Math 2584, Spring 2021

The final exam will occur:
- Monday, May 3 at 8:00, in KIMP 102.
- Wednesday, May 5 at 10:15, classroom TBD.

You are allowed a non-graphing calculator.

Please complete the online course evaluation at `courseval.uark.edu` on or before Friday, April 30. If at least 80% of the class completes the course evaluation before the deadline, I will drop your 2 lowest group project scores; otherwise, I will drop your 1 lowest group project score.

Please check your final exam schedule. If you have 3 or more final exams scheduled for the same day, and you need to reschedule the final exam for this class, please let me know by email immediately.

The following Laplace transforms will be written on the last page of the exam.

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0,$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \qquad s > 0,$$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}, \qquad s > 0,$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad s > 0, \qquad n \geq 0$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \qquad s > a$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}, \qquad s > 0$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}, \qquad s > 0$$

$$\text{If } \mathcal{L}\{f(t)\} = F(s) \text{ then } \mathcal{L}\{e^{at} f(t)\} = F(s-a),$$

$$\mathcal{L}^{-1}\{F(s)\} = e^{at}\,\mathcal{L}^{-1}\{F(s+a)\},$$

$$\mathcal{L}\{\delta(t-c)\} = e^{-cs}, \qquad s > 0, \qquad c \geq 0$$

$$\mathcal{L}\{\mathcal{U}(t-c) f(t)\} = e^{-cs}\,\mathcal{L}\{f(t+c)\}, \qquad c \geq 0$$

$$\mathcal{L}\{\mathcal{U}(t-c) g(t-c)\} = e^{-cs}\,\mathcal{L}\{g(t)\}, \qquad c \geq 0$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = s\,\mathcal{L}\{y(t)\} - y(0).$$

$$\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}\,\mathcal{L}\{f(t)\},$$

$$\mathcal{L}\left\{\int_0^t f(r)\,g(t-r)\,dr\right\} = \mathcal{L}\left\{\int_0^t f(t-r)\,g(r)\,dr\right\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}.$$

(AB 1) Is $y = e^t$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 2) Is $y = e^{2t}$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 3) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 1$?

(AB 4) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 2$?

(AB 5) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 1$?

(AB 6) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 2$?

(AB 7) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + \arctan(t)\, y = e^t$, $y(3) = 7$.
(b) $\frac{1}{1+t^2}\frac{dy}{dt} - t^5 y = \cos(6t)$, $y(2) = -1$, $y'(2) = 3$.
(c) $\frac{d^2y}{dt^2} - 5\sin(t)\, y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.
(d) $e^t \frac{d^2y}{dt^2} + 3(t-4)\frac{dy}{dt} + 4t^6 y = 2$, $y(3) = 1$, $y'(3) = -1$.
(e) $\frac{d^2y}{dt^2} + \cos(t)\frac{dy}{dt} + 3\ln(1+t^2)\, y = 0$, $y(2) = 3$.
(f) $\frac{d^3y}{dt^3} - t^7\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\, y = t^3$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$, $y'''(1) = 3$.
(g) $(1+t^2)\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = t^3$, $y(3) = 9$, $y'(3) = 7$, $y''(3) = 5$.
(h) $\frac{d^3y}{dt^3} - e^t\frac{d^2y}{dt^2} + e^{2t}\frac{dy}{dt} + e^{3t}y = e^{4t}$, $y(-1) = 1$, $y'(-1) = 3$.
(i) $(2+\sin t)\frac{d^3y}{dt^3} + \cos t\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\, y = t^3$, $y(7) = 2$.

(AB 8) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + e^t y = \cos t$, $y(0) = 3$, $y'(0) = -2$.
(b) $t^2\frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + 6y = 7t^3$, $y(2) = 4$, $y'(2) = 4$, $y''(2) = 2$.
(c) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0$, $y(4) = 3$, $y'(4) = -2$, $y''(4) = 0$, $y'''(4) = 3$.

(AB 9) A lake initially has a population of 600 trout. The birth rate of trout is proportional to the number of trout living in the lake. Fishermen are allowed to harvest 30 trout/year from the lake. Write an initial value problem for the fish population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 10) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write an initial value problem for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 11) A tank contains nitrogen dioxide. Initially, there are 200 grams of NO_2. The gas decays (converts to molecular oxygen and nitrogen) at a rate proportional to the square of the amount of NO_2 remaining. Write an initial value problem for the amount of NO_2 in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 12) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen. Write an initial value problem for the amount of hydrogen in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 13) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C. Write an initial value problem for the temperature of the cup of coffee. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 14) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Write an initial value problem for the amount of salt in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 15) I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write an initial value problem for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 16) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write an initial value problem for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 17) According to the Stefan-Boltzmann law, a black body in a dark vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

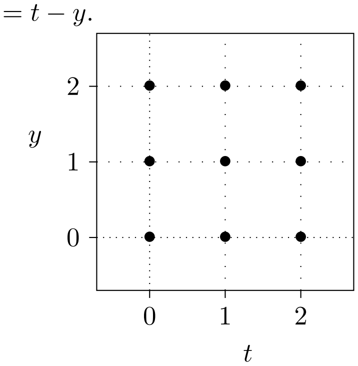
(AB 18) According to the Stefan-Boltzmann law, a black body in a vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). If there is some ambient light in the vacuum, then the black body absorbs energy at a rate proportional to the fourth power of the effective temperature of the light. The proportionality constant is the same in both cases.

Suppose that a planet in space is currently at a temperature of 400 K. The effective temperature of its surroundings is 3 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 19) A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

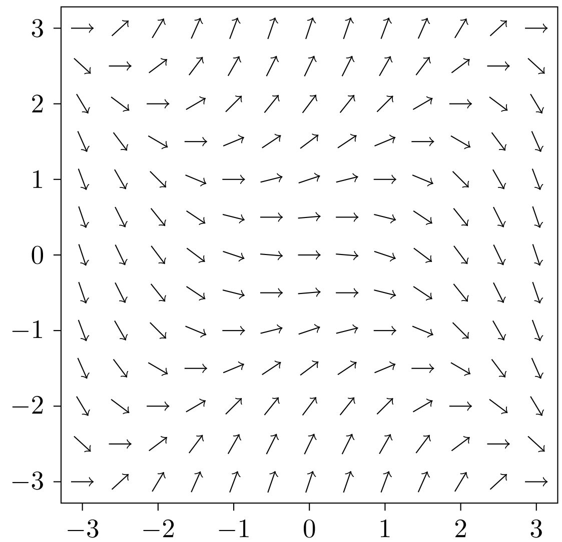
(AB 20) A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 21) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.



(AB 22) Consider the differential equation $\frac{dy}{dx} = (y^2 - x^2)/3$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

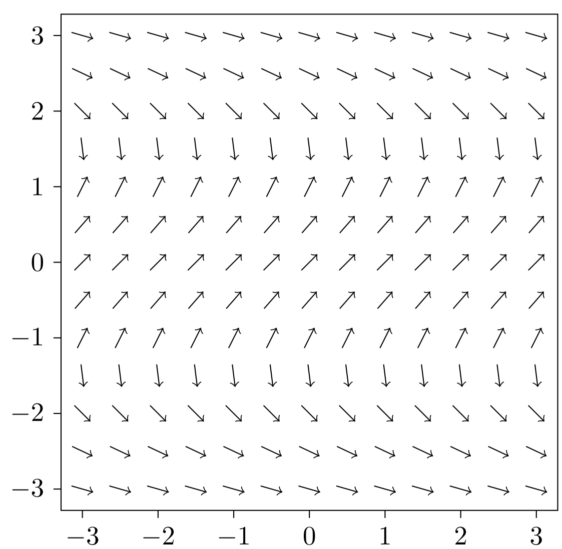
$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-2) = 1.$$



(AB 23) Consider the differential equation $\frac{dy}{dx} = \frac{2}{2-y^2}$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{2}{2 - y^2}, \quad y(1) = 0.$$

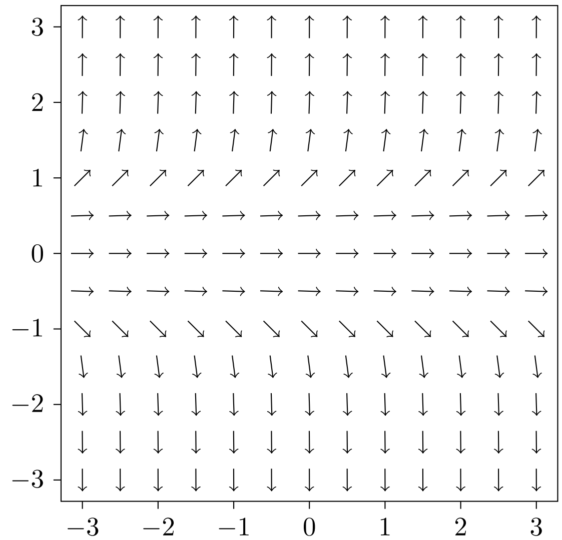
Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 24) Consider the differential equation $\frac{dy}{dx} = y^5$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = y^5, \quad y(1) = -1.$$

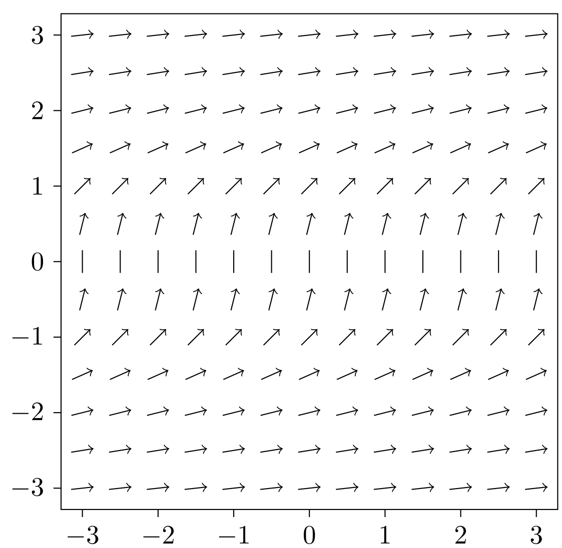
Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 25) Consider the differential equation $\frac{dy}{dx} = 1/y^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{1}{y^2}, \quad y(3) = 2.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 26) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y-2)(y+1)^2$. By hand, sketch some typical solutions.

(AB 27) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 28) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = y^2(y-2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 29) Trout live in a lake. Each trout produces, on average, one baby trout every two years. Fishermen are allowed to harvest 100 trout/year from the lake. You may neglect other reasons why trout die.

(a) Formulate a differential equation for the number of trout in the lake.
(b) Find the critical points of this differential equation and classify them as to stability.
(c) What is the real-world meaning of the critical points?

(AB 30) I owe my bank a debt of B dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

(a) Formulate a differential equation for the amount of money I owe.
(b) Find the critical points of this differential equation and classify them as to stability.
(c) What is the real-world meaning of the critical points?

(AB 31) A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

(a) Write the differential equation for the amount of salt in the tank.
(b) Find the critical points of this differential equation and classify them as to stability.
(c) What is the real-world meaning of the critical points?

(AB 32) A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where v is her velocity in meters per second.

(a) Write a differential equation for her velocity. Assume her velocity is always downwards.
(b) Find the critical points of this differential equation and classify them as to stability. Be sure to include units.
(c) What is the real-world meaning of these critical points?

(AB 33) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen.

(a) Write a differential equation for the amount of hydrogen left in the tank.
(b) Find the critical points of this differential equation and classify them as to stability. Be sure to include units.
(c) What is the real-world meaning of these critical points?

(AB 34) Young oak trees grow in a large field with area 1 square kilometer. Initially, there are 30 trees in the field. Each tree shades 20 square meters. The trees produce acorns. Squirrels scatter the acorns across the field at random. Squirrels eat most of the acorns; on average, each tree produces two acorns per year that are not eaten by squirrels. Every acorn not eaten by squirrels and planted in sunlight sprouts.

(a) Write a differential equation for the number of trees in the field.
(b) Find the critical points of this differential equation and classify them as to stability.
(c) What is the real-world meaning of these critical points?

(AB 35) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate of 3 cm^3 per exposed cm2 per hour.

