Contents

1.1. Elementary Properties of the Complex Numbers

[Definition: The complex numbers] The set of complex numbers is $\mathbb{R}^2$, denoted $\mathbb{C}$. (In this class, you may use everything you know about $\mathbb{R}$ and $\mathbb{R}^2$—in particular, that $\mathbb{R}^2$ is an abelian group and a normed vector space.)

[Definition: Real and imaginary parts] If (x, y) is a complex number, then $\mathrm{Re}(x, y) = x$ and $\mathrm{Im}(x, y) = y$.

[Definition: Addition and multiplication] If (x, y) and (ξ, η) are two complex numbers, we define

$$(x, y) + (\xi, \eta) = (x + \xi, y + \eta),$$
$$(x, y) \cdot (\xi, \eta) = (x\xi - y\eta, x\eta + y\xi).$$

(Problem 10) Show that multiplication in the complex numbers is commutative.

(Fact 20) This notion of addition and multiplication makes the complex numbers a ring—thus, multiplication is also associative and distributes over addition.

(Problem 30) What is the multiplicative identity?

(Problem 40) Let r be a real number. Recall that $\mathbb{C} = \mathbb{R}^2$ is a vector space over $\mathbb{R}$, so we can multiply vectors (complex numbers) by scalars (real numbers). Is there a complex number (ξ, η) such that $r(x, y) = (\xi, \eta) \cdot (x, y)$ for all $(x, y) \in \mathbb{C}$?

[Definition: Notation for the complex numbers]

- If $r \in \mathbb{R}$, we identify r with the number $(r, 0) \in \mathbb{C}$.
- We let i denote $(0, 1)$.

(Problem 50) If x, y are real numbers, what complex number is $x + iy$?

(Problem 60) If $z = x + iy$ for x, y real, what are $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$?

(Problem 70) If $z \in \mathbb{C}$ and r is real, what are $\mathrm{Re}(zr)$ and $\mathrm{Im}(zr)$?

(Problem 80) If z, $w \in \mathbb{C}$, what are $\mathrm{Re}(zw)$, $\mathrm{Im}(zw)$ in terms of $\mathrm{Re}\, z$, $\mathrm{Re}\, w$, $\mathrm{Im}\, z$, and $\mathrm{Im}\, w$?

[Definition: Conjugate] The conjugate to the complex number $x + iy$, where x, y are real, is $\overline{x + iy} = x - iy$.[1]

(Problem 90) If z and w are complex numbers, show that $\overline{z} + \overline{w} = \overline{z + w}$.

(Problem 100) Show that $\overline{z} \cdot \overline{w} = \overline{zw}$.

(Problem 110) Write $\mathrm{Re}\, z$ in terms of z and $\overline{z}$.

(Problem 120) Write $\mathrm{Im}\, z$ in terms of z and $\overline{z}$.

(Problem 130) Show that $z\overline{z}$ is always real and nonnegative.

(Problem 140) If $z\overline{z} = 0$, what can you say about z?

(Clayton, Problem 150) If z is a complex number with $z \neq 0$, show that there exists another complex number w such that $zw = 1$. Give a formula for w in terms of z. We will write $w = \frac{1}{z}$.

[Definition: Modulus] If z is a complex number, we define its modulus $|z|$ as $|z| = \sqrt{z\overline{z}}$.

[1] Some authors, especially in physics, write z^* instead of $\overline{z}$ for the complex conjugate of z.

(David, Problem 160) Show that $|\operatorname{Re} z| \leq |z|$ (where the first $|\cdot|$ denotes the absolute value in the real numbers and the second $|\cdot|$ denotes the modulus in the complex numbers.)

(Problem 170) Show that $|\operatorname{Im} z| \leq |z|$.

(Elliot, Problem 180) If z and w are complex numbers, show that $|zw| = |z|\,|w|$.

(Emily, Problem 190) Give an example of a non-constant polynomial that has no roots (solutions) that are real numbers.

(Katie, Problem 200) Find a root (solution) to your polynomial that is a complex number.

1.2. Real Analysis

(Michael, Problem 210) If $z = x + iy = (x, y)$, verify that the complex modulus $|z|$ is equal to the vector space norm $\|(x, y)\|$ in $\mathbb{R}^2$.

(Nicholas, Problem 220) Show that $\mathbb{C}$ is complete as a metric space if we use the expected metric $d(z, w) = |z - w|$.

