\psi'(0) > 1$ (in particular is real).

Note: You must prove $\psi'(0) > 1$; this is much harder than proving $\psi'(0) > 0$!

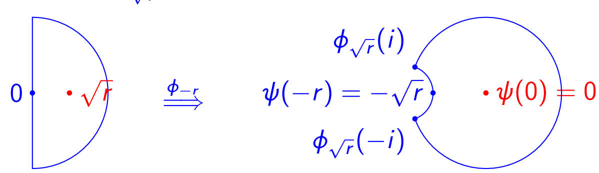
Recall that ϕ_{-r} is a bijection from $\mathbb{D}$ to itself. We compute that $\phi_{-r}(r) = 0$, $\phi_{-r}(-1) = -1$ $\phi_{-r}(1) = 1$, and $\phi_{-r}(0) = r$. Thus by Theorem 6.3.4 and Theorem 6.3.7, ϕ_{-r} is a bijection from $\mathbb{R} \cup \{\infty\}$ to itself; furthermore it is continuous in the metric of Problem 2780, and so we must have that ϕ_{-r} maps $[-1, -r]$ to $[-1, 0]$. Thus $\Omega = \phi_{-r}(W) = \mathbb{D} \setminus (-1, 0]$.



By definition of $\sqrt{\ }$, we have that $\sqrt{z}$ exists for all $z \in \Omega$ and is holomorphic in Ω by Problem 2660. Observe that $\sqrt{\ }$ is also one-to-one. Thus $g(z) = \sqrt{\phi_{-r}(z)}$ is holomorphic in W. Furthermore, $g(w)$ is the right half circle and thus is contained in $\mathbb{D}$:



Finally $\phi_{\sqrt{r}}$ is holomorphic and injective in $\mathbb{D}$, and so $\psi = \phi_{\sqrt{r}} \circ g$ is holomorphic and injective in W.



We compute $\varphi_{-r}(0) = r$, $\sqrt{\varphi_{-r}(0)} = \sqrt{r}$, and finally $\varphi_{\sqrt{r}}(\sqrt{r} = 0$, so $\psi(0) = 0$.
Finally, by the chain rule, Problem 2660, and Problem 3310,

$$\begin{aligned}
\varphi'(0) &= \varphi'_{\sqrt{r}}(\sqrt{r}) \cdot \frac{1}{2\sqrt{r}} \cdot \varphi'_{-r}(0) \\
&= \frac{1-r}{(1-r)^2} \frac{1}{2\sqrt{r}}(1-r^2) \\
&= \frac{(1+r)/2}{\sqrt{1 \cdot r}}.
\end{aligned}$$

Because r and 1 are unequal positive real numbers, by the arithmetic-geometric mean inequality this quantity is greater than 1.

(David, Problem 3780) Let $W \subsetneq \mathbb{D}$ be open and holomorphically simply connected with $0 \in W$. Show that there exists a ψ such that

- ψ is holomorphic on W,
- ψ is one-to-one on W,

- $\psi(W) \subseteq \mathbb{D}$,
- $\psi(0) = 0$,
- $\psi'(0) > 1$ (in particular is real).

Hint: The construction will be similar to the previous problem with Corollary 6.6.5 in place of the explicit square root function. *Note*: You must prove $\psi'(0) > 1$; this is much harder than proving $\psi'(0) > 0$!

Let $P \in \mathbb{D} \setminus W$. By assumption some such P exists. Thus $\phi_P : W \to \mathbb{D}$ is holomorphic, and because $\phi_P(P) = 0$, $P \in \mathbb{D} \setminus W$, and ϕ_P is a bijection from $\mathbb{D}$ to itself, we have that $\phi_P \neq 0$ in W. Thus by Corollary 6.6.5 there is a holomorphic function $g : W \to \mathbb{C}$ such that $g(z)^2 = \phi_P(z)$ for all $z \in W$. It is left as an exercise to the student that $g(W) \subseteq \mathbb{D}$. Let $\psi = e^{i\theta}\phi_b(g(z))$ where $b = g(0)$ and where $\theta \in \mathbb{R}$. It is left as an exercise to the student to show that ψ satisfies the desired conditions.

(Emily, Problem 3790) Let $\Omega \subseteq \mathbb{C}$ be open and holomorphically simply connected and let $f : \Omega \to \mathbb{C}$ be a holomorphic injection. Show that $f(\Omega)$ is also open and holomorphically simply connected.

We begin by showing that $f(\Omega)$ is open. Let $z \in f(\Omega)$. Then there is some $P \in \Omega$ such that $f(P) = z$. Because Ω is open, there is an $r > 0$ such that $D(P, r) \subseteq \Omega$ and so $f(D(P, r)) \subseteq f(\Omega)$. By the Open Mapping Theorem (Theorem 5.2.1, $f(D(P, r))$ is either a single point or open; because f is injective we have that $f(D(P, r))$ must be open. Thus for every $z \in f(\Omega)$ there is an open set U with $z \in U \subseteq f(\Omega)$; thus $f(\Omega)$ must be open.

Now, let $g : f(\Omega) \to \mathbb{C}$ be a holomorphic function. Then $g \circ f$ is a holomorphic function from Ω to $\mathbb{C}$ and so is f', so their product $h(z) = g(f(z)) \cdot f'(z)$ is also holomorphic. Because Ω is holomorphically simply connected, we have that there is some $H : \Omega \to \mathbb{C}$ holomorphic such that $H'(z) = h(z)$.

Recall from Problem 5.7 that f^{-1} is a holomorphic function $f^{-1} : f(\Omega) \to \Omega$. Let $G(z) = H(f^{-1}(z))$. By the chain rule, $1 = f'(f^{-1}(z)) \cdot (f^{-1})'(z)$ and $G'(z) = H'(f^{-1}(z)) \cdot (f^{-1})'(z) = g(f(f^{-1}(z))) \cdot f'(f^{-1}(z)) \cdot \frac{1}{f'(f^{-1}(z))} = g(z)$, as desired.

(While the function $g \circ f$ indeed has a holomorphic antiderivative, it is very difficult to derive a holomorphic antiderivative for g alone from the holomorphic antiderivative for $g \circ f$.)

(Irina, Problem 3800) Prove Claim 6.7.3 by showing that if $f \in \mathcal{F}$ is not surjective, then there is a $g \in \mathcal{F}$ such that $f'(a) < g'(a)$.

6.5. Real Analysis

(Problem 3810) State the Bolzano-Weierstraß theorem in $\mathbb{R}^p$. What does this tell you about bounded sequences in $\mathbb{C}$?

(Problem 3820) Show that every sequence in a compact set has a convergent subsequence.

[Definition: Equicontinuous] Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions from (X, d) to (Y, ϱ), where (X, d) and (Y, ϱ) are two metric spaces. Suppose that for each $x \in X$ and each $\varepsilon > 0$ there is a $\delta = \delta_{\varepsilon,x} > 0$ such that if $y \in X$ with $d(x, y) < \delta_{\varepsilon,x}$ then $\sup_{n\in\mathbb{N}} \varrho(f_n(x), f_n(y)) < \varepsilon$ (that is, $\varrho(f_n(x), f_n(y)) < \varepsilon$ for all $n \in \mathbb{N}$, and $\delta_{\varepsilon,x}$ cannot depend on n.) Then we say that the sequence $\{f_n\}_{n=1}^\infty$ is *equicontinuous*.

If $\delta = \delta_\varepsilon$ may be taken to be independent of x, then the sequence is *uniformly equicontinuous*.

(Bonus Problem 3821) Suppose that $\{f_n\}_{n=1}^\infty$ is an equicontinuous sequence of functions and that their common domain (X, d) is compact. Show that $\{f_n\}_{n=1}^\infty$ is uniformly equicontinuous.

(Problem 3822) Give an example of a bounded sequence in a complete metric space that does not have a convergent subsequence.

Definition 6.5.1. If $\{f_j\}_{j=1}^{\infty}$ is a sequence of functions from an open set $\Omega \subseteq \mathbb{C}$ to $\mathbb{C}$, we say that $\{f_j\}_{j=1}^{\infty}$ converges normally to f if $\{f_j\}_{j=1}^{\infty}$ converges to f uniformly on compact subsets K of Ω.

If $\mathcal{F}$ is a family of functions such that $f : \Omega \to \mathbb{C}$ for each $f \in \mathcal{F}$, and if every sequence in $\mathcal{F}$ has a subsequence that converges normally, we say that $\mathcal{F}$ is a normal family.

Recall [Problem 1900]: If each f_j is holomorphic and $f_j \to f$ normally then f is holomorphic.

Recall [Problem 1920]: If each f_j is holomorphic and $f_j \to f$ normally then $f_j' \to f'$ normally.

Theorem 6.5.3. [Montel's theorem, first version.] Suppose that $\mathcal{F}$ is a family of functions that are holomorphic on some open set Ω. Suppose that there is a constant $M > 0$ such that, if $f \in \mathcal{F}$ and $z \in \Omega$, then $|f(z)| \leq M$. Then $\mathcal{F}$ is a normal family.

The Arzelà-Ascoli Theorem. Let (Ψ, d) and (Y, ρ) be compact metric spaces.[3] Let $\{f_n\}_{n=1}^{\infty}$ be an equicontinuous sequence of functions from X to Y. Then there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ that converges uniformly on Ψ.

(Michael, Problem 3830) In this problem we begin the proof of the Arzelà-Ascoli theorem. This problem is a strengthening of the known fact that compact sets are separable. Suppose that Ψ is compact and that for each positive number $\varepsilon > 0$ and each $z \in \Psi$ we are given a positive number $\delta_{\varepsilon,z} > 0$. Show that there exists a sequence $\{z_m\}_{m=1}^{\infty} \subset \Psi$ such that, for each $\varepsilon > 0$, there is a $M \in \mathbb{N}$ such that $\Psi \subseteq \bigcup_{m=1}^{M} B(z_m, \min(\varepsilon, \delta_{\varepsilon,z_m}))$.

For each $k \in \mathbb{N}$, the set $\{B(z, \min(2^{-k}, \delta_{2^{-k},z}) : z \in \Psi\}$ is an open cover of Ψ. Thus it has a finite subcover. Let $\{z_{k,j} : 1 \leq j \leq N_k\}$ be a set of finitely many points such that $\Psi = \bigcup_{j=1}^{N_k} B(z_{k,j}, \min(2^{-k}, \delta_{2^{-k},z}))$.

We may then define z_m such that $z_m = z_{1,m}$ if $1 \leq m \leq N_1$, $z_m = z_{2,m-N_1}$ if $N_1 < m \leq N_2$, and in general $z_m = z_{k,m-N_1-\dots-N_{k-1}}$ for k the unique number such that $N_1 + \dots + N_{k-1} < m \leq N_1 + \dots + N_k$.

If $\varepsilon > 0$, there is then a k such that $2^{-k} < \varepsilon$; choosing $M = N_1 + \dots + N_k$ completes the proof.

(Timmy, Problem 3840) Let Ψ, Y, and f_n be as in the Arzelà-Ascoli theorem. Let $\{z_m\}_{m=1}^{\infty} \subseteq \Psi$ be a sequence. Show that there is one subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ (with n_k independent of m) such that $\{f_{n_k}(z_m)\}_{k=1}^{\infty}$ is a convergent sequence for each $m \in \mathbb{N}$.

Because $\{f_n(z_1) : n \in \mathbb{N}\}$ is bounded and (closed bounded subsets of) Y are compact, we have that some subsequence $\{f_{n_{1,k}}(z_1)\}_{k=1}^{\infty}$ is convergent.

Now, suppose that the strictly increasing sequence of natural numbers $\{n_{j,k}\}_{k=1}^{\infty}$ has been defined and that $\{f_{n_{j,k}}(z_m)\}_{k=1}^{\infty}$ converges for all $m \leq j$. Then the sequence $\{f_{n_{j,k}}(z_{j+1})\}_{k=1}^{\infty}$ is bounded, and so some subsequence converges. Define $\{n_{j+1,k}\}_{k=1}^{\infty}$ to be a subsequence such that $\{f_{n_{j+1,k}}(z_{j+1})\}_{k=1}^{\infty}$ converges; because $\{n_{j+1,k}\}_{k=1}^{\infty}$ is a subsequence of $\{n_{j,k}\}_{k=1}^{\infty}$, we have that $\{f_{n_{j,k}}(z_m)\}_{k=1}^{\infty}$ converges for all $m \leq j$ and thus all $m \leq j+1$.

Define n_ℓ by $n_\ell = n_{\ell,\ell}$. If $m \in \mathbb{N}$, then $\{n_\ell\}_{\ell=m}^{\infty}$ is a subsequence of $\{n_{m,k}\}_{k=1}^{\infty}$, and therefore we must have that $\{f_{n_\ell}(z_m)\}_{\ell=1}^{\infty}$ converges.

(Zach, Problem 3850) Suppose in addition that z_m is as in Problem 3830. Show that $\{f_{n_k}\}_{k=1}^{\infty}$ is uniformly convergent on $\{z_m : m \in \mathbb{N}\}$.

Pick $\varepsilon > 0$.

Let $M \in \mathbb{N}$ be such that $\Psi = \bigcup_{m=1}^{M} B(z_m, \min(\varepsilon, \delta_{\varepsilon,z_m}))$; by Problem 3830, such an M exists.

Now, for each m, the sequence $\{f_{n_k}(z_m)\}_{k=1}^{\infty}$ converges to some element of Y; let us call this element $f(z_m)$. By the definition of convergence, there is a N_m such that if $k \geq N_m$ then $\rho(f_{n_k}(z_m), f(z_m)) < \varepsilon$. Because M is finite, so is $N = \max\{N_1, \dots, N_M\}$.

Pick some $k \geq N$ and some $m \in \mathbb{N}$. Then $z_m \in B(z_\mu, \min(\varepsilon, \delta_{\varepsilon,z_\mu}))$ for some $\mu \leq M$ by definition of M. Because $\{f_{n_k}(z_m)\}_{k=1}^{\infty}$ converges, there is some $\lambda \in \mathbb{N}$ (possibly much bigger than k) with $\lambda \geq N$ such that $\rho(f_{n_\lambda}(z_m), f(z_m)) < \varepsilon$.

[3] We may weaken the condition that (Y, ρ) be compact to the condition that closed and bounded subsets of (Y, ρ) are compact if we in addition impose the condition that if $z \in \Psi$, then $\{f_n(z) : n \in \mathbb{N}\}$ is bounded.

We may then compute that

$$\begin{aligned}\rho(f_{n_k}(z_m), f(z_m)) &\leq \rho(f_{n_k}(z_m), f_{n_k}(z_\mu)) \\ &\quad + \rho(f_{n_k}(z_\mu), f(z_\mu)) \\ &\quad + \rho(f(z_\mu), f_{n_\lambda}(z_\mu)) \\ &\quad + \rho(f_{n_\lambda}(z_\mu), f_{n_\lambda}(z_m)) \\ &\quad + \rho(f_{n_\lambda}(z_m), f(z_m)) \\ &< 5\varepsilon.\end{aligned}$$

Thus, for any $\varepsilon > 0$ there is a $N \in \mathbb{N}$ (depending on ε alone) such that if $k \geq N$ then $\rho(f_{n_k}(z_m), f(z_m)) < 5\varepsilon$ for all $m \in \mathbb{N}$; thus $f_{n_k} \to f$ uniformly on $\{z_m : m \in \mathbb{N}\}$.

(This is somewhat simpler if you show that the sequence $\{f_{n_k}\}_{k=1}^{\infty}$ is uniformly Cauchy on $\{z_m : m \in \mathbb{N}\}$ and then invoke the known fact from real analysis that a uniformly Cauchy sequence of functions is uniformly convergent.)

(Alex, Problem 3860) Prove the Arzelà-Ascoli theorem by showing that $\{f_{n_k}\}_{k=1}^{\infty}$ is uniformly convergent on Ψ.

Let $\varepsilon > 0$.

Because $\{f_{n_k}\}_{k=1}^{\infty}$ converges uniformly on $\{z_m : m \in \mathbb{N}\}$, it is uniformly Cauchy. Let $K \in \mathbb{N}$ be such that if $j, k \geq K$ then $\rho(f_{n_k}(z_m), f_{n_j}(z_m)) < \varepsilon$ for all $m \in \mathbb{N}$.

Let M be such that $\Psi = \bigcup_{m=1}^{M} B(z_m, \min(\varepsilon, \delta_{\varepsilon,z_m}))$; by Problem 3830, such an M exists.

If $z \in \Psi$, then $z \in B(z_m, \delta_{\varepsilon,z_m})$ for some m. Then

$$\begin{aligned}\rho(f_{n_k}(z), f_{n_j}(z)) &\leq \rho(f_{n_k}(z), f_{n_k}(z_m)) \\ &\quad + \rho(f_{n_k}(z_m), f_{n_j}(z_m)) \\ &\quad + \rho(f_{n_j}(z_m), f_{n_j}(z))\end{aligned}$$

by the triangle inequality. The first and third terms are less than ε because $d(z, z_m) < \delta_{\varepsilon,z_m}$, and the third term is less than ε by definition of K; thus,

$$\rho(f_{n_k}(z), f_{n_j}(z)) < 3\varepsilon$$

for all $k, j \geq K$ and all $z \in \Psi$. Thus $\{f_{n_k}\}_{k=1}^{\infty}$ is uniformly Cauchy, and thus uniformly convergent, on Ψ.

(Clayton, Problem 3870) In this problem we begin the proof of Montel's theorem. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of holomorphic functions defined on an open set $\Omega \subset \mathbb{C}$. Suppose that there is a $M \in \mathbb{R}$ such that $|f_j(z)| \leq M$ for all $z \in \Omega$ and $j \in \mathbb{N}$. Let $\Psi \subset \Omega$ be compact. Show that the functions f_j satisfy the conditions of the Arzelà-Ascoli theorem on Ψ.

(Problem 3880) Show that if $\Omega \subset \mathbb{C}$ is open, then there exist compact sets $\Psi_1 \subset \Psi_2 \subset \Psi_3 \subset \dots$ such that $\cup_m \Psi_m = \Omega$ and such that every compact set $K \subset \Omega$ is contained in Ψ_m for some $m \in \mathbb{N}$. (This property is called *σ-compactness*.)

(Problem 3890) Prove Montel's theorem, first version.

6.7. The Proof of the Analytic Form of the Riemann Mapping Theorem

Recall [Problem 3740]: Let $\Omega \subsetneq \mathbb{C}$ be holomorphically simply connected with $a \in \Omega$. Let $\mathcal{F}$ be the set of all functions f such that

- f is holomorphic on Ω,
- f is one-to-one,
- $|f(z)| < 1$ for all $z \in \Omega$, so $f(\Omega) \subseteq \mathbb{D}$,
- $f(a) = 0$,
- $f'(a) > 0$.

Recall [Problem 3750]: $\mathcal{F}$ is nonempty.

Recall [Problem 3760]: $0 < \sup\{g'(a) : g \in \mathcal{F}\} < \infty$.

Recall [Problem 3800]: (Claim 6.7.3.) If $f \in \mathcal{F}$ and $f'(a) = \sup\{g'(a) : g \in \mathcal{F}\}$, then $f : \Omega \to \mathbb{D}$ is onto.

(David, Problem 3900) Prove that there is a function $f \in \mathcal{F}$ with $f'(a) = \sup\{g'(a) : g \in \mathcal{F}\}$. (This proves the Riemann mapping theorem.)

By definition of supremum, if $n \in \mathbb{N}$, then there is a $f_n \in \mathcal{F}$ such that $f_n'(a) > \sup\{g'(a) : g \in \mathcal{F}\} - \frac{1}{n}$. Observe that $\mathcal{F}$ is a family of holomorphic functions with a common domain Ω whose range is contained in $\mathbb{D}$; thus if $f \in \mathcal{F}$ then $\sup_\Omega |f| \leq 1$. Thus, by Montel's theorem, $\mathcal{F}$ is a normal family. Thus there is subsequence $\{f_{n_k}\}_{k=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ which converges normally. Let $f = \lim_{k\to\infty} f_{n_k}$.

We claim that $f \in \mathcal{F}$ with $f'(a) = \sup\{g'(a) : g \in \mathcal{F}\}$.

By Theorem 3.5.1, f is holomorphic on Ω. By definition of limit $f(a) = \lim_{k\to\infty} f_{n_k}(a) = 0$. By Corollary 3.5.2, $f'(a) = \lim_{k\to\infty} f_{n_k}'(a) = \sup\{g'(a) : g \in \mathcal{F}\}$. In particular $f'(a) > 0$, and so f is not a constant.

If $z \in \Omega$, then $|f(z)| = \lim_{k\to\infty} |f_{n_k}(z)| \leq 1$ because $f_{n_k}(\Omega) \subseteq \mathbb{D}$. By the maximum modulus principle Theorem 5.4.2, we must have that $|f(z)| < 1$ for all $z \in \Omega$, and thus $f(\Omega) \subseteq \mathbb{D}$.

It remains to show that f is injective. Let $w \in \Omega$. Let $h_k(z) = f_{n_k}(z) - f_{n_k}(w)$. Let $h(z) = f(z) - f(w)$. We view the domain of h_k, h as $\Omega \setminus \{w\}$ rather than all of Ω.

Then each h_k is holomorphic on $\Omega \setminus \{w\}$. Furthermore, $h_k \to h$, uniformly on compact subsets of Ω (and thus of $\Omega \setminus \{w\}$). Finally, because f_{n_k} is one-to-one, we have that $h_k(z) \neq 0$ for all $z \in \Omega \setminus \{w\}$.

By Hurwitz's theorem (Theorem 5.3.3), we have that either $h(z) = 0$ for all $z \in \Omega \setminus \{w\}$ or $h(z) \neq 0$ for all $z \in \Omega \setminus \{w\}$. But $h'(a) = f'(a) > 0$, so h cannot be constant and thus we must have that $0 \neq h(z) = f(z) - f(w)$ and so $f(z) \neq f(w)$ for all $z \in \Omega \setminus \{w\}$. Because w was arbitrary, this implies that f is one-to-one. This completes the proof.

7.1. Basic Properties of Harmonic Functions

Definition 7.1.1. Harmonic function. We say that u is harmonic in a domain $\Omega \subseteq \mathbb{C}$ if u is C^2 in Ω and if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in Ω.

(Emily, Problem 3910) Let $\Omega \subseteq \mathbb{C}$ be open and let $u \in C^2(\Omega)$. Show that u is harmonic if and only if $\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial \bar z}\right) = 0$ and also that u is harmonic if and only if $\frac{\partial}{\partial \bar z}\left(\frac{\partial u}{\partial z}\right) = 0$.

Let $\Omega \subseteq \mathbb{C}$ be open and let $u \in C^2(\Omega)$.
Recall that $\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial \bar z} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}$.
Then

$$\begin{aligned}
\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar z}u\right) &= \frac{\partial}{\partial z}\left(\frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial u}{\partial y}\right) \\
&= \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y}\right) + \frac{1}{2i}\frac{\partial}{\partial y}\left(\frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial u}{\partial y}\right) \\
&= \frac{1}{4}\frac{\partial^2 u}{\partial x^2} + \frac{1}{4}\frac{\partial^2 u}{\partial y^2} - \frac{1}{4i}\frac{\partial^2 u}{\partial x \partial y} + \frac{1}{4i}\frac{\partial^2 u}{\partial y \partial x}.
\end{aligned}$$

By equality of mixed partials the final two terms cancel and we see that u is harmonic if and only if $\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial \bar z}\right) = 0$. Because $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar z}$ commute, we see that u is harmonic if and only if $\frac{\partial}{\partial \bar z}\left(\frac{\partial u}{\partial z}\right) = 0$.

(Problem 3911) Suppose that $\Omega \subseteq \mathbb{C}$ is open and that u, $v : \Omega \to \mathbb{C}$ are both harmonic. Let α, $\beta \in \mathbb{C}$. Show that $\alpha u + \beta v$ is also harmonic.

[Chapter 7, Problem 12] If u is a real-valued harmonic function on a connected open set, and if u^2 is also harmonic, then u is a constant.

[Chapter 7, Problem 13] If u is a complex-valued harmonic function on a connected open set Ω, and if u^2 is also harmonic, then u is either holomorphic or conjugate-holomorphic (meaning that either u or $\overline{u}$ is holomorphic, or, equivalently, either $\frac{\partial u}{\partial z} \equiv 0$ or $\frac{\partial u}{\partial \bar z} \equiv 0$ in Ω).

(Irina, Problem 3920) Prove that if F is holomorphic in an open set Ω and $u = \operatorname{Re} F$ then u is harmonic.

Because $F' = \frac{\partial}{\partial z} F$ is also holomorphic, we have that $0 = \frac{\partial}{\partial \bar{z}} F' = \frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z} F = \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} F$. Thus F is harmonic. So

$$0 = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \left(\frac{\partial^2 \operatorname{Re} F}{\partial x^2} + \frac{\partial^2 \operatorname{Re} F}{\partial y^2}\right) + i\left(\frac{\partial^2 \operatorname{Im} F}{\partial x^2} + \frac{\partial^2 \operatorname{Im} F}{\partial y^2}\right).$$

Because the two quantities $\left(\frac{\partial^2 \operatorname{Re} F}{\partial x^2} + \frac{\partial^2 \operatorname{Re} F}{\partial y^2}\right)$ and $\left(\frac{\partial^2 \operatorname{Im} F}{\partial x^2} + \frac{\partial^2 \operatorname{Im} F}{\partial y^2}\right)$ are both real, they must both be zero and $\operatorname{Re} F$ and $\operatorname{Im} F$ are both harmonic.

Lemma 7.1.4. If u is real-valued and harmonic in a simply connected open set Ω, then there is a holomorphic function f such that $\operatorname{Re} f = u$.

(Michael, Problem 3930) Prove Lemma 7.1.4.

Corollary 7.1.3. If $\Omega \subseteq \mathbb{C}$ is open and $u : \Omega \to \mathbb{R}$ is harmonic, then u is smooth.

(Timmy, Problem 3940) Prove Corollary 7.1.3.

Let u be harmonic in an open set Ω. Let $D(P, r) \subseteq \Omega$ for some $P \in \mathbb{C}$ and some $r > 0$. Then by Problem 3930, there is some function f holomorphic in $D(P, r)$ such that $u = \operatorname{Re} f$ in $D(P, r)$.

By Theorem 3.1.1, we have that f is smooth (infinitely differentiable) in $D(P, r)$. Therefore, $u = \operatorname{Re} f$ is infinitely differentiable in $D(P, r)$. Since this is true in every disc contained in Ω, u is smooth throughout Ω.

[Definition: Harmonic conjugate] Let u and v be two real-valued functions. If $F = u + iv$ is holomorphic, then we say that v is a harmonic conjugate of u.

[Chapter 7, Problem 4] Suppose that v_1 and v_2 are both conjugates of the (real harmonic) function u. What can you say about v_1 and v_2?

We can say that $v_1 - v_2$ is locally constant, that is, constant on any connected component of the domain of u.

(Problem 3941) Let u be a harmonic function. Suppose that v is a harmonic conjugate of u. Is u also a harmonic conjugate of v?

No, but $-u$ is a harmonic conjugate of v.

7.2. The Maximum Principle

Theorem 7.2.1. (The maximum principle for harmonic functions.) If $\Omega \subseteq \mathbb{C}$ is open and connected, if $u : \Omega \to \mathbb{R}$ is harmonic, and if there is a $P \in \Omega$ such that $u(P) \geq u(z)$ for all $z \in \Omega$, then u is constant in Ω.

(Zach, Problem 3950) Prove the maximum principle for harmonic functions.

Let $M = u(P)$ and let $E = u^{-1}(\{M\}) = \{z \in \Omega : u(z) = M\}$. Then $E \subseteq \Omega$ is not empty because $P \in E$. The set $\{M\}$ is closed, so E must also be (relatively) closed in Ω because u is continuous. Furthermore, $P \in E$ and so E is nonempty.

Let $Q \in E$. Because Ω is open, there is an $r > 0$ such that $D(Q, r) \subseteq \Omega$. By Problem 3930, there is a $f : D(Q, r) \to \mathbb{C}$ that is holomorphic in Ω such that $u = \operatorname{Re} f$ in $D(Q, r)$. Let $h = e^f$. Then $|h| = e^u$, and because the exponential function is strictly increasing on the reals and $u(Q) = u(P) \geq u(z)$ for all $z \in D(Q, r)$, we have that $|h|$ attains a maximum at Q. Thus h must be constant on $D(Q, r)$. But $u = \ln |h|$, and so u is also constant in $D(Q, r)$. Since $Q \in E$, we have $u(Q) = M$ and so $u(z) = M$ for all $z \in D(Q, r)$; thus $D(Q, r) \subseteq E$. This is true for all $Q \in E$, and so E is open.

Because Ω is connected and $E \subseteq \Omega$ is nonempty, open, and relatively closed, we must have that $E = \Omega$ and so u is constant in Ω.

(Alex, Problem 3960) State and prove the minimum principle and corollaries involving the values of u on $\partial\Omega$.

The minimum principle is as follows:

If $\Omega \subseteq \mathbb{C}$ is open and connected, if $u : \Omega \to \mathbb{R}$ is harmonic, and if there is a $P \in \Omega$ such that $u(P) \leq u(z)$ for all $z \in \Omega$, then u is constant in Ω.

It may be proven by noting that, if u is harmonic, then so is $v = -u$, and if $u(P) \leq u(z)$ then $v(P) \geq v(z)$.

Here is the corollary:

Let $\Omega \subset \mathbb{C}$ be open and bounded. Let $u : \overline{\Omega} \to \mathbb{R}$ be a continuous function such that u is harmonic in Ω. Then there are points z_0 and z_1 in $\partial\Omega$ such that $u(z_0) \leq u(z) \leq u(z_1)$ for all $z \in \overline{\Omega}$. Furthermore, if Ω is connected then either $u(z_0) = u(z_1) = u(z)$ for all $z \in \overline{\Omega}$ or $u(z_0) < u(z) < u(z_1)$ for all $z \in \Omega$.

It may be proven by recalling the well known fact that a continuous function on a compact set (in particular, on any closed and bounded subset of $\mathbb{C}$) attains its maximum and minimum.

