Math 2584, Spring 2024

Exam 2 will occur on Friday, March 8, 2024, at 2:00 p.m., in PHYS 133.

You are allowed a non-graphing calculator and a double-sided 3 inch by 5 inch card of notes.

(AB 1) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + \arctan(t)\, y = e^t$, $y(3) = 7$.

(b) $\frac{1}{1+t^2}\frac{dy}{dt} - t^5 y = \cos(6t)$, $y(2) = -1$, $y'(2) = 3$.

(c) $\frac{d^2y}{dt^2} - 5\sin(t)\, y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.

(d) $e^t\frac{d^2y}{dt^2} + 3(t-4)\frac{dy}{dt} + 4t^6 y = 2$, $y(3) = 1$, $y'(3) = -1$.

(e) $\frac{d^2y}{dt^2} + \cos(t)\frac{dy}{dt} + 3\ln(1+t^2)\, y = 0$, $y(2) = 3$.

(f) $\frac{d^3y}{dt^3} - t^7\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\, y = t^3$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$, $y'''(1) = 3$.

(g) $(1+t^2)\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = t^3$, $y(3) = 9$, $y'(3) = 7$, $y''(3) = 5$.

(h) $\frac{d^3y}{dt^3} - e^t\frac{d^2y}{dt^2} + e^{2t}\frac{dy}{dt} + e^{3t}y = e^{4t}$, $y(-1) = 1$, $y'(-1) = 3$.

(i) $(2+\sin t)\frac{d^3y}{dt^3} + \cos t\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\, y = t^3$, $y(7) = 2$.

(AB 2) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $e^t \frac{dy}{dt} + y = \cos t$, $y(0) = 3$, $y'(0) = -2$.

(b) $(t^2+4)\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + 6y = 7t^3$, $y(2) = 4$, $y'(2) = 4$, $y''(2) = 1$.

(c) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0$, $y(4) = 3$, $y'(4) = -2$, $y''(4) = 0$, $y'''(4) = 3$.

(AB 3) Find the general solution to the following differential equations.

(a) $\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 85x = 0$.

(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.

(c) $\frac{d^4z}{dt^4} + 7\frac{d^2z}{dt^2} - 144z = 0$.

(d) $\frac{d^4w}{dt^4} - 8\frac{d^2w}{dt^2} + 16w = 0$.

(AB 4) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $9\frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 2v = 0$, $v(0) = 3$, $v'(0) = 2$.

(b) $\frac{d^2u}{dt^2} + 10\frac{du}{dt} + 25u = 0$, $u(0) = 1$, $u'(0) = 4$.

(c) $8\frac{d^2f}{dt^2} - 6\frac{df}{dt} + f = 0$, $f(0) = 3$, $f'(0) = 1$.

(AB 5) A spring is suspended vertically. When an object with mass 5 kg is attached, it stretches the spring to a new equilibrium 4 cm lower. The system moves in a medium which damps it with damping constant 16 newton $\cdot$ seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Write the differential equation and initial conditions that describe the position of the object. You may use 9.8 meters/second^2 for the acceleration of gravity.

(AB 6) A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant β. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20$ N $\cdot$ s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30$ N $\cdot$ s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(AB 7) A 3-kg object is attached to a spring with constant k and to a viscous damper with damping constant 42 N$\cdot$sec/m. The object is set in motion from its equilibrium position with initial velocity 5 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object.

Find the values of k for which the system is underdamped, overdamped, and critically damped. Be sure to include units for k.

(AB 8) A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant β. The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