(Zachary, Problem 230) Recall that $(\mathbb{R}^2, d)$ is a metric space, where $d(u, v) = \|u - v\|$. In particular, this metric satisfies the triangle inequality. Write the triangle inequality as a statement about moduli of complex numbers. Simplify your statement as much as possible.

(Problem 240) If $\{a_n\}_{n=1}^{\infty}$ is a sequence of points in $\mathbb{R}^p$, $a \in \mathbb{R}^p$, and we write $a_n = (a_n^1, a_n^2, \dots, a_n^p)$, $a = (a^1, \dots a^p)$, show that $a_n \to a$ (in the metric space sense) if and only if $a_n^k \to a^k$ for each $1 \leq k \leq p$.

(Clayton, Problem 250) What does this tell you about the complex numbers?

[Definition: Maclaurin series] If $f : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable function, the Maclaurin series for f is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

with the convention that $0^0 = 1$.

(Problem 260) Find the Maclaurin series for the exp function.

(Problem 270) Find the Maclaurin series for the sin function.

(Problem 280) Find the Maclaurin series for the cos function.

(Problem 290) Show that if x is real, then the Maclaurin series for $\exp x$, $\sin x$, or $\cos x$ converges to $\exp x$, $\sin x$, or $\cos x$, respectively.

(Problem 300) Write down the sum angle identities for sine and cosine and prove that they are correct.

(Problem 310) Write down and prove the Cauchy-Schwarz inequality for real numbers.

1.2. Further Properties of the Complex Numbers

(Problem 320) State the Cauchy-Schwarz inequality for complex numbers and prove that it is valid.

(David, Problem 330) Let $z \in \mathbb{C}$. Consider the series $\sum_{j=0}^{\infty} \frac{z^j}{j!}$, that is, the sequence of complex numbers $\{\sum_{j=0}^{n} \frac{z^j}{j!}\}_{n=0}^{\infty}$. Show that this sequence is a Cauchy sequence.

(Problem 340) Since $\mathbb{C}$ is complete, the series converges. If $z = x$ is a real number, to what number does the series converge?

(Elliot, Problem 350) If $z = iy$ is purely imaginary, to what (complex) number does the series converge?

(Bonus Problem 360) If $z = x + iy$, show that $\sum_{j=0}^{\infty} \frac{z^j}{j!}$ converges to the product $\left(\sum_{j=0}^{\infty} \frac{x^j}{j!}\right)\left(\sum_{j=0}^{\infty} \frac{(iy)^j}{j!}\right)$.

[Definition: The complex exponential] If x is real, we define

$$\exp(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} \quad \text{and} \quad \exp(ix) = \sum_{j=0}^{\infty} \frac{(ix)^j}{j!}.$$

If $z = x + iy$ is a complex number, we define

$$\exp(z) = \exp(x) \cdot \exp(iy).$$

(Emily, Problem 370) If y, η are real, show that $\exp(iy + i\eta) = \exp(iy) \cdot \exp(i\eta)$.

(Katie, Problem 380) If z, w are any complex numbers, show that $\exp(z + w) = \exp(z) \cdot \exp(w)$.

(Michael, Problem 390) Suppose that z is a complex number and that $|z| = 1$. Show that there is a number $\theta \in \mathbb{R}$ with $\exp(i\theta) = z$.

(Nicholas, Problem 400) How many such numbers θ exist?

(Zachary, Problem 410) Suppose that z is a complex number. Show that there exist numbers $r \in [0, \infty)$ and $\theta \in \mathbb{R}$ such that $z = r \exp(i\theta)$.

(Clayton, Problem 420) How many possible values of r exist?

(Problem 430) How many possible values of θ exist?

(David, Problem 440) Find all solutions to the equation $z^6 = i$.

1.3. Real Analysis

(Problem 450) Give an example of a function that can be written in two different ways.

[Definition: Ring of polynomials] Let $\mathbb{R}[z]$ be the ring of polynomials in one variable with real coefficients, that is,

$$\mathbb{R}[z] = \Bigl\{p : p(z) = \sum_{k=0}^{n} a_k \, z^k \text{ for some } n \in \mathbb{N}_0, a_k \in \mathbb{R}\Bigr\}.$$

Let $\mathbb{R}[x, y]$ be the ring of polynomials in two variables with real coefficients, that is,

$$\mathbb{R}[x, y] = \Bigl\{p : p(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{n} a_{j,k} \, x^j \, y^k \text{ for some } n \in \mathbb{N}_0, a_{j,k} \in \mathbb{R}\Bigr\}.$$

[Definition: Degree] If $p \in \mathbb{R}[z]$ and $p(z) = \sum_{k=0}^{n} a_k \, z^k$, then the degree of p is the smallest nonnegative integer m such that $a_m \neq 0$.