7.2. The Mean Value Property

Theorem 7.2.5. (The mean value property.) If u is harmonic in a neighborhood of $\overline{D}(P, r)$, then

$$u(P) = \frac{1}{2\pi} \int_0^{2\pi} u(P + re^{i\theta})\, d\theta.$$

(Clayton, Problem 3970) Prove the mean value property.

Let $\Omega \supset \overline{D}(P, r)$ be the indicated neighborhood. By a standard real analysis argument, there is a $R > r$ such that $\overline{D}(P, r) \subset D(P, R) \subseteq \Omega$.

Because $D(P, R)$ is simply connected and u is harmonic in $\Omega \supseteq D(P, R)$, by Lemma 7.1.4 there is a holomorphic function $f : D(P, R) \to \mathbb{C}$ such that $u = \operatorname{Re} f$ in $D(P, R)$. By the Cauchy integral formula,

$$f(P) = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f(\zeta)}{\zeta - P}\, d\zeta.$$

We use the standard parameterization $\gamma(t) = P + re^{it}$ and see that, by the definition of line integral,

$$\begin{aligned} f(P) &= \frac{1}{2\pi i} \oint_\gamma \frac{f(\zeta)}{\zeta - P}\, d\zeta = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma(t))}{\gamma(t) - P} \gamma'(t)\, dt \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(P + re^{it})}{re^{it}}\, ire^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{it})\, dt. \end{aligned}$$

But by definition of the integral of a complex function over a real interval,

$$\begin{aligned} u(P) = \operatorname{Re} f(P) &= \operatorname{Re} \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{it})\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} f(P + re^{it})\, dt = \frac{1}{2\pi} \int_0^{2\pi} u(P + re^{it})\, dt. \end{aligned}$$

(David, Problem 3980) Suppose that u is harmonic in $D(P, r)$ and continuous on $\overline{D}(P, r)$ (without necessarily being harmonic in any larger set). Is the mean value property still valid?

Yes. Let $0 < \rho < r$. By the previous problem,

$$u(P) = \frac{1}{2\pi} \int_0^{2\pi} u(P + \rho e^{i\theta})\, d\theta.$$

Thus,

$$u(P) = \frac{1}{2\pi} \lim_{\rho \to r^-} \int_0^{2\pi} u(P + \rho e^{i\theta})\, d\theta.$$

Because u is continuous on the compact set $\overline{D}(P, r)$, it is uniformly continuous. Choose $\varepsilon > 0$. There is a $\delta > 0$ such that if $z, w \in \overline{D}(P, r)$ then $|u(z) - u(w)| < \varepsilon$.

In particular, if $\rho > r - \delta$ then $|re^{i\theta} - \rho e^{i\theta}| = r - \rho < \delta$ and so $|u(re^{i\theta}) - u(\rho e^{i\theta})| < \varepsilon$.

If we let $u_\rho(\theta) = u(P + \rho e^{i\theta})$, then $u_\rho \to u_r$ uniformly as $\rho \to r$ from the left, and so by Problem 722, we have that
$$\int_0^{2\pi} u_r(\theta)\,d\theta = \lim_{\rho\to r^-} \int_0^{2\pi} u_\rho(\theta)\,d\theta$$
and so
$$\frac{1}{2\pi}\int_0^{2\pi} u(P + re^{i\theta})\,d\theta = \lim_{\rho\to r^-} u(P) = u(P),$$
as desired.

7.3. The Poisson Integral Formula

Lemma 7.3.2. Let Ω, $W \subseteq \mathbb{C}$ be open sets, let $\psi : \Omega \to W$ be holomorphic, and let $u : W \to \mathbb{R}$ be harmonic. Then $u \circ \psi$ is harmonic.

(Emily, Problem 3990) Prove Lemma 7.3.2.

Denote by u_z, $u_{\bar z}$ the functions $\frac{\partial u}{\partial z}$, $\frac{\partial u}{\partial \bar z}$. Then by Problem 1.49, we have that
$$\frac{\partial}{\partial z}(u\circ\psi) = (u_z \circ \psi)\,\frac{\partial\psi}{\partial z} + u_{\bar z}\circ\psi\,\frac{\partial\overline{\psi}}{\partial z}.$$
Because ψ is holomorphic, by Problem 600 we have that $\frac{\partial\overline{\psi}}{\partial z} = \overline{\left(\frac{\partial\psi}{\partial\bar z}\right)} = 0$ and so
$$\frac{\partial}{\partial z}(u\circ\psi) = (u_z \circ \psi)\,\frac{\partial\psi}{\partial z}.$$
Similarly, by the Leibniz rule (Problem 490) we have that
$$\frac{\partial}{\partial \bar z}\frac{\partial}{\partial z}(u\circ\psi) = (u_{zz}\circ\psi)\,\frac{\partial\psi}{\partial\bar z}\,\frac{\partial\psi}{\partial z} + (u_{\bar z z}\circ\psi)\,\frac{\partial\overline{\psi}}{\partial\bar z}\,\frac{\partial\psi}{\partial z} + (u_z\circ\psi)\,\frac{\partial}{\partial\bar z}\frac{\partial\psi}{\partial z}.$$
The first term vanishes because ψ is holomorphic and so $\frac{\partial\psi}{\partial\bar z} = 0$. The second term vanishes because u is harmonic and so $u_{\bar z z} = 0$. Finally, the third term vanishes because ψ and thus $\frac{\partial\psi}{\partial z}$ is holomorphic.

[Definition: Poisson integral kernel] Let $P(z,\zeta) = \frac{|\zeta|^2-|z|^2}{2\pi|\zeta-z|^2}$; P is called the Poisson kernel.

Theorem 7.3.3. (The Poisson integral formula.) Suppose that u is harmonic in $D(0,R) \supset \overline{\mathbb{D}}$ for some $R > 1$. If $z \in \mathbb{D}$, then
$$u(w) = \int_0^{2\pi} u(e^{i\psi})\,P(w,e^{i\psi})\,d\psi = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\psi})\,\frac{1-|w|^2}{|e^{i\psi}-w|^2}\,d\psi.$$

(Irina, Problem 4000) Prove Theorem 7.3.3.

Recall [Problem 1340]: Let $\Omega \subseteq \mathbb{C}$ be open and let $\overline{D}(P,r) \subset \Omega$. Let f be holomorphic in Ω. Then $f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$ for all $z \in D(P,r)$.

Recall [Problem 1460]: Let f be continuous on $\partial D(P,r)$. Define F by $F(z) = \frac{1}{2\pi i}\oint_{\partial D(P,r)} \frac{f(\zeta)}{\zeta - z}\,d\zeta$ for all $z \in D(P,r)$. Then F is C^1 and holomorphic in $D(P,r)$.

[Chapter 2, Problem 21] It is not necessarily true that $\lim_{w\to z} F(w) = f(z)$ for $z \in \partial D(P,r)$.

Theorem 7.3.4. Let $f : \partial\mathbb{D} \to \mathbb{R}$ be continuous. If $z \in \mathbb{D}$, define
$$u(z) = \int_0^{2\pi} f(e^{i\psi})\,P(z,e^{i\psi})\,d\psi = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\psi})\,\frac{1-|z|^2}{|e^{i\psi}-z|^2}\,d\psi.$$
Then u is harmonic in $\mathbb{D}$, and $\lim_{z\to e^{i\theta}} u(z) = f(e^{i\theta})$ for all $\theta \in [0,2\pi]$. In particular, if we define $u(z) = f(z)$ for all $z \in \partial\mathbb{D}$, then u is continuous on $\overline{\mathbb{D}}$.

(Michael, Problem 4010) Let $P_r(\theta) = \frac{1}{2\pi}\frac{1-r^2}{1-2r\cos(\theta)+r^2}$. Show that $P_r(\theta-\psi) = P(re^{i\theta}, e^{i\psi})$, so that

$$v(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\psi})\,P_r(\theta-\psi)\,d\psi$$

for all $0 \leq r < 1$, all $\theta \in \mathbb{R}$ and all $v : \overline{\mathbb{D}} \to \mathbb{R}$ continuous and harmonic in $\mathbb{D}$.

By the Poisson integral formula, we have that

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\psi})\,\frac{1-|re^{i\theta}|^2}{|e^{i\psi}-re^{i\theta}|^2}\,d\psi.$$

First observe that $|re^{i\theta}| = r$. Next, observe that

$$\begin{aligned}|e^{i\psi}-re^{i\theta}|^2 &= (e^{i\psi}-re^{i\theta})(e^{-i\psi}-re^{-i\theta})\\ &= 1-re^{i(\theta-\psi)}-re^{i(\psi-\theta)}+r^2\\ &= 1-2r\cos(\theta-\psi)+r^2\end{aligned}$$

where we have used the variant $2\cos\eta = e^{i\eta}+e^{-i\eta}$ of Euler's identity $e^{i\eta} = \cos\eta + i\sin\eta$. Thus

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\psi})\,\frac{1-r^2}{1-2r\cos(\theta-\psi)+r^2}\,d\psi.$$

(Timmy, Problem 4020) Prove that if θ is real and $0 \leq r < 1$ then $0 < P_r(\theta) < \infty$ (in particular, the denominator is never zero).

(Zach, Problem 4030) Show that $p(z) = P(z,\zeta)$ is harmonic on $\mathbb{C}\setminus\{\zeta\}$; in particular, if $\zeta = e^{i\theta}$ then $p(z)$ is harmonic in $\mathbb{D}$.

(Alex, Problem 4040) Prove that u is harmonic in $\mathbb{D}$.

Define $F(\psi,x,y) = f(e^{i\psi})P(x+iy,e^{i\psi}) = f(e^{i\psi})\frac{1-x^2-y^2}{|e^{i\psi}-x-iy|^2}$.

Observe that F is continuous in ψ and twice continuously differentiable in both x and y in the region $\{(\psi,x,y) : \psi\in\mathbb{R},\ x^2+y^2<1\}$. Observe that for fixed ψ and y (or ψ and x) this region contains a set of x-values (or y-values) that is open in $\mathbb{R}$.

By Problem 730, this implies that for (ψ,x,y) in the same region,

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\int_0^{2\pi} F(\psi,x,y)\,d\psi &= \int_0^{2\pi}\frac{\partial^2}{\partial x^2}F(\psi,x,y)\,d\psi,\\ \frac{\partial^2}{\partial y^2}\int_0^{2\pi} F(\psi,x,y)\,d\psi &= \int_0^{2\pi}\frac{\partial^2}{\partial y^2}F(\psi,x,y)\,d\psi\end{aligned}$$

and so

$$\Delta u(x+iy) = \int_0^{2\pi}\frac{\partial^2}{\partial x^2}F(\psi,x,y)+\frac{\partial^2}{\partial y^2}F(\psi,x,y)\,d\psi.$$

But $F(\psi,x,y) = f(e^{i\psi})P(x+iy,e^{i\psi})$ and so

$$\frac{\partial^2}{\partial x^2}F(\psi,x,y)+\frac{\partial^2}{\partial y^2}F(\psi,x,y) = f(e^{i\psi})\Delta P(x+iy,e^{i\psi}) = 0$$

and so $\Delta u(x+iy) = 0$, as desired.

(Clayton, Problem 4050) Prove that if $0 \leq r < 1$ then $\int_0^{2\pi} P_r(\theta)\,d\theta = 1$.

The function $v(z) = 1$ for all z is harmonic in $\mathbb{C}$. Therefore, by Lemma 7.3.2 and Problem 4010 we have that

$$1 = v(re^{i\theta}) = \int_0^{2\pi} v(e^{i\theta})\,P_r(\theta)\,d\theta = \int_0^{2\pi} P_r(\theta)\,d\theta$$

for all $0 \leq r < 1$ and all $\theta \in \mathbb{R}$.

(David, Problem 4060) Prove that $\lim_{r\to 1^-} P_r(\theta) = 0$ for all $\theta \neq 2n\pi$.

If $\theta \neq 0$ then $-1 \leq \cos\theta < 1$. Thus $1 - 2(1)\cos\theta + 1^2 > 0$ and so the rational function $f(r) = P_r(\theta) = \frac{1-r^2}{2\pi(1-2r\cos\theta+r^2)}$ is continuous at $r = 1$. Thus $\lim_{r\to 1^-} P_r(\theta) = P_1(\theta) = 0$.

(Emily, Problem 4070) Let $0 < \delta < \pi$ be a small positive number. Prove that $\lim_{r\to 1^-} P_r(\theta) = 0$ uniformly for all $\delta < \theta < 2\pi - \delta$.

Because cos is monontonically decreasing on $[0, \pi]$, and because $\cos(2\pi - \theta) = \cos\theta$, we have that if $\delta < \theta < 2\pi - \delta$ then $\cos\theta < \cos\delta < 1$. Thus $\sup_{\delta<\theta<2\pi-\delta} P_r(\theta) \leq P_r(\delta)$. Recall $P_r(\theta) > 0$ for all $\theta \in \mathbb{R}$ and all $0 \leq r < 1$.

Because $\lim_{r\to 1^-} P_r(\delta) = 0$, we have that $\lim_{r\to 1^-} \sup_{\delta<\theta<2\pi-\delta} |P_r(\theta)| = 0$ and so we must have that $P_r \to 0$ as $r \to 1^-$ uniformly for $\theta \in (\delta, 2\pi - \delta)$.

(Irina, Problem 4080) Let $0 < \delta < \pi$. Prove that $\lim_{r\to 1^-} \int_{-\delta}^{\delta} P_r(\theta)\,d\theta = 1$.

(Michael, Problem 4090) Let u, f be as in Theorem 7.3.4. Show that $u(re^{i\theta})$ converges to $f(e^{i\theta})$ as $r \to 1^-$ uniformly in θ.

(Timmy, Problem 4100) Let u, f be as in Theorem 7.3.4. Show that u is continuous on $\overline{\mathbb{D}}$. (This completes the proof of Theorem 7.3.4. This differs from the previous problem in that the previous problem considers $u(z)$ as $z \to e^{i\theta}$ along a ray through the origin, while this problem considers $u(z)$ as $z \to e^{i\theta}$ along arbitrary paths.)

(Zach, Problem 4110) Write an analogue to Theorem 7.3.4 in an arbitrary disc. That is, let $P \in \mathbb{C}$, $r > 0$, and $f : \partial D(P, r) \to \mathbb{R}$ be continuous. Show that there is a function u that is harmonic in $D(P, r)$, continuous on $\overline{D}(P, r)$, and satisfies $u(z) = f(z)$ for all $z \in \partial D(P, r)$. Can you write a formula for u in $D(P, r)$?

If $z \in \mathbb{D}$, define

$$v(z) = \frac{1}{2\pi}\int_0^{2\pi} f(P + re^{i\theta})\frac{1-|z|^2}{|e^{i\theta} - z|^2}\,d\theta.$$

By Theorem 7.3.4 we have that v is continuous on $\overline{D}$, harmonic in $\mathbb{D}$ and satisfies $v(e^{i\theta}) = f(P + re^{i\theta})$.

If we let $u(z) = v(\frac{1}{r}(z - P))$, a straightforward computation shows that u is harmonic in $D(P, r) = \{z \in \mathbb{C} : \frac{1}{r}(z - P) \in \mathbb{D}\}$, continuous on $\overline{D}(P, r)$, and satisfies $u(P + re^{i\theta}) = v(e^{i\theta}) = f(P + re^{i\theta})$, that is, $u = f$ on $\partial D(P, r)$. We compute that

$$u(z) = v\left(\frac{1}{r}(z-P)\right) = \frac{1}{2\pi}\int_0^{2\pi} f(P + re^{i\theta})\frac{1 - |(z-P)/r|^2}{|e^{i\theta} - (z-P)/r|^2}\,d\theta$$

which simplifies to

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} f(P + re^{i\theta})\frac{r^2 - |z-P|^2}{|P + re^{i\theta} - z|^2}\,d\theta.$$

(Problem 4111) Show that Theorem 7.3.3 is valid if u is merely harmonic in $\mathbb{D}$ and continuous on $\overline{D}$ (rather than being harmonic in an open superset of $\overline{D}$). Also, write and prove an analogue in an arbitrary disc. That is, if u is harmonic in $D(P, r)$ and continuous on $\overline{D}(P, r)$, write a formula for u in $D(P, r)$ in terms of u on $\partial D(P, r)$.

As in the previous problem, we use Theorem 7.3.3 and a change of variables to see that

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(P + re^{i\theta})\frac{r^2 - |z-P|^2}{|P + re^{i\theta} - z|^2}\,d\theta$$

if u is harmonic in $D(P, R)$ for some $R > r$. Taking the limit as $r \to R^-$ completes the proof.

[Chapter 7, Problem 25] If u is harmonic in $\mathbb{H} = \{x + iy : x \in \mathbb{R},\ y > 0\}$ and continuous on $\overline{\mathbb{H}}$, and if $\lim_{|z|\to\infty} u(z)$ exists (and is finite), then

$$u(x + iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} u(t)\frac{y}{(x-t)^2 + y^2}\,dt$$

for all $x \in \mathbb{R}$ and all $y \in (0, \infty)$.

Conversely, if $f : \mathbb{R} \to \mathbb{R}$ is continuous, if $\lim_{t\to\infty} f(t) = \lim_{t\to-\infty} f(t) \in \mathbb{R}$, and if we define

$$u(x+iy) = \begin{cases} \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{y}{(x-t)^2+y^2}\,dt, & y > 0, \\ f(x), & y = 0, \end{cases}$$

then u is harmonic in $\mathbb{H}$ and continuous on $\overline{\mathbb{H}}$.

(Problem 4112) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous. However, we do not require that $\lim_{t\to\infty} f(t)$ or $\lim_{t\to-\infty} f(t)$ exist. Show that if

$$u(x+iy) = \begin{cases} \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{y}{(x-t)^2+y^2}\,dt, & y > 0, \\ f(x), & y = 0, \end{cases}$$

then we still have that u is harmonic in $\mathbb{H} = \{x + iy : x \in \mathbb{R},\ y > 0\}$ and continuous on $\overline{\mathbb{H}}$.

(Problem 4113) Suppose that u and v are harmonic in some open set Ω, that $\overline{D}(P, r) \subset \Omega$, and that $u(z) = v(z)$ for all $z \in \partial D(P, r)$. Show that $u(z) = v(z)$ for all $z \in D(P, r)$.

[Chapter 7, Problem 22] If u is harmonic in a connected open set Ω and $u = 0$ in $D(P, r)$ for some $D(P, r) \subseteq \Omega$ with $r > 0$, then $u = 0$ everywhere in Ω.

(Problem 4114) Give an example of two harmonic functions that are equal on a set with an accumulation point but are not equal everywhere.

(Problem 4115) Show that the zeroes of a real harmonic function cannot be isolated. That is, let $\Omega \subseteq \mathbb{C}$ be open and let $u : \Omega \to \mathbb{R}$ be harmonic. Let $P \in \Omega$ with $u(P) = 0$ and let $r > 0$. Show that $u(z) = 0$ for some $z \in D(P, r) \setminus \{P\}$.

7.4. Real Analysis

(Problem 4120) Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be a nonnegative continuous function that satisfies $\int_a^b f = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

7.4. Regularity of Harmonic Functions

Definition 7.4.1. Let $\Omega \subset \mathbb{C}$ be open and let $h : \Omega \to \mathbb{R}$ be continuous. We say that h has the *small circle mean value property* (SCMV property) if, for every $P \in \Omega$, there is some number $\varepsilon_P > 0$ such that $D(P, \varepsilon_P) \subset \Omega$ and such that $h(P) = \frac{1}{2\pi}\int_0^{2\pi} h(P + \varepsilon e^{i\theta})\,d\theta$ for all $0 < \varepsilon < \varepsilon_P$.

Recall [Theorem 7.2.5]: If u is harmonic in an open set Ω, then u has the small circle mean value property, and if $P \in \Omega$ then ε_P is the largest number such that $D(P, \varepsilon_P) \subseteq \Omega$.

Lemma 7.4.4. Let $\Omega \subset \mathbb{C}$ be open and connected. Let g be continuous on Ω and satisfy the "small circle" mean value property. Suppose furthermore that there is some $P \in \Omega$ such that $g(P) \geq g(z)$ for all $z \in \Omega$. Then g is constant.

(Alex, Problem 4130) Prove Lemma 7.4.4.

Let $E = \{z \in \Omega : g(z) = g(P)\} = g^{-1}(\{g(P)\})$. Then E is relatively closed in Ω because g is continuous.

Let $z \in E$. We then know that

$$g(P) = g(z) = \frac{1}{2\pi}\int_0^{2\pi} g(z + re^{i\theta})\,d\theta$$

for all $0 < r < \varepsilon_z$. Rewriting, we have that

$$0 = \int_0^{2\pi} g(P) - g(z + re^{i\theta})\,d\theta$$

for all $0 < r < \varepsilon_z$.

Observe that $f(\theta) = g(P) - g(z + re^{i\theta})$ is continuous and nonnegative because $g(P) \geq g(w)$ for all $w \in \Omega$. Thus by Problem 4120, we have that $f(\theta) = 0$ for all $\theta \in [0, 2\pi]$ and so $g(z + re^{i\theta}) = g(P)$ for all

$0 < r < \varepsilon_z$ and all $\theta \in \mathbb{R}$. By assumption $g(z) = g(P)$, so we have that $D(z, \varepsilon_z) \subseteq E$. Thus E is open. Because $E \subseteq \Omega$ is nonempty, open, and (relatively) closed, and Ω is connected, we have that $E = \Omega$ and so $u(z) = u(P)$ for all $z \in \Omega$.

(Clayton, Problem 4140) Suppose that g is continuous on $\overline{D}(P, r)$ and has the "small circle" mean value property in $D(P, r)$. Suppose further that $g = 0$ on $\partial D(P, r)$. Show that $g = 0$ in $D(P, r)$.

Because $\overline{D}(P, r)$ is compact and g is continuous, there are points ζ, $\omega \in \overline{D}(P, r)$ such that $g(\zeta) \geq g(z) \geq g(\omega)$ for all $z \in \overline{D}(P, r)$.

If $\zeta \in D(P, r)$ then g is constant by Lemma 7.4.4. Because $g(z) = 0$ for some $z \in \overline{D}(P, r)$ (in fact all $z \in \partial D(P, r)$), if g is constant then $g \equiv 0$.

If $\omega \in D(P, r)$, observe that $h = -g$ is also continuous and has the SCMVP; thus, by Lemma 7.4.4, h and thus g is constant. Again if g is constant then $g \equiv 0$.

Finally, if ζ, $\omega \in \partial D(P, r)$ then

$$0 = g(\omega) \leq g(z) \leq g(\zeta) = 0$$

for all $z \in \overline{D}(P, r)$, and so $g \equiv 0$.

(David, Problem 4150) Suppose that g and h are continuous on $\overline{D}(P, r)$ and that $g = h$ on $\partial D(P, r)$. Suppose that h is harmonic in $D(P, r)$ and that g has the "small circle" mean value property in $D(P, r)$. Show that $g = h$ in $D(P, r)$ as well.

Let $f = g - h$. Then f is clearly continuous on $\overline{D}(P, r)$. Because h is harmonic, by Theorem 7.2.5 it has the mean value property in $D(P, r)$. Thus, if $z \in D(P, r)$ and ε_z is as in the definition of SCMVP for g, we have that

$$f(z) = g(z) - h(z) = \frac{1}{2\pi} \int_0^{2\pi} g(z + \rho e^{i\theta})\, d\theta - \frac{1}{2\pi} \int_0^{2\pi} h(z + \rho e^{i\theta})\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(z + \rho e^{i\theta})\, d\theta$$

for all $0 < \rho < \varepsilon_z$. Thus f has the SCMVP in $D(P, r)$. Furthermore, $f = 0$ on $\partial D(P, r)$, and so by the previous problem, $f = 0$ in $D(P, r)$ as well.

Theorem 7.4.2. Let $\Omega \subset \mathbb{C}$ be open. Suppose that $g : \Omega \to \mathbb{R}$ is continuous and has the "small circle" mean value property in Ω. Then g is harmonic in Ω.

(Emily, Problem 4160) Prove Theorem 7.4.2.

Let $P \in \Omega$. Then there is a $r > 0$ such that $\overline{D}(P, r) \subset \Omega$. Observe that h is continuous on $\partial D(P, r)$. By Theorem 7.3.4 (or rather, by Problem 4110) there is a function $u : \overline{D}(P, r) \to \mathbb{R}$ that is continuous on $\overline{D}(P, r)$, harmonic in $D(P, r)$, and satisfies $u = h$ on $\partial D(P, r)$. By Problem 4150, we have that $h = u$ in $D(P, r)$ and so h must be harmonic in $D(P, r)$ as well. This is true for every $P \in \Omega$, so h must be harmonic in all of Ω.

Corollary 7.4.3. Let $\Omega \subset \mathbb{C}$ be open. Suppose that $\{h_j\}_{j=1}^{\infty}$ is a sequence of functions, each harmonic on Ω, and that $h_j \to h$ uniformly on compact subsets of Ω. Then h is also harmonic.

(Irina, Problem 4170) Prove Corollary 7.4.3. *Hint*: Show that h has the "small circle" mean value property.

Lemma 7.5.1. Let $\Psi \subset \mathbb{C}$ be open and connected. Suppose that Ψ is symmetric about the real axis; that is, $z \in \Psi$ if and only if $\overline{z} \in \Psi$. Let $\Omega = \{z \in \Psi : \operatorname{Im} z > 0\}$.

Let $v : \overline{\Omega} \cap \Psi \to \mathbb{R}$ be continuous. Suppose that v is harmonic in Ω and that $v(x) = 0$ for all $x \in \partial\Omega \cap \Psi = \mathbb{R} \cap \Psi$. Then there is a function $\widehat{v} : \Psi \to \mathbb{R}$ that is harmonic in Ψ and satisfies $\widehat{v} = v$ in Ω. Furthermore,

$$\widehat{v}(z) = \begin{cases} v(z), & z \in \Omega, \\ 0, & z \in \partial\Omega \cap \Psi, \\ -v(\overline{z}), & z \in \widehat{\Omega} = \{w \in \mathbb{C} : \overline{w} \in \Omega\}. \end{cases}$$

(Problem 4180) Let Ψ, Ω, and v be as in Lemma 7.5.1.

Sketch Ψ. Label Ω, $\widehat{\Omega}$, the set where v is harmonic, and the set where v is equal to zero.

(Michael, Problem 4190) Suppose $\Omega \subset \mathbb{C}$ is open and that v is harmonic on Ω. Let $w(z) = v(\overline{z})$. Show that w is harmonic on $\widehat{\Omega} = \{z \in \mathbb{C} : \overline{z} \in \Omega\}$.

This is an easy consequence of Problems 600 and 3910.

(Timmy, Problem 4200) Let Ψ, Ω, $\widehat{\Omega}$, v, and $\widehat{v}$ be as in Lemma 7.5.1. Show that $\widehat{v}$ is continuous in Ψ.

Let $\Phi = \overline{\Omega} \cap \Psi$ and let $\widehat{\Phi} = \{z \in \Psi : \overline{z} \in \Phi\}$. Then Φ and $\widehat{\Phi}$ are relatively closed in Ψ. Furthermore, $v : \Phi \to \mathbb{R}$ is continuous by assumption. Let $w(z) = -v(\overline{z})$; then $w : \widehat{\Phi} \to \mathbb{R}$ is continuous, and we have that

$$\widehat{v}(z) = \begin{cases} v(z), & z \in \Phi, \\ w(z), & z \in \widehat{\Phi}. \end{cases}$$

If $F \subseteq \mathbb{R}$ is closed, then because v and w are continuous we must have that $v^{-1}(F)$ and $w^{-1}(F)$ are relatively closed in Φ and $\widehat{\Phi}$, respectively. Because Φ and $\widehat{\Phi}$ are (relatively) closed in Ψ we have that $v^{-1}(F)$ and $w^{-1}(F)$ are relatively closed in Ψ. Thus their union is relatively closed. But $\widehat{v}^{-1}(F) = v^{-1}(F) \cup w^{-1}(F)$, and so $\widehat{v}^{-1}(F)$ is relatively closed in Ψ. Recalling that this is equivalent to continuity completes the proof.

(Zach, Problem 4210) Suppose that v, $\widehat{v}$, and Ψ are as in Lemma 7.5.1. Show that $\widehat{v}$ is harmonic in Ψ. This completes the proof of Lemma 7.5.1.

Let $z \in \Psi$. Then either $z \in \Omega$, $z \in \widehat{\Omega}$, or $z \in \mathbb{R} \cap \Psi$.

If $z \in \Omega$ or $z \in \widehat{\Omega}$, let $r > 0$ be such that $D(z, r) \subsetneq \Omega$ or $D(z, r) \subsetneq \widehat{\Omega}$. By assumption, or by Problem 4190, $\widehat{v}$ is harmonic in Ω and in $\widehat{\Omega}$, and thus in either case is harmonic in $D(P, r)$. By Theorem 7.2.5, we have that for every ρ with $0 < \rho < r$, it holds $\widehat{v}(z) = \frac{1}{2\pi}\int_0^{2\pi} \widehat{v}(z + \rho e^{i\theta})\,d\theta$.

Now consider the case where $z \in \mathbb{R}$. By assumption Ψ is open and so there is a $r > 0$ such that $D(z, r) \subseteq \Psi$. Let $0 < \rho < r$. Because $\widehat{v}$ is continuous on Ψ, it is integrable, and so

$$\frac{1}{2\pi}\int_0^{2\pi} \widehat{v}(z+\rho e^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{\pi} \widehat{v}(z+\rho e^{i\theta})\,d\theta + \frac{1}{2\pi}\int_\pi^{2\pi} \widehat{v}(z+\rho e^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{\pi} v(z+\rho e^{i\theta})\,d\theta + \frac{1}{2\pi}\int_\pi^{2\pi} -v(z+\rho e^{-i\theta})\,d\theta.$$

Making the change of variables $\psi = 2\pi - \theta$ in the second integral, we have that

$$\frac{1}{2\pi}\int_0^{2\pi} \widehat{v}(z+\rho e^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{\pi} v(z+\rho e^{i\theta})\,d\theta - \frac{1}{2\pi}\int_0^{\pi} v(z+\rho e^{-i(2\pi-\psi)})\,d\theta = 0 = \widehat{v}(z).$$

Thus v satisfies the SCMVP in Ψ, and so must be harmonic by Theorem 7.4.2. This completes the proof of Lemma 7.5.1.