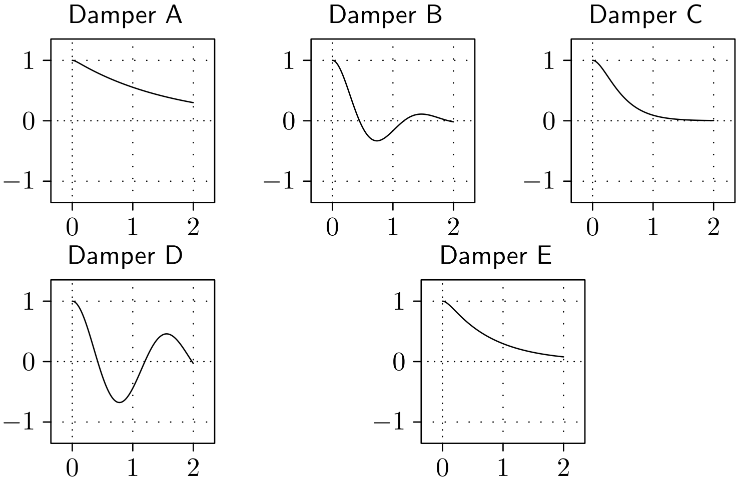
Write the differential equation and initial conditions that describe the position of the object. Then find the values of β for which the system is underdamped, overdamped, and critically damped. Be sure to include units for β.

(AB 9) An object of mass m is attached to a spring with constant 80 N/m and to a viscous damper with damping constant 20 N·s/m. The object is pulled down to 5cm below its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of m for which the system is underdamped, overdamped, or critically damped. Be sure to include units for m.

(AB 10) Five objects, each with mass 3 kg, are attached to five springs, each with constant 48 N/m. Five dampers with unknown constants are attached to the objects. In each case, the object is pulled down to a distance 1 cm below the equilibrium position and released from rest. You are given that the system is critically damped in exactly one of the five cases.

Here are the graphs of the objects' positions with respect to time:



(*a*) For which damper is the system critically damped?
(*b*) For which dampers is the system overdamped?
(*c*) For which dampers is the system underdamped?
(*d*) Which damper has the highest damping constant? Which damper has the lowest damping constant?

(AB 11) An object with mass 5 kg stretches a spring 4 cm. It is attached to a viscous damper with damping constant 16 newton · seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Using only first derivatives, write the differential equations and initial conditions for the object's position. That is, write a first-order system involving the object's position. You may use 9.8 meters/second^2 for the acceleration of gravity.

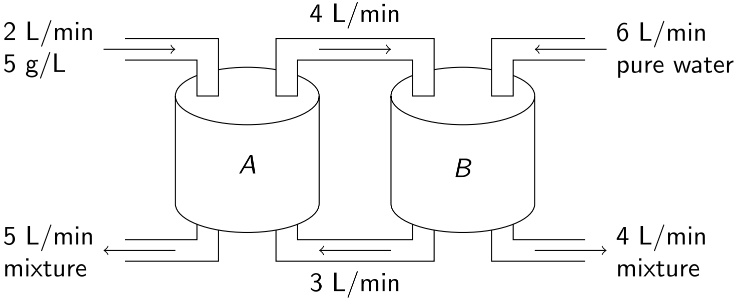
(AB 12) Imperial stormtroopers and Rebel Alliance fighters battle each other on an open plain, where both groups can easily see and aim at all members of the other group. Every minute, each stormtrooper has a 2% chance of killing a rebel, and each rebel has a 5% chance of killing a stormtrooper. There are initially 4000 stormtroopers and 1000 rebels.

Write the initial value problem for the number of stormtroopers and rebels still alive.

(AB 13) Jedi knights and Sith lords battle in a dense forest. Every minute, each Jedi has a 1% chance of finding each Sith lord. If a Jedi finds a Sith lord, they fight; the Jedi has a 60% chance of dying and the Sith lord has a 40% chance of dying. Initially there are 90 Jedi and 50 Sith lords.

Write the initial value problem for the number of Jedi and Sith lords still alive.

(AB 14) Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



Write the differential equations and initial conditions that describe the amount of salt in each tank.

(AB 15) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(AB 16) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(AB 17) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

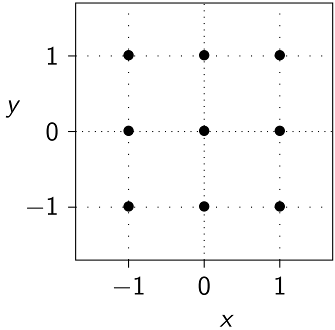
Set up the system of differential equations that describes the number of susceptible, infected, and resistant people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(AB 18) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers.

Each infected person encounters an average of 20 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. If an infected person encounters a recovered person, the recovered person has a 1% chance of contracting the disease again. Each infected person has a 12% chance of recovering from the disease on any given day.

Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(AB 19) Here is a grid. Draw a small phase plane (vector field) with nine arrows for the autonomous system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = x$.

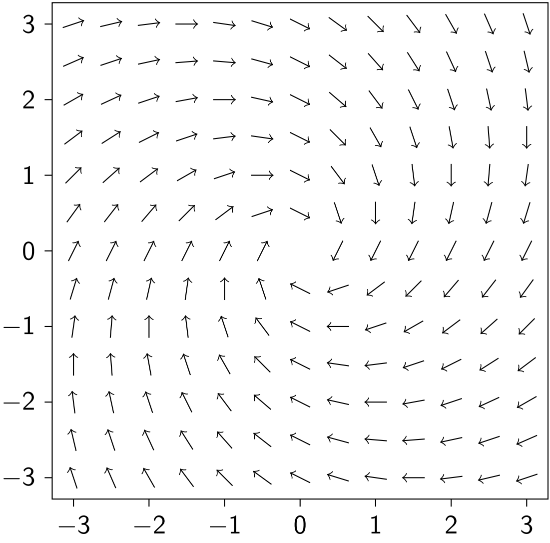


(AB 20) Here is the phase plane for the system

$$\frac{dx}{dt} = 2y - x, \quad \frac{dy}{dt} = -2x - y$$

Sketch the solution to the initial value problem

$$\frac{dx}{dt} = 2y - x, \quad \frac{dy}{dt} = -2x - y, \quad x(0) = 2, \quad y(0) = 1.$$



(AB 21) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Note: On the exam I may ask you to find the general solution instead. However, for a problem like this there are always many ways to write the general solution, not all of which are obviously equivalent; solutions to initial value problems take much more predictable forms and therefore make the answer key much easier to read.

(AB 22) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 23) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 24) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 25) You are given that the general solution to the system

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 11 & -4 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

may be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{5t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{5t} \begin{pmatrix} 2t+1 \\ 3t+1 \end{pmatrix}.$$

Find the solution to the initial value problem

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 11 & -4 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

Answer key

(AB 1) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + \arctan(t)\,y = e^t$, $y(3) = 7$.
We expect a unique solution.

(b) $\frac{1}{1+t^2}\frac{dy}{dt} - t^5 y = \cos(6t)$, $y(2) = -1$, $y'(2) = 3$.
We do not expect any solutions.

(c) $\frac{d^2y}{dt^2} - 5\sin(t)\,y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.
We do not expect any solutions.

(d) $e^t\frac{d^2y}{dt^2} + 3(t-4)\frac{dy}{dt} + 4t^6 y = 2$, $y(3) = 1$, $y'(3) = -1$.
We expect a unique solution.

(e) $\frac{d^2y}{dt^2} + \cos(t)\frac{dy}{dt} + 3\ln(1+t^2)\,y = 0$, $y(2) = 3$.
We expect an infinite family of solutions.

(f) $\frac{d^3y}{dt^3} - t^7\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\,y = t^3$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$, $y'''(1) = 3$.
We do not expect any solutions.

(g) $(1+t^2)\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = t^3$, $y(3) = 9$, $y'(3) = 7$, $y''(3) = 5$.
We expect a unique solution.

(h) $\frac{d^3y}{dt^3} - e^t\frac{d^2y}{dt^2} + e^{2t}\frac{dy}{dt} + e^{3t}y = e^{4t}$, $y(-1) = 1$, $y'(-1) = 3$.
We expect an infinite family of solutions.

(i) $(2+\sin t)\frac{d^3y}{dt^3} + \cos t\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\,y = t^3$, $y(7) = 2$.
We expect an infinite family of solutions.

(AB 2) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $e^t\frac{dy}{dt}+y=\cos t$, $y(0)=3$, $y'(0)=-2$.
We expect a unique solution.

(b) $(t^2+4)\frac{d^2y}{dt^2}+3t\frac{dy}{dt}+6y=7t^3$, $y(2)=4$, $y'(2)=4$, $y''(2)=1$.
We expect a unique solution.

