Major definitions from last semester

[Definition: Outer measure] Let $A \subseteq \mathbf{R}$. The *outer measure* $|A|$ of A is defined to be

$$|A| = \inf\Big\{\sum_{k=1}^{\infty} \ell(I_k) : \text{Each } I_k \text{ is an open interval and } A \subseteq \bigcup_{k=1}^{\infty} I_k\Big\}.$$

[Definition: σ-algebra] Let X be a set. Let $\mathcal{S}$ be a collection of subsets of $\mathcal{S}$. We say that $(X, \mathcal{S})$ is a measurable space if:

- $\emptyset \in \mathcal{S}$,
- If $E \in \mathcal{S}$, then $X \setminus E \in \mathcal{S}$,
- If $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$ then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$.

We call the elements of $\mathcal{S}$ measurable sets.

[Definition: measure] Let X be a set. Let $\mathcal{S}$ be a σ-algebra on X. We say that μ is a measure on $(X, \mathcal{S})$ if:

- $\mu : \mathcal{S} \to [0, \infty]$
- $\mu(\emptyset) = 0$.
- If $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$ is a sequence of pairwise-disjoint subsets of X then $\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{k=1}^{\infty} \mu(E_k)$.

We call $(X, \mathcal{S}, \mu)$ a measure space.

[Definition: Integral of a nonnegative function] Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $f : X \to [0, \infty]$ be $\mathcal{S}$-measurable. Then

$$\int f \, d\mu = \sup\Big\{\sum_{j=1}^{m} \mu(A_j) \inf_{A_j} f : X = \bigsqcup_{j=1}^{m} A_j\Big\}.$$

[Definition: The Lebesgue space] Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $f : X \to [-\infty, \infty]$ be $\mathcal{S}$-measurable. Then the $\mathcal{L}^1$-norm of f is defined by

$$\|f\|_1 = \int |f| \, d\mu.$$

The Lebesgue space $\mathcal{L}^1(\mu)$ is defined by

$$\mathcal{L}^1(\mu) = \{f : f \text{ is an } \mathcal{S}\text{-measurable function } f : X \to \mathbf{R} \text{ and } \|f\|_1 < \infty\}.$$

[Definition: Hardy-Littlewood maximal function] Let $h : \mathbf{R} \to \mathbf{R}$ be measurable. Then

$$h^*(b) = \sup_{t>0} \frac{1}{2t} \int_{b-t}^{b+t} |h|.$$

[Definition: Product σ-algebra] Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces. Then $\mathcal{S} \otimes \mathcal{T}$ is the smallest σ-algebra on $X \times Y$ that contains $A \times B$ for all $A \in \mathcal{S}$ and all $B \in \mathcal{T}$.

[Definition: Cross section of a set] Let X and Y be sets and let $E \subseteq X \times Y$. If $a \in X$ or $b \in Y$, then the cross sections $[E]_a$ and $[E]^b$ are defined by

$$[E]_a = \{y \in Y : (a, y) \in E\}, \quad [E]^b = \{x \in X : (x, b) \in E\}.$$

[Definition: Cross sections of functions] Let $f : X \times Y \to \mathbf{R}$ be a function. If $a \in X$ and $b \in Y$, we let

$$[f]_a(y) = f(a, y), \quad [f]^b(x) = f(x, b)$$

for all $y \in Y$, $x \in X$.

[Definition: Algebra] Let W be a set and let $\mathcal{A}$ be a set of subsets of W. We say that $\mathcal{A}$ is an algebra on W if:

- $\emptyset \in \mathcal{A}$
- If $E \in \mathcal{A}$ then $W \setminus E \in \mathcal{A}$
- If E, $F \in \mathcal{A}$, then $E \cup F \in \mathcal{A}$.

[Definition: Monotone class] Let W be a set and let $\mathcal{M}$ be a set of subsets of W. We say that $\mathcal{M}$ is a monotone class on W if:

- If $E_1 \subseteq E_2 \subseteq \dots$ is an increasing sequence of sets in $\mathcal{M}$ then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$.

- If $E_1 \supseteq E_2 \supseteq \dots$ is a decreasing sequence of sets in $\mathcal{M}$ then $\bigcap_{k=1}^{\infty} E_k \in \mathcal{M}$.

5C. Undergraduate analysis

[Definition: The norm $\|\cdot\|_\infty$] Let $x = (x_1, \dots, x_n) \in \mathbf{R}^n$. Then $\|x\| = \max\{|x_1|, |x_2|, \dots, |x_n|\}$.

(Problem 10) Let $d(x, y) = \|x - y\|_\infty$. Show that $(\mathbf{R}^n, d)$ is a metric space.

[Definition: Ball] The open ball with radius δ is

$$B(x, \delta) = \{y \in \mathbf{R}^n : d(x, y) < \delta\}.$$

(Problem 11) Describe the set $B((3, 4), 1) \subseteq \mathbf{R}^2$, where the metric is $d(x, y) = \|x - y\|_\infty$.

[Definition: Cube] We will let

$$Q(x, \delta) = \{y \in \mathbf{R}^n : \|x - y\|_\infty < \delta\}$$

that is, $Q(x, \delta)$ is the ball in the metric based on $\|\cdot\|_\infty$.

5C. Lebesgue Integration on $\mathbf{R}^n$

[Definition: Open set] A subset $G \subseteq \mathbf{R}^n$ is open if for all $x \in G$, there is a $\delta > 0$ with $Q(x, \delta) \subseteq G$.

[Exercise 5C.1] Show that $G \subseteq \mathbf{R}^n$ is open if and only if, for all $x \in G$, there is a $\rho > 0$ with $\{y \in \mathbf{R}^n : \|x - y\|_2 < \rho\} \subseteq G$, where $\|x\|_2$ is the usual Euclidean distance in $\mathbf{R}^n$.

[Definition: Natural identification] We identify the point $((x_1, \dots, x_n), (y_1, \dots, y_m))$ in $\mathbf{R}^n \times \mathbf{R}^m$ with the point $(x_1, \dots, x_n, y_1, \dots, y_m)$ in $\mathbf{R}^{n+m}$.

(Problem 30) Show that $Q(x, \delta) \times Q(y, \delta) = Q((x, y), \delta)$ for all $x \in \mathbf{R}^n$ and all $y \in \mathbf{R}^m$.

(Problem 40) Let $G_1 \subseteq \mathbf{R}^n$ and $G_2 \subseteq \mathbf{R}^m$. Suppose that G_1 and G_2 are both open. Show that $G_1 \times G_2$ is also open.

[Definition: Borel sets] The smallest σ-algebra on $\mathbf{R}^n$ containing all open subsets of $\mathbf{R}^n$ is denoted by $\mathcal{B}_n$, and its elements are called Borel subsets of $\mathbf{R}^n$.

(Problem 50) Let $G \subseteq \mathbf{R}^n$. Suppose that G is the union of countably many open cubes. Show that G is open.

(Problem 60) Let $G \subseteq \mathbf{R}^n$. Suppose that G is open. Show that G is the union of countably many open cubes.

(Problem 70) Show that $\mathcal{B}_n$ is the smallest σ-algebra containing all open cubes.

(Problem 80) Show that $\mathcal{B}_{m+n} \subseteq \mathcal{B}_m \otimes \mathcal{B}_n$.

(Problem 90) Show that if $D \in \mathcal{B}_m$ and $G \subseteq \mathbf{R}^n$ is open, then $D \times G \in \mathcal{B}_{m+n}$.

(Problem 100) Show that if $D \in \mathcal{B}_m$ and $E \in B_n$, then $D \times E \in \mathcal{B}_{m+n}$.

(Problem 110) What does this tell you about $\mathcal{B}_m \otimes \mathcal{B}_n$ and $\mathcal{B}_{m+n}$?

(Problem 120) Show that $(\mathcal{B}_m \otimes \mathcal{B}_n) \otimes \mathcal{B}_p = \mathcal{B}_m \otimes (\mathcal{B}_n \otimes \mathcal{B}_p)$.

[Definition: Lebesgue measure] We define λ_1 to be Lebesgue measure on $\mathcal{B}_1$, and $\lambda_n = \lambda_{n-1} \times \lambda_1$.

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(Problem 121) Let $G \subseteq \mathbf{R}^n$ be open. Show that $\{D \in \mathcal{B}_n : D \subseteq G\} = \{D \cap G : D \in \mathcal{B}_n\}$ is the smallest σ-algebra on G containing all open subsets of G.

(Problem 130) Let $t > 0$. Let $E \in \mathcal{B}_n$. Show that $tE \in B_n$.

(Problem 140) Suppose $E \in \mathcal{B}_n$ and $E \subseteq Q(0, m)$ for some $m \in \mathbf{N}$. Show that $\lambda_n(tE) = t^n \lambda_n(E)$.

(Problem 150) Show that $\lambda_n(tE) = t^n \lambda_n(E)$ even if E is unbounded.

[Definition: Open unit ball in $\mathbf{R}^n$] The open unit ball in $\mathbf{R}^n$ is defined by

$$\mathbf{B}_n = \{(x_1, \dots, x_n) \in \mathbf{R}^n : {x_1}^2 + \dots + {x_n}^2 < 1\}.$$

[Definition: π] We define $\pi = \lambda_2(\mathbf{B}_2)$.

[Exercise 5C.8] Show that $\mathbf{B}_n$ is open.

(Problem 170) Show that if $n \geq 3$, then $\lambda_n(\mathbf{B}_n) = \frac{2\pi}{n}\lambda_{n-2}(\mathbf{B}_{n-2})$.

[Definition: $n!!$] We define $1!! = 1$, $2!! = 2$, and $n!! = n\big((n-2)!!\big)$ if $n \in \mathbf{N}$ with $n \geq 3$.

(Problem 180) Suppose that n is even. Show that $\lambda_n(\mathbf{B}_n) = \frac{2^{n/2}\pi^{n/2}}{n!!}$.

(Problem 190) Suppose that n is odd. Show that $\lambda_n(\mathbf{B}_n) = \frac{2^{(n+1)/2}\pi^{(n-1)/2}}{n!!}$.

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[Definition: Partial derivatives] Let $G \subseteq \mathbf{R}^2$ be open. Let $f : G \to \mathbf{R}$ be a function. If $(x, y) \in G$, then

$$(D_1 f)(x,y) = \lim_{t\to 0} \frac{f(x+t,y) - f(x,y)}{t} = ([f]^y)'(x), \qquad (D_2 f)(x,y) = \lim_{t\to 0} \frac{f(x,y+t) - f(x,y)}{t} = ([f]_x)'(y)$$

provided the given limits (or derivatives) exist.

(Problem 200) Suppose $G \subseteq \mathbf{R}^2$ is open. Let $f : G \to \mathbf{R}$ be a function. Assume that $D_1 f$, $D_2 f$, $D_1(D_2 f)$, and $D_2(D_1 f)$ all exist and are continuous on G. Show that $D_1(D_2 f) = D_2(D_1 f)$.

6A. Metric spaces

Please review this section.

6B. Vector spaces

Please review this section.

[Definition: Measurable complex-valued function] Let $(X, \mathcal{S})$ be a measurable space and let $f : X \to \mathbf{C}$. We say that f is $\mathcal{S}$-measurable if $\operatorname{Re} f$ and $\operatorname{Im} f$ are both $\mathcal{S}$-measurable functions.

(Problem 210) Suppose that f is $\mathcal{S}$-measurable and that $p > 0$. Show that $|f|^p$ is $\mathcal{S}$-measurable.