[Chapter 7, Problem 1] Liouville's theorem is true for harmonic functions: if $u : \mathbb{C} \to \mathbb{C}$ is both bounded and harmonic, then u is constant.

(Problem 4211) If u is harmonic in $\mathbb{H} = \{x + iy : x \in \mathbb{R},\ y > 0\}$ and continuous on $\overline{\mathbb{H}}$, and in addition is bounded, show that

$$u(x + iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} u(t)\frac{y}{(x-t)^2 + y^2}\,dt$$

for all $x \in \mathbb{R}$ and all $y \in (0, \infty)$.

Theorem 7.5.2. Let Ψ and Ω be as in Lemma 7.5.1. That is, let $\Psi \subset \mathbb{C}$ be open, connected, and symmetric about the real axis. Let $\Omega = \{z \in \Psi : \operatorname{Im} z > 0\}$.

Let $f : \overline{\Omega} \cap \Psi \to \mathbb{R}$ be continuous. Suppose that f is holomorphic in Ω and that $\operatorname{Im} f(x) = 0$ for all $x \in \partial\Omega \cap \Psi$. Then there is a function $\widehat{f} : \Psi \to \mathbb{R}$ that is holomorphic in Ψ and satisfies $\widehat{f} = f$ in Ω. Furthermore,

$$\widehat{f}(z) = \begin{cases} f(z), & z \in \overline{\Omega} \cap \Psi, \\ \overline{f(\overline{z})}, & z \in \widehat{\Omega} = \{w \in \mathbb{C} : \overline{w} \in \Omega\}. \end{cases}$$

(Alex, Problem 4220) Suppose $\Omega \subset \mathbb{C}$ is open and that f is holomorphic on Ω. Let $g(z) = \overline{f(\overline{z})}$. Show that g is holomorphic on $\widehat{\Omega} = \{z \in \mathbb{C} : \overline{z} \in \Omega\}$.

Let $f = u + iv$, where u and v are real functions. Let $z = x + iy$. If $g(z) = \overline{f}(\overline{z})$, then $h(x + iy) = u(x - iy) - iv(x - iy) = U(x + iy) + iV(x + iy)$. Using the fact that the Cauchy-Riemann equations hold for f, we have that

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial V}{\partial y}$$

and

$$\frac{\partial U}{\partial y} = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -\frac{\partial V}{\partial x}.$$

Thus g satisfies the Cauchy-Riemann equations and so is holomorphic in Ω.

(Clayton, Problem 4230) Suppose that f is holomorphic in $D(x_0, r)$ for some $x_0 \in \mathbb{R}$ and some $r > 0$. Suppose further that $f(x)$ is real for all $x \in (x_0 - r, x_0 + r) = D(x_0, r) \cap \mathbb{R}$. Show that $f(z) = \overline{f(\overline{z})}$ for all $z \in D(x_0, r)$.

By the previous problem, $g(z) = f(z) - \overline{f(\overline{z})}$ is holomorphic in $D(x_0, r)$. Because $f(x)$ is real for all $x \in \mathbb{R} \cap D(x_0, r)$, we have that $g(x) = 0$ on $\mathbb{R} \cap D(x_0, r)$, a set with an accumulation point. Thus $g \equiv 0$ in $D(x_0, r)$ and so $f(z) = \overline{f(\overline{z})}$.

(David, Problem 4240) Prove Theorem 7.5.2. *Hint*: Start with the special case where Ψ is a disc centered at a point on the real axis.

Let $\widehat{f}$ be as in the statement of Theorem 7.5.2. By assumption $\widehat{f}$ is holomorphic in Ω, and by Problem 4220 $\widehat{f}$ is holomorphic in $\widehat{\Omega} = \{z \in \Psi : \operatorname{Im} z < 0\}$. We need only show that $\widehat{f}$ is holomorphic in a neighborhood of every $x \in \Psi \cap \mathbb{R}$.

Let $f = u + iv$ where u and v are real functions; they are then harmonic in $\Omega = \{z \in \Psi : \operatorname{Im} z > 0\}$ and $v = 0$ on $\mathbb{R} \cap \Psi$. By Lemma 7.5.1, we have that there is a function $\widehat{v}$ harmonic in Ψ that equals v in Ω.

If $x_0 \in \mathbb{R} \cap \Psi$, there is a $r > 0$ with $D(x_0, r) \subseteq \Psi$. Then there is a holomorphic function $g : D(x_0, r) \to \mathbb{C}$ with $\operatorname{Im} g = \widehat{v}$ in $D(x_0, r)$. Then $\widehat{f} - g$ is holomorphic and purely real in $\{z \in D(x_0, r) : \operatorname{Im} z > 0\}$ and also in $\{z \in D(x_0, r) : \operatorname{Im} z < 0\}$, and so must be constant in each of those sets. Because $\widehat{f}$ and g are continuous on $D(x_0, r)$ we must have that $\widehat{f} - g$ is equal to one constant in all of $D(x_0, r)$; thus $\widehat{f}$ must be holomorphic in $D(x_0, r)$, as desired.

(Emily, Problem 4250) Suppose that f is holomorphic on $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and continuous on $\overline{\mathbb{H}}$ and that $f(x) = 0$ for all $0 < x < 1$. Show that $f(z) = 0$ for all $z \in \mathbb{H}$.

(Memory 4260) Let $\{x_j\}_{j=1}^\infty$ be an increasing sequence of real numbers, that is, $x_j \in \mathbb{R}$ and $x_j \leq x_{j+1}$ for all $j \in \mathbb{N}$. Show that either $x_j \to \infty$ or $x_j \to x$ for some $x \in \mathbb{R}$.

Recall [Problem 1590]: If E is a set, (X, d) is a complete metric space, $f_k : E \to X$, and $\{f_k\}_{k=1}^\infty$ is uniformly Cauchy, then $\{f_k\}_{k=1}^\infty$ converges uniformly to some $f : E \to X$.

7.6. Harnack's Principle

(Irina, Problem 4270) Recall that if u is harmonic in $D(P, R)$ and continuous on $\overline{D}(P, R)$, then for any $0 \leq r < R$ and any $0 \leq \theta \leq 2\pi$,

$$u(P + re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} u(P + Re^{i\psi}) \frac{R^2 - r^2}{|Re^{i\psi} - re^{i\theta}|^2}\, d\psi.$$

Show that

$$\min_{0 \leq \theta \leq 2\pi, 0 \leq \psi \leq 2\pi} \frac{R^2 - r^2}{|Re^{i\psi} - re^{i\theta}|^2} = \frac{R - r}{R + r}$$

and

$$\max_{0 \leq \theta \leq 2\pi, 0 \leq \psi \leq 2\pi} \frac{R^2 - r^2}{|Re^{i\psi} - re^{i\theta}|^2} = \frac{R + r}{R - r}.$$

First, if R, r, θ, and ψ are real, then

$$\begin{aligned} |Re^{i\psi} - re^{i\theta}| &= |Re^{i(\psi-\theta)} - r| \\ &= \sqrt{(R\cos(\psi - \theta) - r)^2 + R^2 \sin^2(\psi - \theta)} \\ &= \sqrt{R^2 + r^2 - 2Rr\cos(\psi - \theta)} \end{aligned}$$

which clearly has a minimum value of $\sqrt{R^2 + r^2 - 2Rr} = R - r$ (attained if $\theta = \psi$) and a maximum value of $\sqrt{R^2 + r^2 + 2Rr} = R + r$. (This is attained if $\psi = \theta \pm \pi$; if $0 \leq \theta \leq 2\pi$ then either $\theta \leq \pi$ and so $\psi = \theta + \pi \in [0, 2\pi]$ or $\theta \geq \pi$ and so $\psi = \theta - \pi \in [0, 2\pi]$. In either case the maximum value $R + r$ is attained for some ψ in the given range.)

Thus,

$$\frac{R - r}{R + r} = \frac{R^2 - r^2}{(R + r)^2} \leq \frac{R^2 - r^2}{|Re^{i(\psi-\theta)} - r|^2} \leq \frac{R^2 - r^2}{(R - r)^2} = \frac{R + r}{R - r}$$

and the left-hand and right-hand bound, respectively, is attained at $\theta = \psi \pm \pi$ and $\theta = \psi$.

Corollary 7.6.2. (The Harnack inequality.) Suppose that u is nonnegative and harmonic in $D(P, R)$ and continuous on $\overline{D}(P, R)$. If $z \in D(P, R)$ with $r = |z - P|$, then

$$\frac{R - r}{R + r} u(P) \leq u(z) \leq \frac{R + r}{R - r} u(P).$$

(Michael, Problem 4280) Prove Corollary 7.6.2.

(Timmy, Problem 4290) Did we need the assumption that u was continuous on $\overline{D}(P, R)$?

No. If u is nonnegative and harmonic in $D(P, R)$, then u is nonnegative and harmonic in $D(P, \rho)$ and continuous on $\overline{D}(P, \rho)$ for all $0 < \rho < R$. We then have that

$$\frac{\rho - r}{\rho + r} u(P) \leq u(z) \leq \frac{\rho + r}{\rho - r} u(P)$$

for all ρ with $r < \rho < R$, and taking the limit as $\rho \to R^-$ completes the proof.

Theorem 7.6.3. (Harnack's principle.) Let $\{u_k\}_{k=1}^\infty$ be a sequence of real-valued functions harmonic in a connected open set $\Omega \subseteq \mathbb{C}$ such that $u_1(z) \leq u_2(z) \leq u_3(z) \leq \cdots$ for each $z \in \Omega$. Then either $u_k \to \infty$ uniformly on compact sets or there is a function $u : \Omega \to \mathbb{R}$ such that $u_k \to u$ uniformly on compact sets.

(By Corollary 7.4.3, u is harmonic.)

(Zach, Problem 4300) In this problem we begin the proof of Theorem 7.6.3. Let Ω and u_k be as in Theorem 7.6.3. Let $D(P,R) \subseteq \Omega$. Suppose that $\lim_{j\to\infty} u_k(P) = \infty$. Show that $u_k \to \infty$ uniformly on $\overline{D}(P,r)$ for any $0 < r < R$.

Define $v_k = u_k - u_1$. Then $v_k : \Omega \to \mathbb{R}$ is harmonic, and $v_k \geq 0$.

Fix some $r \in (0,R)$. Let $M \in \mathbb{R}$. By definition of $u_k(P) \to \infty$, there is some $K \in \mathbb{N}$ such that, if $k \geq K$, then $u_k(P) \geq M$.

We have that each v_k is harmonic and nonnegative in $D(P,R)$. Then, if $z \in \overline{D}(P,r)$, then by Harnack's inequality, if $k \geq K$ then

$$\begin{aligned} u_k(z) = v_k(z) + u_1(z) &\geq \frac{R - |z-P|}{R + |z-P|} v_k(P) + u_1(z) \\ &\geq M\frac{R-r}{R+r} + \min_{\overline{D}(P,r)} u_1 \geq M\frac{R-r}{R+r} - \max_{\overline{D}(P,r)} |u_1| \end{aligned}$$

where the final term is finite because u_1 is continuous and $\overline{D}(P,r)$ is compact, and where we have used that if $|z-P| \leq r < R$ then $\frac{R-|z-P|}{R+|z-P|} \leq \frac{R-r}{R+r}$. Thus $u_k \to \infty$ uniformly on $\overline{D}(P,r)$.

(Alex, Problem 4310) Suppose that $\lim_{k\to\infty} u_k(P) < \infty$. Show that $\{u_k\}_{k=1}^{\infty}$ converges to some (finite) harmonic function, uniformly on $D(P,r)$ for any $0 < r < R$.

(Clayton, Problem 4320) Show that either $\lim_{k\to\infty} u_k(z) = \infty$ for all $z \in \Omega$ or $\lim_{k\to\infty} u_k(z) < \infty$ for all $z \in \Omega$.

(David, Problem 4330) Prove Theorem 7.6.3.

Suppose that $\lim_{k\to\infty} u_k(z) = \infty$ for all $z \in \Omega$. Then if $D(P,R) \subseteq \Omega$ then $u_k \to \infty$ uniformly on $D(P,R/2)$.

Let $K \subset \Omega$ be compact. Then $\{D(P,R/2) : P \in K,\ R > 0, D(P,R) \subseteq \Omega\}$ is an open cover of K. It thus has a finite subcover $K \subset \bigcup_{n=1}^{N} D(P_n, R_n/2)$. We have that $u_k \to \infty$ uniformly on $D(P_n, R_n/2)$ for all $1 \leq n \leq N$, and thus for all $L \in \mathbb{R}$ there is a $M_n \in \mathbb{N}$ such that if $k \geq M_n$ then $u_k \geq L$ in $D(P_n, R_n/2)$. Letting $M = \max\{M_1, \dots, M_n\}$, we see that if $k \geq M$ then $u_k \geq L$ in $\bigcup_{n=1}^{N} D(P_n, R_n/2) \supset K$, as desired.

7.7. Real Analysis

(Problem 4331) Suppose that u and v are both real-valued and continuous in a set Ω. Let $f(z) = \max(u(z), v(z))$. Show that f is continuous in Ω.

7.7. Subharmonic Functions

[Definition: Subharmonic functions] Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{R}$ be continuous. Suppose that for every $\overline{D}(P,r) \subset \Omega$, we have that

$$f(P) \leq \frac{1}{2\pi}\int_0^{2\pi} f(P + re^{i\theta})\,d\theta.$$

Then we say that f is subharmonic in Ω.

[Definition: Superharmonic functions] Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{R}$ be continuous. Suppose that for every $\overline{D}(P,r) \subset \Omega$, we have that

$$f(P) \geq \frac{1}{2\pi}\int_0^{2\pi} f(P + re^{i\theta})\,d\theta.$$

Then we say that f is superharmonic in Ω.

(Problem 4340) Show that f is subharmonic if and only if $-f$ is superharmonic.

(Problem 4350) Suppose that f is a continuous, real-valued function in an open set $\Omega \subseteq \mathbb{C}$. Show that f is harmonic if and only if f is both subharmonic and superharmonic.

(Emily, Problem 4360) Suppose that f is subharmonic in an open set Ω and that $\alpha \neq 0$ is a nonzero real number. Show that αf is subharmonic if $\alpha > 0$ and that αf is superharmonic if $\alpha < 0$.

(Irina, Problem 4370) Suppose that f and g are both subharmonic in an open set Ω. Show that $f + g$ is subharmonic in Ω. Is $f - g$ subharmonic in Ω?

(Michael, Problem 4380) Suppose that f is subharmonic and g is superharmonic in an open set $\Omega \subseteq \mathbb{C}$. Show that $f - g$ is subharmonic in Ω.

(Timmy, Problem 4390) Suppose that u and v are both subharmonic in an open set Ω. Let $f(z) = \max(u(z), v(z))$. Show that f is subharmonic in Ω. (In particular, if u and v are real and harmonic then f is subharmonic.)

Let $\overline{D}(P, r) \subset \Omega$. Without loss of generality we have that $f(P) = u(P)$. Then

$$f(P) = u(P) \leq \frac{1}{2\pi} \int_0^{2\pi} u(P + re^{i\theta})\, d\theta$$

because u is subharmonic. But $u(P + re^{i\theta}) \leq \max() = f()$ by definition of maximum, and so

$$\int_0^{2\pi} u(P + re^{i\theta})\, d\theta \leq \int_0^{2\pi} f(P + re^{i\theta})\, d\theta.$$

Combining the two estimates yields that

$$f(P) \leq \int_0^{2\pi} f(P + re^{i\theta})\, d\theta.$$

This is true for all $P \in \Omega$ and all $r > 0$ such that $\overline{D}(P, r) \subset \Omega$, so f is subharmonic in Ω.

(Zach, Problem 4400) Let $\Omega \subset \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C}$ be holomorphic. Show that $u(z) = |f(z)|$ is subharmonic in Ω.

By the Cauchy integral formula (Theorem 2.4.2), if $\overline{D}(P, r) \subset \Omega$ then

$$f(P) = \frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

By definition of line integral

$$f(P) = \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta})\, d\theta.$$

By Proposition 2.1.7, we have that

$$|f(P)| = \frac{1}{2\pi} \left| \int_0^{2\pi} f(P + re^{i\theta})\, d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(P + re^{i\theta})|\, d\theta$$

and so $|f|$ is subharmonic.

(Alex, Problem 4410) Let $\Omega \subset \mathbb{C}$ be open and let $u : \Omega \to \mathbb{C}$ be subharmonic. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be nondecreasing and convex, so that if $0 < t < 1$ and a, $b \in \mathbb{R}$ then $\varphi(ta + (1-t)b) \leq t\varphi(a) + (1-t)\varphi(b)$. Show that $v(z) = \varphi(u(z))$ is subharmonic in Ω.

Suppose that $\overline{D}(P, r) \subset \Omega$. Because φ is nondecreasing and u is subharmonic, we have that

$$v(P) = \varphi(u(P)) \leq \varphi\left(\frac{1}{2\pi} \int_0^{2\pi} u(P + re^{i\theta})\, d\theta \right).$$

By Jensen's inequality,

$$\begin{aligned} \varphi\left(\frac{1}{2\pi} \int_0^{2\pi} u(P + re^{i\theta})\, d\theta \right) &\leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(u(P + re^{i\theta}))\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} v(P + re^{i\theta})\, d\theta. \end{aligned}$$

This completes the proof.

[Chapter 7, Problem 46] If u is harmonic and φ is convex, then $\varphi \circ u$ is subharmonic even if φ is not nondecreasing.

(Clayton, Problem 4420) Give an example of a function that is subharmonic in a domain Ω but is not harmonic in that domain.

Let $f(z) = |z|$. Then f is subharmonic in $\mathbb{C}$ by Problem 4400. But f is not harmonic in $\mathbb{C}$ because f is not differentiable at 0, and harmonic functions by definition are C^2.

(David, Problem 4430) [Redacted]

Proposition 7.7.7. Subharmonic functions satisfy the maximum principle. That is, suppose that $\Omega \subseteq \mathbb{C}$ is open and connected, that $f : \Omega \to \mathbb{R}$ is subharmonic, and that there is a $P \in \Omega$ such that $f(P) \geq f(z)$ for all $z \in \Omega$. Then f is constant in Ω.

(Emily, Problem 4440) Is there a minimum principle for subharmonic functions?

No; observe that the subharmonic function $|z|$ has a minimum at $z = 0$ but is not constant.

(Irina, Problem 4450) Prove the following generalization of Proposition 7.7.7. Let $\Omega \subseteq \mathbb{C}$ be open and connected. Suppose that $f : \Omega \to \mathbb{R}$ is continuous and satisfies the small circle sub-mean-value property: for every $P \in \Omega$, there is some $\varepsilon_P > 0$ such that $D(P, \varepsilon_P) \subset \Omega$ and such that

$$f(P) \leq \frac{1}{2\pi} \int_0^{2\pi} f(P + \varepsilon e^{i\theta})\, d\theta \quad \text{for all } 0 < \varepsilon < \varepsilon_P.$$

Show that f satisfies the maximum principle in Ω.

Suppose that there is a $P \in \Omega$ such that $f(P) \geq f(z)$ for all $z \in \Omega$. We claim that f is constant.

To prove this, let $E = \{z \in \Omega : f(z) = f(P)\} = f^{-1}(\{f(P)\})$. Then E is relatively closed in Ω because f is continuous and $\{f(P)\}$ is closed in $\mathbb{R}$.

Conversely, suppose that $z \in E$. Let $0 < r < \varepsilon_z$. By the small circle sub-mean-value property,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, d\theta.$$

Because $z + re^{i\theta} \in \Omega$, we have that $f(z + re^{i\theta}) \leq f(P)$ by assumption on P, so

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} f(P)\, d\theta = f(P).$$

Since $f(P) = f(z)$, we must have that all terms are equal and so in particular

$$\frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} f(P)\, d\theta$$

or

$$\int_0^{2\pi} f(P) - f(z + re^{i\theta})\, d\theta = 0.$$

Let $g(\theta) = f(P) - f(z + re^{i\theta})$. Then g is nonnegative, continuous, and integrates to zero, so by Problem 4120 we have that $g(\theta) = 0$ for all θ.

Thus $f(z + re^{i\theta}) = f(P)$ and so $z + re^{i\theta} \in E$ for all $0 < r < \varepsilon_z$ and all $\theta \in \mathbb{R}$. Because $z \in E$ by assumption, we have that $D(z, \varepsilon_z) \subset E$ and so E is open. Thus, E is nonempty, open, and relatively closed, and so because Ω is connected we must have that $\Omega = E$.

Proposition 7.7.4. Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{R}$ be continuous. Then f is subharmonic (in the sense given in these lecture notes) if and only if, whenever $\overline{D}(P, r) \subset \Omega$, h is harmonic in $D(P, r)$ and continuous on $\overline{D}(P, r)$, and $h \geq f$ on $\partial D(P, r)$, we have that $h \geq f$ in $D(P, r)$.

(Michael, Problem 4460) [Redacted]

(Timmy, Problem 4470) Begin the proof of Proposition 7.7.4 as follows. Suppose that $\Omega \subseteq \mathbb{C}$ is open and that $f : \Omega \to \mathbb{R}$ is continuous. Suppose further that whenever $\overline{D}(P, r) \subset \Omega$, h is harmonic in $D(P, r)$ and continuous on $\overline{D}(P,r)$, and $h \geq f$ on $\partial D(P,r)$, we have that $h \geq f$ in $D(P,r)$. Prove that f is subharmonic in the sense that f satisfies the sub-mean-value property

$$f(P) \leq \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{i\theta})\, d\theta$$

whenever $\overline{D}(P,r) \subset \Omega$.

(Zach, Problem 4480) Complete the proof of Proposition 7.7.4 and strengthen the result as follows. Let $\Omega \subseteq \mathbb{C}$ be open. Suppose that $f : \Omega \to \mathbb{R}$ is continuous and satisfies the small circle sub-mean-value property in Ω (as in Problem 4450). Show that whenever $\overline{D}(P,r) \subset \Omega$, h is harmonic in $D(P,r)$ and continuous on $\overline{D}(P,r)$, and $h \geq f$ on $\partial D(P,r)$, we have that $h \geq f$ in $D(P,r)$.

Choose some such P, r, and h. Then $f - h$ satisfies the small circle sub-mean-value property because h satisfies the full mean value property. Therefore, by Problem 4450, we have that $f - h$ satisfies the maximum principle. Because $\overline{D}(P,r)$ is compact, this implies that $\sup_{D(P,r)}(f-h) \leq \max_{\partial D(P,r)}(f-h) \leq 0$, as desired.

7.7. The Dirichlet Problem

[Definition: The Dirichlet problem] Let $\Omega \subsetneq \mathbb{C}$ be a nonempty connected open set. We say that the Dirichlet problem is well posed on Ω if, for every function f defined and continuous on $\partial\Omega$, there is exactly one function u that is harmonic in Ω, continuous on $\overline{\Omega}$, and such that $u = f$ on $\partial\Omega$.

(Alex, Problem 4490) Give an example of an unbounded domain Ω and two distinct functions u and v that are harmonic in Ω, continuous on $\overline{\Omega}$ and equal zero on $\partial\Omega$.

Let $\Omega = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, let $u(z) = 0$, and let $v(z) = \operatorname{Re} z$. It is straightforward to verify that u, v, and Ω satisfy the desired conditions.

(Clayton, Problem 4500) Prove that we have uniqueness for the Dirichlet problem in any bounded domain; that is, show that if $\Omega \subsetneq \mathbb{C}$ is bounded, if u and v are both harmonic in Ω and continuous on $\overline{\Omega}$, and if $u = v$ on $\partial\Omega$, then $u = v$ in Ω. Clearly explain how you used the fact that Ω is bounded.

(David, Problem 4510) Let $0 < r < 1$. Let $u(z) = \frac{1}{\log r} \log|z|$. Show that u is harmonic in the annulus $\Omega = D(0,1) \setminus D(0,r)$, continuous on $\overline{\Omega}$, and satisfies $u(e^{i\theta}) = 0$, $u(re^{i\theta}) = 1$ for any $0 \leq \theta \leq 2\pi$.

(Emily, Problem 4520) [Redacted]

(Irina, Problem 4530) Let $\Omega = D(0,1) \setminus \{0\}$. Suppose that u is harmonic in Ω, continuous on $\overline{\Omega}$, and that $u = 0$ on $\partial D(0,1)$. Prove that $u(0) = 0$. Is the Dirichlet problem well posed in Ω?

Definition 7.7.8. Let $\Omega \subsetneq \mathbb{C}$ be open and let $P \in \partial\Omega$. Suppose that the function $b : \overline{\Omega} \to \mathbb{R}$ has the following properties.

- **(i)** b is continuous on $\overline{\Omega}$,
- **(ii)** b is subharmonic in Ω,
- **(iii)** $b(z) \leq 0$ for all $z \in \overline{\Omega}$,
- **(iv)** $\{z \in \partial\Omega : b(z) = 0\} = \{P\}$. (That is, $b(P) = 0$ but $b(z) < 0$ for all other values of $z \in \partial\Omega$.)

Then we say that b is a *barrier* for Ω at P.

Theorem 7.8.1. Let $\Omega \subset \mathbb{C}$ be a nonempty bounded connected open set. The Dirichlet problem is well posed in Ω if and only if, for every $P \in \partial\Omega$, there is a function b_P that is a barrier for Ω at P.

(Problem 4540) If b is a barrier for Ω at P, show that $b(z) < 0$ for all $z \in \Omega$.

Example 7.7.9. The function $b(z) = \operatorname{Re} z - 1$ is a barrier for $\mathbb{D}$ at 1.

Example 7.7.10. Let $\Omega \subsetneq \mathbb{C}$ be a nonempty bounded open set. Let $P \in \partial\Omega$ satisfy $|P| \geq |z|$ for all $z \in \overline{\Omega}$. Then the function $b(z) = \operatorname{Re}(\frac{1}{P} z) - 1$ is a barrier for Ω at P.

Example 7.7.12. Suppose that $\Omega \subsetneq \mathbb{C}$ is open and $P \in \partial\Omega$. Suppose that there is an $r > 0$ such that there exists a barrier b_1 for $\Omega \cap D(P, r)$ at P. If $\varepsilon > 0$ is small enough, then the function

$$b_2(z) = \begin{cases} -\varepsilon & \text{if } z \in \overline{\Omega} \setminus D(P, r), \\ \max(-\varepsilon, b_1(z)) & \text{if } z \in \overline{\Omega} \cap D(P, r) \end{cases}$$

is a barrier for Ω at P.

(Michael, Problem 4550) Prove that the function b_2 in Example 7.7.12 is indeed a barrier for Ω at P.

(Timmy, Problem 4560) State and prove the converse to Example 7.7.12.

(Zach, Problem 4570) Show that there is a barrier for the domain $\Omega = \mathbb{C} \setminus [0, \infty) = \{re^{i\theta} : r > 0, 0 < \theta < 2\pi\}$ at the point $P = 0$.

Example 7.7.11. If $\Omega \subsetneq \mathbb{C}$ is an open set, $P \in \partial\Omega$, and $\mathbb{C} \setminus \Omega$ contains a line segment with one end point at P, then there exists a barrier for Ω at P.

(Alex, Problem 4580) Show that the barrier of Example 7.7.11 exists. You don't have to give an explicit formula for the barrier.

(Clayton, Problem 4590) Let $\Omega = \mathbb{D} \setminus \{0\}$. Show that there is no barrier for Ω at 0. *Hint*: Suppose that b satisfies all of the properties of a barrier except that $b(0) \leq 0$ instead of $b(0) = 0$. Show that b may be bounded above by a suitable modification of the harmonic function in Problem 4510 and see what you can conclude about b.

7.8. Real Analysis

(Memory 4600) Let (X, d) and (Y, ϱ) be metric spaces. Suppose that $X = F \cup D$, where F and D are (relatively) closed. Let $f : X \to Y$ be a function. Suppose that $f\big|_F$ and $f\big|_D$ are both continuous. Then f is continuous.

7.8. The Perròn Method and the Solution of the Dirichlet Problem

Recall [Theorem 7.8.1]: Let $\Omega \subset \mathbb{C}$ be a nonempty bounded connected open set. The Dirichlet problem is well posed in Ω if and only if, for every $P \in \partial\Omega$, there is a function b_P that is a barrier for Ω at P.

(David, Problem 4610) Prove the "only if" direction of Theorem 7.8.1: Suppose that the Dirichlet problem is well posed in Ω and $P \in \partial\Omega$ and construct a barrier for Ω at P.

Let $f(z) = -|z - P|$. Then $f : \partial\Omega \to \mathbb{R}$ is continuous, $f \leq 0$ on $\partial\Omega$, and $f(z) = 0$ if and only if $z = P$.

Let b_P be the solution to the Dirichlet problem with boundary data f; by assumption the Dirichlet problem is well posed and so such a solution must exist. By definition of the Dirichlet problem, b_P is continuous on $\overline{\Omega}$, harmonic (therefore subharmonic) in Ω, and satisfies $b_P = f$ on $\partial\Omega$ and therefore $b_P(P) = 0$, $b_P(z) < 0$ for all $z \in \partial\Omega \setminus \{P\}$. Since Ω is bounded, by the maximum principle we have that $b_P \leq 0$ in Ω as well as on the boundary. Thus b_P is a barrier for Ω at P.

(Problem 4620) Let $\Omega \subset \mathbb{C}$ be a nonempty bounded connected open set and let $f : \partial\Omega \to \mathbb{R}$ be continuous. Let $M = \max_{\partial\Omega} |f|$. Let

$$S = \{\psi \in C^0(\overline{\Omega}) : \psi \text{ is subharmonic in } \Omega \text{ and satisfies } \psi \leq f \text{ on } \partial\Omega\}$$

where $C^0(\overline{\Omega})$ is the set of all continuous functions on $\overline{\Omega}$. Then:

- The constant function $-M$ is in S.
- $\psi(z) \leq M$ for all $\psi \in S$ and all $z \in \overline{\Omega}$.

(Problem 4630) Let S, f, Ω be as in Problem 4620 and define

$$u(z) = \sup\{\psi(z) : \psi \in S\}.$$

Then $-M \leq u(z) \leq M$ (and $u(z)$ exists) for all $z \in \overline{\Omega}$. Show that $u(P) \leq f(P)$ for all $P \in \partial\Omega$.