(c) $\frac{d^3y}{dt^3}+3\frac{d^2y}{dt^2}+3\frac{dy}{dt}+y=0$, $y(4)=3$, $y'(4)=-2$, $y''(4)=0$, $y'''(4)=3$.
We expect a unique solution.

(AB 3) Find the general solution to the following differential equations.

(a) $\frac{d^2x}{dt^2}+12\frac{dx}{dt}+85x=0$.
If $\frac{d^2x}{dt^2}+12\frac{dx}{dt}+85x=0$, then $x=C_1e^{-6t}\cos(7t)+C_2e^{-6t}\sin(7t$

(b) $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+2y=0$.
If $\frac{d^2y}{dt^2}+4\frac{dy}{dt}+2y=0$, then $y=C_1e^{(-2+\sqrt{2})t}+C_2e^{(-2-\sqrt{2})t}$.

(c) $\frac{d^4z}{dt^4}+7\frac{d^2z}{dt^2}-144z=0$.
If $\frac{d^4z}{dt^4}+7\frac{d^2z}{dt^2}-144z=0$, then $z=C_1e^{3t}+C_2e^{-3t}+C_3\cos 4t+C_4\sin 4t$.

(d) $\frac{d^4w}{dt^4}-8\frac{d^2w}{dt^2}+16w=0$.
If $\frac{d^4w}{dt^4}-8\frac{d^2w}{dt^2}+16w=0$, then $w=C_1e^{2t}+C_2te^{2t}+C_3e^{-2t}+C_4te^{-2t}$.

(AB 4) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $9\frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 2v = 0$, $v(0) = 3$, $v'(0) = 2$.
If $9\frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 2v = 0$, $v(0) = 3$, $v'(0) = 2$, then $v = 3e^{-t/3}\cos(t/3) + 9e^{-t/3}\sin(t/3)$.

(b) $\frac{d^2u}{dt^2} + 10\frac{du}{dt} + 25u = 0$, $u(0) = 1$, $u'(0) = 4$.
If $\frac{d^2u}{dt^2} + 10\frac{du}{dt} + 25u = 0$, $u(0) = 1$, $u'(0) = 4$, then $u = e^{-5t} + 9te^{-5t}$.

(c) $8\frac{d^2f}{dt^2} - 6\frac{df}{dt} + f = 0$, $f(0) = 3$, $f'(0) = 1$.
If $8\frac{d^2f}{dt^2} - 6\frac{df}{dt} + f = 0$, then $f = e^{t/2} + 2e^{t/4}$.

(AB 5) A spring is suspended vertically. When an object with mass 5 kg is attached, it stretches the spring to a new equilibrium 4 cm lower. The system moves in a medium which damps it with damping constant 16 newton $\cdot$ seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Write the differential equation and initial conditions that describe the position of the object. You may use 9.8 meters/second^2 for the acceleration of gravity.

(Answer 5) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$5\frac{d^2x}{dt^2} + 16\frac{dx}{dt} + 1225x = 0, \qquad x(0) = -0.04, \quad x'(0) = 3.$$

(AB 6) A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant β. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20$ N $\cdot$ s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30$ N $\cdot$ s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(Answer 6) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$2\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.1, \quad x'(0) = 0.$$

If $\beta = 20$ N $\cdot$ s/m, then the system is underdamped, and we do expect to see decaying oscillations.

If $\beta = 30$ N $\cdot$ s/m, then the system overdamped, and we do not expect to see decaying oscillations.

(AB 7) A 3-kg object is attached to a spring with constant k and to a viscous damper with damping constant 42 N·sec/m. The object is set in motion from its equilibrium position with initial velocity 5 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object.

Find the values of k for which the system is underdamped, overdamped, and critically damped. Be sure to include units for k.

(Answer 7) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$3\frac{d^2x}{dt^2} + 42\frac{dx}{dt} + kx = 0, \qquad x(0) = 0, \quad x'(0) = -5.$$

The system is critically damped if $k = 147$ newtons/meter. It is underdamped if $k > 147$ newtons/meter and overdamped if $0 < k < 147$ newtons/meter.

(AB 8) A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant β. The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the values of β for which the system is underdamped, overdamped, and critically damped. Be sure to include units for β.