(Elliot, Problem 460) Let $p(x) = \sum_{k=0}^{n} a_k \, x^k$ and let $q(x) = \sum_{k=0}^{n} b_k \, x^k$ be two polynomials in $\mathbb{R}[x]$, with a_k, $b_k \in \mathbb{R}$. Show that if $p(x) = q(x)$ for all $x \in \mathbb{R}$ then $a_k = b_k$ for all $k \in \mathbb{N}_0$.

(Emily, Problem 470) Let $p \in \mathbb{R}[x]$ be a polynomial. Suppose that $x_0 \in \mathbb{R}$ and that $p(x_0) = 0$. Show that there is a polynomial $q \in \mathbb{R}[x]$ such that $p(x) = (x - x_0)q(x)$ for all $x \in \mathbb{R}$. What can you say about the degree of q?

(Katie, Problem 480) Let $p(x) = \sum_{k=0}^{n} a_k \, x^k$ and let $q(x) = \sum_{k=0}^{n} b_k \, x^k$ be two polynomials of degree at most n in $\mathbb{R}[x]$, with a_k, $b_k \in \mathbb{R}$. Suppose that there are $n + 1$ distinct numbers $x_0, x_1, \ldots, x_n \in \mathbb{R}$ such that $p(x_j) = q(x_j)$ for all $0 \leq j \leq n$. Show that $a_k = b_k$ for all $k \in \mathbb{N}_0$. *Hint*: Consider the polynomial $r(x) = p(x) - q(x)$.

(Problem 490) Let $p(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{n} a_{j,k} \, x^j \, y^k$ and let $q(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{n} b_{j,k} \, x^j \, y^k$ be two polynomials in $\mathbb{R}[x, y]$, with $a_{j,k}$, $b_{j,k} \in \mathbb{R}$. Show that if $p(x, y) = q(x, y)$ for all $(x, y) \in \mathbb{R}^2$ then $a_{j,k} = b_{j,k}$ for all j, $k \in \mathbb{N}_0$.

Definition 1.3.1 (part 1). Let $\Omega \subseteq \mathbb{R}^2$ be open. Suppose that $f : \Omega \to \mathbb{R}$. We say that f is continuously differentiable, or $f \in C^1(\Omega)$, if the two partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere in Ω and are continuous on Ω.

(Michael, Problem 500) Suppose that $\Omega \subseteq \mathbb{R}^2$ is open and connected. Let $f \in C^1(\Omega)$ and suppose that $\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} = 0$ everywhere in Ω. Show that f is a constant.

[Definition: Ring of polynomials] Let $\mathbb{C}[z]$ be the ring of polynomials in one variable with complex coefficients, that is,

$$\mathbb{C}[z] = \left\{p : p(z) = \sum_{k=0}^{n} a_k\, z^k \text{ for some } n \in \mathbb{N}_0, a_k \in \mathbb{C}\right\}.$$

Let $\mathbb{C}[x,y]$ be the ring of polynomials in two variables with complex coefficients, that is,

$$\mathbb{C}[x,y] = \left\{p : p(x,y) = \sum_{j=0}^{n}\sum_{k=0}^{n} a_{j,k}\, x^j\, y^k \text{ for some } n \in \mathbb{N}_0, a_{j,k} \in \mathbb{C}\right\}.$$

(Nicholas, Problem 510) Let $p(z) = \sum_{k=0}^{n} a_k\, z^k$ and let $q(z) = \sum_{k=0}^{n} b_k\, z^k$ be two polynomials in $\mathbb{C}[z]$. Show that if $p(x) = q(x)$ for all $x \in \mathbb{R}$ then $a_k = b_k$ for all k, and so $p(z) = q(z)$ for all $z \in \mathbb{C}$.

(Zachary, Problem 520) Show that Problems 470 and 480 are valid for polynomials in $\mathbb{C}[z]$ with complex roots.