(Emily, Problem 4640) Let $P \in \partial\Omega$ and let b_P be a barrier for Ω at P. Let $\varepsilon > 0$. Show that there exists a $C > 0$ large enough that

$$f(P) - \varepsilon/2 + C b_P(z) \leq f(z) \leq f(P) + \varepsilon/2 - C b_P(z)$$

for all $z \in \partial\Omega$.

Because f is continuous, there is a $\delta > 0$ such that if $z \in \partial\Omega$ and $|z - P| < \delta$, then $|f(z) - f(P)| < \varepsilon/2$. The set $\partial\Omega \setminus D(P,\delta)$ is closed and bounded, thus compact; because b_P is continuous, it attains its maximum on this set. Let $\mu = -\max_{\partial\Omega\setminus D(P,\delta)} b_P$; because $P \notin \partial\Omega \setminus D(P,\delta)$, we have that b_P is negative on this set and so $\mu > 0$.

Let $C \geq \frac{2M}{\mu}$.

If $z \in \partial\Omega \cap D(P,\delta)$, then

$$|f(z) - f(P)| < \varepsilon/2 \leq \varepsilon/2 - Cb_P(z)$$

by definition of δ and because $C > 0$ and $b_P(z) \leq 0$.

If $z \in \partial\Omega \setminus D(P,\delta)$, then

$$|f(z) - f(P)| \leq |f(z)| + |f(P)| \leq 2M \leq 2M\frac{-b_P(z)}{\mu} \leq -Cb_P(z) \leq \varepsilon/2 - Cb_P(z)$$

by definition of μ.

In either case $|f(z) - f(P)| < \varepsilon/2 - Cb_P(z)$, as desired.

(Irina, Problem 4650) Let $P \in \partial\Omega$ and let $\varepsilon > 0$. Show that there is a $\delta > 0$ such that if $|z - P| < \delta$ and $z \in \overline{\Omega}$ then $u(z) > f(P) - \varepsilon$. (This implies in particular that $u(P) = f(P)$ but also implies that u is lower semicontinuous at P.)

(Michael, Problem 4660) Show that u is continuous at P for all $P \in \partial\Omega$.

(Timmy, Problem 4670) Let $\psi \in S$ and let $\overline{D}(P,r) \subset \Omega$. Define

$$\varphi(w) = \begin{cases} \psi(w), & w \notin D(P,r), \\ \frac{1}{2\pi}\int_0^{2\pi} \psi(P + re^{i\theta}) \frac{r^2 - |w-P|^2}{|P+re^{i\theta}-w|^2}\,d\theta, & w \in D(P,r). \end{cases}$$

Show that φ is also in S and that $\varphi(\zeta) \geq \psi(\zeta)$ for all $\zeta \in \overline{\Omega}$.

(Zach, Problem 4680) Let $\overline{D}(P,r) \subset \Omega$ for some $r > 0$. For each $w \in D(P,r)$, show that there is a sequence of functions $\{\varphi_{w,n}\}_{n=1}^{\infty}$, all of which are in S, harmonic in $D(P,r)$, and nondecreasing on Ω, such that $u(w) = \lim_{n\to\infty} \varphi_{w,n}(w)$.

By definition of supremum, for each $n \in \mathbb{N}$ there is a $\theta_{w,n} \in S$ such that $u(w) - \frac{1}{n} < \theta_{w,n}(w) \leq u(w)$. Define $\psi_{w,n}(\zeta) = \max\{\theta_{w,k}(\zeta) : 1 \leq k \leq n\}$. Then $\psi_{w,n}$ is continuous on $\overline{\Omega}$ by Problem 4331 and is subharmonic in Ω by Problem 4390. Furthermore, $\psi_{w,n}(\zeta) \leq f(\zeta)$ for all $\zeta \in \partial\Omega$ because $\theta_{w,k}(\zeta) \leq f(\zeta)$ for all $\zeta \in \partial\Omega$. Thus $\psi_{w,n} \in S$.

Construct $\varphi_{w,n}$ from $\psi_{w,n}$ as in Problem 4670 with $\psi = \psi_{w,n}$. Then each $\varphi_{w,n}$ is harmonic in $D(P,r)$ by construction, $\varphi_{w,n} \in S$ by Problem 4670.

Because $\varphi_{w,n} \in S$, we have that $u(w) = \sup_{\psi\in S} \psi(w) \geq \varphi_{w,n}(w)$. But $\varphi_{w,n}(w) \geq \psi_{w,n}(w)$ by Problem 4670, and $\psi_{w,n}(w) \geq \theta_{w,n}(w)$ by definition of $\psi_{w,n}$. Thus $u(w) - \frac{1}{n} < \varphi_{w,n}(w) \leq u(w)$ for all n, and so $\varphi_{w,n}(w) \to u(w)$.

Finally, $\varphi_{w,n} \leq \varphi_{w,n+1}$ on $\overline{\Omega} \setminus D(P,r)$ because $\varphi_{w,n} = \psi_{w,n}$ on that set. But $\varphi_{w,n}$ and $\varphi_{w,n+1}$ are both harmonic in $D(P,r)$ and continuous on $\overline{D}(P,r)$, and $\varphi_{w,n} \leq \varphi_{w,n+1}$ on $\partial D(P,r)$, so by the maximum principle $\varphi_{w,n} \leq \varphi_{w,n+1}$ in $D(P,r)$ as well, and thus on all of $\overline{\Omega}$, as desired.

(Alex, Problem 4690) For each w, $z \in D(P,r)$, show that there is a sequence of functions $\{\varphi_{w,z,n}\}_{n=1}^{\infty}$, all of which are in S, harmonic in $D(P,r)$, and nondecreasing on Ω, such that $u(w) = \lim_{n\to\infty} \varphi_{w,z,n}(w)$, $u(z) = \lim_{n\to\infty} \varphi_{w,z,n}(z)$, and if $\zeta \in \Omega$ then $\varphi_{z,w}(\zeta) \geq \varphi_z(\zeta)$ and $\varphi_{z,w}(\zeta) \geq \varphi_w(\zeta)$.

Recall [Theorem 7.6.3]: (Harnack's principle.) Let $\{u_k\}_{k=1}^{\infty}$ be a sequence of real-valued functions harmonic in a connected open set $\Omega \subseteq \mathbb{C}$ such that $u_1(z) \leq u_2(z) \leq u_3(z) \leq \cdots$ for each $z \in \Omega$. Then either $u_k \to \infty$ uniformly on compact sets or there is a harmonic function $u : \Omega \to \mathbb{R}$ such that $u_k \to u$ uniformly on compact sets.

(Clayton, Problem 4700) If w, $z \in D(P, r)$, let

$$u_w(\zeta) = \lim_{n\to\infty} \varphi_{w,n}(\zeta), \qquad u_{z,w}(\zeta) = \lim_{n\to\infty} \varphi_{z,w,n}(\zeta).$$

Then u_w and $u_{z,w}$ are both harmonic in $D(P, r)$. Show that $u_w(\zeta) = u_{z,w}(\zeta)$ for all $\zeta \in D(P, r)$.

(David, Problem 4710) Complete the proof of Theorem 7.8.1.

We have uniqueness of solutions to the Dirichlet problem by Problem 4500. We need only establish existence. Let $f : \partial\Omega \to \mathbb{R}$ be continuous and let u be as in Problem 4630. Then $u = f$ on $\partial\Omega$ by Problem 4650, and u is continuous at P for every $P \in \partial\Omega$ by Problem 4660.

Thus, we need only show that u is harmonic (thus continuous) in Ω. Let $\overline{D}(P, r) \subset \Omega$. Because harmonicity is a local property, it suffices to show that u is harmonic in $D(P, r)$ for all such $D(P, r)$.

For any z, $w \in D(P, r)$, let u_z and $u_{z,w}$ be as in Problem 4700. Then $u(w) = u_w(w)$ by Problem 4680 and definition of u_w. But $u_w(w) = u_{w,P}(w) = u_P(w)$ by Problem 4700. Thus $u = u_P$ in $D(P, r)$. But u_P is harmonic in $D(P, r)$ by Harnack's principle (as observed in Problem 4700), and so u is harmonic in $D(P, r)$, as desired.

7.9. Real Analysis

(Memory 4720) (The Bolzano-Weierstraß theorem.) Suppose that $\{z_n\}_{n=1}^{\infty} \subset \mathbb{R}^d$ is bounded. Then there is a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ that converges as $k \to \infty$.

7.9. Conformal Mappings of Annuli

Theorem 7.9.1. Let $0 < r_1 < R_1 < \infty$ and $0 < r_2 < R_2 < \infty$. Let $P_1 \in \mathbb{C}$ and $P_2 \in \mathbb{C}$. Let $A_1 = \{z \in \mathbb{C} : r_1 < |z - P_1| < R_1\}$ and $A_2 = \{z \in \mathbb{C} : r_2 < |z - P_2| < R_2\}$. Then A_1 and A_2 are conformally equivalent (meaning that there is a holomorphic bijection $\varphi : A_1 \to A_2$) if and only if $R_1/r_1 = R_2/r_2$.

[Chapter 7, Problem 37] Let $A = \{z \in \mathbb{C} : 1/R < |z| < R\}$ be an annulus for some $R > 1$. Find all conformal self-maps of A.

(Emily, Problem 4730) Prove the straightforward direction of Theorem 7.9.1; that is, assume $R_1/r_1 = R_2/r_2$ and prove that A_1 and A_2 are conformally equivalent.

Define $\varphi(z) = P_2 + \frac{R_2}{R_1}(z - P_1)$. φ is clearly a holomorphic bijection from $\mathbb{C}$ to $\mathbb{C}$, so we need only show that $\varphi(A_1) = A_2$.

Because R_2/R_1 is a positive real number, we have that $r_1 < |z - P_1| < R_1$ if and only if $\frac{R_2}{R_1} r_1 < \frac{R_2}{R_1}|z - P_1| < R_2$. But $|\varphi(z) - P_2| = \frac{R_2}{R_1}|z - P_1|$, and because $\frac{r_1}{R_1} = \frac{r_2}{R_2}$, we have that $r_1 < |z - P_1| < R_1$ if and only if $r_2 < |\varphi(z) - P_2| < R_2$, that is, $z \in A_1$ if and only if $\varphi(z) \in A_2$, as desired.

(Problem 4740) Suppose that Theorem 7.9.1 is true in the special case where and $P_1 = P_2 = 0$. Show that Theorem 7.9.1 is true for any P_1, $P_2 \in \mathbb{C}$.

(Irina, Problem 4750) Suppose that there exists some $\varphi : A_1 \to A_2$ that is a holomorphic bijection. Let $z_\infty \in \partial A_1$. Let $\{z_n\}_{n=1}^{\infty} \subset A_1$ with $z_n \to z_\infty$. Then $\{\varphi(z_n)\}_{n=1}^{\infty} \subset A_2$ is bounded, so by the Bolzano-Weierstraß theorem there is a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ such that $\varphi(z_{n_k})$ converges. Show that $\varphi(z_{n_k})$ converges to a point in ∂A_2. *Hint*: You only need that φ and φ^{-1} are continuous, not that they are holomorphic.

Let $\varphi(z_{n_k}) \to w_\infty$. By definition of $\overline{A}_2$, because $\{\varphi(z_n)\} \subset A_2$, we must have that $w_\infty \in \overline{A}_2$.

Suppose $w_\infty \notin \partial A_2$. Then $w_\infty \in A_2$, and so there is a $v \in A_1$ with $\varphi(v) = w_\infty$. We have that $\varphi^{-1} : A_2 \to A_1$ is continuous. Therefore, $\varphi^{-1}(\varphi(z_n)) \to \varphi^{-1}(w_\infty)$ because $\varphi(z_n) \to w_\infty$. But $\varphi^{-1}(\varphi(z_n)) = z_n$ and so $z_n \to v$. But $z_n \to z_\infty$ and so $v = z_\infty$. This is a contradiction because $v \in A_1$ and $z_\infty \in \partial A_1$ and A_1 is open.

(Lemma 4760) Let

$$A_1 = \{z \in \mathbb{C} : r_1 < |z| < R_1\}, \qquad A_2 = \{z \in \mathbb{C} : r_2 < |z| < R_2\}$$

for some $0 < r_1 < R_1 < \infty$, $0 < r_2 < R_2 < \infty$. Suppose that $\varphi : A_1 \to A_2$ is a holomorphic bijection.

Let $\{z_n\}_{n=1}^{\infty} \subset A_1$ and $\{\zeta_m\}_{m=1}^{\infty} \subset A_1$ with

$$z_n \to R_1, \qquad \zeta_m \to r_1.$$

By passing to subsequences, we may assume that $\{\varphi(z_n)\}_{n=1}^{\infty}$ and $\{\varphi(\zeta_m)\}_{m=1}^{\infty}$ also converge. By Problem 4750,

$$\text{either } |\varphi(z_n)| \to R_2 \text{ or } |\varphi(z_n)| \to r_2,$$
$$\text{either } |\varphi(\zeta_m)| \to R_2 \text{ or } |\varphi(\zeta_m)| \to r_2.$$

We then have that

(a) If $|\varphi(z_n)| \to R_2$, then $\lim\limits_{z\to w} |\varphi(z)| = R_2$ for all $w \in \partial D(0, R_1)$.

(b) If $|\varphi(z_n)| \to r_2$, then $\lim\limits_{z\to w} |\varphi(z)| = r_2$ for all $w \in \partial D(0, R_1)$.

(c) If $|\varphi(\zeta_m)| \to R_2$, then $\lim\limits_{z\to w} |\varphi(z)| = R_2$ for all $w \in \partial D(0, r_1)$.

(d) If $|\varphi(\zeta_m)| \to r_2$, then $\lim\limits_{z\to w} |\varphi(z)| = r_2$ for all $w \in \partial D(0, r_1)$.

(Problem 4770) We now begin the proof of Lemma 4760a. Let $r_2 < \rho < R_2$. Let $s_\rho = \sup\{|\varphi^{-1}(\rho e^{i\theta})| : 0 \leq \theta \leq 2\pi\}$. Show that $r_1 < s_\rho < R_1$.

(Michael, Problem 4780) Let $r_2 < \rho < R_2$. Let $\Omega_\rho^- = \{z \in A_1 : |\varphi(z)| < \rho\}$ and let $\Omega_\rho^+ = \{z \in A_1 : |\varphi(z)| > \rho\}$. Show that Ω_ρ^+ and Ω_ρ^- are disjoint connected open sets.

(Timmy, Problem 4790) Suppose that $|\varphi(z_n)| \to R_2$. Show that $|\varphi(z)| > \rho$ for all $z \in A_1$ with $|z| > s_\rho$.

(Zach, Problem 4800) Suppose that $|\varphi(z_n)| \to R_2$. Show that $\lim_{z\to w} |\varphi(z)| = R_2$ for all $w \in \partial D(0, R_1)$.

Let $w \in \partial D(0, R_1)$. Choose $\varepsilon > 0$; we may assume without loss of generality that $\varepsilon < R_2 - r_2$. Let $\rho = R_2 - \varepsilon$; then $r_2 < \rho < R_2$. Let s_ρ be as in Problem 4770; then $r_1 < s_\rho < R_1$. Let $\delta = R_1 - s_\rho$, so $\delta > 0$.

If $z \in A_1$ and $|w - z| < \delta$, then $|z| > |w| - \delta = R_1 - (R_1 - s_\rho) = s_\rho$ by the triangle inequality, and so $|\varphi(z)| > \rho = R_2 - \varepsilon$ by the previous problem. Because $\varphi : A_1 \to A_2$, we have that $|\varphi(z)| \leq \sup_{\zeta\in A_2} |\zeta| = R_2$, and so $R_2 - \varepsilon < |\varphi(z)| \leq R_2$ for all $z \in A_1$ with $|z - w| < \delta$. This completes the proof.

(Fact 4810) We can similarly prove the other three cases of Lemma 4760.

(Alex, Problem 4820) Let $\Omega \subseteq \mathbb{C}$ be open and let $f : \Omega \to \mathbb{C} \setminus \{0\}$ be holomorphic. Show that $u(z) = \log |f(z)|$ is harmonic in Ω. (Note that Ω may not be simply connected, and so Lemma 6.6.4 cannot be applied in Ω.)

Suppose that $D(P, r) \subseteq \Omega$. Then $D(P, r)$ is simply connected, and so we can apply Lemma 6.6.4 in $D(P, r)$. Thus there is an $h : D(P, r) \to \mathbb{C}$ holomorphic such that $f(z) = e^{h(z)}$ for all $z \in D(P, r)$. Taking the modulus and then the logarithm, we have that $\log |f(z)| = \log |e^{h(z)}| = \log e^{\operatorname{Re} h(z)} = \operatorname{Re} h(z)$ for all $z \in D(P, r)$. Because h is holomorphic we have that $\operatorname{Re} h(z)$ is harmonic in $D(P, r)$. Thus u is harmonic in $D(P, r)$ for all $D(P, r) \subseteq \Omega$. Because harmonicity is a local property, this implies that u is harmonic in Ω, as desired.

(Clayton, Problem 4830) Let A_1, A_2, and φ be as in Lemma 4760. Let $u(z) = \log |\varphi(z)|$ for all $z \in A_1$, $u(z) = \lim_{\zeta\to z} |\varphi(\zeta)|$ for all $z \in \partial A_1$. Then u is continuous on $\overline{A_1}$ and harmonic in A_1. Show that there exist constants α and β such that

$$\log |\varphi(z)| = u(z) = \alpha + \beta \log |z|$$

for all $z \in \overline{A_1}$.

(David, Problem 4840) Let $\Omega = A_1 \setminus (-\infty, 0)$ be the annulus with the negative real numbers deleted. Then the function log can be defined such that it is holomorphic on Ω. Show that there is a $\omega \in \mathbb{C}$ such that $\varphi(z) = \omega e^{\beta \log z}$ for all $z \in \Omega$, where β is as in the previous problem.

(Emily, Problem 4850) Given that φ is continuous on A_1 (and so continuous up to the negative reals), what must be true about the number β? Given that φ is injective on A_1 (and in particular on $\{e^{i\theta} : -\pi < \theta \leq \pi\}$), what must be true about the number β?

[Chapter 6, Problem 31] If f is a fractional linear transformation, c and C are concentric circles, and $f(c)$ and $f(C)$ are also concentric circles, then the ratio of the radii of c and C is equal to the ratio of the radii of $f(c)$ and $f(C)$.

(Problem 4860) Prove Theorem 7.9.1.

8.1. Basic Concepts Concerning Infinite Sums and Products

[Chapter 8, Problem 2] Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of complex numbers. Suppose $A_k \neq 0$ for all k and that $\lim_{N\to\infty} \prod_{k=1}^{N} A_k$ exists and is a finite nonzero complex number. Show that $\lim_{k\to\infty} A_k = 1$.

(Alex, Problem 4870) Do we still have that $\lim_{k\to\infty} A_k = 1$ if we allow $\lim_{N\to\infty} \prod_{k=1}^{N} A_k$ to equal zero? Do we still have that $\lim_{k\to\infty} A_k = 1$ if we allow A_k to equal zero for some k?

No. For example, if we take $A_k = 1/2$ for all k then $\lim_{N\to\infty} \prod_{k=1}^{N} A_k = 0$ but $\lim_{k\to\infty} A_k = 1/2 \neq 1$. If $A_n = 0$ for some n then $\lim_{N\to\infty} \prod_{k=1}^{N} A_k = 0$ regardless of the values of A_k for $k \neq n$.

Definition 8.1.1. Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence of complex numbers. We say that the infinite product

$$\prod_{n=1}^{\infty} (1 + a_n)$$

converges if there is a $N_0 \in \mathbb{N}$ such that

$$\lim_{N\to\infty} \prod_{n=N_0}^{N} (1 + a_n) \in \mathbb{C} \setminus \{0\}.$$

(David, Problem 4880) Show that if $\prod_{n=1}^{\infty}(1 + a_n)$ converges, then $(1 + a_n) \neq 0$ for all but finitely many n, and $a_n \to 0$.

Lemma 8.1.2. If $0 \leq x \leq 1$, then $1 + x \leq e^x \leq 1 + 2x$.

(Emily, Problem 4890) Prove Lemma 8.1.2.

Let $f(x) = 1 + x$, $g(x) = 1 + 2x$. Observe that $f(0) = \exp(0) = g(0)$.

We compute that $f'(x) = 1$, $g'(x) = 2$, and $\exp'(x) = \exp x$. Because exp is increasing, we have that $\exp'(x) \geq f'(x)$ for all $x \geq 0$, and so by the mean value theorem, $\exp(x) - f(x) \geq 0$ or $\exp(x) \geq 1 + x$ for all $x \geq 1$.

We compute that $\exp(0) = 1 = g(0)$, $\exp(1) = e < 3 = g(1)$, and $\exp(2) = e^2 > 5 = g(2)$. Because g and exp are continuous, by the intermediate value theorem there is an $x_0 \in (1, 2)$ with $\exp(x_0) = g(x_0)$.

Now, $\exp''(x) - g''(x) = \exp(x) > 0$ for all x. This implies that $\exp'(x) - g'(x)$ has at most one zero. If $\exp(y) = g(y)$ and $\exp(z) = g(z)$, then by Rolle's theorem there is an x between y and z such that $\exp'(x) = g'(x)$. Thus, we can have $\exp(z) = g(z)$ for at most two values of z. In particular, $\exp - g \neq 0$ on $(0, x_0)$. Again by the intermediate value theorem we must have either that $\exp(x) > g(x)$ or $\exp(x) < g(x)$ for all $x \in (0, x_0)$; since $\exp(1) < g(1)$ we must have that $\exp(x) < g(x)$ for all $x \in (0, x_0) \supset (0, 1]$. Since $\exp(0) \leq g(0)$ we have $\exp(x) \leq g(x) = 1 + 2x$ on $[0, 1]$, as desired.

Corollary 8.1.3. If $\{a_k\}_{k=1}^{n} \subset \mathbb{C}$ with $|a_k| < 1$ for each $1 \leq k \leq n$, then $\exp\left(\frac{1}{2}\sum_{k=1}^{n} |a_k|\right) \leq \prod_{k=1}^{n}(1 + |a_k|) \leq \exp\left(\sum_{k=1}^{n} |a_k|\right)$.

(Irina, Problem 4900) Prove Corollary 8.1.3.

By Lemma 8.1.2, and by monotonicity of products of positive real numbers,

$$\prod_{k=1}^{n} \exp\left(\frac{1}{2}|a_k|\right) \leq \prod_{k=1}^{n}(1+|a_k|) \leq \prod_{k=1}^{n} \exp(|a_k|).$$

Applying properties of exponents completes the proof.

(Problem 4910) Suppose that $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}$. Show that either $\prod_{k=1}^{\infty}(1+|a_k|)$ converges to a positive real number or $\lim_{N\to\infty} \prod_{k=1}^{N}(1+|a_k|) = \infty$.

Corollary 8.1.4. If $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}$ with $\sum_{k=1}^{\infty} |a_k| < \infty$, then $\prod_{k=1}^{\infty}(1+|a_k|)$ converges.

(Michael, Problem 4920) Prove Corollary 8.1.4.

Recall from real analysis that, if $\sum_{k=1}^{\infty} |a_k| < \infty$, then $\lim_{k\to\infty} a_k = 0$. In particular there is a $N_0 \in \mathbb{N}$ such that if $k \geq N_0$ then $|a_k| = |a_k - 0| < 1$.

By Corollary 8.1.3, if $N \geq N_0$ then

$$\prod_{k=N_0}^{N} (1+|a_k|) \leq \exp\Big(\sum_{k=N_0}^{N} |a_k|\Big) \leq \exp\Big(\sum_{k=1}^{\infty} |a_k|\Big) < \infty$$

and so the partial products are uniformly bounded by a finite number. This means that they cannot converge to ∞, and so must converge to a finite positive number.

Corollary 8.1.5. If $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}$ and $\prod_{k=1}^{\infty}(1+|a_k|)$ converges, then $\sum_{k=1}^{\infty} |a_k| < \infty$.

(Timmy, Problem 4930) Prove Corollary 8.1.5.

Let $L = \lim_{n\to\infty} \prod_{k=1}^{n}(1+|a_k|)$. By assumption L is finite; because the sequence of partial products is nondecreasing, we must have that $L \geq \prod_{k=1}^{n}(1+|a_k|)$ for any $n \in \mathbb{N}$.

We claim that $\prod_{k=1}^{n}(1+|a_k|) \geq 1 + \sum_{k=1}^{n} |a_k|$. This is clearly true if $n = 1$. If it is true for some n, then

$$\begin{aligned}
\prod_{k=1}^{n+1}(1+|a_k|) = (1+|a_{n+1}|)\prod_{k=1}^{n}(1+|a_k|) &\geq (1+|a_{n+1}|)\Big(1+\sum_{k=1}^{n} |a_k|\Big) \\
&= 1 + \Big(\sum_{k=1}^{n} |a_k|\Big) + |a_{n+1}| + |a_{n+1}|\sum_{k=1}^{n} |a_k| \\
&\geq 1 + \Big(\sum_{k=1}^{n} |a_k|\Big) + |a_{n+1}| = 1 + \sum_{k=1}^{n+1} |a_k|
\end{aligned}$$

and so the claim is true by induction. Thus $\sum_{k=1}^{n} |a_k| \leq L - 1$ for all $n \in \mathbb{N}$, and so the series $\sum_{k=1}^{\infty} |a_k|$ converges.

Theorem 8.1.7. If $\prod_{k=1}^{\infty}(1+|a_k|)$ converges, then $\prod_{k=1}^{\infty}(1+a_k)$ converges.

(Zach, Problem 4940) (This is the first step in the proof of Theorem 8.1.7.) Let $\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}$. Show that if $N \geq M \geq 1$, then

$$\left|\left(\prod_{k=M}^{N} (1+a_k)\right) - 1\right| \leq \left(\prod_{k=M}^{N} (1+|a_k|)\right) - 1.$$

Hint: Use induction.

Fix some $M \geq 1$.

If $N = M$, then

$$\left|\left(\prod_{k=M}^{M} (1+a_k)\right) - 1\right| = |(1+a_M) - 1| = |a_M| = (1+|a_M|) - 1 = \left(\prod_{k=M}^{M} (1+|a_k|)\right) - 1.$$

Now, suppose that $N > M$ and the inequality is true for $N-1$. Then

$$\left|\left(\prod_{k=M}^{N}(1+a_k)\right)-1\right| = \left|(1+a_N)\left(\prod_{k=M}^{N-1}(1+a_k)\right)-1\right|$$
$$\leq \left|(1+a_N)\left(\prod_{k=M}^{N-1}(1+a_k)\right)-1-a_N\right|+|a_N|$$
$$= |1+a_N|\left|\left(\prod_{k=M}^{N-1}(1+a_k)\right)-1\right|+|a_N|$$
$$\leq (1+|a_N|)\left|\left(\prod_{k=M}^{N-1}(1+a_k)\right)-1\right|+|a_N|.$$

By the induction hypothesis,

$$\left|\left(\prod_{k=M}^{N}(1+a_k)\right)-1\right| \leq (1+|a_N|)\left|\left(\prod_{k=M}^{N-1}(1+a_k)\right)-1\right|+|a_N|$$
$$\leq (1+|a_N|)\left(\left(\prod_{k=M}^{N-1}(1+|a_k|)\right)-1\right)+|a_N|$$
$$= \left(\left(\prod_{k=M}^{N}(1+|a_k|)\right)-1-|a_N|\right)+|a_N|$$
$$= \left(\prod_{k=M}^{N}(1+|a_k|)\right)-1$$

as desired; thus, by induction the inequality is true for all $N \geq M$.

(Problem 4950) Show that if $\prod_{k=1}^{\infty}(1+|a_k|)$ converges, then there is a N_0 such that if $n \geq N_0$ then $1+a_n \neq 0$.

(Alex, Problem 4960) Show that if $\prod_{k=1}^{\infty}(1+|a_k|)$ converges and N_0 is as in the previous problem, then

$$\left\{\prod_{k=N_0}^{N}(1+a_k)\right\}_{N=N_0}^{\infty}$$

is a Cauchy sequence and so converges.

Choose some $\varepsilon > 0$.

If $\prod_{k=1}^{\infty}(1+|a_k|)$ converges, then the sequence of partial products is Cauchy and so there is some N_0 such that if $N > M > N_0$ then

$$\left|\prod_{k=1}^{N}(1+|a_k|)-\prod_{k=1}^{M}(1+|a_k|)\right| < \varepsilon.$$

Now, if $N > M > N_0$ then

$$\left|\prod_{k=1}^{N}(1+a_k)-\prod_{k=1}^{M}(1+a_k)\right| = \left|\left(\prod_{k=M+1}^{N}(1+a_k)\right)-1\right|\left|\prod_{k=1}^{M}(1+a_k)\right|$$

by the above remarks. By Problem 4940,

$$\Big|\prod_{k=1}^{N}(1+a_k)-\prod_{k=1}^{M}(1+a_k)\Big| \leq \Big|\Big(\prod_{k=M+1}^{N}(1+a_k)\Big)-1\Big|\Big|\prod_{k=1}^{M}(1+a_k)\Big|$$
$$\leq \Big(\Big(\prod_{k=M+1}^{N}(1+|a_k|)\Big)-1\Big)\Big(\prod_{k=1}^{M}(1+|a_k|)\Big)$$
$$= \Big(\Big(\prod_{k=1}^{N}(1+|a_k|)\Big)-\prod_{k=1}^{M}(1+|a_k|)\Big) < \varepsilon$$

as desired.

(David, Problem 4970) Show that if $\prod_{k=1}^{\infty}(1+|a_k|)$ converges and N_0 is as in the previous problem, then

$$\inf_{N\geq N_0}\Big|\prod_{k=N_0}^{N}(1+a_k)\Big| > 0$$

and so $\prod_{k=N_0}^{N}(1+a_k)$ does not converge to zero. This completes the proof of Theorem 8.1.7.

By Corollary 8.1.5 we have that $\sum_{k=1}^{\infty}|a_k| < \infty$. Let K be such that $\sum_{k=K+1}^{\infty}|a_k| < \ln\frac{3}{2}$.

Let $m = \min\Big\{\Big|\prod_{k=N_0}^{N}(1+a_k)\Big| : N_0 \leq N \leq K\Big\}$; because $1+a_k \neq 0$ for all $k \geq N_0$ we have that $0 < m < \infty$ and

$$\Big|\prod_{k=N_0}^{N}(1+a_k)\Big| \geq m$$

for all $N_0 \leq N \leq K$.