(a) Write a differential equation for the amount of water in the tank.
(b) Find the critical points of this differential equation and classify them as to stability.
(c) What is the real-world meaning of these critical points?

(AB 36) For each of the following differential equations, determine whether it is linear, separable, exact, homogeneous, a Bernoulli equation, or of the form $\frac{dy}{dt} = f(At+By+C)$. Then solve the differential equation.

(a) $t + \cos t + (y - \sin y)\frac{dy}{dt} =$
(b) $\ln y + y + x + \left(\frac{x}{y} + x\right)\frac{dy}{dx} = 0$
(c) $1 + t^2 - ty + (t^2+1)\frac{dy}{dt} = 0$
(d) $ty - y^2 - t^2 + t^2\frac{dy}{dt} = 0$
(e) $4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5)\frac{dy}{dt} = 0$
(f) $y \tan 2x + y^3 \cos 2x + \frac{dy}{dx} = 0$
(g) $3t - 5x + (t + x)\frac{dx}{dt} = 0$
(h) $\frac{dy}{dt} = \frac{1}{3t+2y+7}$.
(i) $y^3 \cos(2t) + \frac{dy}{dt} = 0$
(j) $4ty\frac{dy}{dt} = 3y^2 - 2t^2$
(k) $t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$
(l) $\sin^2(x - t)\frac{dx}{dt} = \csc^2(x - t)$.
(m) $\frac{dy}{dt} = 8y - y^8$
(n) $t\frac{dz}{dt} = -\cos t - 3z$
(o) $\frac{dy}{dt} = \cot(y/t) + y/t$
(p) $\frac{dy}{dt} = ty + t^2\sqrt[3]{y}$

(AB 37) (14 points) For each of the following differential equations, find all the equilibrium solutions or state that no such solutions exist. You do not need to find the nonequilibrium solutions.

(If this problem appears on Exam 2, its points will not count towards your Exam 2 total. Instead, if your score on this problem is better than your score on Problem 4 on Exam 1, it will replace your score on that problem and your Exam 1 score will be adjusted accordingly.)

(a) $\frac{dy}{dt} = y^3 - yt$
(b) $\frac{dy}{dt} = t^2e^y$
(c) $\frac{dy}{dt} = ty + t^3$
(d) $\frac{dy}{dt} = (y^2 + 3y + 2)\sin(t)$
(e) $\frac{dy}{dt} = \ln(y^t)$

(AB 38) For each of the following differential equations, determine whether it is linear, separable, exact, homogeneous, Bernoulli, or a function of a linear term. Then solve the given initial-value problem.

(a) $\cos(t + y^3) + 2t + 3y^2\cos(t + y^3)\frac{dy}{dt} = 0$, $y(\pi/2) = 0$
(b) $\frac{dy}{dt} = \sin t \cos y$, $y(\pi) = \pi/2$
(c) $ty^2 - 4t^3 + 2t^2y\frac{dy}{dt} = 0$, $y(1) = 3$.
(d) $1 + y^2 + t\frac{dy}{dt} = 0$, $y(1) = 1$
(e) $\frac{dy}{dt} = (2y + 2t - 5)^2$, $y(0) = 3$.
(f) $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$.
(g) $3y + e^{-3t}\sin t + \frac{dy}{dt} = 0$, $y(0) = 2$

(AB 39) Solve the initial-value problem $\frac{dy}{dt} = \frac{t-5}{y}$, $y(0) = 3$ and determine the range of t-values in which the solution is valid.

(AB 40) Solve the initial-value problem $\frac{dy}{dt} = y^2$, $y(0) = 1/4$ and determine the range of t-values in which the solution is valid.

(AB 41) Consider the differential equation $\ln(3/4 + t^2)\frac{dy}{dt} = \sin(y)$.

(a) Find all pairs of numbers T_0 and Y_0 such that the initial value problem $\ln(3/4 + t^2)\frac{dy}{dt} = \sin(y)$, $y(T_0) = Y_0$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 1.2.1.)

(b) Find a pair of numbers T_0 and Y_0 such that the initial value problem $\ln(3/4 + t^2)\frac{dy}{dt} = \sin(y)$, $y(T_0) = Y_0$ has no solutions.

(AB 42) Consider the differential equation $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-1)y = 0$.

(a) Find all triples of numbers T_0, Y_0, and Y_1 such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-1)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 4.1.1.)

(b) Find a triple of numbers T_0, Y_0, and Y_1 such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-1)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ has no solutions.

(AB 43) Consider the differential equation $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-3)y = 0$.

(a) Find all triples of numbers T_0, Y_0, and Y_1 such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-3)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 4.1.1.)

(b) Find a triple of numbers T_0, Y_0, and Y_1 such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-3)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ has no solutions.

(AB 44) Consider the differential equation $\frac{dy}{dt} = (y-5)^{3/8}$.

(a) Find all pairs of numbers T_0 and Y_0 such that the initial value problem $\frac{dy}{dt} = (y-5)^{3/8}$, $y(T_0) = Y_0$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 1.2.1.)

(b) Find two solutions to the initial value problem $\frac{dy}{dt} = (y-5)^{3/8}$, $y(16) = 5$. Where are your solutions equal?

(c) Find two solutions to the initial value problem $\frac{dy}{dt} = (y-5)^{3/8}$, $y(16) = 6$. Where are your solutions equal?

(AB 45) Consider the differential equation $\frac{dy}{dt} = 3y(\ln y)^{2/3}$.

(a) Find all pairs of numbers T_0 and Y_0 such that the initial value problem $\frac{dy}{dt} = 3y(\ln y)^{2/3}$, $y(T_0) = Y_0$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 1.2.1.)

(b) Find two solutions to the initial value problem

$$\frac{dy}{dt} = 3y(\ln y)^{2/3}, \quad y(7) = 1.$$

Where are your solutions equal?

(c) Find two solutions to the initial value problem

$$\frac{dy}{dt} = 3y(\ln y)^{2/3}, \quad y(7) = e.$$

Where are your solutions equal?

(AB 46)

(a) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0, \quad y(0)=2, \quad y'(0)=7$$

is guaranteed a unique solution?

(b) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0, \quad y(2)=2, \quad y'(2)=7$$

is guaranteed a unique solution?

(c) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0, \quad y(5)=2, \quad y'(5)=7$$

is guaranteed a unique solution?

(d) How many solutions are there to the initial value problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0, \quad y(3/2)=2, \quad y'(3/2)=7?$$

(AB 47) You are given that $g(t)$ and $q(t)$ are continuous functions. Which of the following functions are possible solutions to the initial value problem $(t-5)^3\frac{d^2y}{dt^2}-(3t-15)^2\frac{dy}{dt}+q(t)y=g(t)$, $y(0)=-5$, $y'(0)=-1$? If a function cannot be a solution, explain why not.

(a) $y(t)=\frac{25}{t-5}$

(b) $y(t)=\frac{5}{t-1}+4t$

(AB 48) You are given that $y_1=t^3+4t^2+4t$ and $y_2=t^2+4t+4$ are both solutions to the differential equation $(t+2)^2\frac{d^2y}{dt^2}-(4t+8)\frac{dy}{dt}+6y=0$ for all t.

What is the longest interval containing the number $t=-3$ on which the general solution to $(t+2)^2\frac{d^2y}{dt^2}-(4t+8)\frac{dy}{dt}+6y=0$ may be written $y=C_1(t^3+4t^2+4t)+C_2(t^2+4t+4)$?

(AB 49) Find a solution to the initial value problem $(t+2)^2\frac{d^2y}{dt^2}-(4t+8)\frac{dy}{dt}+6y=0$, $y(0)=4$, $y'(0)=4$ that is not $y(t)=t^2+4t+4$.

(AB 50) Suppose $y_1(t)$ and $y_2(t)$ are both solutions to the initial value problem $(t+2)^2\frac{d^2y}{dt^2}-(4t+8)\frac{dy}{dt}+6y=0$, $y(5)=3$, $y'(5)=7$ on $(-\infty,\infty)$. What is the longest interval on which you are guaranteed that $y_1=y_2$?

(AB 51) The function $y_1(t)=e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2}-(1+2t)\frac{dy}{dt}+(t+1)y=0$. Find the general solution to this differential equation on the interval $t>0$.

(AB 52) The function $y_1(t)=t$ is a solution to the differential equation $t^2\frac{d^2y}{dt^2}-(t^2+2t)\frac{dy}{dt}+(t+2)y=0$. Find the general solution to this differential equation on the interval $t>0$.

(AB 53) The function $y_1(x)=x^3$ is a solution to the differential equation $x^2\frac{d^2y}{dx^2}+(x^2\tan x-6x)\frac{dy}{dx}+(12-3\,x\tan x)y=0$. Find the general solution to this differential equation on the interval $0<x<\pi/2$.

(AB 54) Find the general solution to the following differential equations.

(a) $\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 85x = 0.$

(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0.$

(c) $\frac{d^4z}{dt^4} + 7\frac{d^2z}{dt^2} - 144z = 0.$

(d) $\frac{d^4w}{dt^4} - 8\frac{d^2w}{dt^2} + 16w = 0.$

(AB 55) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $9\frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 2v = 0$, $v(0) = 3$, $v'(0) = 2$.

(b) $\frac{d^2u}{dt^2} + 10\frac{du}{dt} + 25u = 0$, $u(0) = 1$, $u'(0) = 4$.

(AB 56) Are the functions $f_1(x) = 0$, $f_2(x) = x$, $f_3(x) = x^2$ linearly independent or linearly dependent on the interval $(-\infty, \infty)$? If they are linearly dependent, find constants c_1, c_2, and c_3, not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all $x \in (-\infty, \infty)$.

(AB 57) Are the functions $f_1(x) = 3$, $f_2(x) = \cos^2 x$, $f_3(x) = \sin^2 x$ linearly independent or linearly dependent on the interval $(-\infty, \infty)$? If they are linearly dependent, find constants c_1, c_2, and c_3, not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all $x \in (-\infty, \infty)$.

(AB 58) Are the functions $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$ linearly independent or linearly dependent on the interval $(-\infty, \infty)$? If they are linearly dependent, find constants c_1, c_2, and c_3, not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all $x \in (-\infty, \infty)$.

(AB 59) An object with mass 5 kg stretches a spring 4 cm. It is attached to a viscous damper with damping constant 16 newton · seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Write the differential equation and initial conditions that describe the position of the object. You may use 9.8 meters/second^2 for the acceleration of gravity.

(AB 60) A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant β. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20$ N · s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30$ N · s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(AB 61) A 3-kg object is attached to a spring with constant k and to a viscous damper with damping constant 42 N·sec/m. The object is set in motion from its equilibrium position with initial velocity 5 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object.

If $k = 100$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $k = 200$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(AB 62) A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant β. The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

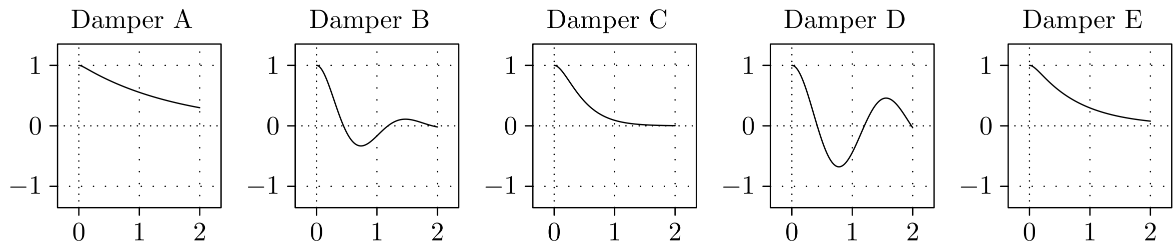
Write the differential equation and initial conditions that describe the position of the object. Then find the value of β for which the system is critically damped. Be sure to include units for β.

(AB 63) An object of mass m is attached to a spring with constant 80 N/m and to a viscous damper with damping constant 20 N·s/m. The object is pulled down to 5cm below its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of m for which the system is critically damped. Be sure to include units for m.

(AB 64) Five objects, each with mass 3 kg, are attached to five springs, each with constant 48 N/m. Five dampers with unknown constants are attached to the objects. In each case, the object is pulled down to a distance 1 cm below the equilibrium position and released from rest. You are given that the system is critically damped in exactly one of the five cases.