(Clayton, Problem 530) Let $p(z,w) = \sum_{j=0}^{n}\sum_{k=0}^{n} a_{j,k}\, z^j\, w^k$ and let $q(z,w) = \sum_{j=0}^{n}\sum_{k=0}^{n} b_{j,k}\, z^j\, w^k$ be two polynomials in $\mathbb{C}[z,w]$, with $a_{j,k}$, $b_{j,k} \in \mathbb{C}$. Show that if $p(x,y) = q(x,y)$ for all $(x,y) \in \mathbb{R}^2$ then $a_{j,k} = b_{j,k}$ for all j, $k \in \mathbb{N}_0$.

(Problem 540) Let $p(z,w) = \sum_{j=0}^{n}\sum_{k=0}^{n} a_{j,k}\, z^j\, w^k$ and let $q(z,w) = \sum_{j=0}^{n}\sum_{k=0}^{n} b_{j,k}\, z^j\, w^k$ be two polynomials in $\mathbb{C}[z,w]$, with $a_{j,k}$, $b_{j,k} \in \mathbb{C}$. Show that if $p(z,\bar{z}) = q(z,\bar{z})$ for all $z \in \mathbb{C}$ then $a_{j,k} = b_{j,k}$ for all j, $k \in \mathbb{N}_0$.

(David, Problem 550) Let $p \in \mathbb{C}[z,w]$ satisfy $p(z,\bar{z}) = z^2 - \bar{z}^3$. Is there a polynomial $q \in \mathbb{C}[z]$ such that $q(z) = p(z,\bar{z})$ for all $z \in \mathbb{C}$?

Definition 1.3.1 (part 2). Let $\Omega \subseteq \mathbb{C}$ be an open set. Recall $\mathbb{C} = \mathbb{R}^2$. Let $f : \Omega \to \mathbb{C}$ be a function. Then $f \in C^1(\Omega)$ if $\operatorname{Re} f$, $\operatorname{Im} f \in C^1(\Omega)$.

[Definition: Derivative of a complex function] Let $f \in C^1(\Omega)$. Let $u(z) = \operatorname{Re} f(z)$ and let $v(z) = \operatorname{Im} f(z)$. Then

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \quad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}.$$

(Elliot, Problem 560) Establish the Leibniz rules

$$\frac{\partial}{\partial x}(fg) = \frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}, \qquad \frac{\partial}{\partial y}(fg) = \frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}.$$

[Definition: Complex derivative] Let $f \in C^1(\Omega)$. Then

$$\frac{\partial f}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{1}{2i}\frac{\partial f}{\partial y}, \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2i}\frac{\partial f}{\partial y}.$$

(Emily, Problem 570) Let $f(z) = z$. Find $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$.

(Katie, Problem 580) Let $g(z) = \bar{z}$. Find $\frac{\partial g}{\partial z}$ and $\frac{\partial g}{\partial \bar{z}}$.

(Problem 590) Show that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ are linear operators.

(Michael, Problem 600) Show that $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ commute in the sense that, if $\Omega \subseteq \mathbb{C}$ is open and $f \in C^2(\Omega)$, then $\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}}f\right) = \frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial z}f\right)$.

(Nicholas, Problem 610) Establish the Leibniz rules

$$\frac{\partial}{\partial z}(fg) = \frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}, \qquad \frac{\partial}{\partial \bar{z}}(fg) = \frac{\partial f}{\partial \bar{z}}g + f\frac{\partial g}{\partial \bar{z}}..$$

(Zachary, Problem 620) Let ℓ and m be nonnegative integers. Find $\frac{\partial}{\partial z}(z^\ell \bar{z}^m)$ and $\frac{\partial}{\partial \bar{z}}(z^\ell \bar{z}^m)$.

(Clayton, Problem 630) Let j, k, ℓ, and m be nonnegative integers. Find $\frac{\partial^j}{\partial z^j}\frac{\partial^k}{\partial \bar{z}^k}(z^\ell \bar{z}^m)$.

(Problem 640) Let $p \in \mathbb{C}[z,w]$. Show that there is a $q \in \mathbb{C}[z]$ such that $p(z,\bar{z}) = q(z)$ for all $z \in \mathbb{C}$ if and only if $\frac{\partial}{\partial \bar{z}}(p(z,\bar{z})) = 0$ everywhere in $\mathbb{C}$.

(David, Problem 650) Let $p \in \mathbb{C}[x, y]$. Show that there is a $q \in \mathbb{C}[z]$ such that $p(x, y) = q(x + iy)$ for all x, $y \in \mathbb{R}$ if and only if $\frac{\partial}{\partial \bar{z}}(p(x, y)) = 0$ everywhere in $\mathbb{C}$.