Suppose that $N > K$. Then by Problem 4940,

$$\Big|\prod_{k=N_0}^{N}(1+a_k)\Big| = \Big|\prod_{k=N_0}^{K}(1+a_k)\Big|\Big|\prod_{k=K+1}^{N}(1+a_k)-1+1\Big|$$
$$\geq m\Big(1-\Big(\prod_{k=K+1}^{N}(1+|a_k|)-1\Big)\Big)$$

By Corollary 8.1.3,

$$\Big|\prod_{k=N_0}^{N}(1+a_k)\Big| \geq m\Big(2-\exp\Big(\sum_{k=K+1}^{N}|a_k|\Big)\Big)$$
$$\geq m\Big(2-\exp\Big(\sum_{k=K+1}^{\infty}|a_k|\Big)\Big) \geq \frac{m}{2}.$$

Thus,

$$\inf_{N\geq N_0}\Big|\prod_{k=N_0}^{N}(1+a_k)\Big| \geq \frac{m}{2} > 0$$

as desired.

Corollary 8.1.8. If $\sum_{k=1}^{\infty}|a_k|$ converges, then $\prod_{k=1}^{\infty}(1+a_k)$ converges. (This follows immediately from Corollary 8.1.5 and Theorem 8.1.7, so you may use the result at once.)

[Definition: Notation for multiplicity] If $\Omega \subseteq \mathbb{C}$ is open, $f : \Omega \to \mathbb{C}$ is holomorphic, $z \in \Omega$, and f is not identically zero in the connected component of Ω containing z, then we let $\mathrm{mult}_f(z)$ be the multiplicity of the zero of f at z. For convenience, if $f(z) \neq 0$ we take $\mathrm{mult}_f(z) = 0$.

Theorem 8.1.9. Let $\Omega \subseteq \mathbb{C}$ be open. Let $f_n : \Omega \to \mathbb{C}$ be holomorphic. Suppose that

$$\sum_{n=1}^{\infty} |f_n|$$

converges uniformly on compact sets.

Then:

- $\prod_{n=1}^{N}(1+f_n(z))$ converges to some function $F : \Omega \to \mathbb{C}$, uniformly on all compact sets.
- The limit function F is holomorphic in Ω.
- If $z_0 \in \Omega$ then $F(z_0) = 0$ if and only if $1 + f_n(z_0) = 0$ for some $n \in \mathbb{N}$.
- If $z_0 \in \Omega$ then $1 + f_n(z_0) = 0$ for at most finitely many $n \in \mathbb{N}$.
- If $F \equiv 0$ then $1 + f_n \equiv 0$ for some $n \in \mathbb{N}$.
- If $F \not\equiv 0$ and $F(z_0) = 0$ then the multiplicity of the zero at z_0 of F is the sum of the multiplicities of the zeroes of the functions $1 + f_n$ at z_0.

(Emily, Problem 4980) Let f_n and Ω be as in Theorem 8.1.9. Show that if $K \subset \Omega$ is compact then $\sup\{|f_n(z)| : n \in \mathbb{N},\ z \in K\}$ is finite (that is, that the functions $\{f_n : n \in \mathbb{N}\}$ are uniformly bounded on K).

(Irina, Problem 4990) (This is the first step of the proof of Theorem 8.1.9.) Let f_n and Ω be as in Theorem 8.1.9 and let $K \subset \Omega$ be compact. By Corollary 8.1.8, if $z \in K$ then $\prod_{n=1}^{\infty}(1+f_n(z))$ converges. Show that there is a $L \in \mathbb{R}$ such that

$$\left|\prod_{n=1}^{N}(1+f_n(z))\right| < L$$

for all $z \in K$ and all $N \in \mathbb{N}$.

(Michael, Problem 5000) Let f_n and K be as in Problem 4990. Show that the sequence $\left\{\prod_{n=1}^{N}(1+f_n(z))\right\}_{N=1}^{\infty}$ is uniformly Cauchy, and thus converges uniformly.

Fix $\varepsilon > 0$. Let P be such that $\sum_{n=P}^{\infty} |f_n(z)| < \varepsilon$ for all $z \in K$; by uniform convergence of $\sum_{n=1}^{\infty} |f_n(z)|$, we have that P exists and does not depend on z.

Suppose $M > N \geq P$. If $z \in K$ then

$$\begin{aligned}
\left|\prod_{n=1}^{N}(1+f_n(z)) - \prod_{n=1}^{M}(1+f_n(z))\right| &= \left|\prod_{n=1}^{N}(1+f_n(z))\right|\left|\prod_{n=N+1}^{M}(1+f_n(z)) - 1\right| \\
&\leq L\Big(\prod_{n=N+1}^{M}(1+|f_n(z)|) - 1\Big) \\
&\leq L\Big(\exp\Big(\sum_{n=N+1}^{M}|f_n(z)|\Big) - 1\Big) \\
&\leq L\Big(\exp\Big(\sum_{n=P}^{\infty}|f_n(z)|\Big) - 1\Big) \leq L(e^{\varepsilon} - 1)
\end{aligned}$$

by Corollary 8.1.3 and Problems 4990 and 4940. Because exp is continuous at 0 and $\exp 0 = 1$, we may make the right hand side arbitrarily small by choosing ε appropriately; thus, we have that $\left\{\prod_{n=1}^{N}(1+f_n(z))\right\}_{N=1}^{\infty}$ is uniformly Cauchy.

(Timmy, Problem 5010) Let f_n and Ω be as in Theorem 8.1.9. What do Corollary 8.1.8 and Definition 8.1.1 tell you about $f_n(z)$?

By Corollary 8.1.8, we have that for each $z \in \Omega$, the limit $\lim_{N\to\infty} \prod_{n=1}^{N}(1+f_n(z))$ exists. Furthermore, there is a $N_0 = N_0(z) \in \mathbb{N}$ such that $1+f_n(z) \neq 0$ for all $n \geq N_0(z)$, and that $\lim_{N\to\infty} \prod_{n=N_0(z)}^{N}(1+f_n(z)) \in \mathbb{C} \setminus \{0\}$. In particular, $\lim_{N\to\infty} \prod_{n=1}^{N}(1+f_n(z)) = 0$ if and only if $1 + f_n(z) = 0$ for at least one $n \in \mathbb{N}$.

(Zach, Problem 5020) Let $\Omega \subseteq \mathbb{C}$ be open and connected and let $S \subset \Omega$ be a set with no accumulation points in Ω. Show that S is a countable set. Conclude that if $f : \Omega \to \mathbb{C}$ is holomorphic and not identically zero, then f has at most countably many zeroes.

Because $S \subset \Omega$, we have in particular that no $z \in S$ is an accumulation point for S. Thus for each $z \in S$ there is a $r_z > 0$ such that $D(z, r_z) \setminus \{z\}$ contains no points of S.

By density of points with rational coordinates, there is some w_z such that $\operatorname{Re} w_z$ and $\operatorname{Im} w_z$ are both rational and such that $|z - w_z| < \frac{1}{2} r_z$.

If $z, \zeta \in S$, then $z \notin D(\zeta, r_\zeta)$ and $\zeta \notin D(z, r_z)$, and so $|z-\zeta| \geq \max(r_\zeta, r_z) \geq \frac{1}{2} r_\zeta + \frac{1}{2} r_z$. By the triangle inequality $|w_z - w_\zeta| > |z - \zeta| - |z - w_z| - |\zeta - w_\zeta| > 0$ and so $w_z \neq w_\zeta$.

Thus there is an injective function φ from S to $\mathbb{Q} + i\mathbb{Q} = \{x + iy : x, y \in \mathbb{Q}\}$ given by $\varphi(z) = w_z$. Because $\mathbb{Q} + i\mathbb{Q}$ is countable, S must be countable as well.

By Theorem 3.6.1, if f is holomorphic in Ω and not identically zero, then $S = \{z \in \Omega : f(z) = 0\}$ has no accumulation points in Ω; thus S contains countably many points, as desired.

(Alex, Problem 5030) Let f_n and Ω be as in Theorem 8.1.9. Let $F(z) = \prod_{n=1}^{\infty}(1 + f_n(z))$. Suppose that $f_n \not\equiv 0$ for any $n \in \mathbb{N}$. Show that F has at most countably many zeroes, and so cannot be identically zero.

By Problem 5010, $F(z) = 0$ if and only if $1 + f_n(z) = 0$ for some $n \in \mathbb{N}$. Thus

$$\{z \in \Omega : F(z) = 0\} = \bigcup_{n=1}^{\infty} \{z \in \Omega : 1 + f_n(z) = 0\}.$$

By the previous problem, each of the sets on the right hand side is countable, and so their union is countable.

(Memory 5040) Let Ω, f_n and F be as in Theorem 8.1.9. Then $F(z_0) = 0$ if and only if $1 + f_n(z_0) = 0$ for some $n \geq 1$. Furthermore $1 + f_n(z_0) = 0$ for at most finitely many values of n.

(David, Problem 5050) Let Ω, f_n and F be as in Theorem 8.1.9. Suppose that F is not identically equal to zero. Let $z_0 \in \Omega$ and suppose that $F(z_0) = 0$. Show that the multiplicity of the zero of F at z_0 is equal to the sum of the multiplicities of the zeros of $1 + f_n$ at z_0.

Because F is holomorphic and $F \not\equiv 0$, there is an open set U with $z_0 \in U \subseteq \Omega$ such that $F \neq 0$ on $U \setminus \{z_0\}$. This implies that $1 + f_n \neq 0$ on $U \setminus \{z_0\}$. Let N_0 be such that $1 + f_n(z_0) \neq 0$ for all $n \geq N_0$; such an N_0 exists by the previous problem.

Thus, we have that in U, $h(z) = \prod_{n=N_0}^{\infty}(1 + f_n(z))$ is holomorphic and nonzero. For each n with $1 \leq n < N_0$, let m_n be the multiplicity of the zero of $1 + f_n$ at z_0 (with $m_n = 0$ if $1 + f_n(z_0) \neq 0$); by definition of multiplicity $1 + f_n(z) = (z - z_0)^{m_n} h_n(z)$ in U where h_n is holomorphic and nonzero in U.

Then, if $z \in U$, then

$$F(z) = \prod_{n=1}^{N_0-1}(1+f_n(z)) \prod_{n=N_0}^{\infty}(1+f_n(z)) = \left(\prod_{n=1}^{N_0-1}(z - z_0)^{m_n}\right)\left(\prod_{n=1}^{N_0-1} h_n(z)\right) h(z) = (z-z_0)^{\sum_{n=1}^{N_0-1} m_n} \left(\prod_{n=1}^{N_0-1} h_n(z)\right) h(z).$$

But the function $H(z) = \left(\prod_{n=1}^{N_0-1} h_n(z)\right) h(z)$ is holomorphic and nonzero in U (as it is the product of N_0 nonzero holomorphic functions), and so the multiplicty of the zero of F at z_0 must be $\sum_{n=1}^{N_0-1} m_n$.

8.2. The Weierstrass Factorization Theorem

(Emily, Problem 5060) Let $\Omega \subseteq \mathbb{C}$ be open, let $S \subset \Omega$ be a set with no accumulation points in Ω, and let $m : S \to \mathbb{N}$ be a function.

Show that there is a list $\{a_n\}_{n=1}^{N}$ or sequence $\{a_n\}_{n=1}^{\infty}$ such that $S = \{a_n : n \in \mathbb{N}\}$ and such that $m(n) = \#\{k : a_k = a_n\}$. Show furthermore that no subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to a point in Ω.

In particular, if $f : \Omega \to \mathbb{C}$ is holomorphic and not identically zero on any connected component of Ω, show that there is a list $\{a_n\}_{n=1}^{N}$ or sequence $\{a_n\}_{n=1}^{\infty}$ such that $\{z \in \Omega : f(z) = 0\} = \{a_n : n \in \mathbb{N}\}$ and such that the multiplicity $\operatorname{mult}_f(a_n)$ of the zero of f at a_n is equal to $\#\{k : a_k = a_n\}$, and furthermore no subsequence of $\{a_n\}_{n=1}^{\infty}$ converges to a point in Ω.

Corollary 8.2.3. Let $\{a_n\}_{n=1}^\infty$ be a sequence of points in $\mathbb{C}$ such that no subsequence converges to a point in $\mathbb{C}$. Then there exists a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $\{z \in \mathbb{C} : f(z) = 0\} = \{a_n : n \in \mathbb{N}\}$ and such that the multiplicity $m(a_n)$ of the zero of f at a_n is equal to $\#\{k : a_k = a_n\}$.

(Problem 5061) Show that this is equivalent to the following. Let $S \subset \mathbb{C}$ be a set with no accumulation points and let $m : S \to \mathbb{N}$ be a function. Then there exists a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that $\{z \in \mathbb{C} : f(z) = 0\} = S$ and such that the multiplicity $\operatorname{mult}_f(z)$ of the zero of f at z is equal to $m(z)$ for all $z \in S$.

(Irina, Problem 5070) Compute the power series (centered at zero) for the function $f(z) = \log \frac{1}{1-z}$. Simplify your answer as much as possible.

[Definition: Elementary factors] If $p \geq 0$ is an integer, we let

$$E_p(z) = (1-z) \exp\left(z + \frac{z^2}{2} + \dots + \frac{z^p}{p}\right).$$

Theorem 8.2.2. A function that satisfies the conditions of Corollary 8.2.3 is

$$f(z) = \prod_{n=1}^{N_0} (z - a_n) \prod_{n=N_0+1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

where $a_n \neq 0$ for all $n > N_0$, and where $\{p_n\}_{n=1}^\infty$ is an appropriate sequence of natural numbers.

(Michael, Problem 5080) We now begin the proof of Theorem 8.2.2 (and thus of Corollary 8.2.3). Show that E_p is an entire function, that $E_p(z) = 0$ if and only if $z = 1$, that the multiplicity of the zero of $E_p(z)$ at 1 is 1, and that $E_p \to 1$ in $\mathbb{D}$.

(Timmy, Problem 5090) Show that $E_p'(z) = -z^p \exp(\sum_{k=1}^p \frac{z^k}{k})$ for all $z \in \mathbb{C}$.

(Zach, Problem 5100) Let b_n be the coefficients of the power series for E_p centered at zero, so

$$E_p(z) = \sum_{n=0}^\infty b_n z^n$$

for all $z \in \mathbb{C}$. Observe that $b_0 = 1$. Show that $b_n = 0$ if $1 \leq n \leq p$. *Hint*: What is the order of the zero of E_p' at 0?

We have that $E_p'(z) = -z^p \exp(\sum_{k=1}^p \frac{z^k}{k})$. But $h(z) = \exp(\sum_{k=1}^p \frac{z^k}{k})$ is holomorphic and never zero. Thus E_p' has a zero of multiplicity p at 0. Because $E_p'(z) = \sum_{n=1}^\infty n b_n z^{n-1}$, this means that $n b_n = 0$ and so $b_n = 0$ whenever $1 \leq n \leq p$.

Recall [Problem 1820]: Suppose that $\sum_{n=0}^\infty a_n z^n$ and $\sum_{n=0}^\infty b_n z^n$ are two power series with radius of convergence at least r. Show that

$$\sum_{n=0}^\infty \left(\sum_{k=0}^n a_k\, b_{n-k}\right) z^n$$

has radius of convergence at least r and that

$$\sum_{n=0}^\infty \left(\sum_{k=0}^n a_k\, b_{n-k}\right) z^n = \left(\sum_{n=0}^\infty a_n z^n\right)\left(\sum_{n=0}^\infty b_n z^n\right)$$

for all $|z| < r$.

(Alex, Problem 5110) Show that $b_n \leq 0$ and is real for all $n \geq p+1$. *Hint*: Write E_p' as a product of functions whose power series you know.

Observe that

$$\begin{aligned} E_p'(z) &= \sum_{n=1}^{\infty} n b_n z^{n-1} \\ &= -z^p \exp\Big(\sum_{m=1}^{p} \frac{z^m}{m}\Big) \\ &= -z^p \prod_{m=1}^{p} \exp\Big(\frac{z^m}{m}\Big) \\ &= -z^p \prod_{m=1}^{p} \sum_{k=0}^{\infty} \frac{z^{mk}}{k! m^k}. \end{aligned}$$

The quantity $\prod_{m=1}^{p} \sum_{k=0}^{\infty} \frac{z^{mk}}{k! m^k}$ is a product of finitely many power series all with nonnegative coefficients all of which converge everywhere. By Problem 1820, the product is a power series which converges everywhere; we may see from Problem 1820 and induction that the coefficients of the power series representation of $\prod_{m=1}^{p} \sum_{k=0}^{\infty} \frac{z^{mk}}{k! m^k}$ must also be real and nonnegative. Thus the coefficients of the power series representation for $-z^p \exp\Big(\sum_{m=1}^{p} \frac{z^m}{m}\Big)$ must be real and nonpositive.

(David, Problem 5120) Show that $\sum_{n=p+1}^{\infty} |b_n| = 1$. *Hint*: Start by computing $E_p(1)$.

Because $b_n \leq 1$ for all $n > 0$ we have that $\sum_{n=p+1}^{\infty} |b_n| = -\sum_{n=p+1}^{\infty} b_n$.
But

$$0 = E_p(1) = \sum_{n=0}^{\infty} b_n 1^n = \sum_{n=0}^{\infty} b_n = b_0 + \sum_{n=1}^{p} b_n + \sum_{n=p+1}^{\infty} b_n = 1 - \sum_{n=p+1}^{\infty} |b_n|$$

by the above results concerning b_n.

(Emily, Problem 5130) Show that if $|z| \leq 1$ then $|E_p(z) - 1| \leq |z|^{p+1}$.

(Irina, Problem 5140) Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence of nonzero complex numbers. Suppose that the a_ns have no accumulation point in the sense that no subsequence converges. (We do not require that the a_ns be distinct.) Show that $\lim_{n\to\infty} |a_n| = \infty$.

Recall that $\lim_{n\to\infty} |a_n| = \infty$ means that, for all $R > 0$, there is a $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $|a_n| > R$.

Suppose not. Then there exists a $R > 0$ such that, for all $N \in \mathbb{N}$, there is a $n \geq N$ such that $|a_n| \leq R$.

Define the subsequence a_{n_k} by letting $n_1 \geq 1$ be such that $|a_{n_1}| \leq R$ and letting $n_{k+1} \geq n_k + 1$ be such that $|a_{n_{k+1}}| \leq R$.

By the Bolzano-Weierstraß theorem, there is then a subsequence $\{a_{n_{k_j}}\}$ which converges to a point in $\mathbb{C}$. Since a sub-subsequence is a subsequence, this contradicts the assumption that the a_ns have no accumulation point. Thus $|a_n| \to \infty$.

(Michael, Problem 5150) Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{C}$ be a sequence of nonzero complex numbers. Suppose that the a_ns have no accumulation point.

Fix an $r > 0$. Show that $\sum_{n=1}^{\infty} |E_n(z/a_n) - 1|$ converges uniformly on $\overline{D}(0, r)$.

Let k be such that, if $n \geq k$, then $|a_n| \geq 2r$; by Problem 5140, some such k must exist. If $m \geq k$ and $|z| \leq r$, then by Problem 5130,

$$\sum_{n=m}^{\infty} |E_n(z/a_n) - 1| \leq \sum_{n=m}^{\infty} |z/a_n|^{n+1} \leq \sum_{n=m}^{\infty} 2^{-n-1} = 2^- m.$$

This converges to zero as $m \to \infty$, uniformly for $z \in \overline{D}(0, r)$.

(Timmy, Problem 5160) Prove Theorem 8.2.2 (and thus Corollary 8.2.3).

Theorem 8.2.4. [The Weierstraß factorization theorem.] Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function that is not identically equal to zero.

If f has finitely many zeros, then there is an entire function $g(z)$, an integer $N_0 \geq 0$, and complex numbers $a_n \in \mathbb{C}$ such that

$$f(z) = e^{g(z)} \prod_{n=1}^{N_0} (z - a_n).$$

Otherwise, there is an entire function $g(z)$, an integer $m \geq 0$, and complex numbers $a_n \in \mathbb{C}$ such that

$$f(z) = e^{g(z)} \, z^m \prod_{n=1}^{\infty} E_n\biggl(\frac{z}{a_n}\biggr)$$

where the infinite product converges uniformly on compact sets.

(Zach, Problem 5170) Prove Theorem 8.2.4.

Let m be the multiplicity of the zero of f at $z = 0$ (or $m = 0$ if $f(0) \neq 0$; in this case we take $z^0 = 1$ for all $z \in \mathbb{C}$, including $z = 1$).

Then $\frac{1}{z^m} f(z)$ has a removable singularity at $z = 0$. Let $k(z) = \frac{1}{z^m} f(z)$ for all $z \neq 0$ and $k(z) = \lim_{z \to 0} \frac{1}{z^m} f(z)$; by Theorem 4.1.1, k is also entire.

Let $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}_{n=1}^{N}$ be the zeroes of k, counted with multiplicity; by Problem 5060, there is such a sequence, and furthermore no subsequence converges. In particular the a_ns are isolated.

If f (and therefore k) has finitely many zeroes, define

$$h(z) = z^m \prod_{n=1}^{N} (z - a_n).$$

Otherwise, let

$$h(z) = z^m \prod_{n=1}^{\infty} E_n\biggl(\frac{z}{a_n}\biggr).$$

In either case h is entire (either by being a polynomial or by Theorem 8.2.2.

Furthermore, $\frac{k}{h}$ has a removable singularity at a_n for each n. By the Riemann removable singularities theorem, there is an entire function ℓ with $\ell(z) = \frac{k(z)}{h(z)}$ for each $z \in \mathbb{C} \setminus \{a_n : n \leq N\}$ or each $z \in \mathbb{C} \setminus \{a_n : n \in \mathbb{N}\}$. Finally, $\ell(a_n) \neq 0$ because k and h have the same multiplicity at a_n, and $\ell(z) \neq 0$ if $z \notin \{a_n : n \in \mathbb{N}\}$ or $z \notin \{a_n : n \leq N\}$ because that is the zero set of k and h.

Thus by Lemma 6.6.4, there is an entire function g with $e^g = \ell$.

Thus

$$f(z) = e^{g(z)} \prod_{n=1}^{N_0} (z - a_n).$$

or

$$f(z) = e^{g(z)} \, z^m \prod_{n=1}^{\infty} E_n\biggl(\frac{z}{a_n}\biggr)$$

as desired.

Theorem 8.3.1. (Weierstraß) Let $\Omega \subseteq \mathbb{C}$ be open and let $\{a_n\}_{n=1}^\infty$ be a sequence of points in Ω such that no subsequence converges to a point in Ω. Then there exists a holomorphic function $f : \Omega \to \mathbb{C}$ such that $\{z \in \Omega : f(z) = 0\} = \{a_n : n \in \mathbb{N}\}$ and such that the multiplicity $\operatorname{mult}_f(a_n)$ of the zero of f at a_n is equal to $\#\{k : a_k = a_n\}$.

(Alex, Problem 5180) (This is the first step in the proof of Theorem 8.3.1.) Let $\Omega \subsetneq \mathbb{C}$ be an open set and let $\{a_n\}_{n=1}^\infty \subset \Omega$ be a sequence with no accumulation points in Ω. Show that for each $n \in \mathbb{N}$ there is a point $\widehat{a}_n \in \mathbb{C} \setminus \Omega$ such that $|a_n - \widehat{a}_n| = \operatorname{dist}(a_n, \partial\Omega)$.

If $a_n \in \Omega$ then $\Omega \neq \emptyset$, and by assumption $\Omega \neq \mathbb{C}$. Because $\mathbb{C}$ is connected, we must have that $\partial\Omega \neq \emptyset$.

Let $w \in \partial\Omega$. Consider the set $K = \overline{D}(a_n, |a_n - w|) \setminus \Omega$. K is closed (because $\overline{D}(a_n, |a_n - w|)$ is closed and Ω is open) and bounded (because $K \subseteq \overline{D}(a_n, |a_n - w|)$). Thus K is compact. Therefore the continuous function $f(z) = |a_n - z|$ attains its minimum on K. Let $\widehat{a}_n \in K$ be the minimizer.

Then $|a_n - \widehat{a}_n| = f(\widehat{a}_n) \leq f(z) = |a_n - z|$ for all $z \in \overline{D}(a_n, |a_n - w| \setminus \Omega$. Observe that $\widehat{a}_n \in \overline{D}(a_n, |a_n - w|)$ and so $|a_n - \widehat{a}_n| \leq |a_n - w|$.

If $z \in \mathbb{C} \setminus \Omega$, then either $z \in \overline{D}(a_n, |a_n - w|)$ or $z \in \mathbb{C} \setminus \overline{D}(a_n, |a_n - w|)$. If $z \in \overline{D}(a_n, |a_n - w|)$, then $|a_n - \widehat{a}_n| = f(\widehat{a}_n) \leq f(z) = |a_n - z|$ because $\widehat{a}_n$ is the minimizer for f.

If $z \in \mathbb{C} \setminus \overline{D}(a_n, |a_n - w|)$, then $|a_n - z| > |a_n - w| \geq |a_n - \widehat{a}_n|$ because $\widehat{a}_n \in \overline{D}(a_n, |a_n - w|)$.

In either case $|a_n - z| \geq |a_n - \widehat{a}_n|$, as desired.

(David, Problem 5190) We will begin with the case where the sequence $\{a_n\}_{n=1}^\infty$ is bounded. Under the conditions of the previous problem, if in addition we have that there is an $R > 0$ such that $a_n \in \overline{D}(0, R)$ for all $n \in \mathbb{N}$, show that $\lim_{n\to\infty} |a_n - \widehat{a}_n| = \lim_{n\to\infty} \operatorname{dist}(a_n, \partial\Omega) = 0$.

Suppose for the sake of contradiction that this is false. Then there exists a $\varepsilon > 0$ such that, if $N \in \mathbb{N}$, then there is some $n \geq N$ such that $|a_n - \widehat{a}_n| \geq \varepsilon$.

Let $n_1 \geq 1$ be such that $|a_{n_1} - \widehat{a}_{n_1}| \geq \varepsilon$. If n_k has been given, let $n_{k+1} \geq n_k + 1$ be such that $|a_{n_{k+1}} - \widehat{a}_{n_{k+1}}| \geq \varepsilon$. This gives a subsequence $\{a_{n_k}\}_{k=1}^\infty$ of $\{a_n\}_{n=1}^\infty$ that satisfies $|a_{n_k} - \widehat{a}_{n_k}| \geq \varepsilon$.

Because $\{a_{n_k}\}_{k=1}^\infty$ is a bounded subsequence in $\mathbb{C} = \mathbb{R}^2$, by the Bolzano-Weierstraß theorem it has a sub-subsequence $\{a_{n_{k_j}}\}_{j=1}^\infty$ which converges to some point $a \in \mathbb{C}$.

Because each $a_{n_{k_j}}$ is in Ω, we must have that their limit a is in $\overline{\Omega}$. By assumption $\{a_n\}$ has no subsequences that converge to points in Ω, and so $a \in \partial\Omega$.

But then $|a_{n_{k_j}} - a| \geq \operatorname{dist}(a_{n_{k_j}}, \partial\Omega) = |a_{n_{k_j}} - \widehat{a}_{n_{k_j}}| \geq \varepsilon$ for all j, and so $a_{n_{k_j}} \not\to a$. This is a contradiction; thus we must have that $\lim_{n\to\infty} |a_n - \widehat{a}_n| = 0$.

(Emily, Problem 5200) Let Ω, $\{a_n\}_{n=1}^\infty$ and $\{\widehat{a}_n\}_{n=1}^\infty$ be as in Problem 5190. Observe that $E_n\Big(\frac{a_n - \widehat{a}_n}{z - \widehat{a}_n}\Big)$ is holomorphic on Ω. Show that

$$\sum_{n=1}^\infty \Big|1 - E_n\Big(\frac{a_n - \widehat{a}_n}{z - \widehat{a}_n}\Big)\Big|$$

converges uniformly on $K_\delta = \{z \in \Omega : \operatorname{dist}(z, \{\widehat{a}_n : n \in \mathbb{N}\}) \geq \delta\}$.

Fix $\varepsilon > 0$. By Problem 5190, there is a $N \in \mathbb{N}$ such that if $n \geq N$, then $|a_n - \widehat{a}_n| < \delta/2$. We may further assume that $\frac{1}{2^N} < \varepsilon$.

We have that if $z \in K_\delta$ then $|z - \widehat{a}_n| \geq \delta$ for all $n \in \mathbb{N}$ and so $|\frac{a_n - \widehat{a}_n}{z - \widehat{a}_n}| \leq \frac{1}{2}$ for all $n \geq N$. Thus by Problem 5130,

$$\sum_{n=N}^\infty \Big|1 - E_n\Big(\frac{a_n - \widehat{a}_n}{z - \widehat{a}_n}\Big)\Big| \leq \sum_{n=N}^\infty \frac{1}{2^{n+1}} = \frac{1}{2^N} < \varepsilon$$

as desired.

(Problem 5201) Let $F(z) = \prod_{n=1}^\infty E_n\big(\frac{a_n - \widehat{a}_n}{z - \widehat{a}_n}\big)$. Show that F satisfies the conditions of Theorem 8.3.1.

(Problem 5202) Show that there is an $M > 0$ such that $\sup_{|z|>M} |F(z)| < \infty$ and $\inf_{|z|>M} |F(z)| > 0$.

Because $\lim_{n\to\infty} |a_n - \widehat{a}_n| = 0$, we have that $\sup_{n\in\mathbb{N}} |a_n - \widehat{a}_n| < \infty$. By the triangle inequality, and because $|a_n| \leq R$, we have that $|\widehat{a}_n| \leq |a_n| + |\widehat{a}_n - a_n| \leq \sup_{n\in\mathbb{N}} |a_n - \widehat{a}_n| + R$. Let $M = 1 + 4R + 4\sup_{n\in\mathbb{N}} |a_n - \widehat{a}_n|$.