(Answer 8) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$4\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 70x = 0, \qquad x(0) = 0.05, \quad x'(0) = -3.$$

Critical damping occurs when $\beta = 4\sqrt{70}$ N $\cdot$ s/m. The system is underdamped if $0 < \beta < 4\sqrt{70}$ N $\cdot$ s/m and is overdamped if $\beta > 4\sqrt{70}$ N $\cdot$ s/m.

(AB 9) An object of mass m is attached to a spring with constant 80 N/m and to a viscous damper with damping constant 20 N·s/m. The object is pulled down to 5cm below its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of m for which the system is underdamped, overdamped, or critically damped. Be sure to include units for m.

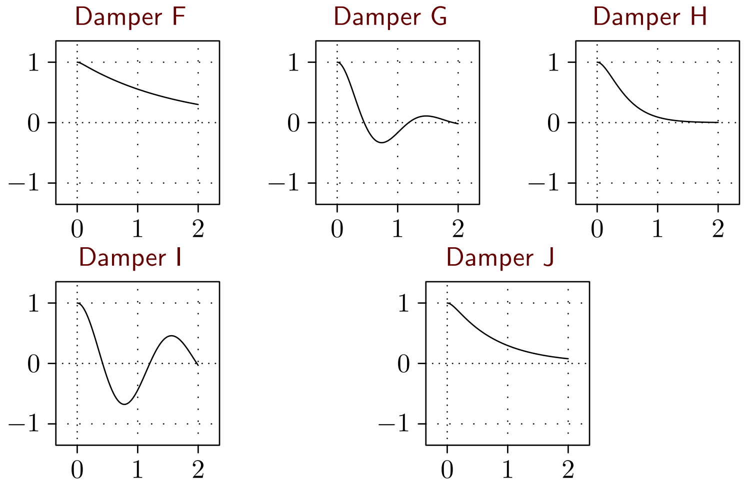
(Answer 9) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$m\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.05, \quad x'(0) = -3.$$

Critical damping occurs when $m = \frac{5}{4}$ kg. The system is underdamped if $m > \frac{5}{4}$ kg and overdamped if $0 < m < \frac{5}{4}$ kg.

(AB 10) Five objects, each with mass 3 kg, are attached to five springs, each with constant 48 N/m. Five dampers with unknown constants are attached to the objects. In each case, the object is pulled down to a distance 1 cm below the equilibrium position and released from rest. You are given that the system is critically damped in exactly one of the five cases.

Here are the graphs of the objects' positions with respect to time:



(*a*) For which damper is the system critically damped?
The system is critically damped for Damper H.

(*b*) For which dampers is the system overdamped?
The system overdamped for Dampers J and F.

(*c*) For which dampers is the system underdamped?
The system underdamped for Dampers G and I.

(*d*) Which damper has the highest damping constant? Which damper has the lowest damping constant?

Damper F has the highest damping constant. Damper I has the lowest damping constant.

(AB 11) An object with mass 5 kg stretches a spring 4 cm. It is attached to a viscous damper with damping constant 16 newton · seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Using only first derivatives, write the differential equations and initial conditions for the object's position. That is, write a first-order system involving the object's position. You may use 9.8 meters/second^2 for the acceleration of gravity.

(Answer 11) Let t denote time (in seconds), let x denote the object's displacement above equilibrium (in meters), and let v denote the object's velocity (in meters per second). Then

$$5\frac{dv}{dt}+16v+1225x=0, \qquad \frac{dx}{dt}=v, \qquad x(0)=-0.04, \quad v(0)=3.$$

(AB 12) Imperial stormtroopers and Rebel Alliance fighters battle each other on an open plain, where both groups can easily see and aim at all members of the other group. Every minute, each stormtrooper has a 2% chance of killing a rebel, and each rebel has a 5% chance of killing a stormtrooper. There are initially 4000 stormtroopers and 1000 rebels.

Write the initial value problem for the number of stormtroopers and rebels still alive.

(Answer 12) Let t denote time (in minutes), let S denote the number of stormtroopers, and let R denote the number of rebels.