Here are the graphs of the objects' positions with respect to time:



(*a*) For which damper is the system critically damped?
(*b*) For which dampers is the system overdamped?
(*c*) For which dampers is the system underdamped?
(*d*) Which damper has the highest damping constant? Which damper has the lowest damping constant?

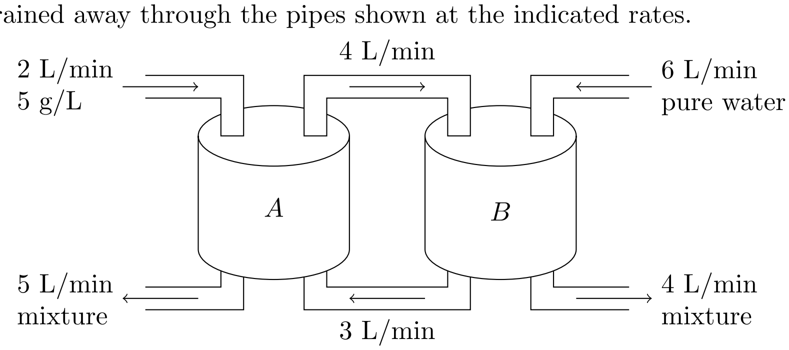
(AB 65) An object with mass 5 kg stretches a spring 4 cm. It is attached to a viscous damper with damping constant 16 newton · seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Using only first derivatives, write the differential equations and initial conditions for the object's position. That is, write a first-order system involving the object's position. You may use 9.8 meters/second^2 for the acceleration of gravity.

(AB 66) A forest is inhabited by rabbits and foxes. Every year, each rabbit gives birth to 3 babies and each fox gives birth to 5 babies on average. Foxes try to catch and eat rabbits; each month, each fox has a 0.02% chance of catching each of the rabbits present in the forest. Each month, every fox that does not eat a rabbit starves to death.

Write the differential equations for the number of foxes and rabbits in the forest.

(AB 67) Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



Write the differential equations and initial conditions that describe the amount of salt in each tank.

(AB 68) Write the system

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y$$

in matrix form.

(AB 69) Write the system

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = 4y - x$$

in matrix form.

(AB 70) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 71) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 72) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 73) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 74) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 75) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 76) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 77) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 4-m & 1 & -1 \\ 1 & 4-m & -1 \\ -1 & -1 & 4-m \end{pmatrix} = -(m-3)^2(m-6).$

(AB 78) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, \qquad x(0) = 3, \quad y(0) = 2, \quad z(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 3-m & 1 & 0 \\ 2 & 3-m & -2 \\ 0 & 1 & 3-m \end{pmatrix} = -(m-3)^3.$

(AB 79) Find the solution to the initial-value problem

$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, x(0) = 4, \quad y(0) = 3, \quad z(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 1-m & 1 & 1 \\ 1 & 3-m & -1 \\ 0 & 2 & 2-m \end{pmatrix} = -(m-2)^3.$

(AB 80) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(b) Use the phase plane method to find an equation relating the number of susceptible and infected people.

(c) Use the phase plane method to find an equation relating the number of resistant and susceptible people.

(d) As $t \to \infty$, $I \to 0$ and the number of people who never contract the disease approaches a limit. Let S_∞ be the limiting number of people who never contract the disease. Find an (algebraic, not differential) equation involving S_∞.

(e) What is the maximum number of people that are infected at any one time?

(AB 81) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(AB 82) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

(a) Set up the system of differential equations that describes the number of susceptible, infected, and resistant people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(b) Assume that there are initially no recovered or vaccinated people and 7 infected people. Use the phase plane method to find an equation relating S and I.

(c) Find an algebraic equation for the maximum number of people that are infected with the virus at any one time.

(AB 83) A small town has a population of 16,000 people. It is expected that soon, one of them will be infected with a contagious disease.

Epidemiologists observe the town and expect each infected person to encounter 10 people per day, who are distributed among susceptible, vaccinated, recovered and infected people according to their proportion of the total population. Each time an infected person encounters a susceptible person, there is a 6% chance that the susceptible person becomes infected. Each infected person has a 25% chance per day of recovering. A recovered person can never contract the disease again.

A (not very effective) vaccine can be distributed before the epidemic starts. Each time an infected person encounters a vaccinated person, there is a 2% chance they become infected. (Once a vaccinated person becomes infected, they are identical to an infected person who was never vaccinated.)

The mayor of the town would like to be sure that no more than 2,000 people are ever infected.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(b) Use the phase plane method to find an equation relating the number of susceptible and recovered people.

(c) Use the phase plane method to find an equation relating the number of recovered people to the number of vaccinated (and never-infected) people.

(d) Find an equation relating the number of infected people to the number of recovered people.

(e) How many people must be vaccinated in order for there to be at most 2000 previously-infected (and thus resistant) people when the epidemic ends?

(AB 84) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers.

Each infected person encounters an average of 20 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. If an infected person encounters a recovered person, the recovered person has a 1% chance of contracting the disease again. Each infected person has a 12% chance of recovering from the disease on any given day.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(b) Use the phase plane method to find an equation relating the number of susceptible and recovered people. Solve for the number of recovered people.

(c) Write an equation relating the number of infected people to the number of susceptible people.

(d) Does the number of infected people ever approach zero? If so, how many susceptible people are left at that time?

(e) Does the number of susceptible people ever approach zero? If so, how many infected people and how many resistant people are there at that time?

(AB 85) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y - 15, \qquad \frac{dy}{dt} = -5x + 6y - 8, \qquad x(0) = 2, \quad y(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 86) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 4, \quad y(0) = -2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 87) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y + e^{2t}\cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 5, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 88) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 89) Find the general solution to the equation $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3}\ln t$ on the interval $0 < t < \infty$.

(AB 90) Find the general solution to the equation $4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 8e^{-t/2}\arctan t$.

(AB 91) Find the general solution to the equation $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$.

(AB 92) Solve the initial-value problem $9\frac{d^2y}{dt^2} + y = \sec^2(t/3)$, $y(0) = 4$, $y'(0) = 2$, on the interval $-3\pi/2 < t < 3\pi/2$.

(AB 93) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - 2x = 0$, $t > 0$, is $x(t) = C_1t^2 + C_2t^{-1}$. Solve the initial-value problem $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(AB 94) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - t\frac{dx}{dt} - 3x = 0$, $t > 0$, is $x(t) = C_1t^3 + C_2t^{-1}$. Find the general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

(AB 95) Suppose that $\frac{d^2x}{dt^2} = 18x^3$, $x(0) = 1$, $x'(0) = 3$. Let $v = \frac{dx}{dt}$. Find a formula for v in terms of x. Then find a formula for x in terms of t.

(AB 96) Suppose that a rocket of mass $m = 1000$ kg is launched straight up from the surface of the planet Trantor with initial velocity 10 km/sec. The radius of Trantor is 3,000 km. When the rocket is r meters from the center of the earth, it experiences a force due to gravity of magnitude GMm/r^2, where $GM = 6 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the rocket's position.
(b) Find the velocity of the rocket as a function of position.
(c) How far away from the earth is the rocket when it stops moving and starts to fall back?

(AB 97) A 5-kg toolbox is dropped (from rest) out of a spaceship at an altitude of 9,000 km above the surface of Trantor. The radius of Trantor is 3,000 km. When the toolbox is r meters from the center of the earth, it experiences a force due to gravity of magnitude $5GM/r^2$, where $GM = 6 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the toolbox's position.
(b) Find the velocity of the toolbox as a function of position.
(c) How fast is the toolbox moving when it strikes the surface of Trantor?

(AB 98) A particle of mass $m = 3$ kg a distance r from an infinitely long string experiences a force due to gravity of magnitude Gm/r, where $G = 2000$ meters^2/second^2, directed directly toward the string. Suppose that the particle is initially 1000 meters from the string and takes off with initial velocity 200 meters/second directly away from the string.

(a) Formulate the initial value problem for the particle's position.
(b) Find the velocity of the particle as a function of position.
(c) How far away from the string is the particle when it stops moving and starts to fall back?

(AB 99) A charged particle of mass $m = 20$ g a distance r meters from an electric dipole experiences a force of $3/r^3$ newtons, directed directly toward the dipole. Suppose that the particle is initially 3 meters from the dipole and is set in motion with initial velocity 5 meters/second away from the dipole.

(a) Formulate the initial value problem for the particle's position.
(b) Find the velocity of the particle as a function of position.
(c) What is the limiting velocity of the particle?

(AB 100) Suppose that a bob of mass 300g hangs from a pendulum of length 15cm. The pendulum is set in motion from its equilibrium point with an initial velocity of 3 meters/second.

If θ denotes the angle between the pendulum and the vertical, then the pendulum satisfies the equation of motion $m\frac{d^2\theta}{dt^2} = -\frac{mg}{\ell}\sin\theta$, where m is the mass of the pendulum bob, ℓ is the length of the pendulum and g is the acceleration of gravity (which you may take to be 9.8 meters/second^2).

Find a formula for $\omega = \frac{d\theta}{dt}$ in terms of θ.

(AB 101) Find the general solution to the following differential equations.

(a) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 3e^{4t}$.

(b) $16\frac{d^2y}{dt^2} - y = e^{t/4}\sin t$.

(c) $\frac{d^2y}{dt^2} + 49y = 3t\sin 7t$.

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$.

(e) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = t\sin(3t)$.

(f) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 7e^{-5t}$.

(g) $16\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 9y = 6t^2 + \cos(2t)$.

(h) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 25y = t^2e^{3t}$.

(i) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 3e^{-5t}$.

(j) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 3\cos(2t)$.

(k) $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 5\sin(4t)$.

(l) $\frac{d^2y}{dt^2} + 9y = 5\sin(3t)$.

(m) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$.

(n) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$.

(o) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$.

(p) $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$.

(q) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$.

(AB 102) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $16\frac{d^2y}{dt^2} - y = 3e^t$, $y(0) = 1$, $y'(0) = 0$.

(b) $\frac{d^2y}{dt^2} + 49y = \sin 7t$, $y(\pi) = 3$, $y'(\pi) = 4$.

(c) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 25t^2$, $y(2) = 0$, $y'(2) = 3$.

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$.

(AB 103) A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 N·s/m and is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(20t)$ N upwards.

Write the differential equation and initial conditions that describe the position of the object.

(AB 104) An object weighing 4 lbs stretches a spring 2 inches. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(\omega t)$ pounds, directed upwards. There is no damping.

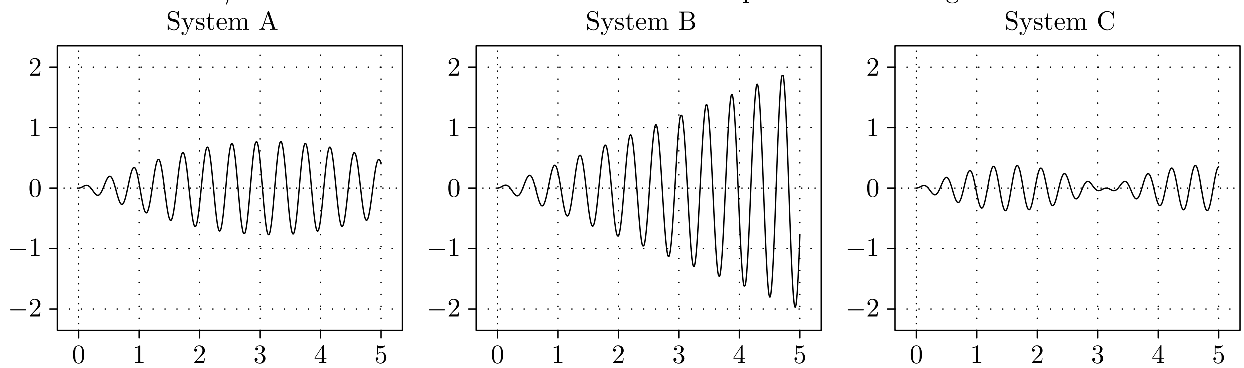
Write the differential equation and initial conditions that describe the position of the object. Then find the value of ω for which resonance occurs. Be sure to include units for ω.