If $|z| > M$, then $|z - \widehat{a}_n| \geq |z| - |\widehat{a}_n| \geq 3M/4$. Also $|a_n - \widehat{a}_n| \leq M/4$. Thus by Problem 5130

$$\left|1 - E_n\Big(\frac{a_n - \widehat{a}_n}{z - \widehat{a}_n}\Big)\right| \leq \frac{1}{3^{n+1}}.$$

Therefore by Problem (4940),

$$\left|\left(\prod_{n=1}^N E_n\Big(\frac{a_n - \widehat{a}_n}{z - \widehat{a}_n}\Big)\right) - 1\right| \leq \left(\prod_{n=1}^N \Big(1 + \Big|E_n\Big(\frac{a_n - \widehat{a}_n}{z - \widehat{a}_n}\Big) - 1\Big|\Big)\right) - 1$$

By Corollary 8.1.3,

$$\begin{aligned}\left|\left(\prod_{n=1}^N E_n\Big(\frac{a_n - \widehat{a}_n}{z - \widehat{a}_n}\Big)\right) - 1\right| &\leq \exp\left(\sum_{n=1}^N \Big|E_n\Big(\frac{a_n - \widehat{a}_n}{z - \widehat{a}_n}\Big) - 1\Big|\right) - 1 \\ &\leq \exp\left(\sum_{n=1}^N \frac{1}{3^{n+1}}\right) - 1.\end{aligned}$$

Taking the limit as $N \to \infty$, we see that

$$|F(z) - 1| \leq e^{1/2} - 1 < 1.$$

In particular $0 < 2 - e^{1/2} < |F(z)| \leq e^{1/2}$ for all such z, as desired.

(Irina, Problem 5210) Prove Theorem 8.3.1 in the case where $\{a_n\}_{n=1}^\infty$ is unbounded.

Let $z_0 \in \Omega \setminus \{a_n : n \in \mathbb{N}\}$; because Ω is uncountable some such z_0 must exist. Because $z_0 \in \Omega$, z_0 is not an accumulation point of $\{a_n : n \in \mathbb{N}\}$, so there is a $r > 0$ such that $D(z_0, r) \subset \Omega \setminus \{a_n : n \in \mathbb{N}\}$.

Let $W = \{\frac{1}{z - z_0} : z \in \Omega\}$. Let $b_n = \frac{1}{a_n - z_0}$. Then $\{b_n\}_{n=1}^\infty \subset W$ and has no accumulation points in W; furthermore, $|b_n| \leq 1/r$ for all n and so we are in the situation of Problem 5190.

Let F be the function given by Problem 5201 defined on W with zeroes at the points b_n; let $f(z) = F(\frac{1}{z - z_0})$. Then f is holomorphic in $\Omega \setminus \{z_0\}$. By Problem 5202, f and is bounded in a neighborhood of z_0; by the Riemann removable singularities theorem, we may extend f to a function holomorphic on Ω. Observe further that $f(z_0) \neq 0$.

We compute $f(a_n) = F(\frac{1}{a_n - z_0}) = F(b_n) = 0$ for all n. Furthermore, if F has a zero at b_n of multiplicity k, then $F^{(j)}(b_n) = 0$ for all $j < k$ and $F^{(k)}(b_n) \neq 0$; a straightforward induction argument shows that $f^{(j)}(a_n) = 0$ for all $j < k$ and $f^{(k)}(a_n) \neq 0$.

Corollary 8.3.4. Let $\Omega \subseteq \mathbb{C}$ be open. Let $m : \Omega \to \mathbb{C}$ be meromorphic on Ω. Then there exist two holomorphic functions $f, g : \Omega \to \mathbb{C}$ such that $m = \frac{f}{g}$.

(Michael, Problem 5220) Prove Corollary 8.3.4.

8.3. The Mittag-Leffler theorem

(Lemma 5221) Let $\{\beta_n\}_{n=1}^\infty \subset \mathbb{C}$ and $\{\gamma_n\}_{n=1}^\infty \subset \mathbb{C}$ be two sequences of complex numbers and let $\{a_n\}_{n=1}^\infty$ be a sequence of distinct complex numbers such that no subsequence has an accumulation point in $\mathbb{C}$. Then there exists an entire function f such that $f(a_n) = \beta_n$ and $f'(a_n) = \gamma_n$ for all $n \geq 1$.

Recall [Problem 5140]: We have that $\lim_{n\to\infty} |a_n| = \infty$.

(Timmy, Problem 5230) In this problem we begin the proof of Lemma 5221. For each $n \in \mathbb{N}$, show that there is an entire function g_n such that

$$\{z \in \mathbb{C} : g_n(z) = 0\} = \{a_k : k \in \mathbb{N} \setminus \{n\}\},$$

$g_n(a_n) = \beta_n$, and such that $|g_n(z)| < \frac{1}{2^n}$ for all z with $|z| < \frac{1}{2}|a_n|$.

Let ψ_n be the entire function given by Corollary 8.2.3 with $\{a_k : k \in \mathbb{N} \setminus \{n\}\} = \{z \in \mathbb{C} : \psi_n(z) = 0\}$. Observe that $\psi_n(a_n) \neq 0$ because $a_k \neq a_n$ if $k \neq n$.

Then let

$$\varphi_n(z) = \frac{\beta_n}{\psi_n(a_n)} \psi_n(z).$$

If $a_n = 0$ we let $g_n = \varphi_n$. Otherwise, φ_n is entire and therefore continuous, so $M_n = \sup_{\overline{D(0,|a_n|/2)}}$ is finite. Let $m_n \in \mathbb{N}$ satisfy $2^{m_n} > M_n$. Letting

$$g_n(z) = \frac{\beta_n}{\psi_n(a_n)} \left(\frac{z}{a_n}\right)^{m_n} \psi_n(z)$$

we see that g_n satisfies all the desired conditions.

(Zach, Problem 5240) Show that $g = \sum_{n=1}^\infty g_n$ converges normally on $\mathbb{C}$.

Let $K \subset \mathbb{C}$ be compact. Then K is bounded. By Problem 5140, there is an N such that if $n \geq N$, then $|a_n|/2 > |z|$ for all $z \in K$. Thus

$$\sum_{n=1}^{\infty} \sup_K |g_n| \leq \sum_{n=1}^{N-1} \sup_K |g_n| + \sum_{n=N}^{\infty} \frac{1}{2^n}.$$

Because g_n is continuous and K is compact, $\sup_K |g_n| < \infty$ for any fixed n. Thus

$$\sum_{n=1}^{\infty} \sup_K |g_n| < \infty$$

and so by the Weierstraß M-test, $\sum_{n=1}^\infty g_n$ converges uniformly on K.

(Alex, Problem 5250) For each $n \in \mathbb{N}$, show that there is an entire function h_n such that

$$\{z \in \mathbb{C} : h_n(z) = 0\} = \{a_k : k \in \mathbb{N}\},$$

$h_n'(a_k) = 0$ if $k \neq n$, $h_n'(a_n) = \gamma_n - g'(a_n)$, and such that $|h_n(z)| < \frac{1}{2^n}$ for all z with $|z| < \frac{1}{2}|a_n|$. Then show that $f(z) = g(z) + \sum_{n=1}^\infty h_n(z)$ converges normally on $\mathbb{C}$ and satisfies the conditions of Lemma 5221.

Let ϑ_n be the entire function given by Corollary 8.2.3 with $\{a_k : k \in \mathbb{N}\} = \{z \in \mathbb{C} : \vartheta_(z) = 0\}$, with the zero at a_n being of multiplicity 1, and with the zero at a_k for $k \neq n$ being of order 2. In particular $\vartheta_n'(a_n) \neq 0$.

Define

$$h_n(z) = \frac{\gamma_n - g'(a_n)}{\vartheta'(a_n)} \left(\frac{z}{a_n}\right)^{m_n} \vartheta_n(z)$$

where $m_n \in \mathbb{N}$. Because $\lambda(z) = \frac{\gamma_n - g'(a_n)}{\vartheta'(a_n)} \vartheta_n(z)$ is entire (hence continuous), as before we may choose m_n large enough that $|h_n(z)| < \frac{1}{2^n}$ for all z with $|z| < \frac{1}{2}|a_n|$.

By the Leibniz rule

$$h_n'(z) = \frac{\gamma_n - g'(a_n)}{\vartheta'(a_n)} m_n \left(\frac{z}{a_n}\right)^{m_n - 1} \vartheta_n(z) + \frac{\gamma_n - g'(a_n)}{\vartheta'(a_n)} \left(\frac{z}{a_n}\right)^{m_n} \vartheta_n'(z).$$

Recalling that $\vartheta_n(a_n) = 0$, we have that

$$h_n'(a_n) = \gamma_n - g'(a_n)$$

as desired.

As in the previous problem, normal convergence of $\sum_{n=1}^\infty h_n$ follows by the Weierstraß M-test. Thus f is entire by Theorem 3.5.1, $f(a_n) = g(a_n) = \beta_n$, and $f'(a_n) = g'(a_n) + h_n'(a_n) = \gamma_n$ by Corollary 3.5.2.

Theorem 8.3.6. (Mittag-Leffler) Let $\Omega \subseteq \mathbb{C}$ be open. Let $\{\alpha_n\}_{n=1}^\infty \subset \Omega$ be a sequence of distinct elements with no accumulation points in Ω. For each n, let $p_n \in \mathbb{N}$ and suppose that $a_{\ell,n} \in \mathbb{C}$ is defined for each $n \in \mathbb{N}$ and each $\ell \in \mathbb{N}$ with $-p_n \leq \ell \leq -1$.

Then there is a function f that is meromorphic in Ω, whose singular set is $\{\alpha_n : n \in \mathbb{N}\}$, and such that

$$f(z) - \sum_{\ell=-p_n}^{-1} a_{\ell,n}(z-\alpha_n)^\ell$$

has a removable singularity at α_n.

Lemma 8.3.5. [Pole-pushing lemma]. Let α, $\beta \in \mathbb{C}$. Suppose that

$$A(z) = \sum_{j=-M}^{-1} a_j(z-\alpha)^j$$

for some $M \in \mathbb{N}$ and some $a_j \in \mathbb{C}$. Then for all $r > |\alpha - \beta|$ and all $\varepsilon > 0$ we have that there is a $N \in \mathbb{N}$ and numbers $b_j \in \mathbb{C}$ such that

$$\left|A(z) - \sum_{j=-N}^{-1} b_j(z-\beta)^j\right| < \varepsilon$$

for all $z \in \widehat{\mathbb{C}} \setminus D(\beta, r)$.

(David, Problem 5260) Prove Lemma 8.3.5.

$A(z)$ is holomorphic in the set $\mathbb{C} \setminus \overline{D}(\beta, |\alpha - \beta|)$ and so by Theorem 4.3.2 has a Laurent series $\sum_{j=-\infty}^{\infty} b_j(z-\beta)^j$ in $\mathbb{C} \setminus \overline{D}(\beta, |\alpha - |\beta|)$.

We have that $|A(z)| \to 0$ as $|z| \to \infty$. By Problem 2230,

$$b_k = \frac{1}{2\pi i} \oint_{\partial D(\beta,R)} \frac{A(\zeta)}{(\zeta-\beta)^{k+1}}\,d\zeta$$

for all $R > |\alpha - \beta|$; if $k \geq 0$ then we may take the limit as $R \to \infty$ to see $b_k = 0$.

Thus, for all $z \in \widehat{\mathbb{C}} \setminus \overline{D}(\beta, |\alpha - \beta|)$, we have that

$$A(z) = \sum_{j=-\infty}^{-1} b_j(z-\beta)^j.$$

Furthermore, by Proposition 3.2.9, if $r > |\beta - \alpha|$ then $\sum_{j=-\infty}^{-1} b_j(z-\beta)^j$ converges uniformly on $\mathbb{C} \setminus D(0, r)$. Thus, for every $\varepsilon > 0$, there is an N such that

$$\left|A(z) - \sum_{j=-N}^{-1} b_j(z-\beta)^j\right| < \varepsilon$$

for all $z \in \widehat{\mathbb{C}} \setminus D(\beta, r)$, as desired.

(Emily, Problem 5270) Prove Theorem 8.3.6 in the case where $\{\alpha_n\}_{n=1}^\infty$ is bounded.

[Chapter 8, Problem 23] Theorem 8.3.6 is still true even if $\{\alpha_n\}_{n=1}^\infty$ is unbounded.

Theorem 8.3.8. Let $\Omega \subseteq \mathbb{C}$ be open. Let $\{\alpha_n\}_{n=1}^\infty \subset \Omega$ be a sequence of distinct elements with no accumulation points in Ω. For each n, let p_n, $q_n \in \mathbb{N}$ and suppose that $a_{\ell,n} \in \mathbb{C}$ is defined for each $n \in \mathbb{N}$ and each $\ell \in \mathbb{N}$ with $-p_n \leq \ell \leq q_n$.

Then there is a function f that is meromorphic in Ω, whose singular set is (a subset of) $\{\alpha_n : n \in \mathbb{N}\}$, and such that

$$f(z) - \sum_{\ell=-p_n}^{q_n} a_{\ell,n}(z-\alpha_n)^\ell$$

has a removable singularity at α_n and whose Laurent series expansion about α_n may be written as

$$f(z) = \sum_{\ell=-p_n}^{\infty} c_{\ell,n}(z-\alpha_n)^\ell$$

satisfies $c_{\ell,n} = a_{\ell,n}$ for all $\ell \leq q_n$.

We may equivalently say that, for each α_n, there is an $r_n > 0$ such that $D(\alpha_n, r_n) \subseteq \Omega$ contains no other α_ks and such that

$$f(z) = \sum_{\ell=-p_n}^{q_n} a_{\ell,n}(z-\alpha_n)^\ell + \sum_{\ell=q_n+1}^{\infty} c_{\ell,n}(z-\alpha_n)^\ell$$

in $D(\alpha_n, r_n) \setminus \{\alpha_n\}$ for some $c_{\ell,n} \in \mathbb{C}$.

Lemma 8.3.7. Suppose that the numbers p, q, and a_ℓ for $-p \le \ell \le q$ are given. Let $\Omega \subseteq \mathbb{C}$ be open, let $\alpha \in \Omega$, and let $g : \Omega \to \mathbb{C}$ be holomorphic. Suppose further that $p, q \ge 0$ and that g has a zero of multiplicity $q+1$ at α. Then there exist numbers b_ℓ, $c_\ell \in \mathbb{C}$ such that

$$g(z) \sum_{\ell=-1-p-q}^{-1} b_\ell(z-\alpha)^\ell = \sum_{\ell=-p}^{q} a_\ell(z-\alpha)^\ell + \sum_{\ell=q+1}^{\infty} c_\ell(z-\alpha)^\ell$$

for all z in a suitable punctured neighborhood of α.

(Irina, Problem 5280) (This is the first step in the proof of Lemma 8.3.7.) Suppose that $\sum_{\ell=q+1}^\infty d_\ell(z-\alpha)^\ell$ and $\sum_{\ell=-p-q-1}^\infty b_\ell(z-\alpha)^\ell$ are two Laurent series that converge in $D(\alpha, r) \setminus \{\alpha\}$. Show that

$$\sum_{n=-p}^{\infty} \Big(\sum_{k=q+1}^{n+p+q+1} b_{n-k} d_k \Big)(z-\alpha)^n$$

also converges in $D(\alpha, r) \setminus \{\alpha\}$ and that

$$\sum_{n=-p}^{\infty} \Big(\sum_{k=q+1}^{n+p+q+1} b_{n-k} d_k \Big)(z-\alpha)^n = \Big(\sum_{\ell=q+1}^{\infty} d_\ell(z-\alpha)^\ell \Big) \Big(\sum_{\ell=-p-q-1}^{\infty} b_\ell(z-\alpha)^\ell \Big)$$

for all $z \in D(\alpha, r) \setminus \{\alpha\}$.

If $z \in D(\alpha, r) \setminus \{\alpha\}$, then by Theorem 4.3.2 $\sum_{\ell=q+1}^\infty d_\ell(z-\alpha)^\ell$ and $\sum_{\ell=-p-q-1}^\infty b_\ell(z-\alpha)^\ell$ converge absolutely. Reindexing and factoring out powers of $z-\alpha$, we have that $\sum_{k=0}^\infty d_{k+q+1}(z-\alpha)^k$ and $\sum_{k=0}^\infty b_{k-p-q-1}(z-\alpha)^k$ also converge absolutely and

$$\sum_{\ell=q+1}^{\infty} d_\ell(z-\alpha)^\ell = (z-\alpha)^{q+1} \sum_{k=0}^{\infty} d_{k+q+1}(z-\alpha)^k,$$

$$\sum_{\ell=-p-q-1}^{\infty} b_\ell(z-\alpha)^\ell = (z-\alpha)^{-p-q-1} \sum_{k=0}^{\infty} b_{k-p-q-1}(z-\alpha)^k.$$

By Theorem 3.3.1 the power series on the right hand side of the above formulas have radius of convergence at least r. Thus by Problem 1820

$$\sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} b_{n-k-p-q-1} d_{k+q+1} \Big)(z-\alpha)^n$$

has radius of convergence at least r and converges to

$$\Big(\sum_{k=0}^{\infty} d_{k+q+1}(z-\alpha)^k \Big) \Big(\sum_{k=0}^{\infty} b_{k-p-q-1}(z-\alpha)^k \Big)$$

and so if $z \in D(\alpha, r) \setminus \{\alpha\}$, then

$$\Big(\sum_{\ell=q+1}^{\infty} d_\ell(z-\alpha)^\ell \Big) \Big(\sum_{\ell=-p-q-1}^{\infty} b_\ell(z-\alpha)^\ell \Big) = \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} b_{n-k-p-q-1} d_{k+q+1} \Big)(z-\alpha)^{n-p}$$

and the right hand side converges absolutely.

Reindexing, we see that

$$\Big(\sum_{\ell=q+1}^{\infty} d_\ell(z-\alpha)^\ell \Big) \Big(\sum_{\ell=-p-q-1}^{\infty} b_\ell(z-\alpha)^\ell \Big) = \sum_{n=-p}^{\infty} \Big(\sum_{k=q+1}^{n+p+q+1} b_{n-k} d_k \Big)(z-\alpha)^n$$

as desired.

(Michael, Problem 5290) Prove Lemma 8.3.7.

Let $D(\alpha, r) \subseteq \Omega$. Because g has a zero of multiplicity $q+1$ at α, there are constants d_ℓ such that $d_{q+1} \neq 0$ and such that

$$g(z) = \sum_{\ell=q+1}^{\infty} d_\ell (z-\alpha)^\ell$$

for all $z \in D(\alpha, r)$.

Define the b_ℓ as follows.

$$b_{-p-q-1} = \frac{a_{-p}}{d_{q+1}}, \quad b_{n-q-1} = \frac{a_n}{d_{q+1}} - \sum_{k=q+2}^{n+p+q+1} b_{n-k} d_k$$

for each $-p < n \leq q$. Observe that if $k \geq q+2$ then $n-k < n-q-1$ and so the sum on the right hand side involves only values of b_ℓ for $\ell < n-q-1$.

Let $\widetilde{b}_\ell = b_\ell$ if $\ell \leq -1$ and $\widetilde{b}_\ell = 0$ otherwise. By induction and Lemma 8.3.7, if $z \in D(\alpha, r) \setminus \{\alpha\}$ we have that

$$\begin{aligned}
g(z) \sum_{\ell=-1-p-q}^{-1} b_\ell (z-\alpha)^\ell &= \Big(\sum_{\ell=q+1}^{\infty} d_\ell (z-\alpha)^\ell \Big) \Big(\sum_{\ell=-1-p-q}^{\infty} \widetilde{b}_\ell (z-\alpha)^\ell \Big) \\
&= \sum_{n=-p}^{\infty} \Big(\sum_{k=q+1}^{n+p+q+1} \widetilde{b}_{n-k} d_k \Big) (z-\alpha)^n \\
&= \sum_{n=-p}^{q} \Big(\sum_{k=q+1}^{n+p+q+1} b_{n-k} d_k \Big) (z-\alpha)^n + \sum_{n=q+1}^{\infty} \Big(\sum_{k=n+1}^{n+p+q+1} b_{n-k} d_k \Big) (z-\alpha)^n.
\end{aligned}$$

A straightforward induction argument establishes that

$$\Big(\sum_{k=q+1}^{n+p+q+1} b_{n-k} d_k \Big) = a_n$$

if $-p \leq n \leq q$. Choosing $c_n = \big(\sum_{k=n+1}^{n+p+q+1} b_{n-k} d_k \big)$ completes the proof.

(Timmy, Problem 5300) Prove Theorem 8.3.8.

Let g be the holomorphic function given by Theorem 8.3.1 with a zero of multiplicity q_n at each α_n. Let $D(\alpha_n, r_n) \subseteq \Omega$ contain no other α_ks. Let $b_{\ell,n}$ be given by Lemma 8.3.7 such that

$$g(z) \sum_{\ell=-1-p-q}^{-1} b_{\ell,n} (z-\alpha_n)^\ell = \sum_{\ell=-p}^{q} a_{\ell,n} (z-\alpha_n)^\ell + \sum_{\ell=q+1}^{\infty} c_{\ell,n} (z-\alpha_n)^\ell$$

in $D(\alpha_n, r_n) \setminus \{\alpha_n\}$. Let f be the function given by Theorem 8.3.6 with

$$f(z) = h_n(z) + \sum_{\ell=-1-p-q}^{-1} b_{\ell,n} (z-\alpha_n)^\ell$$

in $D(\alpha_n, r_n) \setminus \{\alpha_n\}$ for some h_n holomorphic in $D(\alpha_n, r_n)$.

Then

$$g(z) f(z) = g(z) h_n(z) + g(z) \sum_{\ell=-1-p-q}^{-1} b_{\ell,n} (z-\alpha_n)^\ell = g(z) h_n(z) + \sum_{\ell=-p_n}^{q_n} a_{\ell,n} (z-\alpha_n)^\ell + \sum_{\ell=q_n+1}^{\infty} c_{\ell,n} (z-\alpha_n)^\ell$$

in $D(\alpha_n, r_n) \setminus \{\alpha_n\}$. But h_n is holomorphic in $D(\alpha_n, r_n)$ and g has a zero of order $q_n + 1$ at α_n, so

$$g(z)h_n(z) = \sum_{\ell=q_n+1}^{\infty} e_{\ell,n}(z-\alpha_n)^\ell$$

for some $e_{\ell,n} \in \mathbb{C}$. Thus

$$g(z)f(z) = \sum_{\ell=-p_n}^{q_n} a_{\ell,n}(z-\alpha_n)^\ell + \sum_{\ell=q_n+1}^{\infty} (c_{\ell,n} + e_{\ell,n})(z-\alpha_n)^\ell$$

is the desired function meromorphic in Ω with the desired Laurent series at the poles α_n.

8.3. Maximal domains of existence of holomorphic functions

Corollary 8.3.3. (Special case) There is a function $f : \mathbb{D} \to \mathbb{C}$ that is holomorphic in $\mathbb{D}$ such that, if $\mathbb{D} \subseteq W$, W is open and connected, $F : W \to \mathbb{C}$ is holomorphic, and $F = f$ in $\mathbb{D}$, then $W = \mathbb{D}$. (That is, there is a function f holomorphic in $\mathbb{D}$ that cannot be extended to a function holomorphic in any larger open set.)

(Zach, Problem 5310) Let $g(e^{i\theta}) = \sum_{n=0}^\infty \frac{5^n}{6^n}\cos(7^n\theta)$. (This is a special case of the *Weierstraß function*.) Show that g is well-defined (the sum converges) for all $0 \leq \theta \leq 2\pi$ and that g is continuous on $\partial\mathbb{D}$.

The terms $\frac{5^n}{6^n}\cos(7^n\theta)$ are well defined functions of $e^{i\theta}$ because, if $e^{i\theta} = e^{i\vartheta}$, then $\vartheta = \theta + 2\pi k$ for some $k \in \mathbb{Z}$ and so $\cos(7^n\theta) = \cos(7^n\vartheta)$.

Because $|\frac{5^n}{6^n}\cos(7^n\theta)| \leq \frac{5^n}{6^n}$ for all θ and $\sum_{n=0}^\infty \frac{5^n}{6^n} < \infty$, by the Weierstraß M-test (Memory 1611) we have that the sum converges uniformly. Because the uniform limit of continuous functions is continuous (see Memory (1600)), we must have that g is continuous.

(Problem 5311) Plot the first few partial sums for the Weierstraß function.

A plot may be found at the following link:
`https://www.desmos.com/calculator/kywnvjzjg7`

(Alex, Problem 5320) Let u be the function that is harmonic in $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ and with $u(e^{i\theta}) = g(e^{i\theta})$ for $0 \leq \theta \leq 2\pi$. Let f be the function that is holomorphic in $\mathbb{D}$ with real part u. (u and f exist by Theorem 7.3.4 and Lemma 7.1.4.)

Show that f satisfies the conditions of Corollary 8.3.3. *Hint*: Use the fact (proven by Weierstraß in 1872) that $g(\theta)$ is nowhere differentiable.

Suppose that $\mathbb{D} \subsetneq W$, that W is open and connected, and that $F : W \to \mathbb{C}$ is holomorphic.

Because $W \supsetneq \mathbb{D}$ is connected, W contains a point $e^{i\theta_0} \in \partial\mathbb{D}$. Because W is open, there is some small $\varrho > 0$ such that if $|\theta - \theta_0| < \varrho$, then $e^{i\theta} \in W$.

Then

$$\frac{d}{d\theta}\operatorname{Re}(F(e^{i\theta})) = \operatorname{Re}(F'(e^{i\theta})\, ie^{i\theta})$$

by Problem 1140 and the definition of $\frac{d}{d\theta}$. In particular this derivative exists.

But we must have that $\operatorname{Re} f(e^{i\theta}) = g(\theta)$ and therefore $\frac{d}{d\theta}\operatorname{Re}(f(e^{i\theta}))$ does not exist for any $\theta \in \mathbb{R}$. Thus we cannot have that $\operatorname{Re} F(e^{i\theta}) = \operatorname{Re} f(e^{i\theta})$ for all θ with $|\theta - \theta_0| < \varrho$. Since f is continuous on $\overline{\mathbb{D}}$ and F is continuous in W, this means that F and f cannot be equal everywhere in $\mathbb{D}$.

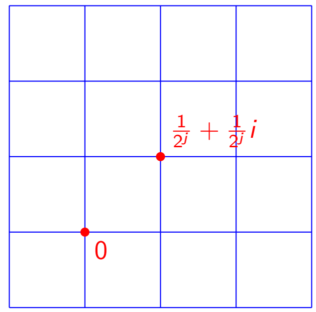
Conversely, if F is holomorphic in $W \supseteq \mathbb{D}$, $F = f$ in $\mathbb{D}$, and W is open and connected, then $W = \mathbb{D}$.

Corollary 8.3.3. Let $\Omega \subsetneq \mathbb{C}$ be any nonempty open set that is not all of $\mathbb{C}$. Then there is a function $f : \Omega \to \mathbb{C}$ that is holomorphic in Ω such that, if $\Omega \subseteq W$, W is open and connected, $F : W \to \mathbb{C}$ is holomorphic, and $F = f$ in Ω, then $\Omega = W$. (That is, there is a function f holomorphic in Ω that cannot be extended to a function holomorphic in any larger open set.)

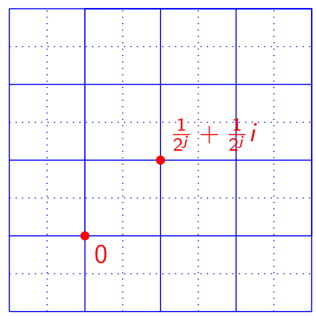
(Problem 5321) Let

$$\mathcal{Q}_j = \{[k2^j, (k+1)2^j) \times [\ell 2^j, (\ell+1)2^j) : k,\, \ell \in \mathbb{Z}\}$$

be the grid of squares in $\mathbb{C}$ with side-length 2^j aligned with the axes. Show that if $z \in \mathbb{C}$ and $j \in \mathbb{Z}$ then $x \in S$ for exactly one $S \in \mathcal{Q}_j$. Here is a sketch of (the cubes in) $\mathcal{Q}_j$.

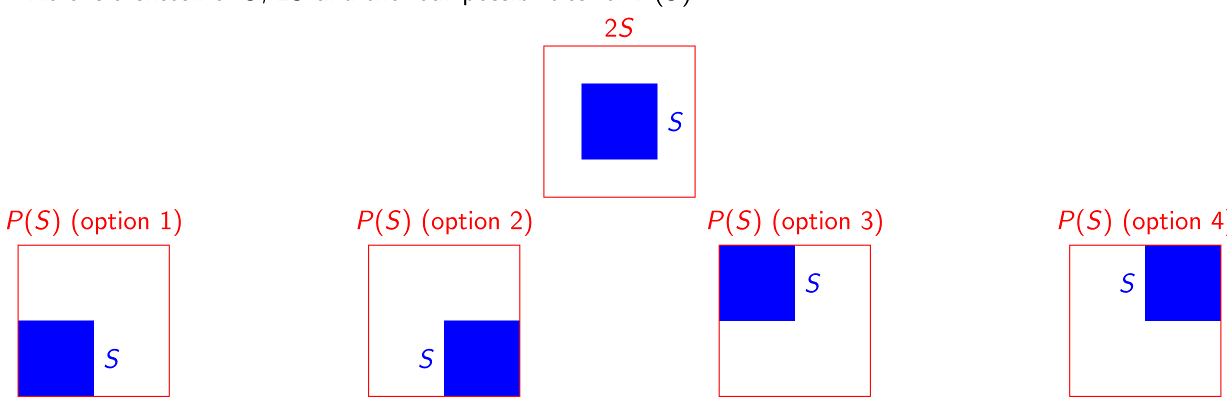


Here is a sketch of (the cubes in) $\mathcal{Q}_{j-1}$.



(Problem 5322) Suppose that $S \in \mathcal{Q}_j$. Let $P(S)$ be the "dyadic parent" of S, so $S \subsetneq P(S) \in \mathcal{Q}_{j+1}$. Let $2S$ be the square concentric to S of side-length 2^{j+1}.

Here is a sketch of S, $2S$ and the four possibilities for $P(S)$.



(David, Problem 5330) If $S \in \mathcal{Q}_j$, let $\ell(S) = 2^j$ be the side-length of S. Show that if $S \in \mathcal{Q}_j$ and $z \in S$, then $D(z, \ell(S)/2) \subset 2S$ and $2P(S) \subset \overline{D}(z, 3\sqrt{2}\ell(S))$.