[Definition: Integral of a complex-valued function] Let $(X, \mathcal{S}, \mu)$ be a measure space and $f : X \to \mathbf{C}$ be measurable. Then

$$\int_X f\,d\mu = \int_X \operatorname{Re} f\,d\mu + i\int_X \operatorname{Im} f\,d\mu.$$

[Exercise 6B.8] If f is $\mathcal{S}$-measurable, $\int |f|\,d\mu < \infty$, and $\alpha \in C$, then $\int \alpha f\,d\mu = \alpha \int f\,d\mu$.

(Problem 220) Suppose that $\int_X f\,d\mu$ exists and is finite. Show that $\left|\int_X f\,d\mu\right| \leq \int_X |f|\,d\mu$.

(Problem 221) Is this still true if $\int_X f\,d\mu = \pm\infty$?

(Problem 222) If f is measurable but $\int_X f\,d\mu$ does not exist, what can you say about $\int_X |f|\,d\mu$?

(Problem 230) Show that $\overline{\int_X f\,d\mu} = \int_X \overline{f}\,d\mu$.

[Definition: F] We let $\mathbf{F}$ be either $\mathbf{R}$ or $\mathbf{C}$.

[Definition: Subspace] A subset U of a vector space V is a subspace if it contains 0 and is closed under addition and scalar multiplication.

[Definition: $\mathbf{F}^X$] Let X be a set. Let $\mathbf{F}^X = \{f : X \to \mathbf{F}\}$ be the set of all functions from X to $\mathbf{F}$.

(Problem 231) Show that $\mathbf{F}^X$ is a vector space for any set X.

[Exercise 6B.9] If $\mathcal{E}$ is a set of subspaces of a vector space V, show that $\cap_{U\in\mathcal{E}} U$ is a subspace.

(Problem 240) Let (X, d) be a metric space and let $C(X)$ be the space of all continuous functions from X to $\mathbf{F}$. Show that $C(X)$ is a subspace of $\mathbf{F}^X$.

(Problem 250) Let $(X, \mathcal{S})$ be a measurable space. Let $U = \{f : X \to \mathbf{F} | f \text{ is } \mathcal{S}\text{-measurable}\}$. Show that U is a subspace of $\mathbf{F}^X$.

(Problem 260) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $U = \{f : X \to \mathbf{F} | f \text{ is } \mathcal{S}\text{-measurable}, \mu(\{x : f(x) \neq 0\}) = 0\}$. Show that U is a subspace of $\mathbf{F}^X$.

(Problem 270) Let $(X, \mathcal{S})$ be a measurable space. Let $U = \{f : X \to \mathbf{F} | f \text{ is bounded}\}$. Show that U is a subspace of $\mathbf{F}^X$.

(Problem 280) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $U = \{f : X \to \mathbf{F} | f$ is $\mathcal{S}$-measurable and $\int_X f\,d\mu = 0\}$. Show that U is a subspace of $\mathbf{F}^X$.

(Problem 290) Let $(X, \mathcal{S}, \mu)$ be a measure space. Show that $\mathcal{L}^1(d\mu)$ is a subspace of $\mathbf{F}^X$.

(Problem 300) Let $\ell^1 = \mathcal{L}^1(d\mu)$, where μ represents the counting measure on $\mathbf{N}$. Show that ℓ^1 is a subspace of $\mathbf{F}^{\mathbf{N}}$.

6C. Undergraduate analysis

[Definition: Norm] We say that $\|\cdot\|$ is a norm on the $\mathbf{F}$-vector space V if for all f, $g \in V$ and $\alpha \in \mathbf{F}$,

- $\|f\|$ exists,
- $0 \le \|f\| < \infty$,
- $\|f\| = 0$ if and only if $f = 0$,
- $\|\alpha f\| = |\alpha|\,\|f\|$,
- $\|f + g\| \le \|f\| + \|g\|$.

(Problem 310) Let V be a normed vector space. Let $d(f, g) = \|f - g\|$. Show that (V, d) is a metric space.

6C. Normed vector spaces

(Problem 320) Let ℓ^1 be the space of all sequences of numbers $\{a_n\}_{n=1}^\infty$ with $\sum_{n=1}^\infty |a_n| < \infty$. Let $\|\{a_n\}_{n=1}^\infty\|_1 = \sum_{n=1}^\infty |a_n|$. Show that $\|\cdot\|_1$ is a norm on ℓ^1.

(Problem 330) Let $\|\{a_n\}_{n=1}^\infty\|_\infty = \sup_{n\in\mathbf{N}} |a_n|$. Show that $\|\cdot\|_\infty$ is a norm on ℓ^1.

(Problem 340) Show that $\|f\|_1 = \int_X |f|\,d\lambda$ is not a norm on $\mathcal{L}^1(\mathbf{R})$.

(Problem 350) Let $C[0, 1]$ be the space of all continuous functions defined on $[0, 1]$. Show that $\|f\|_1 = \int_X |f|\,d\lambda$ is a norm on $C[0, 1]$.

[Definition: Banach space] A normed vector space that is complete (as a metric space with $d(x, y) = \|x - y\|$) is a Banach space.

(Problem 360) Let V be a nontrivial finite-dimensional normed vector space. Show that there exists a basis of unit vectors.

[Exercise 6D.8] Let V be a nontrivial finite-dimensional normed vector space. Show that V is complete. *Hint*: Let $\{e_1, \dots e_n\}$ be a basis of unit vectors. Show that there is a constant $C < \infty$ such that if the a_ks are real numbers, then $|a_j| \le C\|\sum_{k=1}^n a_k e_k\|$ for each $1 \le j \le n$.

(Problem 390) Give an example of a vector space that is not a Banach space.

[Definition: Infinite sum] Let $\{g_k\}_{k=1}^\infty$ be a sequence in a normed vector space V. If $\{\sum_{k=1}^n g_k\}_{n=1}^\infty$ is a convergent sequence, we say that the series converges and let $\sum_{k=1}^\infty g_k = \lim_{n\to\infty} \sum_{k=1}^n g_k$.

(Problem 430) Suppose that $\sum_{k=1}^\infty \|g_k\| < \infty$. Show that $\{\sum_{k=1}^n g_k\}_{n=1}^\infty$ is Cauchy.

(Problem 440) Suppose V is a Banach space and that $\{g_k\}_{k=1}^\infty \subseteq V$, $\sum_{k=1}^\infty \|g_k\| < \infty$. Show that $\sum_{k=1}^\infty g_k$ converges.

[Exercise 6A.14] Any Cauchy sequence with a convergent subsequence is convergent.

(Problem 450) Let V be a normed vector space. Suppose that $\sum_{k=1}^\infty g_k$ converges whenever $\sum_{k=1}^\infty \|g_k\| < \infty$. Show that V is a Banach space.

* * *

[Definition: Linear map] Let V and W be vector spaces over $\mathbf{F}$. We say that T is a linear function or linear map from V to W if

- $T : V \to W$,
- $T(f + g) = T(f) + T(g)$ for all f, $g \in V$,
- $T(\alpha f) = \alpha T(f)$ for all $\alpha \in \mathbf{F}$, $f \in V$.

We often write $Tf = T(f)$.

(Problem 451) Suppose that T, $S : V \to W$ are two linear maps. Suppose that $Tf = Sf$ for all $f \in V$ such that $\|f\| = 1$. Show that $Tg = Sg$ for all $g \in V$.

[Definition: Sum of linear maps] If S, T are two linear maps from V to W and $\alpha \in F$, then αT and $S + T$ are the functions defined by

- $(S + T)f = Sf + Tf$ for all $f \in V$,
- $(\alpha T)f = \alpha(Tf)$ for all $f \in V$.

(Problem 452) Let V and W be vector spaces over $\mathbf{F}$. Show that the set of all linear functions from V to W is also a vector space over $\mathbf{F}$.

[Definition: Norm of linear maps] Let V and W be normed vector spaces over $\mathbf{F}$. If $T : V \to W$ is a linear map, then

$$\|T\| = \sup\{\|Tf\| : f \in V,\ \|f\| \le 1\}.$$

If $\|T\| < \infty$, we say that T is bounded. We let $\mathcal{B}(V, W)$ be the set of all bounded linear maps from V to W.

(Problem 453) Recall that a (possibly nonlinear) function $\varphi : X \to W$ is usually described as bounded if $\sup_{f\in X} \|\varphi(f)\| < \infty$. Show that T is bounded as a linear map if and only if it is bounded in the usual way on the unit ball, that is, $\sup_{\{f\in V:\|f\|\le 1\}} \|\varphi\| < \infty$.

(Problem 454) Let T be a linear map that is bounded in the usual way on all of V, that is, $\|Tf\| \le C < \infty$ for all $f \in V$, where C is independent of f. What can you say about T?

(Problem 455) Let $T \in \mathcal{B}(V, W)$ and let $f \in V$. Show that $\|Tf\|_W \le \|T\|\,\|f\|_V$.

(Problem 456) Show that the bound you found in the previous problem is the best possible (that is, there is some sequence $\{f_k\}_{k=1}^{\infty} \subseteq V$ with $\|Tf\|_W > \frac{n}{n-1}\|T\|\,\|f\|_V$).

(Problem 457) Show that a linear map $T : V \to W$ is unbounded if and only if there is a sequence $\{f_k\}_{k=1}^{\infty} \subseteq V$ with $\|f_k\| = 1$ and $\|Tf_k\| > k$.

(Problem 460) Show that $\mathcal{B}(V, W)$ is a normed vector space.

[Exercise 6D.7] Show that if V is finite-dimensional and $T : V \to W$ is a linear map, then T is bounded.

(Problem 470) Give me three examples of linear maps that are not bounded.

(Problem 471) Were any of your examples defined on a Banach space?

(Problem 480) Suppose that W is a Banach space. Show that $\mathcal{B}(V, W)$ is a Banach space.

(Problem 490) Let $T : V \to W$ be a linear map. Show that T is continuous if and only if it is bounded.

[Exercise 6C.16] Let V be a normed vector space and $U \subseteq V$ be a subspace. Define

$$\|f + U\| = \inf\{\|f + g\| : g \in U\}.$$

Then $\|\cdot\|$ is a norm on V/U if and only if U is closed. If V is a Banach space and U is closed, then V/U is a Banach space. If U and V/U are both Banach spaces, so is V.

6D. Linear Functionals

[Definition: Linear functional] A linear functional on a vector space V over the field $\mathbf{F}$ is a linear map from V to $\mathbf{F}$.

(Problem 500) Let $\varphi : \ell^1 \to \mathbf{F}$ be given by $\varphi(a_1, a_2, \dots) = \sum_{k=1}^{\infty} a_k$. Show that φ is a linear functional.

(Problem 510) If $\|(a_1, a_2, \dots)\| = \sum_{n=1}^{\infty} |a_n|$, show that φ is a bounded linear functional.

(Problem 520) If $\|(a_1, a_2, \dots)\| = \sup_{1\le n<\infty} |a_n|$, show that φ is an unbounded linear functional.

[Definition: Null space] Let $T : V \to W$ be a linear map. Then the null space of T is

$$\operatorname{null} T = \{f \in V : Tf = 0\}.$$

(Problem 530) Show that $\operatorname{null} T$ is a subspace of V.

(Problem 540) [null:closed] Show that if T is a bounded linear map, then $\operatorname{null} T$ is a closed subspace of V.