(Elliot, Problem 660) Suppose that $\Omega \subseteq \mathbb{C}$ is open and connected, that $f \in C^1(\Omega)$, and that $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \bar{z}} = 0$ in Ω. Show that f is constant in Ω.

(Emily, Problem 670) Show that

$$\frac{\partial f}{\partial z} = \overline{\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)}.$$

(Katie, Problem 680) Find $\frac{\partial}{\partial z} \frac{1}{z}$.

(Problem 690) Find $\frac{\partial}{\partial \bar{z}} \frac{1}{z}$.

(Michael, Problem 700) Find $\frac{\partial}{\partial z} \frac{1}{z^n}$ and $\frac{\partial}{\partial \bar{z}} \frac{1}{z^n}$ for any positive integer n.

1.4. Holomorphic Functions, the Cauchy-Riemann Equations, and Harmonic Functions

Definition 1.4.1. Let $\Omega \subseteq \mathbb{C}$ be open and let $f \in C^1(\Omega)$. We say that f is holomorphic in Ω if

$$\frac{\partial f}{\partial \bar{z}} = 0$$

everywhere in Ω.

Lemma 1.4.2. Let $f \in C^1(\Omega)$, let $u = \operatorname{Re} f$, and let $v = \operatorname{Im} f$. Then f is holomorphic in Ω if and only if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

everywhere in Ω. (These equations are called the Cauchy-Riemann equations.)

(Nicholas, Problem 710) Prove the "only if" direction of Lemma 1.4.2: Suppose that f is holomorphic in Ω, $\Omega \subseteq \mathbb{C}$ open, then the Cauchy-Riemann equations hold for $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$.

(Zachary, Problem 720) Prove the "if" direction of Lemma 1.4.2: suppose that $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ are C^1 in Ω and satisfy the Cauchy-Riemann equations. Show that f is holomorphic in Ω.

Proposition 1.4.3. Let $f \in C^1(\Omega)$. Then f is holomorphic in Ω if and only if $\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}$ and that in this case

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}.$$

(Clayton, Problem 730) Begin the proof of Proposition 1.4.3 by showing that if f is holomorphic then $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}$.

(Problem 740) Complete the proof of Proposition 1.4.3 by showing that if $f \in C^1(\Omega)$ and $\frac{\partial f}{\partial x} = \frac{1}{i} \frac{\partial f}{\partial y}$, then f is holomorphic.

Definition 1.4.4. We let $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. If $\Omega \subseteq \mathbb{C}$ is open and $u \in C^2(\Omega)$, then u is harmonic if

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

everywhere in Ω.

(David, Problem 750) Suppose that f is holomorphic in an open set Ω and that $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. Compute $\triangle u$ and $\triangle v$.

(Elliot, Problem 760) Let $f \in \mathbb{C}[z]$ be a holomorphic polynomial. Show that there is a polynomial $F \in \mathbb{C}[z]$ such that $\frac{\partial F}{\partial z} = f$. How many such polynomials are there?

Lemma 1.4.5. Let u be harmonic and real valued in $\mathbb{C}$. Suppose in addition that $u \in \mathbb{R}[x, y]$, that is, that u is a polynomial. Then here is a holomorphic polynomial $f \in \mathbb{C}[z]$ such that $u(x, y) = \operatorname{Re} f(x + iy)$.

(Emily, Problem 770) Prove Lemma 1.4.5. *Hint*: Start by computing $\frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$ and $\frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial z}$.

(Problem 780) Let $a < c < b$ and let $f : (a, b) \to \mathbb{R}$ be continuous. Show that $\lim_{t\to 0} \frac{1}{t} \int_c^{c+t} f(x)\,dx = f(c)$.

(Problem 790) State Green's theorem.

(Katie, Problem 800) Let $f : [a, b]\times[c, d] \to \mathbb{R}$. Suppose that f is continuous on $[a, b]\times[c, d]$ and the function $\partial_x f = \frac{\partial f}{\partial x}$ is continuous on $(a, b) \times [c, d]$. Show that $\partial_x \int_c^d f(x, y)\,dy = \int_c^d \partial_x f(x, y)\,dy$ for all $a < x < b$.

(Problem 810) Let f be a C^2 function in an open set in $\mathbb{R}^2$. Show that $\frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x}$.