There are x_0, $y_0 \in \mathbb{R}$ such that

$$S = [x_0 - \frac{1}{2}\ell(S), x_0 + \frac{1}{2}\ell(S)) \times [y_0 - \frac{1}{2}\ell(S), y_0 + \frac{1}{2}\ell(S)).$$

It is easy to see that

$$2S = [x_0 - \ell(S), x + \ell(S)) \times [y_0 - \ell(S), y + \ell(S)).$$

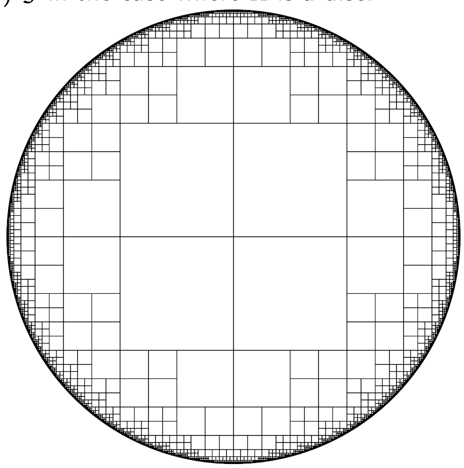
If $z = (x, y) \in S$, then $x_0 - \frac{1}{2}\ell(S) \le x < x_0 + \frac{1}{2}\ell(S)$ and $y_0 - \frac{1}{2}\ell(S) \le y < y_0 + \frac{1}{2}\ell(S)$, and so

$$D(z, \frac{1}{2}\ell(s)) \subset (x - \frac{1}{2}\ell(S), x + \frac{1}{2}\ell(S)) \times (y - \frac{1}{2}\ell(S), y + \frac{1}{2}\ell(S)) \subset [x_0 - \ell(S), x + \ell(S)) \times [y_0 - \ell(S), y + \ell(S)) = 2S.$$

Observe that $2P(S) \subset [x_0 - \frac{5}{2}\ell(S), x_0 + \frac{5}{2}\ell(S)) \times [y_0 - \frac{5}{2}\ell(S), y_0 + \frac{5}{2}\ell(S))$. Thus, if $(x, y) \in S$ and $(\xi, \eta) \in 2P(S)$ then $|x - \xi| \leq 3\ell(S)$ and $|y - \eta| \leq 3\ell(S)$, and so the result follows by the Pythagorean theorem.

(Emily, Problem 5340) Let $\mathcal{Q} = \cup_{j=-\infty}^{\infty} \mathcal{Q}_j$. Let $\Omega \subsetneq \mathbb{C}$ be open. Let $\mathcal{G} = \{S \in \mathcal{Q} : 2S \subset \Omega,\ 2P(S) \not\subset \Omega\}$. We call $\mathcal{G}$ a *dyadic Whitney decomposition* of Ω. Show that $\cup_{S\in\mathcal{G}} S = \Omega$.

Here is a sketch of (the cubes in) $\mathcal{G}$ in the case where Ω is a disc:



(Problem 5341) If $k < j$ and $j, k \in \mathbb{Z}$, and if $S \in \mathcal{Q}_k$, show that there is exactly one $T \in \mathcal{Q}_j$ with $S \subset T$ and that $S \cap R = \emptyset$ for every other $R \in \mathcal{Q}_j$.

(Irina, Problem 5350) Show that if $S \in \mathcal{G}$ and $T \in \mathcal{G}$, then either $S = T$ or $S \cap T = \emptyset$. If $z \in \Omega$, then how many cubes $S \in \mathcal{G}$ can satisfy $z \in S$?

Exactly one.

(Michael, Problem 5360) Show that $\mathcal{G}$ is a countable set.

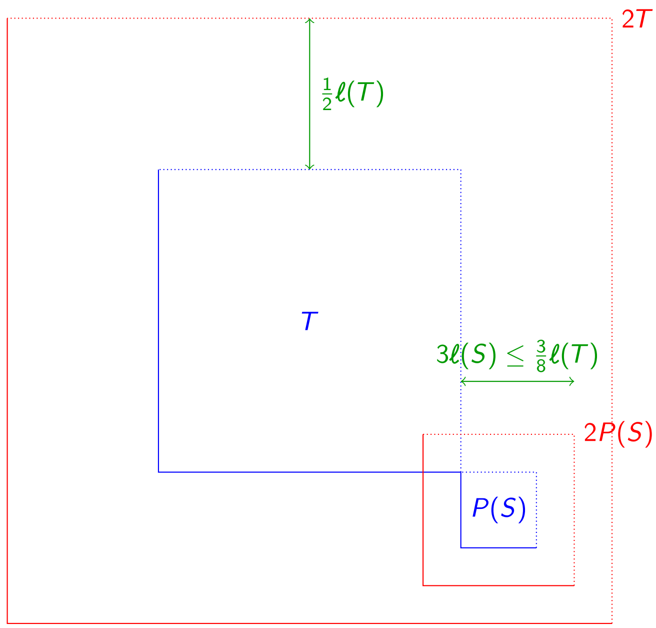
(Timmy, Problem 5370) Suppose that S, $T \in \mathcal{G}$ and that $\operatorname{dist}(S,T) = 0$; that is, the closures of S and T intersect. Show that $\ell(S) \leq 4\ell(T)$ and that $\ell(T) \leq 4\ell(S)$.

Without loss of generality $\ell(S) \leq \ell(T)$; thus we need only show that $\ell(T) \leq 4\ell(S)$.

Suppose for the sake of contradiction that $\ell(T) > 4\ell(S)$, that is, $\ell(S) < \frac{1}{4}\ell(T)$. Because $\ell(S) = 2^j$ and $\ell(T) = 2^k$ for some j, $k \in \mathbb{Z}$, we have that $\ell(S) \leq \frac{1}{8}\ell(T)$.

Thus $\ell(P(S)) = 2\ell(S) \leq \frac{1}{4}\ell(T)$. Furthermore, $S \subset P(S)$, so $\overline{S} \subset \overline{P(S)}$ and so $\operatorname{dist}(T, P(S)) = 0$.

Then $2P(S) \subset 2T$:



But by definition of $\mathcal{G}$, $2T \subset \Omega$ and so $2P(S) \subset 2T \subset \Omega$. Shus $S \notin \mathcal{G}$. Shis is a contradiction; therefore we must have $\ell(T) \leq 4\ell(S)$.

(Zach, Problem 5380) If $S \in \mathcal{G}$, let z_S be the midpoint of S. Let $A = \{z_S : S \in \mathcal{G}\}$.

Let $z \in \Omega$. Show that z is not an accumulation point for A. *Hint*: if $z \in T \in \mathcal{G}$, then how many midpoints z_S can appear in $D(z, \ell(T)/8)$?

Let $z \in \Omega$. Then $z \in T$ for some $T \in \mathcal{G}$. Let $\varepsilon = \ell(T)/8$.

If $\operatorname{dist}(S,T) = 0$, then $\ell(S) \geq \ell(T)/4$ and so $|z - z_S| \geq \ell(T)/8$. If $\operatorname{dist}(S,T) \geq 0$, then there must be at least one square Q between S and T with $\operatorname{dist}(S,Q) = 0$ and so $\ell(Q) \geq \ell(T)/4$, and so $\operatorname{dist}(T,Q) \geq \ell(T)/4$.

In any case, if $S \neq T$ then $z_S \notin D(z, \ell(T)/8)$, and so $A \cap D(z, \ell(T)/8) \subseteq \{z_T\}$. Thus z cannot be an accumulation point for A.

(Alex, Problem 5390) Let $z \in \partial\Omega$. Show that z is an accumulation point for A.

(David, Problem 5400) Prove Corollary 8.3.3. *Hint*: Let f be the function holomorphic in Ω and such that $f(z) = 0$ (with multiplicity one) if and only if $z \in A$ given by Theorem 8.3.1. Show that the domain of existence of f is Ω; that is, if $\widetilde{f} = f$ in Ω and $\widetilde{f}$ is holomorphic on some open set $\Psi \supseteq \Omega$, then $\Psi = \Omega$.

Let f be as in the hint, let $\Psi \supsetneq \Omega$ be open and connected, and let $\widetilde{f}$ be holomorphic in Ψ. By connectivity $\Psi \cap \partial\Omega$ contains at least one point. Let $w \in \Psi \cap \partial\Omega$.

Then w is an accumulation point of the zeroes of f. If $\widetilde{f}(z) = f(z) = 0$ whenever $z \in A$, then the zeroes of $\widetilde{f}$ have an accumulation point in Ψ, and so by Theorem 3.6.1 (or Problem 2041) we have that $\widetilde{f} \equiv 0$ in Ψ. But $\Omega \subset \Psi$ and so $\widetilde{f} \equiv 0$ in Ω. Because $f \not\equiv 0$ in Ω, we cannot have $f = \widetilde{f}$ in Ω.

Recall [Theorem 8.3.1]: If $\{a_n\}_{n=1}^\infty \subset \mathbb{D}$ and no subsequence converges to a point in $\mathbb{D}$, then there is a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ such that $\{z \in \Omega : f(z) = 0\} = \{a_n : n \in \mathbb{N}\}$ and such that the multiplicity $\mathrm{mult}_f(a_n)$ of the zero of f at a_n is equal to $\#\{k : a_k = a_n\}$.

(Question 5401) When can we also require that f be *bounded*?

Theorem 9.1.4. Suppose that $f : \mathbb{D} \to \mathbb{C}$ is a bounded nonconstant holomorphic function. Let $\{a_k\}_{k=1}^\infty$ be the zeroes of f (with multiplicity). Then

$$\sum_{k=1}^{\infty}(1 - |a_k|) < \infty.$$

(Emily, Problem 5410) Give an example of a sequence $\{a_k\}_{k=1}^\infty \subset \mathbb{D}$ such that no subsequence converges to a point in $\mathbb{D}$ and such that $\sum_{k=1}^\infty (1 - |a_k|) = \infty$. Give an example of a function holomorphic in $\mathbb{D}$ that satisfies $\{z \in \mathbb{D} : f(z) = 0\} = \{a_k : k \in \mathbb{N}\}$.

Let $f(z) = \sin \frac{\pi}{1-z}$. Then f is holomorphic on $\mathbb{C}\setminus\{1\} \supset \mathbb{D}$. Furthermore, if $a_k = 1 - \frac{1}{k}$ then $\sin(a_k) = 0$. Also,

$$\sum_{k=1}^{\infty}(1 - |a_k|) = \sum_{k=1}^{\infty}\frac{1}{k} = \infty.$$

Recall [Problem 3260]: If $a \in \mathbb{D}$ and we define

$$\phi_a(z) = \frac{z - a}{1 - \bar{a}z}$$

then ϕ_a is a holomorphic bijection from $\mathbb{D}$ to itself, a continuous bijection from $\partial\mathbb{D}$ to itself, and a continuous bijection from $\overline{\mathbb{D}}$ to itself.

Proposition 9.1.1. ϕ_a is holomorphic on an open neighborhood of $\overline{\mathbb{D}}$. $\phi_a(z) = 0$ if and only if $z = a$ and the zero at a is simple. Finally, $|\phi_a(z)| = 1$ if $|z| = 1$.

Theorem 9.1.2. (Jensen's formula.) Let f be holomorphic in a neighborhood of $\overline{D}(0, r)$ and suppose $f(0) \neq 0$. Let $a_1, a_2, \dots, a_n$ be the zeros of f in $D(0, r)$ counted with multiplicity. Assume that f has no zeroes on $\partial D(0, r)$.
Then

$$\log|f(0)| + \sum_{k=1}^{n}\log\left|\frac{r}{a_k}\right| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta.$$

(Irina, Problem 5420) Show that Theorem 9.1.2 is valid in the following two special cases:

- f has no zeroes in $\overline{D}(0, r)$.
- $f = \phi_{a/r}(z/r)$ for some $a \in D(0, r)$.

If f has no zeroes in $\overline{D}(0, r)$, then $\log|f|$ is harmonic in $D(0, r)$ by Problem 4820 and clearly is continuous on $\overline{D}(0, r)$. Thus the result follows from the mean value property (Theorem 7.2.5 and Problem 3980).

If $f = \phi_{a/r}(z/r)$, then $f(0) = -a/r$, f has one zero at $z = a/r$, and $|f(re^{i\theta})| = 1$ for all $\theta \in \mathbb{R}$ by Proposition 9.1.1. The result is a simple computation.

(Michael, Problem 5430) Justify the (implicit) claim in Theorem 9.1.2 that f has at most finitely many zeroes in $\overline{D}(0, r)$. Then prove Theorem 9.1.2.

(Timmy, Problem 5440) Suppose that f has a zero of multiplicity $m \geq 1$ at 0. What does Jensen's formula tell you about $\log|\lim_{z\to 0}\frac{f(z)}{z^m}|$?

It tells us that

$$\log|\lim_{z\to 0}\frac{f(z)}{z^m}| + m\log r + \sum_{k=1}^{n}\log\left|\frac{r}{a_k}\right| = \frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{i\theta})|\,d\theta.$$

(Zach, Problem 5450) Prove Theorem 9.1.4 in the case where $f(0) \neq 0$.

(Problem 5451) Prove Theorem 9.1.4 in the case where $f(0) = 0$.

If f has finitely many zeroes the result is obvious. Recall from Problem 5020 that f has countably many zeroes. Thus there is an increasing sequence $\{r_\ell\}_{\ell=1}^\infty$ with $\lim_{\ell\to\infty} r_\ell = \sup_{\ell\in\mathbb{N}} r_\ell = 1$ and such that $|a_n| \neq r_\ell$ for all n, $\ell \in \mathbb{N}$. We may further require that $r_\ell \geq 1/2$ for all ℓ.

Order the zeroes such that $|a_1| \leq |a_2| \leq |a_3| \leq \dots$; because the zeroes have no accumulation points in $\mathbb{D}$, at most finitely many a_j can satisfy $|a_j| \leq |a_k|$ and so this may be done without omitting any a_ks.

For any such r_ℓ, let n_ℓ be the number of zeroes a_k with $|a_k| < r_\ell$. By Jensen's formula

$$\sum_{k=1}^{n_\ell} \log\left|\frac{r_\ell}{a_k}\right| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta + m\log\frac{1}{r_\ell} - \log|\lim_{z\to0}\frac{f(z)}{z^m}|$$

where $m \geq 0$ is the multiplicity of the zero of f at 0.

The right hand side is bounded by

$$\sup_{z\in\mathbb{D}}|f| + m\log 2 - \log|\lim_{z\to0}\frac{f(z)}{z^m}|.$$

Taking the limit as $\ell\to\infty$, we see that

$$\sum_{k=1}^{\infty}\log\left|\frac{1}{a_k}\right| < \infty.$$

It is elementary to show that $1 - a - \log\frac{1}{a}$ has a maximum at $a = 1$ and so $\log\frac{1}{a} \geq 1 - a$ for all $a > 0$, and so

$$\sum_{k=1}^{\infty}(1 - |a_k|) < \infty.$$

Theorem 9.1.5. Suppose that $m \in \mathbb{N}_0$, $\{a_k\}_{k=1}^\infty \subset \mathbb{D}\setminus\{0\}$, and that $\sum_{k=1}^\infty(1-|a_k|) < \infty$. Then there is a bounded holomorphic function $f : \mathbb{D} \to \mathbb{D}$ such that f has a zero of multiplicity m at zero (or $f(0) \neq 0$ if $m = 0$), $\{z \in \mathbb{D}\setminus\{0\} : f(z) = 0\} = \{a_n : n \in \mathbb{N}\}$, and such that the multiplicity $\mathrm{mult}_f(a_n)$ of the zero of f at a_n is equal to $\#\{k : a_k = a_n\}$. Furthermore, one such f is given by Problem 5470 below.

(Problem 5452) Show that $\phi_a(0) = -a$ for all $a \in \mathbb{D}$ and thus, if $w \in \partial\mathbb{D}$, then $\lim_{\substack{a\to w\\ a\in\mathbb{D}}} \phi_a(0) = -w$.

(Alex, Problem 5460) Let $w \in \partial\mathbb{D}$ and let $a \in \mathbb{D}$. Show that

$$\frac{|\phi_a(z) + w|}{|w - a|} \leq \frac{1 + |z|}{1 - |z|}$$

for all $z \in \mathbb{D}$. Conclude that if $w \in \partial\mathbb{D}$ then

$$\lim_{\substack{a\to w\\ a\in\mathbb{D}}} \phi_a(z) = -w$$

for any $z \in \mathbb{D}$, and that the convergence is uniform for z in any compact subset of $\mathbb{D}$.

Observe $1 = w\overline{w}$. We compute

$$\frac{|\phi_a(z) + w|}{|w - a|} = \frac{1}{|w-a|}\left|\frac{z - a + w - wz\bar{a}}{1 - \bar{a}z}\right| = \frac{1}{|w-a|}\left|\frac{w-a}{1-\bar{a}z} + zw\frac{\overline{w-a}}{1-\bar{a}z}\right| \leq \frac{1+|z|}{1-|z|}$$

because $|a| \leq 1$.

(David, Problem 5470) Prove Theorem 9.1.5 by showing that the infinite product

$$f(z) = z^m\prod_{k=1}^{\infty}\frac{-\overline{a_k}}{|a_k|}\phi_{a_k}(z)$$

converges normally in $\mathbb{D}$ and satisfies the given conditions.

[Definition: Blaschke product] A Blaschke product is an expression of the form

$$z^m \prod_{k=1}^{\infty} \frac{-\overline{a_k}}{|a_k|}\phi_{a_k}(z)$$

where $m \geq 0$ is an integer (written $m \in \mathbb{N}_0$) and where $a_k \in \mathbb{D} \setminus \{0\}$ for all k. (If $\sum_{k=1}^{\infty}(1-|a_k|) < \infty$, then the Blaschke product converges to a holomorphic function on $\mathbb{D}$.)

Corollary 9.1.6. If $f : \mathbb{D} \to \mathbb{C}$ is a bounded holomorphic function with a zero of multiplicity $m \in \mathbb{N}_0$ at 0 (including the case $m = 0$ where $f(0) \neq 0$), and if $\{a_k\}_{k=1}^N$, $N \in \mathbb{N}_0 \cup \{\infty\}$, is the list of the other zeroes of f counted with multiplicity, then there is a holomorphic function $g : \mathbb{D} \to \mathbb{D}$ such that $\operatorname{Re} g$ is bounded above (meaning e^g is bounded in modulus) and such that

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{N} \frac{-\overline{a_k}}{|a_k|}\phi_{a_k}(z)$$

for all $z \in \mathbb{D}$.

Furthermore,

$$\sup_{\mathbb{D}} |f| = \sup_{\mathbb{D}} |e^g|.$$

(Emily, Problem 5480) Prove that

$$f(z) = z^m e^{g(z)} \prod_{k=1}^{N} \frac{-\overline{a_k}}{|a_k|}\phi_{a_k}(z)$$

for some $g : \mathbb{D} \to \mathbb{D}$ holomorphic.

By Theorem 9.1.4, $\sum_{k=1}^N (1-|a_k|) < \infty$. Thus by Theorem 9.1.5, $h(z) = z^m \prod_{k=1}^N \frac{-\overline{a_k}}{|a_k|}\phi_{a_k}(z)$ is holomorphic and has the same zeroes with the same multiplicity as f. Thus by Theorem 4.1.1 (the Riemann removable singularities theorem) there is a holomorphic function $F : \mathbb{D} \to \mathbb{C}$ with $F(z) = f(z)/h(z)$ for all z such that $f(z) \neq 0$. Furthermore, $F(z) = \lim_{\zeta \to z} \frac{f(\zeta)}{h(\zeta)} \neq 0$ if $f(z) = 0$ because f and h have zeroes of the same multiplicity at z.

Thus $F : \mathbb{D} \to \mathbb{C}$ is holomorphic and never zero, and so $F = e^g$ for a holomorphic function $g : \mathbb{D} \to \mathbb{C}$ by Lemma 6.6.4.

(Irina, Problem 5490) Prove that $\sup_{\mathbb{D}} |f| = \sup_{\mathbb{D}} |e^g|$. This completes the proof of Corollary 9.1.6.

By Lemma 3260, we have that

$$\left| z^m \prod_{k=1}^{n} \frac{-\overline{a_k}}{|a_k|}\phi_{a_k}(z) \right| \leq 1$$

for any $z \in \mathbb{D}$ and any $n \in \mathbb{N}$ (with $n \leq N$ if N is finite). Thus

$$|f(z)| = |e^{g(z)}| \left| z^m \prod_{k=1}^{N} \frac{-\overline{a_k}}{|a_k|}\phi_{a_k}(z) \right| \leq |e^{g(z)}|$$

for any $z \in \mathbb{D}$, and so $\sup_{\mathbb{D}} |f| \leq \sup_{\mathbb{D}} |e^g|$.

We now turn to the converse.

If $n \in \mathbb{N}$ and $n \leq N$, define

$$h_n(z) = z^m \prod_{k=1}^{N} \frac{-\overline{a_k}}{|a_k|}\phi_{a_k}(z).$$

Then every zero of h_n is a zero of f, and $\operatorname{mult}_{h_n}(a_k) \leq \operatorname{mult}_f(a_k)$, Thus f/h_n extends to a holomorphic function on $\mathbb{D}$ (with zeroes $\{a_k : k \geq n\}$). Furthermore, $h_n \to h$ normally in $\mathbb{D}$ (if $N = \infty$) or $h_N = f$ (if f has finitely many zeroes and so $N < \infty$). Thus $f/h_n \to e^g$ pointwise (or equals e^g), and so it suffices to show that $\sup_{\mathbb{D}} |f/h_n| \leq 1$ for all $n \in \mathbb{N}$ with $n \leq N$.

Choose some such n. h_n is continuous on $\overline{D}$ (as a product of continuous functions) and satifies $|h_n(z)| = 1$ if $|z| = 1$ by Lemma 3260. Because $\overline{\mathbb{D}}$ is compact, h_n is uniformly continuous. Thus if $\varepsilon > 0$ there is a $r_0 \in (0,1)$ such that $|1 - h_n(z)| < \varepsilon$ for all z with $r_0 \leq |z| < 1$.

By the maximum modulus principle, if $0 < r < 1$ then

$$\sup_{D(0,r)} |f/h_n| \leq \sup_{D(0,\max(r,r_0))} |f/h_n| = \sup_{|z|=\max(r,r_0)} |f(z)/h_n(z)| \leq \frac{\sup_{|z|=\max(r,r_0)} |f(z)|}{1-\varepsilon} \leq \frac{1}{1-\varepsilon} \sup_{\mathbb{D}} |f|.$$

Since this is true for all $\varepsilon > 0$ we must have that

$$\sup_{D(0,r)} |f/h_n| \leq \sup_{\mathbb{D}} |f|$$

for all $r \in (0,1)$, and so

$$\sup_{\mathbb{D}} |f/h_n| \leq \sup_{\mathbb{D}} |f|$$

as desired.

9.2. The Hadamard Gap Theorem

Please see Professor Barton's video lecture (posted to Blackboard) for material concerning the Hadamard gap theorem.

9.3. Entire Functions of Finite Order

Lemma 9.3.1. Let f be an entire function with $f(0) = 1$. If $r > 0$ and $b > 1$, then

$$n(r) \leq \log_b \max_{|z|=br} |f(z)| = \frac{\log \max_{|z|=br} |f(z)|}{\log b}$$

where $n(r)$ denotes the number of zeroes of f (with multiplicity) in $D(0,r)$.

(Michael, Problem 5500) Prove Lemma 9.3.1.

Recall [Theorem 3.4.4]: If f is entire and there is a constant $C \in \mathbb{R}$ and a $k \in \mathbb{N}_0$ such that $|f(z)| \leq C + C|z|^k$ for all $z \in \mathbb{C}$, then f is a polynomial of degree at most k, and so by Theorem 3.4.5 f has exactly k zeroes counted with multiplicity.

[Definition: Order of an entire function] If $f : \mathbb{C} \to \mathbb{C}$ is entire and there are positive real constants a and r such that

$$|f(z)| \leq \exp(|z|^a) \quad \text{for all } |z| > r$$

then we say that f is of finite order. The order of f is

$$\inf\{a > 0 : \text{there is a } r > 0 \text{ such that } |f(z)| \leq \exp(|z|^a) \text{ for all } |z| > r\}.$$

(Exercise 5501) Show that the order of f is also

$$\inf\{a > 0 : \text{there is a } C > 0 \text{ such that } |f(z)| \leq C\exp(|z|^a) \text{ for all } z \in \mathbb{C}\}.$$

(Exercise 5502) Let f be a function of finite order and let p be a polynomial. Show that $f + p$ is a function of finite order and that its order is equal to that of f.

(Exercise 5503) Let f be a function of finite order and let p be a polynomial. Show that fp is a function of finite order and that its order is equal to that of f.

Theorem 9.3.2. Suppose that f is an entire function of finite order $\lambda \geq 0$, with $f(0) = 1$ and with infinitely many zeroes. If the zeroes of f (with multiplicity) are $\{a_k\}_{k=1}^{\infty}$, and if $\varepsilon > 0$, then

$$\sum_{n=1}^{\infty} |a_n|^{-\varepsilon-\lambda} < \infty.$$

(Timmy, Problem 5510) Prove Theorem 9.3.2.

There are countably many zeroes. If $0 < r < \infty$ then there are at most finitely many zeroes of modulus at most r, and so we have that

$$\sum_{n=1}^{\infty} |a_n|^{-\varepsilon-\lambda} = \lim_{N\to\infty} \sum_{|a_n|<2} |a_n|^{-\varepsilon-\lambda} \sum_{k=1}^{N} \sum_{2^k\leq |a_n|<2^{k+1}} |a_n|^{-\varepsilon-\lambda}.$$

Let $n(r)$ be as in Lemma 9.3.1. Because $\lambda+\varepsilon > 0$, we have that

$$\sum_{2^k\leq |a_n|<2^{k+1}} |a_n|^{-\varepsilon-\lambda} \leq n(2^{k+1})(2^k)^{-\varepsilon-\lambda}.$$

By Lemma 9.3.1,

$$\sum_{2^k\leq |a_n|<2^{k+1}} |a_n|^{-\varepsilon-\lambda} \leq (2^k)^{-\varepsilon-\lambda} \log_2 \max_{|z|=2^{k+2}} |f(z)|.$$

By definition of order, there is a K such that if $k \geq K$ then

$$\max_{|z|=2^{k+2}} |f(z)| \leq \exp(2^{(k+2)a})$$

for some a with $\lambda \leq a < \lambda+\varepsilon$. Thus

$$\begin{aligned}
\sum_{n=1}^{\infty} |a_n|^{-\varepsilon-\lambda} &= \sum_{|a_n|<2^K} |a_n|^{-\varepsilon-\lambda} + \lim_{N\to\infty} \sum_{k=K}^{N} \sum_{2^k\leq |a_n|<2^{k+1}} |a_n|^{-\varepsilon-\lambda} \\
&\leq \sum_{|a_n|<2^K} |a_n|^{-\varepsilon-\lambda} + \lim_{N\to\infty} \sum_{k=K}^{N} (2^k)^{-\varepsilon-\lambda} \log_2 \exp(2^{(k+2)a} \\
&\leq \sum_{|a_n|<2^K} |a_n|^{-\varepsilon-\lambda} + \lim_{N\to\infty} \frac{2^{2a}}{\ln 2} \sum_{k=K}^{N} 2^{k(a-\lambda-\varepsilon)}.
\end{aligned}$$

Because $a-\lambda-\varepsilon < 0$, the geometric series converges and so

$$\sum_{n=1}^{\infty} |a_n|^{-\varepsilon-\lambda} \leq \sum_{|a_n|<2^K} |a_n|^{-\varepsilon-\lambda} + \frac{2^{2a}}{\ln 2} \sum_{k=K}^{\infty} 2^{k(a-\lambda-\varepsilon)} < \infty.$$

(Zach, Problem 5520) Rewrite and prove Theorem 9.3.2 without the assumption $f(0) = 1$.

Statement: Suppose that f is an entire function of finite order $\lambda \geq 0$ with infinitely many zeroes. Let m be the multiplicity of the zero of f at 0 (with $m = 0$ if $f(0) \neq 0$) and let $\{a_n\}_{n=1}^{\infty}$ be the other zeroes of f (with multiplicity). If $\varepsilon > 0$, then $\sum_{n=1}^{\infty} |a_n|^{-\lambda-\varepsilon} < \infty$.

Proof: Let $g(z) = az^{-m}f(z)$ for all $z \neq 0$, with $g(0)$ defined by continuity, and with $a \in \mathbb{C}\setminus\{0\}$ such that $g(0) = 1$; by definition of order of a zero, such an a exists and g is also entire.

By Exercise 5503, g is also of finite order and its order is equal to λ. Furthermore, the zeroes of g consist of $\{a_n\}_{n=1}^{\infty}$ (with multiplicity). By Theorem 9.3.2 applied to g, $\sum_{n=1}^{\infty} |a_n|^{-\lambda-\varepsilon} < \infty$, as desired.

Theorem 9.3.7. (Simplified.) Suppose that f is an entire function of finite order and that f has finitely many zeroes.

Then $f(z) = p(z)e^{q(z)}$, where p is a polynomial and q is a polynomial whose degree is equal to the order of f.

In particular, the order of f must be an integer.

Theorem 9.3.9. If $c \in \mathbb{C}$ and f is an entire function of finite order $\lambda \notin \mathbb{Z}$, then the equation $f(z) = c$ has infinitely many solutions.

Theorem 9.3.10. If f is an entire function of finite order, then $\mathbb{C}\setminus f(\mathbb{C})$ can contain at most one point.

Lemma 9.3.4. Suppose that f is an entire function of finite order λ and that $f(0) = 1$. Suppose that $z \in \mathbb{C}$ and that $p \in \mathbb{Z}$ with $p > \lambda - 1$. Then

$$\lim_{r\to\infty} \frac{1}{2\pi} \int_0^{2\pi} 2re^{i\theta}(re^{i\theta} - z)^{-p-2} \log|f(re^{i\theta})|\, d\theta = 0.$$

(Alex, Problem 5530) Prove Lemma 9.3.4.