(Problem 550) Give an example of an unbounded linear map whose null space is closed.

(Problem 560) Suppose that V is a normed vector space and $\varphi : V \to \mathbf{F}$ is a linear functional. Show that φ is continuous if and only if $\operatorname{null}\varphi$ is closed.

(Problem 570) Suppose that V is a normed vector space and $\varphi : V \to \mathbf{F}$ is a linear functional. Show that if φ is unbounded then $\overline{\operatorname{null}\varphi} = V$.

(Problem 580) Describe all bounded linear functionals $\varphi : V \to \mathbf{F}$ for which $\overline{\operatorname{null}\varphi} = V$.

* * *

[Definition: Family] A family $\{e_k\}_{k\in\Gamma}$ in a set X is a function $e : \Gamma \to X$, where we write e_k instead of $e(k)$. (Compare sequences.)

[Definition: Linear dependence, span, dimension, basis] Let $\{e_k\}_{k\in\Gamma}$ be a family in a vector space V.

- $\{e_k\}_{k\in\Gamma}$ is linearly dependent if there is some finite $\Omega \subseteq \Gamma$ and a family $\{a_j\}_{j\in\Omega}$ in $\mathbf{F}$ such that $\sum_{j\in\Omega} a_j e_j = 0$ and such that $a_j \neq 0$ for at least one $j \in \Omega$.
- $\operatorname{span}\{e_k\}_{k\in\Gamma} = \{\sum_{j\in\Omega} a_j e_j : \Omega \subseteq \Gamma \text{ is finite, } \{a_j\}_{j\in\Omega} \text{ is a family in } F\}$.
- If $V = \operatorname{span}\{e_k\}_{k\in\Gamma}$ for some finite set Γ and family $\{e_k\}_{k\in\Gamma}$, then we say that V is finite-dimensional.
- If $V = \operatorname{span}\{e_k\}_{k\in\Gamma}$ and $\{e_k\}_{k\in\Gamma}$ is linearly independent, we call $\{e_k\}_{k\in\Gamma}$ a basis (or Hamel basis) for V.

(Problem 590) Let $\{e_k\}_{k\in\Gamma}$ be a basis for V. Let $f \in V$. Show that we can write f as a linear combination of basis elements in a unique way.

(Problem 591) Let $\{e_k\}_{k\in\Gamma}$ be a basis for V. Let $g \in \Gamma$ and define T by $T(\sum_{j\in\Omega} a_j e_j) = \begin{cases} a_g, & g \in \Omega, \\ 0, & g \notin \Omega. \end{cases}$ Show that T is well defined on all of V and is a linear functional.

(Problem 600) Let $e_k = (0, \dots, 0, 1, 0, \dots) \in \ell^1$ be the sequence with $(e_k)_j = 0$ if $j \neq k$ and 1 otherwise. Show that $\{e_k\}_{k\in\mathbf{N}}$ is linearly independent.

(Problem 610) Show that $\{e_k\}_{k\in\mathbf{N}}$ is not a basis for ℓ^1.

(Problem 620) What is $\operatorname{span}\{e_k\}_{k\in\mathbf{N}}$?

(Problem 630) Is $\operatorname{span}\{e_k\}_{k\in\mathbf{N}}$ a Banach space?

[Definition: Maximal element] Let V be a set and let $\mathcal{A}$ be a collection of subsets of V. Let $\Gamma \in \mathcal{A}$. Suppose that, if $\Omega \in \mathcal{A}$ and $\Gamma \subseteq \Omega$, we have that $\Gamma = \Omega$. Then we say that Γ is a maximal element of $\mathcal{A}$.

(Problem 640) Let $\mathcal{A}$ be the set of all ideals in $\mathbf{Z}$, that is, $\{\{km : m \in \mathbf{Z}\} : k \in \mathbf{Z}, k \geq 2\}$. Show that $\{km : m \in \mathbf{Z}\}$ is maximal if and only if k is prime.

(Problem 650) Let V be a vector space. Let $\mathcal{A}$ be the set of all linearly independent subsets of V. Let $\Gamma \subset V$ be a maximal element of $\mathcal{A}$. Show that Γ is a basis of V.

(Problem 660) Let Ω be a basis of V. Show that $\Omega \in \mathcal{A}$ and that Ω is a maximal element of $\mathcal{A}$.

[Definition: Chain] Let $\mathcal{C}$ be a collection of subsets of V. Suppose that whenever $\Omega, \Gamma \in \mathcal{C}$ we have that either $\Omega \subseteq \Gamma$ or $\Gamma \subseteq \Omega$. Then we say that $\mathcal{C}$ is a chain.

(Problem 670) Give an example of a chain.

Zorn's Lemma. Let V be a set. Let $\mathcal{A}$ be a collection of subsets of V. Suppose that for every chain $\mathcal{C}$ with $\mathcal{C} \subseteq \mathcal{A}$, we have that $\cup_{\Omega\in\mathcal{C}}\Omega \in \mathcal{A}$. Then $\mathcal{A}$ contains a maximal element.

(Bonus Problem 680) Show that Zorn's Lemma is equivalent to the Axiom of Choice.

(Problem 690) Let V be a vector space. Use Zorn's lemma to show that V has a basis.

(Problem 700) Let V be an infinite-dimensional vector space. Show that there exists a discontinuous linear functional $\varphi : V \to \mathbf{F}$.

* * *

[Exercise 6C.14a] Let V be a vector space, W a Banach space, and $U \subseteq V$ a subspace. Let $S : U \to W$ be a bounded linear map. Then there is a unique continuous function $T : \overline{U} \to W$ such that $Tf = Sf$ for all $f \in U$.

[Exercise 6C.14b] T is a bounded linear map $T : \overline{U} \to W$ and $\|T\| = \|S\|$.

[Definition: Span of an element] If V is a real vector space and $h \in V$, then $\mathbf{R}h = \operatorname{span}\{h\} = \{\alpha h : \alpha \in \mathbf{R}\}$.

[Definition: Sum of subspaces] Let U and Y be two subspaces of a vector space V. Then $U + Y = \{u + y : u \in U, y \in Y\}$.

(Problem 710) Show that $U + Y$ is also a subspace of V.

(Problem 720) Suppose that $U \cap Y = \{0\}$. Show that if $f \in U + Y$ then there is exactly one $u \in U$ and exactly one $y \in Y$ such that $f = u + y$.

(Problem 730) Let V be a real normed vector space and $U \subseteq V$ be a subspace. Let $\psi : U \to \mathbf{R}$ be a bounded linear functional and let $c \in \mathbf{R}$. Let $h \in V$, $h \notin U$. Define $\varphi_c : U + \mathbf{R}h \to \mathbf{R}$ by $\varphi_c(f + \alpha h) = \psi(f) + \alpha c$ for all $f \in U$ and all $\alpha \in \mathbf{R}$. Show that φ_c is well defined and linear.

(Problem 740) Show that there is at least one $c \in \mathbf{R}$ such that φ_c is bounded and $\|\varphi_c\| = \|\varphi\|$.

[Definition: Graph] Let V and W be two sets and $T : V \to W$ be a function. The graph of T is

$$\operatorname{graph}(T) = \{(f, T(f)) : f \in V\} \subset V \times W.$$

(Problem 741) Let $E \subseteq V \times W$. Show that there exists a function $T : V \to W$ such that $E = \operatorname{graph}(T)$ if and only if $[E]_f = \{g \in W : (f, g) \in E\}$ has exactly one element for each $f \in V$.

(Problem 750) [subspace:graph] Suppose that V and W are two linear spaces and $T : V \to W$. Show that if T is a linear map, then $\operatorname{graph}(T)$ is a subspace of $V \times W$.

(Problem 760) [subspace:graph:2] Suppose that V and W are two linear spaces and E is a subspace of $V \times W$. Suppose that for all $f \in V$ there is exactly one $g \in W$ with $(f, g) \in E$. Show that there is a linear map $T : V \to W$ with $E = \operatorname{graph}(T)$.

(Problem 770) Suppose that T is a linear map and $c \in [0, \infty)$. Show that $\|T\| \leq c$ if and only if $\|g\| \leq c\|f\|$ for all $(f, g) \in \operatorname{graph}(T)$.

(Problem 780) Let $U \subseteq V$ and let $S : U \to W$, $T : V \to W$. Show that $Sf = Tf$ for all $f \in U$ if and only if $\operatorname{graph}(S) \subseteq \operatorname{graph}(T)$.

(Problem 790) Let V be a real vector space, $U \subset V$ be a subspace, and $\psi : U \to \mathbf{R}$ be a bounded linear functional. Let $\mathcal{A} = \{\operatorname{graph}(\varphi) : \varphi \text{ is an extension of } \psi \text{ to a subspace } W \text{ with } U \subseteq W \subseteq V \text{ and with } \|\varphi\| = \|\psi\|\}$. Show that $\mathcal{A}$ satisfies the conditions of Zorn's lemma.

(Problem 791) [Another approach] Let V be a real vector space, U be a subspace, and $\psi : U \to \mathbf{R}$ be a bounded linear functional. We say that φ is an extension of ψ if

- The domain W of φ is a subspace of V,
- $U \subseteq W \subseteq V$,
- $\varphi : W \to \mathbf{R}$ is linear,
- $\|\varphi\| \leq \|\psi\|$,
- If $f \in U$ then $\varphi(f) = \psi(f)$.

Let $\mathcal{A} = \{D \subseteq V \times \mathbf{R} : D = \operatorname{graph}(\varphi) \text{ for some extension } \varphi \text{ of } \psi\}$.

(Problem 800) Show that any maximal element of $\mathcal{A}$ must be the graph of a functional defined on all of V.

(Problem 801) We want to show that $\mathcal{A}$ satisfies the conditions of Zorn's lemma. Suppose that $\mathcal{C} \subseteq \mathcal{A}$ is a chain. Let $E = \cup_{\Omega \in \mathcal{C}} \Omega$. Show that $E = \operatorname{graph}(\varphi)$ for some $\varphi : W \to \mathbf{R}$ and some $W \subseteq V$.

(Problem 802) Show that $U \subseteq W$.

(Problem 803) Show that W is a subspace of V and that φ is linear. *Hint*: Show that this is true if and only if E is a subspace of $V \times \mathbf{R}$.

(Problem 804) Show that $\|\varphi\| \leq \|\psi\|$.

(Problem 805) Show that $\varphi(f) = \psi(f)$ for all $f \in U$.

(Problem 806) [real:HB] Conclude that there exists a linear functional $\varphi : V \to \mathbf{R}$ with $\|\varphi\| = \|\psi\|$ and with $\varphi(f) = \psi(f)$ for all $f \in U$.

(Problem 810) Let U be a complex vector space and let $\psi : U \to \mathbf{C}$ be a linear functional. Let $\psi_1 = \operatorname{Re}\psi$. Show that $\psi_1 : U \to \mathbf{R}$ is a linear functional and that $\|\psi_1\| \leq \|\psi\|$.

(Problem 820) Let V be a complex vector space, $U \subset V$ be a subspace, and $\psi : U \to \mathbf{C}$ be a bounded linear functional. Let φ_1 be the extension of ψ_1 given by (806). Show that there is a unique linear functional $\varphi : V \to \mathbf{C}$ with $\varphi_1 = \operatorname{Re}\varphi$ and with $\|\varphi\| = \|\varphi_1\|$.