Then

$$\frac{dR}{dt} = -0.02S, \quad \frac{dS}{dt} = -0.05R, \quad R(0) = 1000, \quad S(0) = 4000.$$

(AB 13) Jedi knights and Sith lords battle in a dense forest. Every minute, each Jedi has a 1% chance of finding each Sith lord. If a Jedi finds a Sith lord, they fight; the Jedi has a 60% chance of dying and the Sith lord has a 40% chance of dying. Initially there are 90 Jedi and 50 Sith lords.

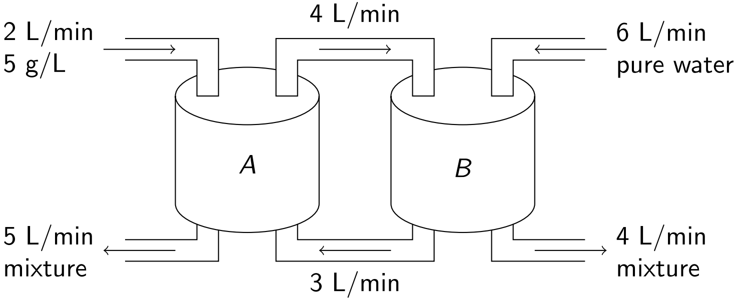
Write the initial value problem for the number of Jedi and Sith lords still alive.

(Answer 13) Let t denote time (in minutes), let S denote the number of Sith lords, and let J denote the number of Jedi.

Then

$$\frac{dR}{dt} = -0.006JS, \quad \frac{dS}{dt} = -0.004JS, \quad J(0) = 90, \quad S(0) = 50.$$

(AB 14) Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



Write the differential equations and initial conditions that describe the amount of salt in each tank.

(Answer 14) Let t denote time (in minutes).

Let x denote the amount of salt (in grams) in tank A.

Let y denote the amount of salt (in grams) in tank B.

Then $x(0) = 3000$ and $y(0) = 2000$.

If $t < 50$, then

$$\frac{dx}{dt} = 10 - \frac{9x}{200 - 4t} + \frac{3y}{300 + 3t}, \qquad \frac{dy}{dt} = \frac{4x}{200 - 4t} - \frac{7y}{300 + 3t}.$$

(AB 15) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(Answer 15) Let t denote time (in days), let S denote the number of susceptible people (who have never had the disease), let I denote the number of infected people (who currently have the disease), and let R denote the number of recovered people (who now are resistant, that is, cannot get the disease again). Then $\boxed{\frac{dS}{dt} = -\frac{1}{22500}SI,\ \frac{dI}{dt} = \frac{1}{22500}SI - 0.17I,\ \frac{dR}{dt} = 0.17I,\ S(0) = 8997,\ I(}$

(AB 16) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(Answer 16) $\frac{dS}{dt} = -\frac{0.2}{5000}SI - 15$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I + 15$.

(AB 17) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

Set up the system of differential equations that describes the number of susceptible, infected, and resistant people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(Answer 17) $\boxed{\frac{dS}{dt} = -\frac{0.2}{5000}SI - 15\frac{S}{5000},\ \frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I,\ \frac{dR}{dt} = 0.}$

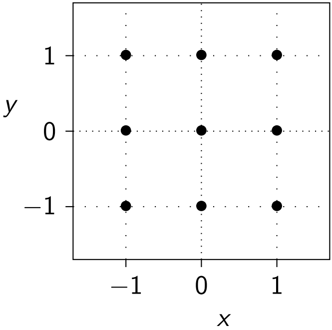
(AB 18) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers.

Each infected person encounters an average of 20 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. If an infected person encounters a recovered person, the recovered person has a 1% chance of contracting the disease again. Each infected person has a 12% chance of recovering from the disease on any given day.

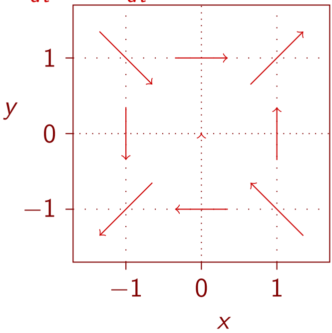
Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(Answer 18) Let t denote time (in days), let S denote the number of susceptible people (who have never had the disease), let I denote the number of infected people (who currently have the disease), and let R denote the number of recovered people (who now are resistant, that is, are less likely to get the disease again). Then $\frac{dS}{dt} = -\frac{1}{9000}SI$, $\frac{dI}{dt} = \frac{1}{9000}SI + \frac{1}{45000}RI - 0.12I$, $\frac{dR}{dt} = 0.12I - \frac{1}{45000}RI$, $S(0) = 8997$, $I(0) = 3$, $R(0) = 0$.