Let $\varphi(w) = \frac{1}{(w-z)^{p+2}}$. Observe that φ is holomorphic in $\mathbb{C} \setminus \{z\}$. Furthermore, $p > \lambda - 1 \geq 0 - 1$ so $p + 2 \geq 2$. Thus $\operatorname{Res}_\varphi(z) = 0$.

Therefore, if $r > 2|z|$ then

$$\begin{aligned}
0 = \operatorname{Res}_\varphi(z) &= \frac{1}{2\pi i} \oint_{\partial D(0,r)} \varphi \\
&= \frac{1}{2\pi i} \int_0^{2\pi} \varphi(re^{i\theta}) ire^{i\theta}\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} re^{i\theta} \frac{1}{(re^{i\theta} - z)^{p+2}}\, d\theta.
\end{aligned}$$

Thus

$$\frac{1}{2\pi} \int_0^{2\pi} 2re^{i\theta}(re^{i\theta} - z)^{-p-2} \log |f(re^{i\theta})|\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} 2re^{i\theta}(re^{i\theta} - z)^{-p-2} \big(\log |f(re^{i\theta})| - C\big)\, d\theta$$

for any $C \in \mathbb{C}$.

We choose $C = \log \sup_{\partial D(0,r)} |f|$. (Observe that $\sup_{\partial D(0,r)} |f| \geq 0$, and if $\sup_{\partial D(0,r)} |f| = 0$ then $f \equiv 0$ on $\partial D(0,r)$, a set with an accumulation point. Thus $f \equiv 0$ in $\mathbb{C}$, contradicting our assumption $f(0) = 1$. Thus $C \in \mathbb{R}$.)

Then

$$\begin{aligned}
\left| \frac{1}{2\pi} \int_0^{2\pi} 2re^{i\theta}(re^{i\theta} - z)^{-p-2} \log |f(re^{i\theta})|\, d\theta \right| &= \left| \frac{1}{2\pi} \int_0^{2\pi} 2re^{i\theta}(re^{i\theta} - z)^{-p-2} \big(\log |f(re^{i\theta})| - \log \sup_{\partial D(0,r)} |f|\big)\, d\theta \right| \\
&\leq \frac{2^{p+3}}{r^{p+1}} \frac{1}{2\pi} \int_0^{2\pi} \big(\log \sup_{\partial D(0,r)} |f| - \log |f(re^{i\theta})|\big)\, d\theta.
\end{aligned}$$

But by Theorem 9.1.2 (Jensen's formula)

$$\int_0^{2\pi} \log |f(re^{i\theta})| d\theta \geq \log |f(0)| = 0$$

and so

$$\left| \frac{1}{2\pi} \int_0^{2\pi} 2re^{i\theta}(re^{i\theta} - z)^{-p-2} \log |f(re^{i\theta})|\, d\theta \right| \leq \frac{2^{p+3}}{r^{p+1}} \frac{1}{2\pi} \int_0^{2\pi} \log \sup_{\partial D(0,r)} |f|\, d\theta.$$

By definition of order and because $p + 1 > \lambda$, if r is large enough then

$$\left| \frac{1}{2\pi} \int_0^{2\pi} 2re^{i\theta}(re^{i\theta} - z)^{-p-2} \log |f(re^{i\theta})|\, d\theta \right| \leq \frac{2^{p+3}}{r^{p+1}} \frac{1}{2\pi} \int_0^{2\pi} \log \exp(r^a)\, d\theta = \frac{2^{p+3}}{r^{p+1-a}}$$

for some a with $\lambda \leq a \leq p + 1$. The right hand side approaches zero as $r \to \infty$, as desired.

[Chapter 9, Problem 1] If f is holomorphic in a neighborhood of $\overline{D}(0,r)$ and $z_0 \in D(0,r)$, and if f has no zeroes in $\overline{D}(0,r)$, then

$$\log |f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta} + z_0}{re^{i\theta} - z_0}\right) \log |f(re^{i\theta})|\, d\theta.$$

Proposition 9.3.5. (Simplified.) Suppose that g is a nonconstant entire function and that $h = e^g$ is a function of finite order λ. If $p > \lambda - 1$ is an integer, then

$$\frac{\partial^{p+1}}{\partial z^{p+1}} g(z) = 0.$$

(David, Problem 5540) Prove Proposition 9.3.5.

By Problem 9.1, if $z \in \mathbb{C}$ and $r > 2|z|$ then

$$\operatorname{Re} g(z) = \log |h(z)| = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta} + z}{re^{i\theta} - z}\right) \log |h(re^{i\theta})|\, d\theta.$$

Now, if f is a holomorphic function then

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z} + \overline{\left(\frac{\partial f}{\partial \bar{z}}\right)} = \frac{\partial f}{\partial z} + \frac{\partial \bar{f}}{\partial z} = \frac{\partial f + \bar{f}}{\partial z} = 2\frac{\partial \operatorname{Re} f}{\partial z}$$

by definition of holomorphic and by Problem 600.

Thus

$$g'(z) = 2\frac{\partial}{\partial z}\operatorname{Re} g(z) = 2\frac{1}{2\pi}\frac{\partial}{\partial z}\int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right)\log|h(re^{i\theta})|\,d\theta.$$

By Problem 730

$$\begin{aligned} g'(z) &= \frac{1}{2\pi}\int_0^{2\pi} 2\frac{\partial}{\partial z}\operatorname{Re}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right)\log|h(re^{i\theta})|\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial z}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right)\log|h(re^{i\theta})|\,d\theta. \end{aligned}$$

Similarly

$$\frac{\partial^{p+1}}{\partial z^{p+1}}g(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial^{p+1}}{\partial z^{p+1}}\left(\frac{re^{i\theta}+z}{re^{i\theta}-z}\right)\log|h(re^{i\theta})|\,d\theta.$$

But

$$\frac{re^{i\theta}+z}{re^{i\theta}-z} = \frac{2re^{i\theta}}{re^{i\theta}-z} - 1$$

and p is an integer with $p > \lambda - 1 \geq -1$ and so $p+1 \geq 1$, and so

$$\frac{\partial^{p+1}}{\partial z^{p+1}}g(z) = (p+1)!\frac{1}{2\pi}\int_0^{2\pi} \frac{2re^{i\theta}}{(re^{i\theta}-z)^{p+2}}\log|h(re^{i\theta})|\,d\theta.$$

By Lemma 9.3.4 the right hand side converges to zero as $r \to \infty$, while the left hand side is independent of r and so we must have that the left hand side is zero, as desired.

(Emily, Problem 5550) Prove Theorem 9.3.7.

Let f be an entire function of finite order with finitely many zeroes. Then there is a nowhere zero holomorphic function h and a polynomial p such that $f = hp$. By Lemma 6.6.4, $h = e^g$ for some holomorphic function g.

Recall that polynomials have poles at ∞ and so if $|z|$ is large enough then $|p(z)| \geq 1$ and so $|h(z)| \leq |f(z)|$. Thus h is a function of finite order. By Exercise 5503, the order of h must equal the order of f.

Let λ be the order of f (and h) and let p be the unique integer with $\lambda - 1 < p \leq \lambda$. By Proposition 9.3.5,

$$\frac{\partial^{p+1}}{\partial z^{p+1}}g(z) = 0$$

and so g is a polynomial of order at most p. Let m be the order of g; then $m \leq p \leq \lambda$.

Conversely, there are constants C and r such that $|g(z)| \leq C|z|^m$ for all z with $|z| > r$, and so $|h(z)| \leq \exp(C|z|^m)$ for all $|z| > r$. If $\varepsilon > 0$, then there is a r_ε such that $(r_\varepsilon)^\varepsilon > C$ and so $|h(z)| \leq \exp(|z|^{m+\varepsilon})$ for all $|z| > \max(r, r_\varepsilon)$. Thus the order λ of h is at most m, and so $\lambda = m$.

(Irina, Problem 5560) Prove Theorem 9.3.9.

Let $h(z) = f(z) - c$. By Exercise 5502, h is a function of finite order and in fact is of order equal to f. If $f(z) = c$ has finitely many solutions, then h has finitely many zeroes and so by Theorem 9.3.7 the order of h is an integer. But the order of h equals the order of f and the order of f is not an integer. This is a contradiction, and so $f(z) = c$ must have infinitely many solutions.

(Michael, Problem 5570) Prove Theorem 9.3.10.

10.1. Definition of an analytic function element

(Problem 5571) Let $\Psi \subsetneq \Omega \subseteq \mathbb{C}$, where Ψ is open and nonempty and Ω is open and connected. Suppose that f is holomorphic in Ψ. Show that there is at most one function F that is holomorphic in Ω and such that $F = f$ in Ψ.

(Timmy, Problem 5580) Suppose that Ω, W are connected open sets with $D(1,1) \subsetneq \Omega \subset \mathbb{C}$ and $D(1,1) \subsetneq W \subset \mathbb{C}$. Suppose that $F : \Omega \to \mathbb{C}$ and $G : W \to \mathbb{C}$ are holomorphic and that $G(z) = F(z) = \ln z = \ln z = \sum_{k=0}^{\infty} \frac{(-1)^k}{k}(z-1)^k$ for all $z \in D(1,1)$.

If U is a connected component of $\Omega \cap W$, show that there is a $n \in \mathbb{Z}$ such that $F(z) = G(z) + 2\pi i n$ for all $z \in U$. Give an example to show that n may not equal zero.

(Zach, Problem 5590) Suppose that Ω, W are connected open sets with $D(1,1) \subsetneq \Omega \subset \mathbb{C}$ and $D(1,1) \subsetneq W \subset \mathbb{C}$. Suppose that $F : \Omega \to \mathbb{C}$ and $G : W \to \mathbb{C}$ are holomorphic and that $G(z) = F(z) = \sqrt{z}$ for all $z \in D(1,1)$. (Here $\sqrt{z}$ is the unique function continuous on $D(1,1)$ such that $\sqrt{z}^2 = z$ and $\sqrt{1} = 1$.)

If U is a connected component of $\Omega \cap W$, show that either $F(z) = G(z)$ for all $z \in U$ or $F(z) = -G(z)$ for all $z \in U$. Give an example to show that the case $F(z) = -G(z)$ can occur.

Definition 10.1.4. A function element is an ordered pair $(f, D(P,r))$ where $P \in \mathbb{C}$, $r > 0$ and f is a holomorphic function defined on $D(P,r)$.

Definition 10.1.5. If $(f, D(P,r))$ and $(g, D(Q,s))$ are function elements, if $D(P,r) \cap D(Q,s) \neq \emptyset$, and if $f = g$ on $D(P,r) \cap D(Q,s)$, we say that $(g, D(Q,s))$ is a direct analytic continuation of $(f, D(P,r))$.

[Definition: Analytic continuation] Suppose that we have a finite sequence of function elements $\{(f_j, D(P_j, r_j))\}_{j=1}^{k}$ such that $(f_j, D(P_j, r_j))$ is a direct analytic continuation of $(f_{j-1}, D(P_{j-1}, r_{j-1}))$ for all $1 < j \leq k$. Then $(f_k, D(P_k, r_k))$ is an analytic continuation of $(f_1, D(P_1, r_1))$.

(Alex, Problem 5600) Find a function element $(f, D(P,r))$ and two distinct function elements $(g, D(Q,s))$ and $(\tilde{g}, D(Q,s))$, with the same disc $D(Q,s)$, such that $(g, D(Q,s))$ and $(\tilde{g}, D(Q,s))$ are both analytic continuations of $(f, D(P,r))$. Can you do this for a direct analytic continuation?

10.2. Analytic continuation along a curve

Definition 10.2.1. Let $\gamma : [0,1] \to \mathbb{C}$ be a continuous function (we will call γ a curve). Let $r > 0$, let $D = D(\gamma(0), r)$ and let (f, D) be a function element. An analytic continuation of (f, D) along γ is a collection of function elements $\{(f_t, D_t)\}_{0 \leq t \leq 1}$ such that:

- If $t \in [0,1]$, then $D_t = D(\gamma(t), r_t)$ for some $r_t > 0$,
- $(f_0, D(\gamma(0), r_0)) = (f, D(\gamma(0), r))$,
- If $t \in [0,1]$, then there is an $\varepsilon = \varepsilon_t > 0$ such that, if $s \in [0,1]$ and $|t - s| < \varepsilon_t$, then (f_s, D_s) is a direct analytic continuation of (f_t, D_t).

Proposition 10.2.2. Let $\gamma : [0,1] \to \mathbb{C}$ be a curve and let (f, D) be a funcion element with $D = D(\gamma(0), r)$ for some $r > 0$. Let $\{(g_t, D_t)\}_{0 \leq t \leq 1}$ and $\{(h_t, B_t)\}_{0 \leq t \leq 1}$ be two analytic continuations of (f, D) along γ.

Then for all $t \in [0,1]$, we have that $g_t = h_t$ in $D_t \cap B_t$.

(Problem 5601) Let $S = \{t \in [0,1] : g_t = h_t \text{ in } D_t \cap B_t\}$. Begin the proof of Proposition 10.2.2 by showing that S is relatively open in $[0,1]$.

(David, Problem 5610) Complete the proof of Proposition 10.2.2 by showing that S is closed.

(Emily, Problem 5620) Suppose that $\gamma : [0,1] \to \mathbb{C}$ is a *closed* curve (so $\gamma(1) = \gamma(0)$). Let $\{(f_t, D(\gamma(t), r_t))\}_{0 \leq t \leq 1}$ be an analytic continuation of $(f, D(\gamma(0), r))$ along γ. Is it necessarily true that $f_1 = f_0$ on $D(\gamma(0), \min(r_0, r_1))$?

No. For example, let $\gamma(t) = \exp(2\pi i t)$, let $r_t = 1/2$ for all t, and let $f(z) = \log z$ (with the branch cut along the negative real numbers). It is easy to see that $f_1(z) = \log z + 2\pi i = f_0(z) + 2\pi i$ for all $z \in D_0 = D_1$.

(Exercise 5621) Let $\gamma(t) = e^{it}$, $0 \leq t \leq 6\pi$. Let $(f, D(P,r)) = (\log z, D(1, 1/2))$ be a function element. Let $\{(f_t, D(\gamma(t), r_t))\}_{0 \leq t \leq 6\pi}$ be an analytic continuation of $(\log z, D(1, 1/2))$ along γ.

Find $f_0(1)$, $f_{2\pi}(1)$, $f_{4\pi}(1)$ and $f_{6\pi}(1)$.

(Exercise 5622) Let $\gamma(t) = e^{it}$, $0 \leq t \leq 6\pi$. Let $(g, D(P,r)) = (\sqrt{z}, D(1, 1/2))$ be a function element. Let $\{(g_t, D(\gamma(t), r_t))\}_{0\leq t\leq 6\pi}$ be an analytic continuation of $(\sqrt{z}, D(1,1/2))$ along γ.

Find $g_0(1)$, $g_{2\pi}(1)$, $g_{4\pi}(1)$ and $g_{6\pi}(1)$.

10.3. The Monodromy Theorem

Definition 10.3.1. Let $a < b$, $c < d$. Let $\Omega \subseteq \mathbb{C}$ be open and connected. Let γ_c, $\gamma_d : [a,b] \to \Omega$ be two continuous[4] curves with the same endpoints (so $\gamma_c(a) = \gamma_d(a)$, $\gamma_c(b) = \gamma_d(b)$).

We say that γ_c and γ_d are homotopic in Ω if there is a function Γ such that:

- $\Gamma : [a,b] \times [c,d] \to \Omega$,
- $\Gamma(t,c) = \gamma_c(t)$, $\Gamma(t,d) = \gamma_d(t)$ for all $t \in [a,b]$,
- $\Gamma(a,s) = \gamma_c(a) = \gamma_d(a)$, $\Gamma(b,s) = \gamma_c(b) = \gamma_d(b)$ for all $s \in [c,d]$,
- Γ is continuous on $[a,b] \times [c,d]$,

We often let $\gamma_s(t) = \Gamma(t,s)$.

Definition 10.3.2. Let $\Omega \subseteq \mathbb{C}$ be open and connected, $D = D(P,r) \subseteq \Omega$, and let (f,D) be a function element. We say that (f,D) admits unrestricted continuation in Ω if, for every continuous curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = P$, there is an analytic continuation of (f,D) along γ.

(Exercise 5623) Prove directly (without Theorem 10.3.3 or Corollary 10.3.4) that $(\ln, D(1,1/2))$ does not admit unrestricted continuation in $\mathbb{C}$.

Theorem 10.3.3. [The monodromy theorem]. Let $\Omega \subseteq \mathbb{C}$ be open and connected, $D = D(P,r) \subseteq \Omega$, and let (f,D) be a function element. Assume that (f,D) admits unrestricted continuation in Ω.

Let γ_0, $\gamma_1 : [0,1] \to \Omega$ be homotopic curves in Ω with $\gamma_0(0) = \gamma_1(0) = P$. If $s \in [0,1]$, let $\{(f_{t,s}, D_{t,s})\}_{0\leq t\leq 1}$ be an analytic continuation of $(f, D(P,r))$ along γ_s.

Then $f_{1,s} = f_{1,\sigma}$ in $D_{1,s} \cap D_{1,\sigma}$. for all $s, \sigma \in [0,1]$. In particular, $f_{1,0} = f_{1,1}$ in $D_{1,0} \cap D_{1,1}$.

(Irina, Problem 5630) Suppose that (f,D) is a function element, $D = D(P,r)$, $\gamma : [0,1] \to \mathbb{C}$ is a curve with $\gamma(0) = P$, and that there exists an analytic continuation $\{(f_t, D(\gamma(t), r_t))\}_{t\in[0,1]}$ of (f,D) along γ.

Show that there is also an analytic continuation along γ such that the radius r_t is a constant. Can you do this in such a way that the $\varepsilon = \varepsilon_t$ in Definition 10.2.1 is also a constant?

(Proposition 5631) Let $\gamma : [0,1] \to \mathbb{C}$ be a curve. Let $D = D(\gamma(0), r)$, let (f,D) be a function element, and let $\{(g_t, D(\gamma(t), r))\}_{t\in[0,1]}$ be an analytic continuation of (f,D) along γ such that the discs in the definition of analytic continuation all have the same radii.

Let $\eta : [0,1] \to \mathbb{C}$ be a curve. Suppose that $|\eta(t) - \gamma(t)| < r$ for all $t \in [0,1]$. Suppose that $\eta(0) = \gamma(0)$ and that $\{(h_t, D(\eta(t), \varrho))\}_{t\in[0,1]}$ is an analytic continuation of (f,D) along η (possibly with a different radius for the discs in the definition of analytic continuation).

Then $(h_t, D(\eta(t), \varrho))$ is a direct analytic continuation of $(g_t, D(\gamma(t), r))$ for all t. In particular, if $\gamma(t) = \eta(t)$ then $g_t = h_t$ in $D(\eta(t), \min(r, \varrho))$.

(Michael, Problem 5640) Begin the proof of Proposition 5631 as follows. Let $S = \{t \in [0,1] : g_t = h_t$ in $D(\gamma(t), r) \cap D(\eta(t), \varrho)\}$. Show that S is relatively open in $[0,1]$.

Let $t \in S$. By definition of analytic continuation, there is a $\varepsilon_\gamma > 0$ such that if $|t - s| < \varepsilon_\gamma$ and $s \in [0,1]$, then $g_t = g_s$ in $D(\gamma(t), r)) \cap D(\gamma(s), r))$. Define ε_η similarly.

Because η is continuous, there is a $\delta_\eta > 0$ such that if $|t - s| < \delta_\eta$ and $s \in [0,1]$ then $|\eta(t) - \eta(s)| < \varrho$. Furthermore, there is a $\delta_\gamma > 0$ such that if $|t - s| < \delta_\gamma$ and $s \in [0,1]$ then $|\gamma(t) - \gamma(s)| < r - |\gamma(t) - \eta(t)|$ and so $|\gamma(s) - \eta(t)| < r$.

Let $\delta = \min(\varepsilon_\gamma, \varepsilon_\eta, \delta_\gamma, \delta_\eta)$.

Suppose $s \in [0,1]$ and $|t - s| < \delta$. We then have that

$$\eta(t) \in D(\gamma(s), r) \cap D(\gamma(t), r) \cap D(\eta(s), \varrho) \cap D(\eta(t), \varrho).$$

In particular, the right hand intersection is nonempty.

Because $|s - t| < \varepsilon_\gamma$, we have that $g_s = g_t$ in $D(\gamma(t), r) \cap D(\gamma(s), r)$.

[4]In Section 2.6 we required γ_c, γ_d, and Γ to be C^1.

Because $t \in S$, we have that $g_t = h_t$ in $D(\gamma(t), r) \cap D(\eta(t), \varrho)$.
Because $|s - t| < \varepsilon_\eta$, we have that $h_s = h_t$ in $D(\eta(t), \varrho) \cap D(\eta(s), \varrho)$.
Thus, $g_s = h_s$ in $D(\gamma(s), r) \cap D(\gamma(t), r) \cap D(\eta(s), \varrho) \cap D(\eta(t), \varrho)$.
The given set is nonempty (it contains $\eta(t)$) and also open, and so has accumulation points. Furthermore, $D(\gamma(s), r) \cap D(\eta(s), \varrho)$ is connected, and so we must have that $g_s = h_s$ in $D(\gamma(s), r) \cap D(\eta(s), \varrho)$. Thus $s \in S$ by definition of S, and so S is relatively open in $[0, 1]$.

(Timmy, Problem 5650) Complete the proof of Proposition 5631 by showing that S is closed.

(Zach, Problem 5660) Use Proposition 5631 to prove Theorem 10.3.3.

Corollary 10.3.4. Let $\Omega \subseteq \mathbb{C}$ be open and simply connected and let $(f, D(P, r))$ be a function element with $D(P, r) \subseteq \Omega$. Assume that $(f, D(P, r))$ admits unrestricted continuation in Ω. Then there exists a unique function $F : \Omega \to \mathbb{C}$ that is holomorphic in Ω and satisfies $F = f$ in $D(P, r)$.

10.4. Topology

[Definition: Covering space] Let W, Ω be two topological spaces. Suppose that there is a continuous function $\pi : W \to \Omega$ such that, if $z \in \Omega$, then there is a connected open set U with $x \in U \subset \Omega$ and such that, if V is a connected component of $\pi^{-1}(U)$, then $\pi : V \to U$ is a homeomorphism. Then we say that W is a covering space for Ω.

(Problem 5661) $\mathbb{R}$ is a covering space for $S^1 = \{(x, y) \in \mathbb{R}^2 : |(x, y)| = 1\}$ under the mapping $\pi(t) = (\cos t, \sin t)$.

(Bonus Problem 5662) If Ω is path connected and also simply connected, show that π must be a homeomorphism from W to Ω.

(Problem 5663) Let $\gamma : [0, 1] \to \mathbb{C}$ be a C^1 curve. Show that $\ell(\gamma) \leq |\gamma(1)-\gamma(0)|$ and that, if $\ell(\gamma) = |\gamma(1)-\gamma(0)|$, then γ must be a parameterization of the straight line segment from $\gamma(0)$ to $\gamma(1)$.

If γ is closed (if $\gamma(0) = \gamma(1)$) then this is clearly true if by the "straight line segment" from a point to itself we mean the corresponding constant path.
Let $\omega = \gamma(1) - \gamma(0)$ and assume $\omega \neq 0$. Then $\frac{\omega}{|\omega|} = e^{i\theta}$ for some $\theta \in \mathbb{R}$.
We compute that
$$\ell(\gamma) = \int_0^1 |\gamma'(t)|\, dt.$$
Observe that
$$|\gamma'(t)| = |\gamma'(t)e^{-i\theta}| \geq \operatorname{Re}(\gamma'(t)e^{-i\theta})$$
and so
$$\ell(\gamma) = \int_0^1 |\gamma'(t)|\, dt \geq \int_0^1 \operatorname{Re}(\gamma'(t)e^{-i\theta})\, dt = \operatorname{Re}\left(e^{-i\theta} \int_0^1 \gamma'(t)\, dt\right) = \operatorname{Re}\left(e^{-i\theta}(\gamma(1)-\gamma(0))\right) = \operatorname{Re}\left(\frac{\overline{\gamma(1) - \gamma(0)}}{|\gamma(1) - \gamma(0)|}(\gamma(1)-\gamma(0))\right)$$
Furthermore, both $|\gamma'(t)|$ and $\operatorname{Re}(\gamma'(t)e^{-i\theta})$ are continuous functions of t because γ is, and so by Problem 4120, if $\ell(\gamma) = |\gamma(1) - \gamma(0)|$ then $\int_0^1 |\gamma'(t)|\, dt = \int_0^1 \operatorname{Re}(\gamma'(t)e^{-i\theta})\, dt$ and so we must have that $|\gamma'(t)| = \operatorname{Re}(\gamma'(t)e^{-i\theta})$ for all t. In particular, $\gamma'(t)e^{-i\theta}$ must be a nonnegative real number. Thus
$$\gamma(t) = \gamma(0) + \int_0^t \gamma'(s)\, ds = \gamma(0) + \int_0^t e^{i\theta}|\gamma'(s)|\, ds = \gamma(0) + e^{i\theta}\ell(\gamma|_[0, t])$$
and so γ is a line segment, as desired.

(Problem 5664) Let $\Omega \subset \mathbb{C}$ be open and connected. If we define $d(z, w) = \inf\{\ell(\gamma) \mid \gamma : [0, 1] \to \Omega$ is $C^1, \gamma(0) = z, \gamma(1) = w\}$, show that (Ω, d) is a metric space.

(Alex, Problem 5670) Let d and Ω be as in the previous problem. Show that if $D(z, r) \subset \Omega$, then $|z - w| < r$ if and only if $d(z, w) < r$, and $d(w_1, w_2) = |w_1 - w_2|$ for all w_1, $w_2 \in D(z, r)$.

By Problem 5663, we have that $d(z, w) \geq |z - w|$ for all z, $w \in \Omega$.
Thus, if $d(z, w) < r$ then $|z - w| < r$ and so $w \in D(z, r)$.

Conversely, suppose that $w_1, w_2 \in D(z,r)$. Let $\gamma(t) = w_1 + t(w_2 - w_1)$. If $0 \leq t \leq 1$ then $\gamma(t) \in D(z,r)$ because $D(z,r)$ is convex, and therefore $\gamma : [0,1] \to \Omega$. Furthermore, $\gamma(0) = w_1$ and $\gamma(1) = w_2$. It is elementary to show that $\ell(\gamma) = |w_2 - w_1|$, and so $d(w_1, w_2) \leq \ell(\gamma) \leq |w_2 - w_1|$. Applying the converse inequality, we see that for such w_1 and w_2 we have that $d(w_1, w_2) = |w_1 - w_2|$. In particular, $z \in D(z,r)$, so if $w \in D(z,r)$ then $|w - z| < r$ and so $d(w,z) = |w - z| < r$.

(Problem 5671) Let $\Omega \subset \mathbb{C}$ be open and connected and let d be as in Problem 5664.

Let $z_0 \in \Omega$ and let

$$W = \{\gamma \mid \gamma : [0,1] \to \Omega \text{ is } C^1, \gamma(0) = z_0\}/\sim$$

where $\gamma \sim \eta$ if $\gamma(1) = \eta(1)$ and γ and η are homotopic.

If $\gamma_1, \eta \in W$, let

$$\delta(\gamma_1, \eta) = \inf\{\ell(\gamma_2) : \gamma_2(0) = \gamma_1(1), \gamma_2(1) = \eta(1), \gamma_3 \text{ is homotopic to } \eta\},$$

where γ_3 is given from γ_1 and γ_2 by Problem 1050.

Show that (W, δ) is a metric space.

(David, Problem 5680) Define $\pi : W \to \Omega$ by $\pi(\gamma) = \gamma(1)$. Suppose $\gamma \in W$ and that $D(\pi(\gamma), r) \subset \Omega$. Show that π is an isomorphism of metric spaces from $B(\gamma, r)$ to $D(\pi(\gamma), r)$, that is, that $\pi(B(\gamma, r)) = D(\pi(\gamma), r)$ and that $\delta(\zeta, \eta) = |\pi(\zeta) - \pi(\eta)|$ for all $\zeta, \eta \in B(\gamma, r)$.

(Emily, Problem 5690) Suppose that $D(z,r) \subset \Omega$. Show that if $\pi(\gamma) = \pi(\eta) = z$, then either $\gamma = \eta$ or $\delta(\gamma, \eta) \geq r$.

(Bonus Problem 5691) Show that (W, δ) is simply connected.

10.4. The Idea of a Riemann Surface

(Irina, Problem 5700) Let $\Omega \subset \mathbb{C}$ be open. Let $D \subset \Omega$ and let (f, D) be a function element that admits unrestricted continuation in Ω.

Let $z_0 \in \Omega$ and let

$$\Psi = \{\gamma \mid \gamma : [0,1] \to \Omega \text{ is } C^1, \gamma(0) = z_0\}/\sim$$

where $\gamma \sim \eta$ if $\eta(1) = \gamma(1)$ and the analytic continuations $\{g_t, D(\gamma(t), r_t)\}_{t\in[0,1]}$ and $\{h_t, D(\eta(t), q_t)\}_{t\in[0,1]}$ of (f, D) along γ and η, respectively, satisfy $g_t = h_t$ in $D(\gamma(1), r_1) \cap D(\eta(1), q_1)$.

Show that if γ and η are homotopic then $\gamma \sim \eta$. Give an example to show that the reverse may not be true.

(Michael, Problem 5710) What metric would you like to impose on Ψ?

(Problem 5711) Show that W is a covering of Ψ and that Ψ is a covering of Ω.

10.5. The Elliptic Modular Function and Picard's Theorem

(Timmy, Problem 5720) Let $f : \mathbb{C} \to \mathbb{C}$ be entire and not constant. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Theorem 10.5.1. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and not constant. Then $\mathbb{C} \setminus f(\mathbb{C})$ contains at most one point.

Recall [Problem 3620]: Let $f : \mathbb{H} \to \mathbb{H}$ be a function. Then f is a biholomorphic self-map if and only if $f(z) = \frac{az+b}{cz+d}$ for some real numbers a, b, c, and d that satisfy $ad - bc > 0$.

(Zach, Problem 5730) Let G be the set of all fractional linear transformations that may be written $f(z) = \frac{az+b}{cz+d}$ where a, b, c, and d are integers that satisfy $ad - bc = 1$. Show that if $f, g \in G$ then $f \circ g \in G$ and $f^{-1} \in G$.