[Definition: Dual space] If V is a normed vector space, then the dual space is $V' = \mathcal{B}(V, \mathbf{F})$.

(Problem 830) Suppose that $f \in V$ and $\varphi \in V'$. Show that $|\varphi(f)| \leq \|\varphi\|\,\|f\|$.

(Problem 840) Let V be a normed vector space and $f \in V$. Show that there is a $\varphi \in V'$ such that $\|\varphi\| = 1$ and $\varphi(f) = \|f\|$.

(Problem 850) Let U be a subspace of a normed vector space V. Let $h \in V$. Show that $h \in \overline{U}$ if and only if, for all $\varphi \in V'$ with $\varphi(f) = 0$ for all $f \in U$, we have $\varphi(h) = 0$.

6E. Undergraduate analysis

(Problem 860) Write the definitions of the interior of a subset of a metric space and dense subset of metric space.

(Problem 870) Show that $f \in \operatorname{int} G$ if and only if $\overline{B}(f, r) \subseteq G$ for some $r > 0$.

6E. Baire's Theorem

(Problem 880) Let $\varepsilon > 0$. Show that there is a countable open dense subset of $\mathbf{R}$ with measure at most ε.

(Problem 890) [Baire's Theorem] Let (V, d) be a complete metric space. Suppose that $\{G_k\}_{k=1}^{\infty}$ is a sequence of subsets of V. Suppose that each G_k is both dense and open. Show that $\bigcap_{k=1}^{\infty} G_k$ is also dense.

(Problem 900) [Baire's Theorem] Let (V, d) be a complete metric space. Suppose that $\{F_k\}_{k=1}^{\infty}$ is a sequence of subsets of V. Suppose that each F_k is closed and has empty interior. Show that $\bigcup_{k=1}^{\infty} F_k$ is not all of V.

(Problem 910) Let (V, d) be a metric space. Suppose that (V, d) is complete and has no isolated points. Prove that V is uncountable. In particular, $\mathbf{R}$ is uncountable.

(Problem 920) Show that $\mathbf{R} \setminus \mathbf{Q}$ cannot be written as a countable union of closed sets.

* * *

(Problem 930) Let V and W be Banach spaces and $T : V \to W$ be bounded, linear, and surjective. Let $B = B(0, 1) \subseteq V$. Show that $\overline{T(B)}$ has a nonempty interior.

(Problem 940) Show that $\overline{T(B)}$ contains a closed ball centered at 0.

(Problem 950) Show that, if $h \in W$, then there is a $f \in V$ with $\|f\| \leq C\|h\|$ and with $\|h - Tf\| < \varepsilon$, where C is a positive constant independent of h and ε.

(Problem 960) [rightinverse] Show that, if $h \in W$, then there is a $f \in V$ with $\|f\| \leq 2C\|h\|$ and with $h = Tf$.

(Problem 970) Show that $B(0, r) \subseteq T(B)$ for some $r > 0$.

(Problem 980) [Open mapping theorem] Let $G \subseteq V$ be open. Show that $T(G)$ is open.

(Problem 990) If T is a linear bijection, show that T^{-1} is linear.

(Problem 1000) [bounded:inverse] [Bounded Inverse Theorem] If T is a bounded linear bijection between Banach spaces, show that T^{-1} is bounded.

* * *

(Problem 1010) Let V and W be normed vector spaces. Give $V \times W$ the usual vector space structure. How could you define a norm on $V \times W$ in terms of the norms on V and W?

(Problem 1020) Show that $(f_k, g_k) \to (f, g)$ if and only if $f_k \to f$ and $g_k \to g$.

(Problem 1030) Show that $\{(f_k, g_k)\}_{k=1}^{\infty}$ is Cauchy if and only if $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Cauchy.

(Problem 1040) Suppose V and W are Banach spaces. Show that $V \times W$ is also a Banach space.

(Problem 1050) Let V and W be Banach spaces and let $T : V \to W$ be a function. Then T is a bounded linear map if and only if graph(T) is a closed subspace of $V \times W$.

* * *

(Problem 1060) Let V be a Banach space and W be a normed vector space. Suppose that $\mathcal{A} \subseteq \mathcal{B}(V, W)$. Suppose that for each $f \in V$, we have that $\sup\{\|Tf\| : T \in \mathcal{A}\} < \infty$. Show that $\sup\{\|T\| : T \in \mathcal{A}\} < \infty$.

7A. $\mathcal{L}^p(\mu)$

[Definition: The p-norm] Let $(X, \mathcal{S}, \mu)$ be a measure space and $f : X \to \mathbf{F}$ be a $\mathcal{S}$-measurable function. If $0 < p < \infty$, then

$$\|f\|_p = \left(\int_X |f|^p\right)^{1/p}.$$

If $p = \infty$, then

$$\|f\|_p = \|f\|_\infty = \operatorname{ess\,sup}|f| = \inf\{t > 0 : \mu\{x \in X : |f(x)| > t\} = 0\}.$$

(Problem 1070) Let f and g be two $\mathcal{S}$-measurable functions. If $0 < p < \infty$, when does $\|f - g\|_p = 0$?

(Problem 1080) When does $\|f - g\|_\infty = 0$?

[Definition: $\mathcal{L}^p(d\mu)$] If $(X, \mathcal{S}, \mu)$ is a measure space, then the Lebesgue space $\mathcal{L}^p(\mu) = \mathcal{L}^p(X, \mathcal{S}, \mu) = \{f : X \to \mathbf{F} | \, \|f\|_p < \infty\}$.

[Definition: $\mathcal{L}^p(E)$] Let $E \subseteq \mathbf{R}^n$ be a Lebesgue or Borel measurable set. Let $\mathcal{S}_E$ be the set of Lebesgue or Borel measurable subsets of E. (By Exercise 2B.11, $\mathcal{S}_E$ is a σ-algebra on E.) Let $\lambda_E = \lambda_n|_E$ be Lebesgue measure restricted to E. Then $\mathcal{L}^p(E) = \mathcal{L}^p(E, \mathcal{S}_E, \lambda_E)$.

[Definition: ℓ^p] Let μ be the counting measure on $\mathbf{N}$. Then $\ell^p = \mathcal{L}^p(\mathbf{N}, 2^{\mathbf{N}}, \mu)$.

(Problem 1090) Let $f \in \mathcal{L}^p(\mu)$ and let $\alpha \in \mathbf{F}$. Prove that $\alpha f \in \mathcal{L}^p(\mu)$.

(Problem 1100) Let $f, g \in \mathcal{L}^p(\mu)$ for some $0 < p < \infty$. Prove that $f + g \in \mathcal{L}^p(\mu)$ (so $\mathcal{L}^p(\mu)$ is a vector space).

[Exercise 7A.1] Prove that $\mathcal{L}^\infty(\mu)$ is a vector space.

[Definition: Dual exponent (conjugate exponent)] If $1 \leq p \leq \infty$, then $p' \in [1, \infty]$ is the unique extended real number that satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.

(Problem 1110) Find $1'$, ∞', $2'$, $3'$, $4'$.

(Problem 1120) [Young's inequality.] Suppose that $1 < p < \infty$, $a \geq 0$, and $b \geq 0$. Show that

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

(Problem 1130) [Hölder's inequality] Let $(X, \mathcal{S}, \mu)$ be a measure space, $1 < p < \infty$, and f, $h : X \to \mathbf{F}$ be $\mathcal{S}$-measurable. Show that $\|fh\|_1 \leq \|f\|_p \|h\|_{p'}$.

[Exercise 7A.4] Hölder's inequality is also true if $p = 1$ or $p = \infty$.

(Problem 1140) Suppose that $\mu(X) < \infty$ and that $0 < p < q < \infty$. Show that $\mathcal{L}^q(\mu) \subseteq \mathcal{L}^p(\mu)$. What can you say about $\|f\|_p$ in terms of $\|f\|_q$?

(Problem 1150) Suppose that $\mu(X) < \infty$ and that $0 < p < \infty$. Show that $\mathcal{L}^\infty(\mu) \subseteq \mathcal{L}^p(\mu)$. What can you say about $\|f\|_p$ in terms of $\|f\|_\infty$?

(Problem 1160) Let $X = (0, 1)$ and let $0 < p < q \leq \infty$. Give an example of a function $f \in \mathcal{L}^p(\lambda) \setminus \mathcal{L}^q(\lambda)$, where λ denotes Lebesgue measure.

[Exercise 7A.10] Let $0 < p < q \leq \infty$. Then $\ell^p \subseteq \ell^q$ and $\|a\|_p \geq \|a\|_q$ for all sequences $a \in \ell^p$.

[Exercise 7A.14] If $0 < p < q \leq \infty$, then $\mathcal{L}^p(\mathbf{R}) \not\subseteq \mathcal{L}^q(\mathbf{R})$ and also $\mathcal{L}^q(\mathbf{R}) \not\subseteq \mathcal{L}^p(\mathbf{R})$.

(Problem 1170) [Holder:reverse] Let $(X, \mathcal{S}, \mu)$ be a measure space and let $f : X \to \mathbf{F}$ be $\mathcal{S}$-measurable. Let $1 \leq p < \infty$. Show that

$$\|f\|_p = \sup\left\{\left|\int f\,h\,d\mu\right| : h \in \mathcal{L}^{p'}(\mu),\ \|h\|_{p'} \leq 1\right\}.$$

(Problem 1180) Let $X = \{b\}$ and $\mu(\emptyset) = 0$, $\mu(X) = \infty$. If $f : X \to \mathbf{F}$ is a function, what is $\|f\|_\infty$ and $\|f\|_p$ for $p < \infty$?

(Problem 1190) Is it necessarily true that

$$\|f\|_\infty = \sup\left\{\left|\int f\,h\,d\mu\right| : h \in \mathcal{L}^1(\mu),\ \|h\|_1 \leq 1\right\}$$

for all $\mathcal{S}$-measurable functions $f : X \to \mathbf{F}$?

[Exercise 7A.9] Recall that a measure space $(X, \mathcal{S}, \mu)$ is σ-finite if there is a sequence of sets $\{X_k\}_{k=1}^\infty \subseteq \mathcal{S}$ with $\sigma(X_k) < \infty$ for all k and $X = \cup_k X_k$. If $(X, \mathcal{S}, \mu)$ is σ-finite, then for all $\mathcal{S}$-measurable functions $f : X \to \mathbf{F}$,

$$\|f\|_\infty = \sup\left\{\left|\int f\,h\,d\mu\right| : h \in \mathcal{L}^1(\mu),\ \|h\|_1 \leq 1\right\}.$$

(Problem 1200) [Minkowski's inequality] Let $(X, \mathcal{S}, \mu)$ be a measure space, $1 \leq p < \infty$, and f, $g \in \mathcal{L}^p(\mu)$. Show that $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

(Problem 1210) Let $(X, \mathcal{S}, \mu)$ be a measure space and f, $g \in \mathcal{L}^\infty(\mu)$. Show that $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

(Problem 1220) Let $0 < p < 1$. Suppose that $(X, \mathcal{S}, \mu)$ is a measure space and that $\mathcal{S}$ contains two disjoint sets with finite positive measure. Show that Minkowski's inequality fails.

(Problem 1230) Suppose $1 \leq p \leq \infty$. When is $\|\cdot\|_p$ a norm on $\mathcal{L}^p(\mu)$?