(AB 19) Here is a grid. Draw a small phase plane (vector field) with nine arrows for the autonomous system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = x$.



(Answer 19) Here is the direction field for the differential equation system $\frac{dx}{dt} = y$, $\frac{dy}{dt} = x$.

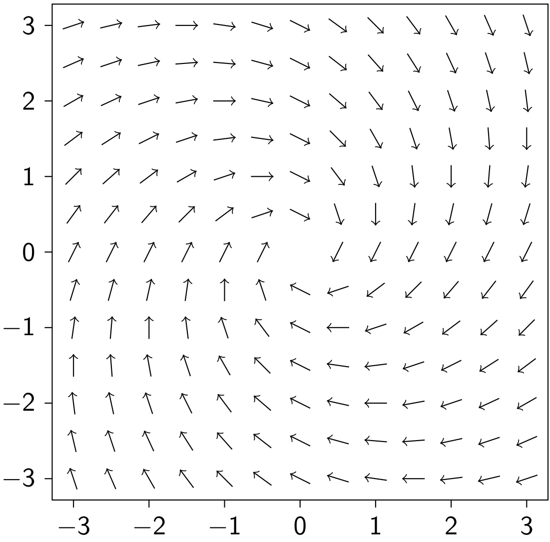


(AB 20) Here is the phase plane for the system

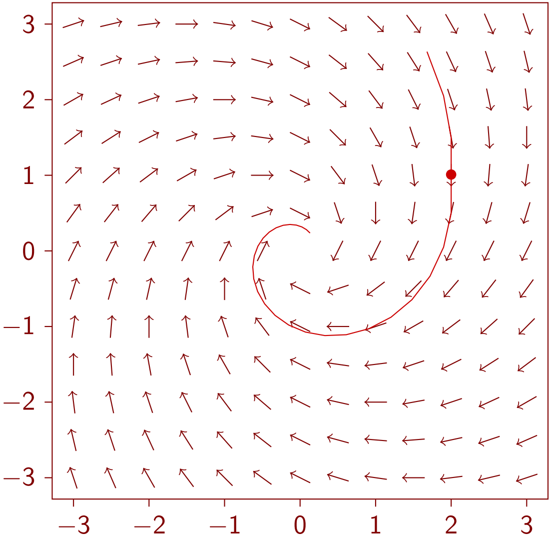
$$\frac{dx}{dt} = 2y - x, \quad \frac{dy}{dt} = -2x - y$$

Sketch the solution to the initial value problem

$$\frac{dx}{dt} = 2y - x, \quad \frac{dy}{dt} = -2x - y, \quad x(0) = 2, \quad y(0) = 1.$$



(Answer 20)



(AB 21) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 21) If

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 4t - 5 \\ -2t + 3 \end{pmatrix}.$$

(AB 22) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 22) If

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-20t} \begin{pmatrix} 3/20 \\ 9/10 \end{pmatrix} + e^{10t} \begin{pmatrix} -3/20 \\ 1/10 \end{pmatrix}.$$

(AB 23) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 23) If

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 15t + 1 \\ (15/4)t + 4 \end{pmatrix}.$$

(AB 24) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 24) If

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} + e^{-6t} \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix}.$$

(AB 25) You are given that the general solution to the system

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 11 & -4 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

may be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{5t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{5t} \begin{pmatrix} 2t+1 \\ 3t+1 \end{pmatrix}.$$

Find the solution to the initial value problem

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 11 & -4 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

(Answer 25) If

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 11 & -4 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix},$$

then

$$\begin{pmatrix} x \\ y \end{pmatrix} = 4e^{5t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} - 7e^{5t} \begin{pmatrix} 2t+1 \\ 3t+1 \end{pmatrix} = e^{5t} \begin{pmatrix} 1-14t \\ 5-21t \end{pmatrix}.$$