(AB 105) A 4-kg object is suspended from a spring. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $7\sin(\omega t)$ newtons, directed upwards.

(a) Write the differential equation and initial conditions that describe the position of the object.

(b) It is observed that resonance occurs when $\omega = 20$ rad/sec. What is the constant of the spring? Be sure to include units.

(AB 106) A 1-kg object is suspended from a spring with constant 225 N/m. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $15\cos(\omega t)$ newtons, directed upwards. Illustrated are the object's position for three different values of ω. You are given that the three values are $\omega = 15$ radians/second, $\omega = 16$ radians/second, and $\omega = 17$ radians/second. Determine the value of ω that will produce each image.



(AB 107) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ (**not** the table in your notes, on Blackboard, or in your book), find the Laplace transforms of the following functions.

(a) $f(t) = e^{-11t}$
(b) $f(t) = t$
(c) $f(t) = \begin{cases} 3e^t, & 0 < t < 4, \\ 0, & 4 \leq t. \end{cases}$

(AB 108) Find the Laplace transforms of the following functions. You may use the table in your notes, on Blackboard, or in your book.

(a) $f(t) = t^4 + 5t^2 + 4$
(b) $f(t) = (t + 2)^3$
(c) $f(t) = 9e^{4t+7}$
(d) $f(t) = -e^{3(t-2)}$
(e) $f(t) = (e^t + 1)^2$
(f) $f(t) = 8\sin(3t) - 4\cos(3t)$
(g) $f(t) = t^2 e^{5t}$
(h) $f(t) = 7e^{3t}\cos 4t$
(i) $f(t) = 4e^{-t}\sin 5t$
(j) $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$
(k) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$
(l) $f(t) = \begin{cases} 0, & t < 1, \\ t - 2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$
(m) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$
(n) $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$
(o) $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi \end{cases}$
(p) $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi \end{cases}$
(q) $f(t) = t\,e^t \sin t$
(r) $f(t) = t^2 \sin 5t$

(s) You are given that $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$. Find $\mathcal{L}\{t\,J_0(t)\}$. (The function J_0 is called a Bessel function and is important in the theory of partial differential equations in polar coordinates.)

(t) You are given that $\mathcal{L}\{\frac{1}{\sqrt{t}}e^{-1/t}\} = \frac{\sqrt{\pi}}{\sqrt{s}}e^{-2\sqrt{s}}$. Find $\mathcal{L}\{\sqrt{t}\,e^{-1/t}\}$.

(AB 109) For each of the following problems, find y.

(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$

(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$

(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$

(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$

(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$

(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$

(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$

(h) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$

(i) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$

(j) $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$

(k) $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$

(l) $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$

(AB 110) Sketch the graph of $y = t^2 - t^2\,\mathcal{U}(t-1) + (2-t)\,\mathcal{U}(t-1)$.

(AB 111) Solve the following initial-value problems using the Laplace transform. Do you expect the graphs of y, $\frac{dy}{dt}$, or $\frac{d^2y}{dt^2}$ to show any corners or jump discontinuities? If so, at what values of t?

(a) $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$

(b) $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$

(c) $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

(e) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

(f) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$

(g) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$

(h) $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

(i) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$

(j) $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \le t < 4, \\ 0, & 4 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 0$

(k) $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

(l) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

(m) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \le t < 2, \\ 4, & 2 \le t \end{cases}$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = 2$.

(n) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$

(o) $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

(p) $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$

(q) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$

(r) $y(t) + \int_0^t r\,y(t-r)\,dr = t$.

(s) $y(t) = te^t + \int_0^t (t-r)\,y(r)\,dr$.

(t) $\frac{dy}{dt} = 1 - \sin t - \int_0^t y(r)\,dr$, $y(0) = 0$.

(u) $\frac{dy}{dt} + 2y + 10\int_0^t e^{4r} y(t-r)\,dr = 0$, $y(0) = 7$.
(v) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$.
(w) $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$.
(x) $\frac{d^2y}{dt^2} + 4y = -2\delta(t-4\pi)$, $y(0) = 1/2$, $y'(0) = 0$
(y) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + \mathcal{U}(t-2)$, $y(0) = 1$, $y'(0) = 0$
(z) $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 5\delta(t-4)$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 2$.

(AB 112) At time $t = 0$, a group of 7 birds, all 1 year old, are blown onto an isolated island. A given bird has a probability of e^{-3t} of being alive after t years. A bird that is t years old and is still alive on average produces te^{-t} chicks per year. Let $B(t)$ be the birth rate at time t years. Find an integrodifferential equation (and, if necessary, an initial condition) for the birth rate.

(AB 113) Let I be the number of people with a rare infections disease. Suppose that 300 people simultaneously contract the disease at time $t = 0$. Each infected person infects three new people per day on average. A person who was infected r days ago has a probability of $4r^2e^{-2r}$ of recovering on day r. Write an integrodifferential equation (and, if necessary, an initial condition) for the number of infected people.

Answer key

(AB 1) Is $y = e^t$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(Answer 1) No, $y = e^t$ is not a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(AB 2) Is $y = e^{2t}$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(Answer 2) Yes, $y = e^{2t}$ is a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t}\frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(AB 3) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 1$?

(Answer 3) No.

(AB 4) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 2$?

(Answer 4) No.

(AB 5) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 1$?

(Answer 5) Yes.

(AB 6) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 2$?

(Answer 6) No.

(AB 7) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + \arctan(t)\,y = e^t$, $y(3) = 7$.
We expect a unique solution.

(b) $\frac{1}{1+t^2}\frac{dy}{dt} - t^5 y = \cos(6t)$, $y(2) = -1$, $y'(2) = 3$.
We do not expect any solutions.

(c) $\frac{d^2y}{dt^2} - 5\sin(t)\,y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.
We do not expect any solutions.

(d) $e^t\frac{d^2y}{dt^2} + 3(t-4)\frac{dy}{dt} + 4t^6 y = 2$, $y(3) = 1$, $y'(3) = -1$.
We expect a unique solution.

(e) $\frac{d^2y}{dt^2} + \cos(t)\frac{dy}{dt} + 3\ln(1+t^2)\,y = 0$, $y(2) = 3$.
We expect an infinite family of solutions.

(f) $\frac{d^3y}{dt^3} - t^7\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\,y = t^3$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$, $y'''(1) = 3$.
We do not expect any solutions.

(g) $(1+t^2)\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = t^3$, $y(3) = 9$, $y'(3) = 7$, $y''(3) = 5$.
We expect a unique solution.

(h) $\frac{d^3y}{dt^3} - e^t\frac{d^2y}{dt^2} + e^{2t}\frac{dy}{dt} + e^{3t}y = e^{4t}$, $y(-1) = 1$, $y'(-1) = 3$.
We expect an infinite family of solutions.

(i) $(2+\sin t)\frac{d^3y}{dt^3} + \cos t\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)\,y = t^3$, $y(7) = 2$.
We expect an infinite family of solutions.

(AB 8) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + e^t y = \cos t$, $y(0) = 3$, $y'(0) = -2$.
We expect a unique solution.

(b) $t^2 \frac{d^2y}{dt^2} + 3t\frac{dy}{dt} + 6y = 7t^3$, $y(2) = 4$, $y'(2) = 4$, $y''(2) = 2$.
We expect a unique solution.

(c) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0$, $y(4) = 3$, $y'(4) = -2$, $y''(4) = 0$, $y'''(4) = 3$.
We expect a unique solution.

(AB 9) A lake initially has a population of 600 trout. The birth rate of trout is proportional to the number of trout living in the lake. Fishermen are allowed to harvest 30 trout/year from the lake. Write an initial value problem for the fish population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 9) Independent variable: t = time (in years).
Dependent variable: P = Number of trout in the lake
Initial condition: $P(0) = 600$.
Parameters: α = birth rate (in 1/years).
Differential equation: $\frac{dP}{dt} = \alpha P - 30$.

(AB 10) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write an initial value problem for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 10) Independent variable: t = time (in years).
Dependent variable: P = Number of birds on the island.
Parameters: α = birth rate parameter (in 1/years); β = death rate parameter (in 1/(bird·years)).
Initial condition: $P(0) = 5$.
Differential equation: $\frac{dP}{dt} = \alpha P - \beta P^2$.

(AB 11) A tank contains nitrogen dioxide. Initially, there are 200 grams of NO_2. The gas decays (converts to molecular oxygen and nitrogen) at a rate proportional to the square of the amount of NO_2 remaining. Write an initial value problem for the amount of NO_2 in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 11) Independent variable: t = time (in minutes).
Dependent variable: M = Amount of NO_2 in the tank (in grams).
Parameters: α = reaction rate parameter (in 1/second·grams).
Initial condition: $M(0) = 200$.
Differential equation: $\frac{dM}{dt} = -\alpha M^2$.

(AB 12) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen. Write an initial value problem for the amount of hydrogen in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 12) Independent variable: t = time (in minutes).

Dependent variable: H = Amount of hydrogen in the tank (in grams).

Parameters: α = reaction rate parameter (in 1/second·grams).

Initial condition: $H(0) = 50$.

Differential equation: $\frac{dH}{dt} = -\alpha H(126.9H - 3345)$.

(AB 13) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C. Write an initial value problem for the temperature of the cup of coffee. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 13) Independent variable: t = time (in seconds).

Dependent variable: T = Temperature of the cup (in degrees Celsius)

Initial condition: $T(0) = 95$.

Differential equation: $\frac{dT}{dt} = -\alpha(T - 25)$, where α is a positive parameter (constant of proportionality) with units of 1/seconds.

(AB 14) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Write an initial value problem for the amount of salt in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 14) Independent variable: t = time (in minutes).

Dependent variable: Q = amount of dissolved salt (in kilograms).

Initial condition: $Q(0) = 3$.

Differential equation: $\frac{dQ}{dt} = 0.01 - \frac{3Q}{600-t}$.

(AB 15) I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write an initial value problem for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 15) Independent variable: t = time (in years).

Dependent variable: B = balance of my loan (in dollars).

Initial condition: $B(0) = 300{,}000$.

Differential equation: $\frac{dB}{dt} = 0.04B - 12 \cdot 1600$

(AB 16) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write an initial value problem for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 16) Independent variable: t = time (in minutes).

Dependent variables:

h = depth of water in the hole (in centimeters)

V = volume of water in the hole (in cubic centimeters); notice that $V = \frac{1}{3}\pi(5h)^2 h$

Initial condition: $V(0) = 0$.

Differential equation: $\frac{dV}{dt} = 1 - \alpha\pi(5h)^2 = 1 - 25\alpha\pi(3V/25\pi)^{2/3}$, where α is a proportionality constant with units of cm/s.

(AB 17) According to the Stefan-Boltzmann law, a black body in a dark vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 17) Independent variable: t = time (in seconds).
Dependent variable: T = object's temperature (in kelvins)
Parameter: σ = proportionality constant (in 1/(seconds·kelvin^3))
Initial condition: $T(0) = 400$.
Differential equation: $\frac{dT}{dt} = -\sigma T^4$.

(AB 18) According to the Stefan-Boltzmann law, a black body in a vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). If there is some ambient light in the vacuum, then the black body absorbs energy at a rate proportional to the fourth power of the effective temperature of the light. The proportionality constant is the same in both cases.

Suppose that a planet in space is currently at a temperature of 400 K. The effective temperature of its surroundings is 3 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 18) Independent variable: t = time (in seconds).
Dependent variable: T = object's temperature (in kelvins)
Parameter: σ = proportionality constant (in 1/(seconds·kelvin^3))
Initial condition: $T(0) = 400$.
Differential equation: $\frac{dT}{dt} = -\sigma T^4 + 81\sigma$.

(AB 19) A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

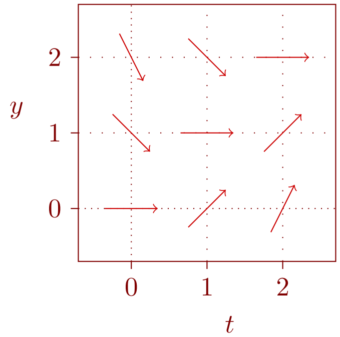
(Answer 19) Independent variable: t = time (in seconds).
Dependent variable: v = velocity of the ball (in meters/second, where a positive velocity denotes upward motion).
Parameters: α = proportionality constant of the drag force (in newton·seconds/meter)
m = mass of the ball (in kilograms)
Initial condition: $v(0) = 10$.
Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/m)v$.