7B. $L^p(\mu)$

(Problem 1240) Let $(X, \mathcal{S}, \mu)$ be a measure space and let $0 < p \leq \infty$. Suppose we define $\sim$ by $f \sim g$ if $\|f - g\|_p = 0$. Show that $\sim$ is an equivalence relation on $\mathcal{L}^p(\mu)$.

(Problem 1250) Suppose that $f \in \mathcal{L}^p(\mu)$ and $f \sim g$. Show that $g \in \mathcal{L}^p(\mu)$.

(Problem 1260) Show that if $f \sim g$ then $\|f\|_p = \|g\|_p$.

(Problem 1270) Show that if $f \sim g$ and $\alpha \in \mathbf{F}$ then $\alpha f \sim \alpha g$.

(Problem 1280) Show that if $f \sim F$ and $g \sim G$ then $f + g \sim F + G$.

[Definition: $L^p(\mu)$] Let $(X, \mathcal{S}, \mu)$ be a measure space and let $0 < p \leq \infty$. If $f \in \mathcal{L}^p(\mu)$, let

$$\widetilde{f} = \{g \in \mathcal{L}^p(\mu) : \|f - g\|_p = 0\}.$$

Let $L^p(\mu) = \{\widetilde{f} : f \in \mathcal{L}^p(\mu)\}$. Let $\|\widetilde{f}\|_p = \|f\|_p$.

(Problem 1290) Show that $L^p(\mu)$ is a normed vector space.

[Definition: $L^p(E)$] Let $E \subseteq \mathbf{R}^n$ be a Lebesgue or Borel measurable set. Let $\mathcal{S}_E$ be the set of Lebesgue or Borel measurable subsets of E. (By Exercise 2B.11, $\mathcal{S}_E$ is a σ-algebra on E.) Let $\lambda_E = \lambda_n|_E$ be Lebesgue measure restricted to E. Then $L^p(E) = L^p(E, \mathcal{S}_E, \lambda_E)$.

(Problem 1300) Let $(X, \mathcal{S}, \mu)$ be a measure space and $1 \leq p \leq \infty$. Why can't we talk about Cauchy sequences in $\mathcal{L}^p(\mu)$?

(Problem 1310) Let $1 \leq p \leq \infty$ and let $\{f_k\}_{k=1}^\infty \subseteq \mathcal{L}^p(\mu)$. Suppose that for every $\varepsilon > 0$ there is a $n \in \mathbf{N}$ such that if j, $k \geq n$ then $\|f_j - f_k\| < \varepsilon$. Suppose that there is a subsequence $\{f_{k_n}\}_{n=1}^\infty$ and a $f \in \mathcal{L}^p(\mu)$ such that $\|f_{k_n} - f\|_p \to 0$. Show that $\|f_k - f\|_p \to 0$.

(Problem 1320) Let $1 \leq p < \infty$ and let $\{f_k\}_{k=1}^\infty \subseteq \mathcal{L}^p(\mu)$. Suppose that for every $\varepsilon > 0$ there is a $n \in \mathbf{N}$ such that if j, $k \geq n$ then $\|f_j - f_k\| < \varepsilon$. Show that there is a $f \in \mathcal{L}^p(\mu)$ such that $\|f_k - f\|_p \to 0$, and a subsequence $\{f_{k_n}\}_{n=1}^\infty$ converges to f pointwise a.e.

[Exercise 7B.8] Prove the above result if $p = \infty$.

(Bonus Problem 1321) Give an example of a convergent sequence in $L^p(\mu)$ with representatives that do not converge pointwise anywhere.

(Problem 1330) Show that $L^p(\mu)$ is a Banach space.

(Problem 1340) Let $1 \le p \le \infty$. Let $h \in \mathcal{L}^{p'}(\mu)$. Define $\phi_h : L^p(\mu) \to \mathbf{F}$ by $\phi_h(\widetilde{f}) = \int f h \, d\mu$. Show that ϕ_h is well defined.

(Problem 1341) Show that $\phi_h \in (L^p(\mu))'$ and that $\|\phi_h\| \le \|h\|_{p'}$.

(Problem 1350) Show that if $p > 1$ or $(X, \mathcal{S}, \mu)$ is σ-finite, then $\|\phi_h\| = \|h\|_{p'}$.

(Problem 1360) Let g, $h \in \mathcal{L}^p(\mu)$. Show that $\phi_g = \phi_h$ if and only if $g \sim h$.

(Problem 1370) Suppose $1 \le p < \infty$. Show that $b \to \phi_b$ is an isomorphism $\ell^{p'} \to (\ell^p)'$. [We will see in Chapter 9 that $b \to \phi_b$ is an isomorphism $L^{p'}(\mu) \to (L^p(\mu))'$ for more general measure spaces.]

[Exercise 6D.14] There is a linear functional $\varphi \in (\ell^\infty)'$ such that $\varphi(a_1, a_2, \dots) = \lim_{k\to\infty} a_k$ whenever the limit exists.

(Problem 1380) Let φ be the functional in Exercise 6D.14. Show that $\varphi \ne \phi_b$ for all $b \in \ell^1$.

[Exercise 7B.15] Let $c_0 = \{(a_1, a_2, \dots) \in \ell^\infty : \lim_{k\to\infty} a_k = 0\}$. Then c_0 is a Banach space and $(c_0)'$ can be identified with ℓ^1.

8A. Undergraduate mathematics

[Definition: Dot product] The dot product on $\mathbf{R}^n$ is given by $\vec{f} \cdot \vec{g} = \langle f, g\rangle = \sum_{k=1}^n f_k g_k$.

(Problem 1390) Write the standard Euclidean norm on $\mathbf{R}^n$ in terms of the dot product.

(Problem 1391) What is $(1, i) \cdot (1, i)$?

[Definition: Inner product] The standard inner product on $\mathbf{C}^n$ is given by $\langle f, g\rangle = \sum_{k=1}^n f_k \overline{g_k}$.

8A. Inner product spaces

[Definition: Inner product] Let V be a vector space. Then $\langle , \rangle : V \times V \to \mathbf{F}$ is an inner product if

- $\langle f, f\rangle \in [0, \infty)$ for all $f \in V$,
- $\langle f, f\rangle = 0$ if and only if $f = 0$.
- $\langle f, g\rangle = \overline{\langle g, f\rangle}$ for all f, $g \in V$,
- If $g \in V$ then the function $\varphi : V \to \mathbf{F}$ given by $\varphi(f) = \langle f, g\rangle$ is linear.

(Problem 1400) Show that

- $\langle f, g + h\rangle = \langle f, g\rangle + \langle f, h\rangle$ for all f, g, $h \in V$,
- $\langle f, \alpha g\rangle = \overline{\alpha}\langle f, g\rangle$ for all $\alpha \in \mathbf{F}$ and all f, $g \in V$,
- $\langle 0, g\rangle = \langle g, 0\rangle = 0$ for all $g \in V$.

[Definition: Norm associated to an inner product] Let V be an inner product space. The associated norm is $\|f\| = \sqrt{\langle f, f\rangle}$.

(Problem 1410) Show that this norm is homogeneous and positive definite. (That is, prove all the norm axioms except the triangle inequality.)

(Problem 1420) Let $(X, \mathcal{S}, \mu)$ be a measure space. Write down an inner product on $L^2(\mu)$ such that the inner product norm coincides with the $L^2(\mu)$ norm.

(Problem 1430) Write down an inner product on ℓ^2 such that the inner product norm coincides with the ℓ^2 norm.

[Exercise 8A.21a] If $p \ne 2$, then there is no inner product on ℓ^p such that the inner product norm coincides with the ℓ^p norm.

[Exercise 8A.21b] If $p \ne 2$, then there is no inner product on $L^p(\mathbf{R})$ such that the inner product norm coincides with the $L^p(\mathbf{R})$ norm.

[Definition: Orthogonal] If f, g are elements of an inner product space, we say they are orthogonal if $\langle f, g\rangle = 0$.

(Problem 1440) Prove the Pythagorean theorem: if $\langle f, g\rangle = 0$ then $\|f + g\|^2 = \|f\|^2 + \|g\|^2$.

(Problem 1450) Let $g \in V$ and let $g^\perp = \{h \in V : \langle g, h\rangle = 0\}$ be the set of orthogonal vectors to g. Show that $g^\perp$ is a subspace of V.

(Problem 1460) Show that $V = g^{\perp} + \mathbf{F}g$.

(Problem 1470) Prove the Cauchy-Schwarz inequality: if f, $g \in V$ then $|\langle f, g\rangle| \leq \|f\|\|g\|$ with equality if and only if f and g are linearly dependent.

(Problem 1480) Prove the triangle inequality for the norm associated to an inner product. Conclude that this is in fact a norm.

8B. Hilbert spaces

[Definition: Hilbert space] A Hilbert space is an inner product space that is a Banach space under the norm determined by the inner product.

(Problem 1600) Show that $L^2(\mu)$ is a Hilbert space.

(Problem 1620) Let $C[0,1] \subseteq L^2(\lambda)$ be the subspace of continuous functions. Show that $C[0,1]$ is not a Hilbert space.

(Problem 1630) Recall that $\ell^1 \subseteq \ell^2$. Equip ℓ^1 with the inner product in ℓ^2. Show that ℓ^1 is not a Hilbert space.

8B. Undergraduate analysis

[Definition: Distance] Let V be a metric space, $f \in V$, $U \subseteq V$. Then $\operatorname{dist}(f, U) = \inf\{d(f,g) : g \in U\}$.

(Problem 1490) Suppose that U is compact. Show that $\operatorname{dist}(f, U) = \min\{d(f,g) : g \in U\}$ (that is, there is some $g \in U$ with $\operatorname{dist}(f, U) = d(f, g)$).

(Problem 1500) Suppose that $V = \mathbf{R}^p$ and that $U \subseteq V$ is closed. Show that $\operatorname{dist}(f, U) = \min\{d(f,g) : g \in U\}$ (that is, there is some $g \in U$ with $\operatorname{dist}(f, U) = d(f, g)$).

(Problem 1510) [dist:subtract] Suppose that V is a vector space, $f \in V$, $U \subseteq V$. Show that $\operatorname{dist}(f, U) = \operatorname{dist}(0, \{g - f : g \in U\})$.

8B. Orthogonality

[Definition: Convex] Let V be a vector space and let $U \subseteq V$. We say that U is convex if, for all f, $g \in U$ and all $t \in [0,1]$ we have that the point $tf + (1-t)g$ is also in U.

(Problem 1520) Let $V = \mathbf{R}^2$. Draw $\{tf + (1-t)g : t \in [0,1]\}$.

(Problem 1530) Show that every subspace U of V is convex.

[Exercise 8B.5] Any open ball $B(f, r)$ is convex.

(Problem 1531) Let V be a Hilbert space, $f \in V$, and let $U \subseteq V$ be nonempty, closed, and convex. Show that if g, $h \in U$ then

$$\|g - h\|^2 \leq 2\|g - f\|^2 + 2\|h - f\|^2 - 4\operatorname{dist}(f, U)^2.$$

(Problem 1650) Let V be a Hilbert space, $f \in V$, and let $U \subseteq V$ be nonempty, closed, and convex. Show that there is a $g \in U$ with $\|f - g\| = \operatorname{dist}(f, U)$.

(Problem 1660) Show that g is unique.

[Exercise 8B.12] There is a nonempty closed but not convex subset U of ℓ^2 and an $a \in \ell^2$ such that $\|a - b\| > \operatorname{dist}(a, U)$ for all $b \in U$.