(AB 20) A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 20) Independent variable: t = time (in seconds).
Dependent variable: v = velocity of the ball (in meters/second, where a positive velocity denotes upward motion).
Parameter: α = proportionality constant of the drag force (in newton·seconds/meter).
Initial condition: $v(0) = 20$.
Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/0.3)v|v|$, or $\frac{dv}{dt} = \begin{cases} -9.8 - (\alpha/0.3)v^2, & v \geq 0, \\ -9.8 + (\alpha/0.3)v^2, & v < 0. \end{cases}$

(AB 21) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.

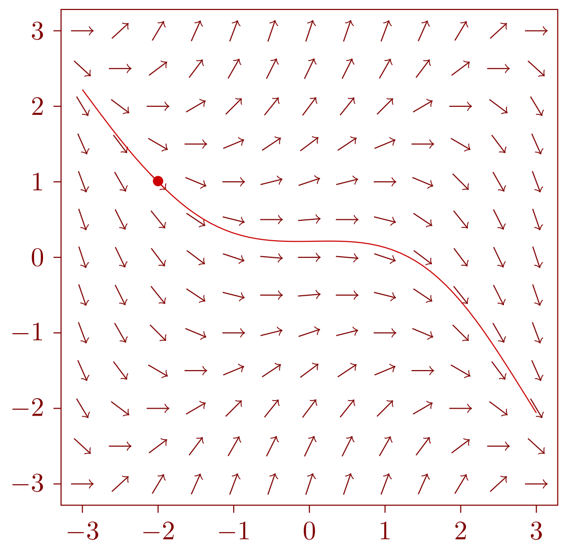
(Answer 21) Here is the direction field for the differential equation $\frac{dy}{dt} = t - y$.



(AB 22) Consider the differential equation $\frac{dy}{dx} = (y^2 - x^2)/3$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-2) = 1.$$

(Answer 22)

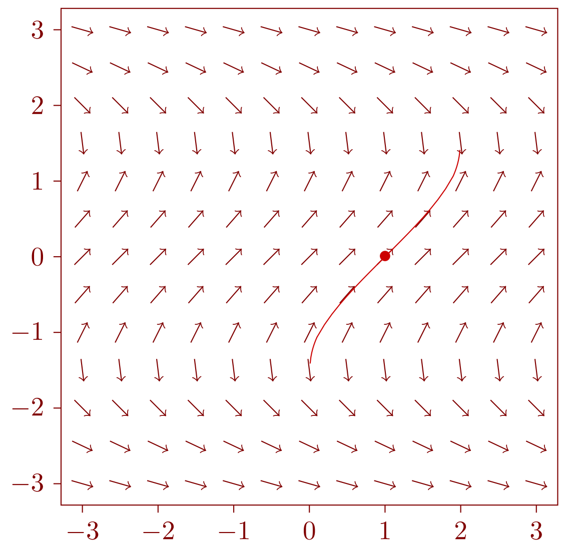


(AB 23) Consider the differential equation $\frac{dy}{dx} = \frac{2}{2-y^2}$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{2}{2 - y^2}, \quad y(1) = 0.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?

(Answer 23)



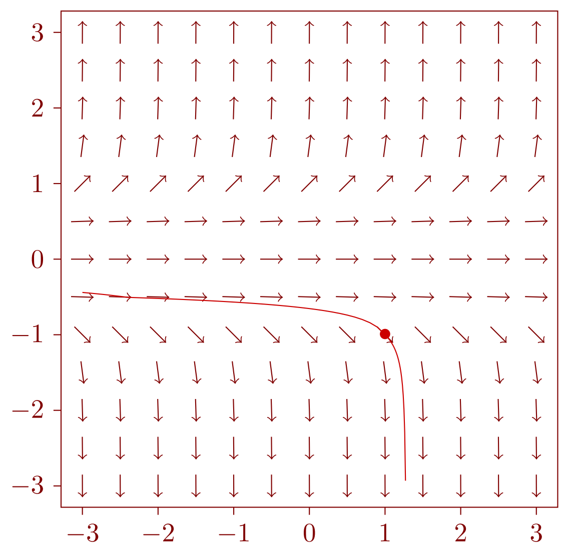
The domain of definition of the solution appears to be $0 < t < 2$.

(AB 24) Consider the differential equation $\frac{dy}{dx} = y^5$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = y^5, \quad y(1) = -1.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?

(Answer 24)



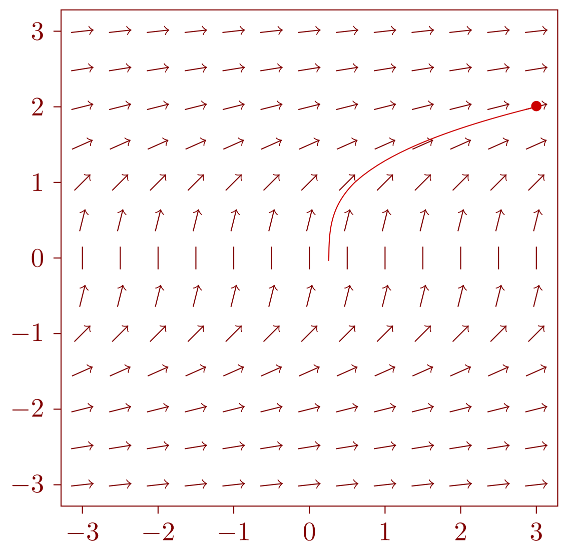
The domain of definition of the solution appears to be approximately $t < 1.3$.

(AB 25) Consider the differential equation $\frac{dy}{dx} = 1/y^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{1}{y^2}, \quad y(3) = 2.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?

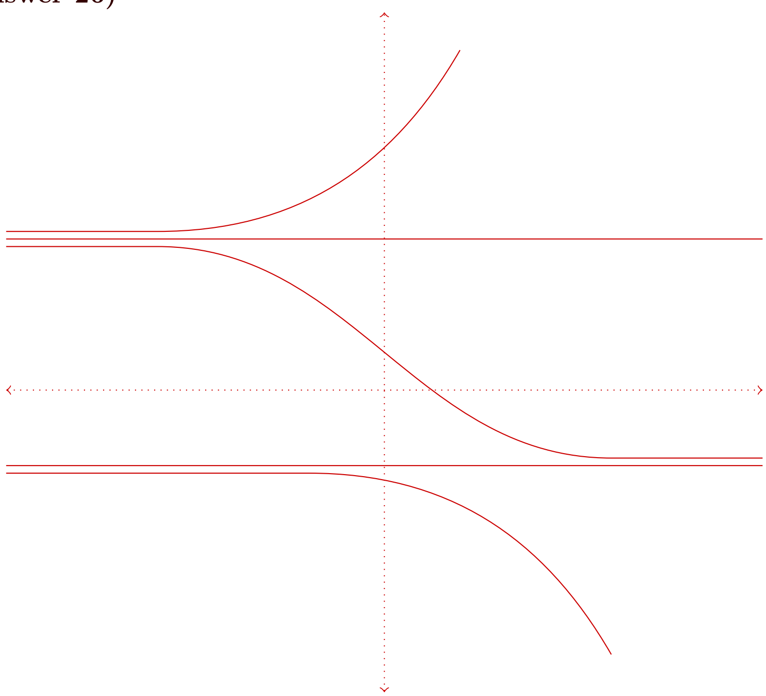
(Answer 25)



The domain of definition of the solution appears to be $\frac{1}{4} < t$.

(AB 26) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y-2)(y+1)^2$. By hand, sketch some typical solutions.

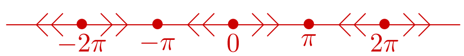
(Answer 26)



(AB 27) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(Answer 27)

Critical points: $y = k\pi$ for any integer k.

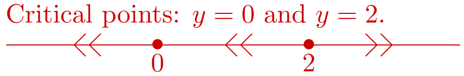


If k is even then $y = k\pi$ is unstable.
If k is odd then $y = k\pi$ is stable.

(AB 28) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = y^2(y-2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(Answer 28)

Critical points: $y = 0$ and $y = 2$.



$y = 0$ is semistable. $y = 2$ is unstable.

(AB 29) Trout live in a lake. Each trout produces, on average, one baby trout every two years. Fishermen are allowed to harvest 100 trout/year from the lake. You may neglect other reasons why trout die.

(*a*) Formulate a differential equation for the number of trout in the lake.
$\frac{dP}{dt} = \frac{1}{2}P - 100$, where t denotes time in years and P denotes the number of trout in the lake.

(*b*) Find the critical points of this differential equation and classify them as to stability.
The critical point is $P = 200$. It is unstable.

(*c*) What is the real-world meaning of the critical points?
If there are initially fewer than 200 trout in the lake, then eventually the trout will go extinct. If there are initially more than 200 trout in the lake, the trout population will grow without limit (or at least, until the present model stops being applicable).

(AB 30) I owe my bank a debt of B dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

(a) Formulate a differential equation for the amount of money I owe.
$\frac{dB}{dt} = 0.05B - 19200$, where t denotes time in years.

(b) Find the critical points of this differential equation and classify them as to stability.
The critical point is $B = \$384{,}000$. It is unstable.

(c) What is the real-world meaning of the critical points?
If my initial debt is less than \$384,000, then I will eventually pay it off (have a debt of zero dollars), but if my initial debt is greater than \$384,000, my debt will grow exponentially. The critical point $B = 384{,}000$ corresponds to the balance that will allow me to make interest-only payments on my debt.

(AB 31) A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

(a) Write the differential equation for the amount of salt in the tank.
$\frac{dQ}{dt} = 10 - Q/300$, where Q denotes the amount of salt in grams and t denotes time in minutes.

(b) Find the critical points of this differential equation and classify them as to stability.
The critical point is $Q = 3000$. It is stable.

(c) What is the real-world meaning of the critical points?
No matter the initial amount of salt in the tank, as time passes the amount of salt will approach 3000 g or 3 kg.

(AB 32) A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where v is her velocity in meters per second.

(a) Write a differential equation for her velocity. Assume her velocity is always downwards.
If $v \leq 0$ then $70\frac{dv}{dt} = -70 * 9.8 + 2v^2$. (If $v > 0$ then $70\frac{dv}{dt} = -70 * 9.8 - 2v^2$.)

(b) Find the critical points of this differential equation and classify them as to stability. Be sure to include units.
$v = -\sqrt{343}$ meters/second is a stable critical point.

(c) What is the real-world meaning of these critical points?
As $t \to \infty$, her velocity will approach the stable critical point of $v = -\sqrt{343}$ meters/second.

(AB 33) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen.

(a) Write a differential equation for the amount of hydrogen left in the tank.
$\frac{dH}{dt} = -\alpha H(126.9H - 3345)$, where H is the amount of hydrogen remaining (in grams), t denotes time in minutes, and α is a positive parameter.

(b) Find the critical points of this differential equation and classify them as to stability. Be sure to include units.
$H = 0$ grams (unstable) and $H = 26.36$ grams (stable).

(c) What is the real-world meaning of these critical points?
As $t \to \infty$, the amount of hydrogen in the tank will tend towards 26.36 grams. The $H = 0$ critical point is not physically meaningful, as if $H = 0$ grams then the model predicts -3345 grams of iodine in the tank.

(AB 34) Young oak trees grow in a large field with area 1 square kilometer. Initially, there are 30 trees in the field. Each tree shades 20 square meters. The trees produce acorns. Squirrels scatter the acorns across the field at random. Squirrels eat most of the acorns; on average, each tree produces two acorns per year that are not eaten by squirrels. Every acorn not eaten by squirrels and planted in sunlight sprouts.

(a) Write a differential equation for the number of trees in the field.
$\frac{dP}{dt} = 2P(1 - 20P/1{,}000{,}000)$, where P is the number of trees in the field and t denotes time in years.

(b) Find the critical points of this differential equation and classify them as to stability.
$P = 0$ (unstable) and $P = 50{,}000$ trees (stable).

(c) What is the real-world meaning of these critical points?
If there are initially no trees in the field, then the number of trees will remain at the $P = 0$ equilibrium. However, if there are initially any trees in the field, then as $t \to \infty$, the number of trees will approach 50,000.

(AB 35) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate of 3 cm^3 per exposed cm2 per hour.

(a) Write a differential equation for the amount of water in the tank.
$\frac{dV}{dt} = 1 - 75\pi(3V/25\pi)^{2/3}$, where t denotes time in hours and V is the volume of water in the hole in cm^3.

(b) Find the critical points of this differential equation and classify them as to stability.
$V = (25/3)(1/75\pi)^{3/2}$. The critical point is stable.

(c) What is the real-world meaning of these critical points?
As $t \to \infty$, the amount of water in the hole will approach $(25/3)(1/75\pi)^{3/2}$ cubic centimeters.

(AB 36) For each of the following differential equations, determine whether it is linear, separable, exact, homogeneous, a Bernoulli equation, or of the form $\frac{dy}{dt} = f(At+By+C)$. Then solve the differential equation.

(a) $t + \cos t + (y - \sin y)\frac{dy}{dt} =$
$t+\cos t+(y-\sin y)\frac{dy}{dt} =$ is separable (and also exact) and has solution $\frac{1}{2}y^2+\cos y = -\frac{1}{2}t^2 - \sin t + C$.

(b) $\ln y + y + x + \left(\frac{x}{y} + x\right)\frac{dy}{dx} = 0$
$\ln y + y + x + \left(\frac{x}{y} + x\right)\frac{dy}{dx} = 0$ is exact and has solution $x\ln y + xy + \frac{1}{2}x^2 = C$.

(c) $1 + t^2 - ty + (t^2+1)\frac{dy}{dt} = 0$
$1 + t^2 - ty + (t^2+1)\frac{dy}{dt} = 0$ is linear and has solution $y = -\sqrt{t^2+1}\ln(t+\sqrt{t^2+1}) + C\sqrt{t^2+1}$.

(d) $ty - y^2 - t^2 + t^2\frac{dy}{dt} = 0$
$ty - y^2 - t^2 + t^2\frac{dy}{dt} = 0$ is homogeneous and has solution $y = \frac{t}{C-\ln|t|} + t$.

(e) $4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5)\frac{dy}{dt} = 0$
$4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5)\frac{dy}{dt} = 0$ is exact and has solution $2t^2y^3 + \frac{3}{2}t^4 + 3y + \frac{1}{6}y^6 = C$.

(f) $y\tan 2x + y^3\cos 2x + \frac{dy}{dx} = 0$
$y\tan 2x + y^3\cos 2x + \frac{dy}{dx} = 0$ is Bernoulli. Let $v = y^{-2}$. Then $\frac{dv}{dx} = 2v\tan 2x + 2\cos 2x$, so $v = \frac{1}{2}\sin(2x) + x\sec(2x) + C\sec(2x)$ and $y = \frac{1}{\sqrt{\frac{1}{2}\sin(2x)+x\sec(2x)+C\sec(2x)}}$.

(g) $3t - 5x + (t+x)\frac{dx}{dt} = 0$
$3t - 5x + (t+x)\frac{dx}{dt} = 0$ is homogeneous and has solution $(x-3t)^2 = C(x-t)$.

(h) $\frac{dy}{dt} = \frac{1}{3t+2y+7}$.
If $\frac{dy}{dt} = \frac{1}{3t+2y+7}$, then $\frac{3t+2y+7}{3} - \frac{2}{9}\ln|3t+2y+7+2/3| = \ln|t| + C$.

(i) $y^3\cos(2t) + \frac{dy}{dt} = 0$
$y^3\cos(2t) + \frac{dy}{dt} = 0$ is separable and has solution $y = \pm\frac{1}{\sqrt{\sin(2t)+C}}$ or $y = 0$.

(j) $4ty\frac{dy}{dt} = 3y^2 - 2t^2$
$4ty\frac{dy}{dt} = 3y^2 - 2t^2$ is homogeneous (and also Bernoulli) and has solution $2\ln(y^2/t^2+2) = -\ln|t| + C$.

(k) $t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$
$t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$ is a Bernoulli equation. Let $v = y^6$. Then $t\frac{dv}{dt} = 18v - 6t^2$, so $v = Ct^{18} + \frac{3}{8}t^2$ and $y = \sqrt[6]{Ct^{18} + \frac{3}{8}t^2}$.

(l) $\sin^2(x-t)\frac{dx}{dt} = \csc^2(x-t)$.
If $\frac{dx}{dt} = \csc^2(x-t)$, then $\tan(x-t) - x = C$.

(m) $\frac{dy}{dt} = 8y - y^8$
$\frac{dy}{dt} = 8y - y^8$ is Bernoulli (and also separable, but separating variables results in an impossible integral). Make the substitution $v = y^{-7}$. Then $\frac{dv}{dt} = -56v + 7$, so $v = Ce^{-56t} + \frac{1}{8}$ and $y = \frac{1}{\sqrt[7]{Ce^{-56t}+1/8}}$.

(n) $t\frac{dz}{dt} = -\cos t - 3z$
$t\frac{dz}{dt} = -\cos t - 3z$ is linear and has solution $z = \frac{1}{t}\sin t + \frac{2}{t^2}\cos t - \frac{2}{t^3}\sin t + \frac{C}{t^3}$.

(o) $\frac{dy}{dt} = \cot(y/t) + y/t$
$\frac{dy}{dt} = \cot(y/t) + y/t$ is homogeneous and has solution $\sec(y/t) = Ct$.

(p) $\frac{dy}{dt} = ty + t^2\sqrt[3]{y}$
$\frac{dy}{dt} = ty + t^2\sqrt[3]{y}$ is Bernoulli. Make the substitution $v = y^{2/3}$. Then $v = -t - \frac{3}{2} + Ce^{t^2/3}$ and so $y = \sqrt{(-t - \frac{3}{2} + Ce^{t^2/3})^3}$.

(AB 37) (14 points) For each of the following differential equations, find all the equilibrium solutions or state that no such solutions exist. You do not need to find the nonequilibrium solutions.

(a) $\frac{dy}{dt} = y^3 - yt$
$y = 0$.

(b) $\frac{dy}{dt} = t^2 e^y$
There are no equilibrium solutions.

(c) $\frac{dy}{dt} = ty + t^3$
There are no equilibrium solutions.

(d) $\frac{dy}{dt} = (y^2 + 3y + 2)\sin(t)$
$y = -1$ and $y = -2$.

(e) $\frac{dy}{dt} = \ln(y^t)$
$y = 1$.

(AB 38) For each of the following differential equations, determine whether it is linear, separable, exact, homogeneous, Bernoulli, or a function of a linear term. Then solve the given initial-value problem.

(a) $\cos(t + y^3) + 2t + 3y^2\cos(t + y^3)\frac{dy}{dt} = 0$, $y(\pi/2) = 0$
If $\cos(t + y^3) + 2t + 3y^2\cos(t + y^3)\frac{dy}{dt} = 0$, $y(\pi/2) = 0$, then $\sin(t + y^3) + t^2 = 1 + \pi^2/4$.

(b) $\frac{dy}{dt} = \sin t \cos y$, $y(\pi) = \pi/2$
If $\frac{dy}{dt} = \sin t \cos y$, $y(\pi) = \pi/2$, then $y = \pi/2$ for all t.

(c) $ty^2 - 4t^3 + 2t^2 y\frac{dy}{dt} = 0$, $y(1) = 3$.
If $ty^2 - 4t^3 + 2t^2 y\frac{dy}{dt} = 0$, $y(1) = 3$, then $y = \sqrt{\frac{4}{3}t^2 + \frac{23}{3t}}$.

(d) $1 + y^2 + t\frac{dy}{dt} = 0$, $y(1) = 1$
If $1 + y^2 + t\frac{dy}{dt} = 0$, $y(1) = 1$, then $y = \tan(\pi/4 - \ln t)$.

(e) $\frac{dy}{dt} = (2y + 2t - 5)^2$, $y(0) = 3$.
If $\frac{dy}{dt} = (2y + 2t - 5)^2$, $y(0) = 3$, then $y = \frac{1}{2}\tan(2t + \pi/4) + \frac{5}{2} - t$.

(f) $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$.
If $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$, then $y = \sqrt[3]{340e^{6t} + 3}$.

(g) $3y + e^{-3t}\sin t + \frac{dy}{dt} = 0$, $y(0) = 2$
If $3y + e^{-3t}\sin t + \frac{dy}{dt} = 0$, $y(0) = 2$, then $y = e^{-3t}\cos t + e^{-3t}$.

(AB 39) Solve the initial-value problem $\frac{dy}{dt} = \frac{t-5}{y}$, $y(0) = 3$ and determine the range of t-values in which the solution is valid.

(Answer 39) $y = \sqrt{(t-5)^2 - 16} = \sqrt{(t-1)(t-9)} = \sqrt{t^2 - 10t + 9}$. The solution is valid for all $t < 1$.

(AB 40) Solve the initial-value problem $\frac{dy}{dt} = y^2$, $y(0) = 1/4$ and determine the range of t-values in which the solution is valid.

(Answer 40) $y = \frac{1}{4-t}$. The solution is valid for all $t < 4$.

(AB 41) Consider the differential equation $\ln(3/4+t^2)\frac{dy}{dt} = \sin(y)$.

(a) Find all pairs of numbers T_0 and Y_0 such that the initial value problem $\ln(3/4+t^2)\frac{dy}{dt} = \sin(y)$, $y(T_0) = Y_0$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 1.2.1.)

(T_0, Y_0) fails to satisfy the conditions if $T_0 = \pm 1/2$.

(b) Find a pair of numbers T_0 and Y_0 such that the initial value problem $\ln(3/4+t^2)\frac{dy}{dt} = \sin(y)$, $y(T_0) = Y_0$ has no solutions.

Here are four possible answers:

- $(T_0, Y_0) = (1/2, 3\pi/2)$,
- $(T_0, Y_0) = (1/2, 0)$,
- $(T_0, Y_0) = (-1/2, \pi/2)$,
- $(T_0, Y_0) = (-1/2, 8)$.

In general, $T_0 = \pm 1/2$, and Y_0 may be any real number you like except for 0, $\pm\pi$, $\pm 2\pi \ldots$

(AB 42) Consider the differential equation $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-1)y = 0$.

(a) Find all triples of numbers T_0, Y_0, and Y_1 such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-1)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 4.1.1.)

(T_0, Y_0, Y_1) fails to satisfy the conditions if $T_0 = 3$.

(b) Find a triple of numbers T_0, Y_0, and Y_1 such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-1)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ has no solutions.

Here are four possible answers:

- $(T_0, Y_0, Y_1) = (3, 2, 0)$,
- $(T_0, Y_0, Y_1) = (3, 0, 1)$,
- $(T_0, Y_0, Y_1) = (3, -2, 2)$,
- $(T_0, Y_0, Y_1) = (3, 4, -5)$.

In general, $T_0 = 3$. Y_0 and Y_1 may be any numbers you like as long as $Y_1 + 2Y_0 \neq 0$.

(AB 43) Consider the differential equation $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-3)y = 0$.

(a) Find all triples of numbers T_0, Y_0, and Y_1 such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-3)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 4.1.1.)

(T_0, Y_0, Y_1) fails to satisfy the conditions if $T_0 = 3$.

(b) Find a triple of numbers T_0, Y_0, and Y_1 such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-3)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ has no solutions.

Here are four possible answers:

- $(T_0, Y_0, Y_1) = (3, 2, 5)$,
- $(T_0, Y_0, Y_1) = (3, 0, 1)$,
- $(T_0, Y_0, Y_1) = (3, -2, 2)$,
- $(T_0, Y_0, Y_1) = (3, 0, -5)$.

In general, $T_0 = 3$. Y_0 and Y_1 may be any numbers you like as long as $Y_1 \neq 0$.

(AB 44) Consider the differential equation $\frac{dy}{dt} = (y-5)^{3/8}$.

(a) Find all pairs of numbers T_0 and Y_0 such that the initial value problem $\frac{dy}{dt} = (y-5)^{3/8}$, $y(T_0) = Y_0$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 1.2.1.)

(T_0, Y_0) fails to satisfy the conditions if $Y_0 = 5$.