[Exercise 8B.13] There is a finite-dimensional Banach space V, a $f \in V$, and a nonempty closed convex subset U of V with $\operatorname{dist}(f, U) = \|f - g\|$ for infinitely many $g \in U$.

(Problem 1550) Let $V = C[0,1]$ with norm $\|g\| = \sup_{[0,1]} |g|$. By Example 6.38, V is a Banach space. Let

$$U_1 = \{g \in C[0,1] : \int_0^1 g = 0\}, \quad U_2 = \{g \in C[0,1] : g(1) = 0\}.$$

Show that U_1 and U_2 are closed.

(Problem 1560) Show that U_1, U_2, and $U_1 \cap U_2$ are convex.

(Problem 1570) Let $f(x) = 1 - x$. Show that $\operatorname{dist}(f, U_1) = 1/2$ and that there is a unique $g \in U_1$ with $d(f, g) = 1/2$.

(Problem 1580) Show that $d(f, g) > 1/2$ for all $g \in U_1 \cap U_2$.

(Problem 1590) Show that $\operatorname{dist}(f, U_1 \cap U_2) = 1/2$.

[Definition: Projection] Let V be a Hilbert space and $U \subseteq V$ be nonempty, closed, and convex. We define P_U by $P_U f = g$ if $g \in U$ and $\|f - g\| = \operatorname{dist}(f, U)$.

(Problem 1670) Let $g \in V$. Find $P_{\mathbf{F}g}$.

(Problem 1680) Let $U = B(0, 1) = \{f \in V : \|f\| \leq 1\}$. Find P_U.

(Problem 1690) Let V be a Hilbert space and let $U \subseteq V$ be a closed convex subset. Show that $P_U f = f$ if and only if $f \in U$.

(Problem 1700) Show that $P_U \circ P_U = P_U$.

(Problem 1710) Suppose in addition that U is a subspace. Show that $f - P_U f$ is orthogonal to g for all $g \in U$.

(Problem 1720) Prove the converse to the above result: if $h \in U$ and $f - h$ is orthogonal to g for all $g \in U$ then $h = P_U f$.

(Problem 1730) Show P_U is a linear map.

[Exercise 8B.15] $\|P_U\| \leq 1$, and if $\|P_U f\| = \|f\|$ then $f \in U$.

(Bonus Problem 1740) Let V be a Hilbert space over $\mathbf{R}$ and let $U \subseteq V$ be a closed convex set. Suppose that $0 \in U$. Show that $\|P_U f\| \leq \|f\|$ for all $f \in V$.

* * *

[Definition: Orthogonal complement] Let V be an inner product space and let $U \subseteq V$. Then

$$U^{\perp} = \{h \in V : \langle g, h \rangle = 0 \text{ for all } g \in U\}.$$

(Problem 1750) Let $V = \mathbf{R}^2$ and let $U = \{(3, 1)\}$. What is $U^{\perp}$?

(Problem 1760) Show that $U^{\perp}$ is closed.

(Problem 1770) Show that $U^{\perp}$ is a subspace of V.

(Problem 1780) [U:perpperp] Show that $U \subseteq (U^{\perp})^{\perp}$.

(Problem 1790) Show that $\overline{U} \subseteq (U^{\perp})^{\perp}$.

(Problem 1800) Show that $\operatorname{span} U \subseteq (U^{\perp})^{\perp}$.

(Problem 1810) [perpcapperp] Show that $U \cap U^{\perp} \subseteq \{0\}$.

(Problem 1820) Show that if $W \subset U$, then $U^{\perp} \subset W^{\perp}$.

(Problem 1830) [closureperp] Show that $\overline{U}^{\perp} = U^{\perp} = ((U^{\perp})^{\perp})^{\perp}$.

(Problem 1840) Suppose that V is a Hilbert space and that U is a subspace. Show that $\overline{U} = (U^{\perp})^{\perp}$.

(Problem 1850) Let U be a subspace of a Hilbert space V. Show that $\overline{U} = V$ if and only if $U^{\perp} = 0$.

(Problem 1860) Let U be a closed subspace of a Hilbert space V. Let $f \in V$. Show that there is a unique $g \in U$ and a unique $h \in U^{\perp}$ with $f = g + h$.

(Problem 1870) Let U be a closed subspace of a Hilbert space V. Show that the range of P_U is U.

(Problem 1880) Show that the nullspace of P_U is $U^{\perp}$.

(Problem 1890) Show that $P_U + P_{U^{\perp}} = I$, where $I : V \to V$ is the identity map.

* * *

(Problem 1900) Let V be an inner product space and let g, $h \in V$. Suppose that $\langle f, g \rangle = \langle f, h \rangle$ for all $f \in V$. Show that $g = h$.

(Problem 1910) Let V be an inner product space, let $g \in V$, and let $\varphi(f) = \langle f, g \rangle$. Show that $\varphi : V \to \mathbf{F}$ is a bounded linear functional and that $\|\varphi\| = \|g\|$.

(Problem 1920) [The Riesz representation theorem.] Let V be a Hilbert space and let $\varphi : V \to \mathbf{F}$ be a bounded linear functional. Show that there is a unique $h \in V$ such that $\varphi(f) = \langle f, h\rangle$ for all $f \in V$.

8C. Orthonormal bases

[Definition: Orthonormal family] A family $\{e_k\}_{k\in\Gamma}$ in an inner product space is an orthonormal family if

$$\langle e_j, e_k\rangle = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}$$

(Problem 1930) Let Γ be a nonempty set and let μ denote the counting measure on Γ. If $k \in \Gamma$, let $e_k : \Gamma \to \mathbf{F}$ be given by $e_k(k) = 1$ and $e_k(j) = 0$ if $j \neq k$. Show that $\{e_k\}_{k\in\Gamma}$ is an orthonormal family in $L^2(\mu)$.

[Exercise 8C.1] Let $e_k(t) = \frac{1}{\sqrt{\pi}} \sin(kt)$ if $k > 0$, $e_0(t) = \frac{1}{\sqrt{2\pi}}$, $e_k(t) = \frac{1}{\sqrt{\pi}} \cos(kt)$ if $k < 0$. Then $\{e_k\}_{k\in\mathbf{Z}}$ is an orthonormal family in $L^2((-\pi, \pi])$.

(Problem 1940) Let

$$e_k(x) = \begin{cases} 1, & \lfloor 2^k x\rfloor \text{ odd}, \\ -1, & \lfloor 2^k x\rfloor \text{ even}. \end{cases}$$

Show that $\{e_k\}_{k\in\mathbf{N}_0}$ is an orthonormal family in $L^2([0, 1))$.

(Problem 1950) Let

$$e_{m,k}(x) = \begin{cases} 1, & m \leq x < m + 1 \text{ and } \lfloor 2^k x\rfloor \text{ odd}, \\ -1, & m \leq x < m + 1 \text{ and } \lfloor 2^k x\rfloor \text{ even}, \\ 0, & x < m \text{ or } x \geq m + 1. \end{cases}$$

Show that $\{e_{m,k}\}_{(m,k)\in\mathbf{Z}\times\mathbf{N}_0}$ is an orthonormal family.

(Problem 1960) [finite:orthonormal] Suppose that Ω is a finite set and $\{e_j\}_{j\in\Omega}$ is an orthonormal family. Show that $\|\sum_{j\in\Omega} \alpha_j e_j\|^2 = \sum_{j\in\Omega} |\alpha_j|^2$ for every family $\{\alpha_j\}_{j\in\Omega}$ in $\mathbf{F}$.

[Definition: Unordered sum] Let $\{f_k\}_{k\in\Gamma}$ be a family in a vector space V. The unordered sum $\sum_{k\in\Gamma} f_k$ converges if there is a $g \in V$ such that for every $\varepsilon > 0$, there is a finite subset $\Omega \subset \Gamma$ such that

$$\Big\| g - \sum_{j\in\Omega'} f_j \Big\| < \varepsilon$$

whenever $\Omega \subseteq \Omega' \subset \Gamma$ and Ω' is finite. In this case we write $g = \sum_{j\in\Gamma} f_j$.

[Exercise 8C.2] If $\{\alpha_k\}_{k\in\Gamma}$ is a family in $\mathbf{R}$, and $\alpha_k \geq 0$ for all k, then $\sum_{k\in\Gamma} \alpha_k = \sup\Big\{ \sum_{j\in\Omega} a_j : \Omega \subseteq \Gamma \text{ finite}\Big\}$ in the sense that the sum on the left converges if and only if the supremum on the right is finite, and if the supremum is finite then the sum converges to the supremum.

[Exercise 8C.6] If $\{\alpha_k\}_{k\in\Gamma}$ is a family in $\mathbf{R}$, then $\sum_{k\in\Gamma} \alpha_k$ converges if and only if $\sum_{k\in\Gamma} |\alpha_k| < \infty$.

(Problem 1970) Let $\{e_k\}_{k\in\Gamma}$ be an orthonormal family in a Hilbert space V. Let $\{\alpha_k\}_{k\in\Gamma}$ be a family in $\mathbf{F}$. Suppose that $\sum_{k\in\Gamma} \alpha_k e_k$ converges. Show that $\sum_{k\in\Gamma} |\alpha_k|^2 < \infty$ and that $\sum_{k\in\Gamma} |\alpha_k|^2 \leq \Big|\sum_{k\in\Gamma} \alpha_k e_k\Big|^2$.

(Problem 1980) [ell2:converge] Let $\{e_k\}_{k\in\Gamma}$ be an orthonormal family in a Hilbert space V. Let $\{\alpha_k\}_{k\in\Gamma}$ be a family in $\mathbf{F}$. Suppose that $\sum_{k\in\Gamma} |\alpha_k|^2 < \infty$. Show that $\sum_{k\in\Gamma} \alpha_k e_k$ converges and that $\sum_{k\in\Gamma} |\alpha_k|^2 \geq \Big|\sum_{k\in\Gamma} \alpha_k e_k\Big|^2$.

(Problem 1981) [e:summand] Let $\{\alpha_k\}_{k\in\Gamma}$ be a family in $\mathbf{F}$ with $\sum_{k\in\Gamma} |\alpha_k|^2 < \infty$. Show that $\Big\langle \sum_{k\in\Gamma} \alpha_k e_k, e_j\Big\rangle = \alpha_j$ for all $j \in \Gamma$.

(Problem 1990) Let $\{\alpha_k\}_{k\in\Gamma}$ and $\{\beta_k\}_{k\in\Gamma}$ be two families in $\mathbf{F}$ with $\sum_{k\in\Gamma} |\alpha_k|^2 < \infty$, $\sum_{k\in\Gamma} |\beta_k|^2 < \infty$, and with $\sum_{k\in\Gamma} \alpha_k e_k = \sum_{k\in\Gamma} \beta_k e_k$. Show that $\alpha_k = \beta_k$ for all k.