(b) Find two solutions to the initial value problem $\frac{dy}{dt} = (y-5)^{3/8}$, $y(16) = 5$. Where are your solutions equal?

$y(t) = 5$ and $y(t) = 5 + (\frac{5}{8}t - 10)^{8/5}$. The solutions are equal only at $t = 16$.

(c) Find two solutions to the initial value problem $\frac{dy}{dt} = (y-5)^{3/8}$, $y(16) = 6$. Where are your solutions equal?

$y(t) = 5 + (\frac{5}{8}t - 9)^{8/5}$ and $y(t) = \begin{cases} 5, & t \leq 14.4, \\ 5 + (\frac{5}{8}t - 9)^{8/5}, & t \geq 14.4. \end{cases}$ The solutions are equal for all $t \geq 14.4$.

(AB 45) Consider the differential equation $\frac{dy}{dt} = 3y(\ln y)^{2/3}$.

(a) Find all pairs of numbers T_0 and Y_0 such that the initial value problem $\frac{dy}{dt} = 3y(\ln y)^{2/3}$, $y(T_0) = Y_0$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 1.2.1.)

(T_0, Y_0) fails to satisfy the conditions if $Y_0 = 1$.

(b) Find two solutions to the initial value problem

$$\frac{dy}{dt} = 3y(\ln y)^{2/3}, \quad y(7) = 1.$$

Where are your solutions equal?

$y(t) = 1$ and $y(t) = e^{(t-7)^3}$. The solutions are equal only at $t = 7$.

(c) Find two solutions to the initial value problem

$$\frac{dy}{dt} = 3y(\ln y)^{2/3}, \quad y(7) = e.$$

Where are your solutions equal?

$y_1 = e^{(t-6)^3}$ and $y_2 = \begin{cases} e^{(t-6)^3}, & t \geq 6, \\ 1, & t \leq 6. \end{cases}$ The two solutions are equal on the interval $[6, \infty)$.

(AB 46)

(a) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0, \quad y(0) = 2, \quad y'(0) = 7$$

is guaranteed a unique solution?
$(-\infty, 3/2)$.

(b) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0, \quad y(2) = 2, \quad y'(2) = 7$$

is guaranteed a unique solution?
$(3/2, 3)$.

(c) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0, \quad y(5) = 2, \quad y'(5) = 7$$

is guaranteed a unique solution?
$(3, \infty)$.

(d) How many solutions are there to the initial value problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0, \quad y(3/2) = 2, \quad y'(3/2) = 7?$$

This initial value problem has no solutions.

(AB 47) You are given that $g(t)$ and $q(t)$ are continuous functions. Which of the following functions are possible solutions to the initial value problem $(t-5)^3\frac{d^2y}{dt^2} - (3t-15)^2\frac{dy}{dt} + q(t)y = g(t)$, $y(0) = -5$, $y'(0) = -1$? If a function cannot be a solution, explain why not.

(a) $y(t) = \frac{25}{t-5}$
This is a possible solution.

(b) $y(t) = \frac{5}{t-1} + 4t$
This is not a possible solution; the solution is guaranteed to exist (and be differentiable) for all $t < 5$, including $t = 1$.

(AB 48) You are given that $y_1 = t^3 + 4t^2 + 4t$ and $y_2 = t^2 + 4t + 4$ are both solutions to the differential equation $(t+2)^2\frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$ for all t.

What is the longest interval containing the number $t = -3$ on which the general solution to $(t+2)^2\frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$ may be written $y = C_1(t^3 + 4t^2 + 4t) + C_2(t^2 + 4t + 4)$?

(Answer 48) $(-\infty, -2)$.

(AB 49) Find a solution to the initial value problem
$(t+2)^2\frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$, $y(0) = 4$, $y'(0) = 4$ that is not $y(t) = t^2 + 4t + 4$.

(Answer 49) There are many possible answers. They include:

$$y(t) = \begin{cases} t^3 + 4t^2 + 4t, & t \le -2, \\ t^2 + 4t + 4, & t \ge -2, \end{cases} \qquad y(t) = \begin{cases} -2t^2 - 8t - 8, & t \le -2, \\ t^2 + 4t + 4, & t \ge -2, \end{cases} \qquad y(t) = \begin{cases} 0, & t \le -2, \\ t^2 + 4t + 4, & t \ge -2. \end{cases}$$

(AB 50) Suppose $y_1(t)$ and $y_2(t)$ are both solutions to the initial value problem $(t+2)^2\frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$, $y(5) = 3$, $y'(5) = 7$ on $(-\infty, \infty)$. What is the longest interval on which you are guaranteed that $y_1 = y_2$?

(Answer 50) $(-2, \infty)$.

(AB 51) The function $y_1(t) = e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2} - (1+2t)\frac{dy}{dt} + (t+1)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

(Answer 51) $y(t) = C_1\, e^t + C_2\, t^2 e^t$.

(AB 52) The function $y_1(t) = t$ is a solution to the differential equation $t^2\frac{d^2y}{dt^2} - (t^2+2t)\frac{dy}{dt} + (t+2)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

(Answer 52) $y(t) = C_1\, t + C_2\, te^t$.

(AB 53) The function $y_1(x) = x^3$ is a solution to the differential equation $x^2\frac{d^2y}{dx^2} + (x^2\tan x - 6x)\frac{dy}{dx} + (12 - 3\,x\tan x)y = 0$. Find the general solution to this differential equation on the interval $0 < x < \pi/2$.

(Answer 53) $y(x) = C_1\, x^3 + C_2\, x^3 \sin x$.

(AB 54) Find the general solution to the following differential equations.

(a) $\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 85x = 0$.
If $\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 85x = 0$, then $x = C_1e^{-6t}\cos(7t) + C_2e^{-6t}\sin(7t)$.

(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.
If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$, then $y = C_1e^{(-2+\sqrt{2})t} + C_2e^{(-2-\sqrt{2})t}$.

(c) $\frac{d^4z}{dt^4} + 7\frac{d^2z}{dt^2} - 144z = 0$.
If $\frac{d^4z}{dt^4} + 7\frac{d^2z}{dt^2} - 144z = 0$, then $z = C_1e^{3t} + C_2e^{-3t} + C_3\cos 4t + C_4\sin 4t$.

(d) $\frac{d^4w}{dt^4} - 8\frac{d^2w}{dt^2} + 16w = 0$.
If $\frac{d^4w}{dt^4} - 8\frac{d^2w}{dt^2} + 16w = 0$, then $w = C_1e^{2t} + C_2te^{2t} + C_3e^{-2t} + C_4te^{-2t}$.

(AB 55) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $9\frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 2v = 0$, $v(0) = 3$, $v'(0) = 2$.
If $9\frac{d^2v}{dt^2} + 6\frac{dv}{dt} + 2v = 0$, $v(0) = 3$, $v'(0) = 2$, then $v = 3e^{-t/3}\cos(t/3) + 9e^{-t/3}\sin(t/3)$.

(b) $\frac{d^2u}{dt^2} + 10\frac{du}{dt} + 25u = 0$, $u(0) = 1$, $u'(0) = 4$.
If $\frac{d^2u}{dt^2} + 10\frac{du}{dt} + 25u = 0$, $u(0) = 1$, $u'(0) = 4$, then $u = e^{-5t} + 9te^{-5t}$.

(AB 56) Are the functions $f_1(x) = 0$, $f_2(x) = x$, $f_3(x) = x^2$ linearly independent or linearly dependent on the interval $(-\infty, \infty)$? If they are linearly dependent, find constants c_1, c_2, and c_3, not all zero, such that $c_1f_1(x) + c_2f_2(x) + c_3f_3(x) = 0$ for all $x \in (-\infty, \infty)$.

(Answer 56) They are linearly dependent. $c_1 = 1$ (or any other nonzero number), $c_2 = 0$, $c_3 = 0$.

(AB 57) Are the functions $f_1(x) = 3$, $f_2(x) = \cos^2 x$, $f_3(x) = \sin^2 x$ linearly independent or linearly dependent on the interval $(-\infty, \infty)$? If they are linearly dependent, find constants c_1, c_2, and c_3, not all zero, such that $c_1f_1(x) + c_2f_2(x) + c_3f_3(x) = 0$ for all $x \in (-\infty, \infty)$.

(Answer 57) They are linearly dependent. $c_1 = 1$, $c_2 = -3$, $c_3 = -3$ is a valid answer (although not the only possible answer).

(AB 58) Are the functions $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$ linearly independent or linearly dependent on the interval $(-\infty, \infty)$? If they are linearly dependent, find constants c_1, c_2, and c_3, not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all $x \in (-\infty, \infty)$.

(Answer 58) They are linearly independent.

(AB 59) An object with mass 5 kg stretches a spring 4 cm. It is attached to a viscous damper with damping constant 16 newton $\cdot$ seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Write the differential equation and initial conditions that describe the position of the object. You may use 9.8 meters/second^2 for the acceleration of gravity.

(Answer 59) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$5\frac{d^2x}{dt^2} + 16\frac{dx}{dt} + 1225x = 0, \qquad x(0) = -0.04, \quad x'(0) = 3.$$

(AB 60) A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant β. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20$ N $\cdot$ s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30$ N $\cdot$ s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(Answer 60) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$2\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.1, \quad x'(0) = 0.$$

If $\beta = 20$ N $\cdot$ s/m, then the system is underdamped, and we do expect to see decaying oscillations.

If $\beta = 30$ N $\cdot$ s/m, then the system overdamped, and we do not expect to see decaying oscillations.

(AB 61) A 3-kg object is attached to a spring with constant k and to a viscous damper with damping constant 42 N$\cdot$sec/m. The object is set in motion from its equilibrium position with initial velocity 5 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object.

If $k = 100$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $k = 200$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

(Answer 61) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$3\frac{d^2x}{dt^2} + 42\frac{dx}{dt} + kx = 0, \qquad x(0) = 0, \quad x'(0) = -5.$$

If $k = 100$ N/m, then the system is overdamped, and we do not expect to see decaying oscillations.

If $k = 200$ N/m, then the system underdamped, and we do expect to see decaying oscillations.

(AB 62) A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant β. The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of β for which the system is critically damped. Be sure to include units for β.

(Answer 62) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$4\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 70x = 0, \qquad x(0) = 0.05, \quad x'(0) = -3.$$

Critical damping occurs when $\beta = 4\sqrt{70}$ N · s/m.

(AB 63) An object of mass m is attached to a spring with constant 80 N/m and to a viscous damper with damping constant 20 N·s/m. The object is pulled down to 5cm below its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of m for which the system is critically damped. Be sure to include units for m.

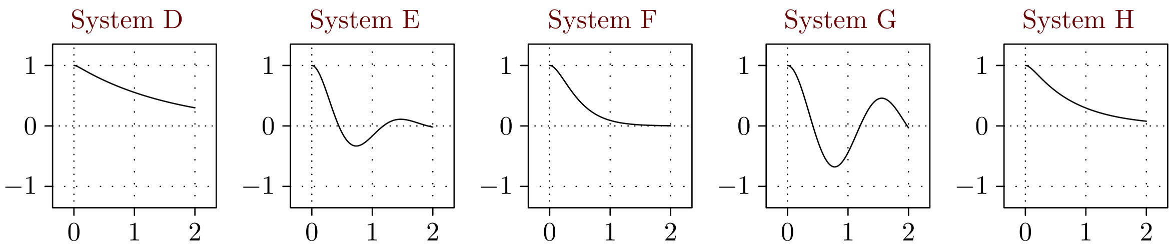
(Answer 63) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters). Then

$$m\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.05, \quad x'(0) = -3.$$

Critical damping occurs when $m = \frac{5}{4}$ kg.

(AB 64) Five objects, each with mass 3 kg, are attached to five springs, each with constant 48 N/m. Five dampers with unknown constants are attached to the objects. In each case, the object is pulled down to a distance 1 cm below the equilibrium position and released from rest. You are given that the system is critically damped in exactly one of the five cases.

Here are the graphs of the objects' positions with respect to time:



(*a*) For which damper is the system critically damped?
The system is critically damped for Damper F.

(*b*) For which dampers is the system overdamped?
The system overdamped for Dampers H and D.