(Problem 1991) [sum:linear] Let $\gamma \in \mathbf{F}$ and let $\{\alpha_k\}_{k\in\Gamma}$ and $\{\beta_k\}_{k\in\Gamma}$ be two families in $\mathbf{F}$ with $\sum_{k\in\Gamma} |\alpha_k|^2 < \infty$, $\sum_{k\in\Gamma} |\beta_k|^2 < \infty$. Show that

$$\sum_{k\in\Gamma} |\alpha_k + \beta_k|^2 < \infty,$$
$$\sum_{k\in\Gamma} |\gamma\alpha_k|^2 < \infty,$$
$$\sum_{k\in\Gamma} (\alpha_k + \beta_k) e_k = \Big(\sum_{k\in\Gamma} \alpha_k e_k\Big) + \Big(\sum_{k\in\Gamma} \beta_k e_k\Big),$$
$$\sum_{k\in\Gamma} (\gamma\alpha_k) e_k = \gamma\Big(\sum_{k\in\Gamma} \alpha_k e_k\Big).$$

(Problem 2000) Let $\{e_k\}_{k\in\mathbf{N}}$ be an orthonormal family. Show that $\sum_{k\in\mathbf{N}} \frac{1}{k} e_k$ converges and that $\sum_{k\in\mathbf{N}} \|\frac{1}{k} e_k\|$ diverges.

(Problem 2010) Let $\{e_k\}_{k\in\Gamma}$ be an orthonormal family in an inner product space V. Let $f \in V$. Show that if $\Omega \subseteq \Gamma$ is finite, then $\sum_{j\in\Omega} \langle f, e_j\rangle e_j$ is orthogonal to $f - \sum_{j\in\Omega} \langle f, e_j\rangle e_j$.

(Problem 2020) [Bessel's inequality.] Show that $\|f\|^2 \geq \sum_{j\in\Omega} |\langle f, e_j\rangle|^2$ for all $\Omega \subseteq \Gamma$ finite.

(Problem 2030) Let $\{e_k\}_{k\in\Gamma}$ be an orthonormal family in a Hilbert space V. Let $\{\alpha_k\}_{k\in\Gamma}$ be a family in $\mathbf{F}$ with $\sum_{k\in\Gamma} |\alpha_k|^2 < \infty$. Show that $\sum_{k\in\Gamma} \alpha_k e_k \in \overline{\operatorname{span}\{e_k\}_{k\in\Gamma}}$.

(Problem 2040) [finbasis] Let $f \in \overline{\operatorname{span}\{e_k\}_{k\in\Gamma}}$. Show that $f = \sum_{j\in\Gamma} \langle f, e_j\rangle e_j$ and that $\sum_{j\in\Gamma} |\langle f, e_j\rangle|^2 < \infty$.

[Definition: Orthonormal basis] An orthonormal family $\{e_k\}_{k\in\Gamma}$ in a Hilbert space V is an orthonormal basis of V if $\overline{\operatorname{span}\{e_k\}_{k\in\Gamma}} = V$.

[Exercise 8C.9] Let V be an infinite-dimensional Hilbert space. Let $B \subseteq V$ be a Hamel basis, that is, a basis in the sense of Definition 6.54. Then B is not an orthonormal set, and therefore is not an orthonormal basis of V.

[Exercise 8C.10] Let

$$e_{m,k}(x) = \begin{cases} 1, & m \leq x < m+1 \text{ and } \lfloor 2^k x\rfloor \text{ odd}, \\ -1, & m \leq x < m+1 \text{ and } \lfloor 2^k x\rfloor \text{ even}, \\ 0, & x < m \text{ or } x \geq m+1. \end{cases}$$

Then $\{e_{m,k}\}_{(m,k)\in\mathbf{Z}\times\mathbf{N}_0}$ is an orthonormal basis of $L^2(\mathbf{R})$.

(Problem 2050) [parseval] Let $\{e_k\}_{k\in\Gamma}$ be an orthonormal basis of a Hilbert space V. Let $f \in V$. Show that $f = \sum_{k\in\Gamma} \langle f, e_k\rangle e_k$ and that $\|f\|^2 = \sum_{k\in\Gamma} |\langle f, e_k\rangle|^2$.

(Problem 2060) [parseval:product] Let $\{e_k\}_{k\in\Gamma}$ be an orthonormal basis of a real Hilbert space V. Let f, $g \in V$. Show that $\langle f, g\rangle = \sum_{k\in\Gamma} \langle f, e_k\rangle \langle e_k, g\rangle$.

(Bonus Problem 2061) Let $\{e_k\}_{k\in\Gamma}$ be an orthonormal basis of a complex Hilbert space V. Let f, $g \in V$. Show that $\langle f, g\rangle = \sum_{k\in\Gamma} \langle f, e_k\rangle \langle e_k, g\rangle$.

* * *

[Definition: Separable] A metric space is separable if it has a countable dense subset.

(Problem 2070) Show that $\mathbf{R}^6$ is separable.

(Problem 2080) Show that ℓ^2 is separable.

(Problem 2090) Show that ℓ^∞ is not separable.

(Problem 2100) Let Γ be an uncountable set and let $\mu : \Gamma \to [0, \infty]$ be the counting measure. Show that $L^2(\mu) = \ell^2(\Gamma)$ is not separable.

[Exercise 8C.13] $L^2([0, 1])$ and $L^2(\mathbf{R})$ are separable.

(Problem 2110) Let V be a separable Hilbert space. Without using the Axiom of Choice or Zorn's lemma, show that V has an orthonormal basis.

(Problem 2120) Let U be a closed subspace of a Hilbert space V and let $\{e_k\}_{k\in\Gamma}$ be an orthonormal basis of U. Show that $P_U f = \sum_{k\in\Gamma}\langle f, e_k\rangle e_k$.

[Definition: Orthonormal subset] Let V be an inner product space and $U \subseteq V$. If $\|f\| = 1$ and $\langle f, g\rangle = 0$ for all f, $g \in U$ with $f \neq g$, we say that U is an orthonormal subset.

(Problem 2130) Let V be a Hilbert space. Let $\mathcal{A} = \{U \subseteq V : U \text{ is orthonormal}\}$. Suppose that $U \in \mathcal{A}$ is maximal. Show that U is an orthonormal basis for V.

(Problem 2140) Let $U \subset V$ be an orthonormal basis. Show that $U \in \mathcal{A}$ is a maximal element.

(Problem 2150) Use Zorn's lemma to show that $\mathcal{A}$ has a maximal element. What is the maximal element?

* * *

(Problem 2160) Let V be a Hilbert space and let $\varphi : V \to \mathbf{F}$ be a bounded linear functional. Let $\{e_j\}_{j\in\Gamma}$ be an orthonormal basis for V. Show that $\sum_{j\in\Gamma}|\varphi(e_j)|^2 < \infty$.

(Problem 2170) By Problem (1980), the sum $\sum_{k\in\Gamma}\overline{\varphi(e_k)}e_k$ converges. Let $h = \sum_{k\in\Gamma}\overline{\varphi(e_k)}e_k$. Show that $\langle f, h\rangle = \varphi(f)$ for all $f \in V$.

(Problem 2180) Show that $\|h\| = \|\varphi\|$. (This provides an alternative proof of the Riesz representation theorem.)

9A. Total Variation

[Definition: (Positive) measure] Let X be a set. Let $\mathcal{S}$ be a σ-algebra on X. We say that μ is a (positive) measure on $(X, \mathcal{S})$ and $(X, \mathcal{S}, \mu)$ is a measure space if:

- $\mu : \mathcal{S} \to [0, \infty]$,
- $\mu(\emptyset) < \infty$,
- If $\{E_k\}_{k=1}^\infty \subseteq \mathcal{S}$ is a sequence of pairwise-disjoint subsets of X then $\mu\Big(\bigcup_{k=1}^\infty E_k\Big) = \sum_{k=1}^\infty \mu(E_k)$.

(Problem 2181) Show that the condition $\mu(\emptyset) < \infty$ and additive summability imply that $\mu(\emptyset) = 0$.

[Definition: Real or complex measure] Let X be a set. Let $\mathcal{S}$ be a σ-algebra on X. We say that ν is a real (or complex) measure on $(X, \mathcal{S})$ if:

- $\nu : \mathcal{S} \to \mathbf{R}$ (or $\mu : \mathcal{S} \to \mathbf{C}$),
- If $\{E_k\}_{k=1}^\infty \subseteq \mathcal{S}$ is a sequence of pairwise-disjoint subsets of X then $\nu\Big(\bigcup_{k=1}^\infty E_k\Big) = \sum_{k=1}^\infty \nu(E_k)$.

(Problem 2190) Let ν be a real or complex measure. Show that $\nu(\emptyset) = 0$.

(Problem 2200) If μ is a (positive) measure, under what circumstances is μ also a real measure?

(Problem 2210) Suppose ν is a complex measure. Show that $\Re\nu$ and $\Im\nu$ are real measures.

(Problem 2220) [measure:finite] Let $\{E_k\}_{k=1}^\infty \subseteq \mathcal{S}$ be a sequence of pairwise-disjoint subsets of X. Show that $\sum_{k=1}^\infty |\nu(E_k)| < \infty$.

(Problem 2230) Let μ be a (positive) measure on a measurable space $(X, \mathcal{S})$ and let $h \in \mathcal{L}^1(\mu)$. Define

$$\nu(E) = \int_E h\,d\mu.$$

Show that ν is a real or complex measure on $(X, \mathcal{S})$. We call this measure $d\nu = h\,d\mu$.

(Problem 2240) Let ν be a complex measure on a measurable space $(X, \mathcal{S})$. Show that if D, $E \in \mathcal{S}$ then $\nu(D \cup E) = \nu(D) + \nu(E) - \nu(D \cap E)$.

(Problem 2250) Show that if $D \subseteq E$ then $\nu(E \setminus D) = \nu(E) \setminus \nu(D)$.

(Problem 2260) [measure:limit:cup] Show that if $\{E_k\}_{k=1}^\infty \subseteq \mathcal{S}$ and $E_k \subseteq E_{k+1}$ then $\nu\left(\bigcup_{k=1}^\infty E_k\right) = \lim_{k\to\infty}\nu(E_k)$.

(Problem 2270) [measure:limit:cap] Show that if $\{E_k\}_{k=1}^\infty \subseteq \mathcal{S}$ and $E_k \supseteq E_{k+1}$ then $\nu\left(\bigcap_{k=1}^\infty E_k\right) = \lim_{k\to\infty} \nu(E_k)$.

[Definition: Total variation] Let ν be a complex measure on a measurable space $(X, \mathcal{S})$. If $E \in \mathcal{S}$, define

$$|\nu|(E) = \sup\Bigl\{\sum_{k=1}^n |\nu(E_k)| : E_k \in \mathcal{S},\ n \in \mathbf{N},\ E_j \subseteq E,\ E_j \cap E_k = \emptyset \text{ if } j \neq k\Bigr\}.$$

(Problem 2271) Show that

$$|\nu|(E) = \sup\Bigl\{\sum_{k=1}^n |\nu(E_k)| : E_k \in \mathcal{S},\ n \in \mathbf{N},\ \bigcup_{k=1}^\infty E_k = E,\ E_j \cap E_k = \emptyset \text{ if } j \neq k\Bigr\}.$$

[Exercise 9A.4]

$$|\nu|(E) = \sup\Bigl\{\sum_{k=1}^\infty |\nu(E_k)| : E_k \in \mathcal{S},\ \bigcup_{k=1}^\infty E_k = E,\ E_j \cap E_k = \emptyset \text{ if } j \neq k\Bigr\}.$$

(Problem 2280) Show that if $E \in \mathcal{S}$ then $|\nu(E)| \leq |\nu|(E)$.

(Problem 2290) [variation:inclusion] Show that if $D \subseteq E$ and D, $E \in \mathcal{S}$ then $|\nu|(D) \leq |\nu|(E)$.

(Problem 2300) [variation:positive] Show that if ν is a (positive) measure then $|\nu| = \nu$.

(Problem 2310) Show that if $|\nu|(E) = 0$ then $\nu(A) = 0$ for all $A \subseteq E$.

(Problem 2320) [variation:zero] Show that if $\nu(A) = 0$ for all $A \subseteq E$ then $|\nu|(E) = 0$.

(Problem 2330) [real:variation:pm] Show that if ν is a real measure, then

$$|\nu|(E) = \sup\bigl\{\nu(A) - \nu(B) : A, B \in \mathcal{S},\ A \cup B \subseteq E,\ A \cap B = \emptyset\bigr\}.$$

(Problem 2340) [variation:integral] Let μ be a (positive) measure on a measurable space $(X, \mathcal{S})$ and let $h \in \mathcal{L}^1(\mu)$. Let $d\nu = h\,d\mu$. Show that

$$|\nu|(E) = \int_E |h|\,d\mu.$$

(Problem 2350) Let ν be a complex measure on a measurable space $(X, \mathcal{S})$ and let D, $E \in \mathcal{S}$ with $D \cap E = \emptyset$. Show that $|\nu|(D \cup E) = |\nu|(D) + |\nu|(E)$.

(Problem 2360) Let ν be a real measure on a measurable space $(X, \mathcal{S})$. Show that $|\nu|$ is a (positive) measure on $(X, \mathcal{S})$.

(Problem 2370) Let ν be a complex measure on a measurable space $(X, \mathcal{S})$. Show that $|\nu|$ is a (positive) measure on $(X, \mathcal{S})$.

(Problem 2380) Show that if ν is a real measure on $(X, \mathcal{S})$ then $|\nu|(X) < \infty$.

(Problem 2390) [variation:finite] Let ν be a complex measure on a measurable space $(X, \mathcal{S})$. Show that $|\nu|(E) \leq |\Re\nu|(E) + |\Im\nu|(E)$ and, in particular, $|\nu|(X) < \infty$.

[Definition: Sum of measures] Let μ and ν be complex measures on a measurable space $(X, \mathcal{S})$. Let $\alpha \in \mathbf{F}$. Define

$$(\mu + \nu)(E) = \mu(E) + \nu(E), \quad (\alpha\nu)(E) = \alpha(\nu(E)).$$

(Problem 2400) Show that $\mu + \nu$ and $\alpha\nu$ are complex measures.

[Definition: Space of measures] Let $(X, \mathcal{S})$ be a measurable space. Then $\mathcal{M}_{\mathbf{R}}(\mathcal{S})$ denotes the vector space of real measures on $(X, \mathcal{S})$, and $\mathcal{M}_{\mathbf{C}}(\mathcal{S})$ denotes the vector space of complex measures on $(X, \mathcal{S})$.

(Problem 2410) Verify that $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ is a vector space.

[Definition: Norm of measures] If ν is a real or complex measure on $(X, \mathcal{S})$ then $\|\nu\| = |\nu|(X)$.

(Problem 2420) Show that if $\nu \in \mathcal{M}_{\mathbf{F}}(\mathcal{S})$ then $\|\nu\| < \infty$.

(Problem 2430) Show that $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ is a normed vector space.

(Problem 2440) Show that $\mathcal{M}_{\mathbf{F}}(\mathcal{S})$ is a Banach space.

9B. Decomposition theorems

(Problem 2450) Let ν be a real measure on a measurable space $(X, \mathcal{S})$. Show that there exists an $A \in \mathcal{S}$ such that $\nu(A) = \sup\{\nu(E) : E \in \mathcal{S}\}$.

(Problem 2460) [hahn] [The Hahn decomposition.] Show that there exist sets A, $B \in \mathcal{S}$ such that

- $A \cup B = X$,
- $A \cap B = \emptyset$,
- $\nu(E) \geq 0$ for all $E \in \mathcal{S}$ with $E \subseteq A$,
- $\nu(E) \leq 0$ for all $E \in \mathcal{S}$ with $E \subseteq B$.

(Problem 2470) Give an example in which A and B are not unique.

[Exercise 9B.1] The Hahn decomposition is almost unique in the sense that, if A, B and A', B' satisfy the conditions of Problem (2460), then $|\nu|(A \setminus A') = |\nu|(A' \setminus A) = |\nu|(B \setminus B') = |\nu|(B' \setminus B) = 0$.

[Definition: Singular measures] If ν and μ are two complex or positive measures on a measure space $(X, \mathcal{S})$, then $\nu \perp \mu$ (or ν and μ are mutually singular) if there are two sets A, $B \in \mathcal{S}$ with $X = A \cup B$, $\emptyset = A \cap B$, and $\nu(E) = \nu(A \cap E)$ and $\mu(E) = \mu(B \cap E)$ for all $E \in \mathcal{S}$.

[Exercise 9B.2] Let $(X, \mathcal{S}, \mu)$ be a measure space and let f, $g \in \mathcal{L}^1(\mu)$. Then $g\,d\mu \perp f\,d\mu$ if and only if $\mu(\{x \in X : f(x) \neq 0\} \cap \{x \in X : g(x) \neq 0\}) = 0$.

(Problem 2480) Give an example of two nonzero measures that are mutually singular.

(Problem 2490) Let ν be a real measure on $(X, \mathcal{S})$. Show that there are two finite (positive) measures $\nu^{\pm}$ on $(X, \mathcal{S})$ such that $\nu^+ \perp \nu^-$ and $\nu = \nu^+ - \nu^-$.

(Problem 2500) Show that if $\nu^{\pm}$ are finite (positive) measures, $\nu^+ \perp \nu^-$, and $\nu = \nu^+ - \nu^-$, then $|\nu| = \nu^+ + \nu^-$.

(Problem 2510) Let ν be a real measure on $(X, \mathcal{S})$. Show that there is exactly one pair of mutually singular finite (positive) measures $\nu^{\pm}$ with $\nu = \nu^+ - \nu^-$.

* * *

[Definition: Absolutely continuous] Let ν be a complex measure on a measurable space $(X, \mathcal{S})$ and let μ be a (positive) measure on $(X, \mathcal{S})$. Suppose that $\nu(E) = 0$ for all $E \in \mathcal{S}$ such that $\mu(E) = 0$. Then we say that $\nu \ll \mu$, or ν is absolutely continuous with respect to μ.

(Problem 2520) Let $h \in \mathcal{L}^1(\mu)$. Show that $h\,d\mu \ll \mu$.

(Problem 2530) If ν is a real measure, show that $\nu^{\pm} \ll |\nu|$.

(Problem 2540) If ν is a complex measure, show that $\nu \ll |\nu|$.

(Problem 2550) If ν is a complex measure, show that $\Re\nu \ll |\nu|$ and $\Im\nu \ll |\nu|$.

(Problem 2570) If ν is any measure on $(X, \mathcal{S})$ and μ is the counting measure on X, show that $\nu \ll \mu$.

[Exercise 8B.10] If ν is a complex measure on $(X, \mathcal{S})$ and μ is a (positive) measure on $(X, \mathcal{S})$, then $\nu \ll \mu$ if and only if $|\nu| \ll \mu$.

(Problem 2580) Let μ be a (positive) measure on $(X, \mathcal{S})$. Suppose that ν is a complex measure on $(X, \mathcal{S})$ and that both $\nu \ll \mu$ and $\nu \perp \mu$. Show that $\nu(E) = 0$ for all $E \in \mathcal{S}$.

(Problem 2590) [The Lebesgue decomposition theorem] Let μ be a (positive) measure on a measurable space $(X, \mathcal{S})$. Let ν be a finite positive, real, or complex measure on $(X, \mathcal{S})$. Show that there are two finite positive, real, or complex measures ν_a and ν_s on $(X, \mathcal{S})$ such that

- $\nu = \nu_a + \nu_s$,
- $\nu_s \perp \mu$,
- $\nu_a \ll \mu$.

(Problem 2600) Show that ν_a and ν_s are unique.

* * *

(Problem 2610) [The Radon-Nikodym Theorem] Let $(X, \mathcal{S}, \mu)$ be a measure space. Suppose that $\mu(X) < \infty$. Let ν be another finite (positive) measure on $(X, \mathcal{S})$. Suppose that $\nu \ll \mu$. Show that there is a $h \in \mathcal{L}^1(\mu)$ with $h \geq 0$ such that $\nu = h\,d\mu$.

(Problem 2620) Let $(X, \mathcal{S}, \mu)$ be a measure space. Suppose that μ is a σ-finite measure. Let ν be a finite (positive) measure on $(X, \mathcal{S})$. Suppose that $\nu \ll \mu$. Show that there is a $h \in \mathcal{L}^1(\mu)$ with $h \geq 0$ such that $\nu = h\,d\mu$.

(Problem 2630) Let $(X, \mathcal{S}, \mu)$ be a measure space. Suppose that μ is a σ-finite measure. Let ν be a real measure on $(X, \mathcal{S})$. Suppose that $\nu \ll \mu$. Show that there is a $h \in \mathcal{L}^1(\mu)$ with h real-valued such that $\nu = h\,d\mu$.

(Problem 2640) Let $(X, \mathcal{S}, \mu)$ be a measure space. Suppose that μ is a σ-finite measure. Let ν be a complex measure on $(X, \mathcal{S})$. Suppose that $\nu \ll \mu$. Show that there is a $h \in \mathcal{L}^1(\mu)$ such that $\nu = h\,d\mu$.

(Problem 2650) Suppose that ν is a complex measure. Show that $d\nu = h\,d|\nu|$ for some h with $|h(x)| = 1$ for all $x \in X$.

(Problem 2660) Suppose that ν is a real measure. Show that $d\nu = h\,d|\nu|$ for some h with $h(x) = \pm 1$ for all $x \in X$.

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(Problem 2670) Suppose that $(X, \mathcal{S}, \mu)$ is a measure space, that $\mu(X) < \infty$, and that $1 < p < \infty$. For each $h \in L^{p'}(\mu)$, define

$$\phi_h(f) = \int f\,h\,d\mu.$$

Recall from Section 7B that ϕ_h is a bounded linear functional on $L^p(\mu)$, that $\|\phi_h\| = \|h\|_{p'}$, and in particular $\phi_g = \phi_h$ if and only if $g = h$ (as elements of $L^{p'}(\mu)$, not $\mathcal{L}^{p'}(\mu)$). Suppose $\phi \in (L^p(\mu))'$. Show that $\phi = \phi_h$ for some $h \in L^{p'}(\mu)$.

(Problem 2680) Suppose that $(X, \mathcal{S}, \mu)$ is a measure space, $1 < p < \infty$, and $\phi \in (L^p(\mu))'$. Suppose that $E \in \mathcal{S}$. Define $\mathcal{S}_E = \{D \in \mathcal{S} : D \subseteq E\}$, $\mu_E = \mu|_{\mathcal{S}_E}$, and if $f \in L^p(\mu_E)$ let $\phi_E(f) = \phi(1_E f)$, where $1_E f(x) = f(x)$ if $x \in E$ and 0 otherwise. Show that $\phi_E \in (L^p(\mu_E))'$ and that $\|\phi_E\| \leq \|\phi\|$.

(Problem 2690) Suppose $\phi \in (L^p(\mu))'$. Show that $\phi = \phi_h$ for some $h \in L^{p'}(\mu)$, even if $\mu(X) = \infty$.

[Exercise 9B.15] This is still true if $p = 1$ under the additional assumption that μ is σ-finite.