(*c*) For which dampers is the system underdamped?
The system underdamped for Dampers E and G.

(*d*) Which damper has the highest damping constant? Which damper has the lowest damping constant?
Damper D has the highest damping constant. Damper G has the lowest damping constant.

(AB 65) An object with mass 5 kg stretches a spring 4 cm. It is attached to a viscous damper with damping constant 16 newton · seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Using only first derivatives, write the differential equations and initial conditions for the object's position. That is, write a first-order system involving the object's position. You may use 9.8 meters/second^2 for the acceleration of gravity.

(Answer 65) Let t denote time (in seconds), let x denote the object's displacement above equilibrium (in meters), and let v denote the object's velocity (in meters per second). Then

$$5\frac{dv}{dt} + 16v + 1225x = 0, \qquad \frac{dx}{dt} = v, \qquad x(0) = -0.04, \quad v(0) = 3.$$

(AB 66) A forest is inhabited by rabbits and foxes. Every year, each rabbit gives birth to 3 babies and each fox gives birth to 5 babies on average. Foxes try to catch and eat rabbits; each month, each fox has a 0.02% chance of catching each of the rabbits present in the forest. Each month, every fox that does not eat a rabbit starves to death.

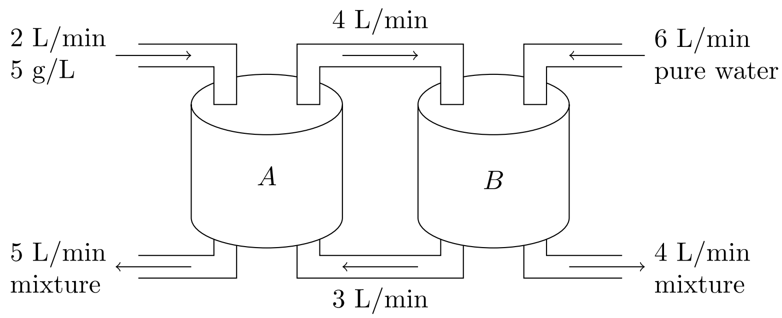
Write the differential equations for the number of foxes and rabbits in the forest.

(Answer 66) Let t denote time (in months), let R denote the number of rabbits, and let F denote the number of foxes.

Then

$$\frac{dR}{dt} = \frac{3}{12}R - 0.0002RF, \quad \frac{dF}{dt} = \frac{5}{12}F - (F - 0.0002RF).$$

(AB 67) Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.



Write the differential equations and initial conditions that describe the amount of salt in each tank.

(Answer 67) Let t denote time (in minutes).

Let x denote the amount of salt (in grams) in tank A.

Let y denote the amount of salt (in grams) in tank B.

Then $x(0) = 3000$ and $y(0) = 2000$.

If $t < 50$, then

$$\frac{dx}{dt} = 10 - \frac{9x}{200-4t} + \frac{3y}{300+3t}, \qquad \frac{dy}{dt} = \frac{4x}{200-4t} - \frac{7y}{300+3t}.$$

(AB 68) Write the system

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y$$

in matrix form.

(Answer 68)

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(AB 69) Write the system

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = 4y - x$$

in matrix form.

(Answer 69)

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(AB 70) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 70) If

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 4t - 5 \\ -2t + 3 \end{pmatrix}.$$

(AB 71) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 71) If

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} -3\cos t + 16\sin t \\ 2\cos t + 7\sin t \end{pmatrix}.$$

(AB 72) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 72) If

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-20t} \begin{pmatrix} 3/20 \\ 9/10 \end{pmatrix} + e^{10t} \begin{pmatrix} -3/20 \\ 1/10 \end{pmatrix}.$$

(AB 73) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 73) If

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 5\cos 12t + \sin 12t \\ 2\cos 12t + 2\sin 12t \end{pmatrix}.$$

(AB 74) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 74) If

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos 3t - 11\sin 3t \\ -3\cos 3t - (23/3)\sin 3t \end{pmatrix}.$$

(AB 75) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 75) If

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 15t + 1 \\ (15/4)t + 4 \end{pmatrix}.$$

(AB 76) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 76) If

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} + e^{-6t} \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix}.$$

(AB 77) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 4-m & 1 & -1 \\ 1 & 4-m & -1 \\ -1 & -1 & 4-m \end{pmatrix} = -(m-3)^2(m-6).$

(Answer 77) If

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{3t} \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} + e^{6t} \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}.$$

(AB 78) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, \qquad x(0) = 3, \quad y(0) = 2, \quad z(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 3-m & 1 & 0 \\ 2 & 3-m & -2 \\ 0 & 1 & 3-m \end{pmatrix} = -(m-3)^3.$

(Answer 78) If

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, \qquad x(0) = 3, \quad y(0) = 2, \quad z(0) = 1$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{3t} \begin{pmatrix} 2t^2 + 2t + 3 \\ 4t + 2 \\ 2t^2 + 2t + 1 \end{pmatrix}.$$

(AB 79) Find the solution to the initial-value problem

$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, x(0) = 4, \quad y(0) = 3, \quad z(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 1-m & 1 & 1 \\ 1 & 3-m & -1 \\ 0 & 2 & 2-m \end{pmatrix} = -(m-2)^3.$

(Answer 79) If

$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, \qquad x(0) = 4, \quad y(0) = 3, \quad z(0) = 5$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 4 + 4t + 2t^2 \\ 2t + 3 \\ 5 + 6t + 2t^2 \end{pmatrix}.$$

(AB 80) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.
Let t denote time (in days), let S denote the number of susceptible people (who have never had the disease), let I denote the number of infected people (who currently have the disease), and let R denote the number of recovered people (who now are resistant, that is, cannot get the disease again). Then
$\boxed{\frac{dS}{dt} = -\frac{1}{22500}SI,\ \frac{dI}{dt} = \frac{1}{22500}SI - 0.17I,\ \frac{dR}{dt} = 0.17I,\ S(0) = 8997,\ I(0) = 3,\ R(0) = 0.}$

(b) Use the phase plane method to find an equation relating the number of susceptible and infected people.
$\frac{dI}{dS} = \frac{3825}{S} - 1$, so $\boxed{I = 9000 - S - 3825 \ln \frac{8997}{S}.}$

(c) Use the phase plane method to find an equation relating the number of resistant and susceptible people.
$\frac{dR}{dS} = -\frac{3825}{S}$, so $\boxed{R = 3825 \ln \frac{8997}{S}.}$

(d) As $t \to \infty$, $I \to 0$ and the number of people who never contract the disease approaches a limit. Let S_∞ be the limiting number of people who never contract the disease. Find an (algebraic, not differential) equation involving S_∞.
$\boxed{9000 - S_\infty - 3825 \ln \frac{8997}{S_\infty} = 0.}$ On the exam you will not be expected to solve this equation.

(e) What is the maximum number of people that are infected at any one time?
When I is maximized, $0 = \frac{dI}{dt} = \frac{1}{22500}SI - 0.17I$, and so $S = 3825$. Thus, the maximum value of I is $\boxed{I = 9000 - 3825 - 3825 \ln \frac{62979}{33750}.}$

(AB 81) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(Answer 81) $\frac{dS}{dt} = -\frac{0.2}{5000}SI - 15$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I + 15$.

(AB 82) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

(*a*) Set up the system of differential equations that describes the number of susceptible, infected, and resistant people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

$\boxed{\frac{dS}{dt} = -\frac{0.2}{5000}SI - 15\frac{S}{5000},\ \frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I,\ \frac{dR}{dt} = 0.1I + 15\frac{S}{5000}.}$

(*b*) Assume that there are initially no recovered or vaccinated people and 7 infected people. Use the phase plane method to find an equation relating S and I.

We compute that $\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{0.2SI - 500I}{-0.2SI - 15S} = \frac{0.2S - 500}{S}\frac{I}{-0.2I - 15}$. This is a separable differential equation, which we solve to see that $-0.2I - 15\ln I = 0.2S - 500\ln S + C$. Applying the initial conditions $I(0) = 7$, $S(0) = 4993$, we see that $\boxed{-0.2(I-7) - 15\ln(I/7) = 0.2(S - 4993) - 500\ln(S/4993).}$

(*c*) Find an algebraic equation for the maximum number of people that are infected with the virus at any one time.

The maximum I value occurs when $0 = \frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, or $S = 2500$. The maximum I value then satisfies

$\boxed{-0.2(I-7) - 15\ln(I/7) = 0.2(2500 - 4993) - 500\ln(2500/4993).}$

(AB 83) A small town has a population of 16,000 people. It is expected that soon, one of them will be infected with a contagious disease.

Epidemiologists observe the town and expect each infected person to encounter 10 people per day, who are distributed among susceptible, vaccinated, recovered and infected people according to their proportion of the total population. Each time an infected person encounters a susceptible person, there is a 6% chance that the susceptible person becomes infected. Each infected person has a 25% chance per day of recovering. A recovered person can never contract the disease again.

A (not very effective) vaccine can be distributed before the epidemic starts. Each time an infected person encounters a vaccinated person, there is a 2% chance they become infected. (Once a vaccinated person becomes infected, they are identical to an infected person who was never vaccinated.)

The mayor of the town would like to be sure that no more than 2,000 people are ever infected.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

Let t denote time (in days). Let V denote the number of never-infected vaccinated people, S denote the number of never-infected susceptible people, I denote the number of infected people, and R denote the number of recovered, disease-resistant people. We have a parameter, v, for the number of people vaccinated at the start of the epidemic. Then

$$\frac{dS}{dt} = -\frac{0.6}{16000}IS, \quad \frac{dV}{dt} = -\frac{0.2}{16000}IV, \quad \frac{dI}{dt} = \frac{0.6}{16000}IS + \frac{0.2}{16000}IV - 0.25I, \quad \frac{dR}{dt} = 0.25I$$

and

$$S(0) = 15999 - v, \quad V(0) = v, \quad I(0) = 1, \quad R(0) = 0.$$

(b) Use the phase plane method to find an equation relating the number of susceptible and recovered people.

We compute that $\frac{dS}{dR} = \frac{dS/dt}{dR/dt} = \frac{-\frac{0.6}{16000}IS}{0.25I} = \frac{-6}{40000}S$. This is a separable differential equation, which we solve to see that $S = Ce^{-6R/40000}$. Applying the initial conditions $S(0) = 15999 - v$, $R(0) = 0$, we see that $\boxed{S = (15999 - v)e^{-6R/40000}}$.

(c) Use the phase plane method to find an equation relating the number of recovered people to the number of vaccinated (and never-infected) people.

We compute that $\frac{dV}{dR} = \frac{dV/dt}{dR/dt} = \frac{-\frac{0.2}{16000}IV}{0.25I} = \frac{-2}{40000}V$. This is a separable differential equation, which we solve to see that $V = Ce^{-2R/40000}$. Applying the initial conditions $V(0) = v$, $R(0) = 0$, we see that $\boxed{V = ve^{-2R/40000}}$.

(d) Find an equation relating the number of infected people to the number of recovered people.

$I + S + V + R = 16{,}000$, so $I = 16000 - R - S - V = 16000 - R - ve^{-2R/40000} - (15999 - v)e^{-6R/40000}$.

(e) How many people must be vaccinated in order for there to be at most 2000 previously-infected (and thus resistant) people when the epidemic ends?

If $I = 0$ and $R = 2000$, then $0 = 14000 - ve^{-1/10} - (15999 - v)e^{-3/10}$. Solving, we see that $\boxed{v = \frac{14000-(15999)e^{-3/10}}{e^{-1/10}+e^{-3/10}}}$. Thus, at least this many people must be vaccinated.

(AB 84) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers.

Each infected person encounters an average of 20 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. If an infected person encounters a recovered person, the recovered person has a 1% chance of contracting the disease again. Each infected person has a 12% chance of recovering from the disease on any given day.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.
Let t denote time (in days), let S denote the number of susceptible people (who have never had the disease), let I denote the number of infected people (who currently have the disease), and let R denote the number of recovered people (who now are resistant, that is, are less likely to get the disease again). Then $\frac{dS}{dt} = -\frac{1}{9000}SI$, $\frac{dI}{dt} = \frac{1}{9000}SI + \frac{1}{45000}RI - 0.12I$, $\frac{dR}{dt} = 0.12I - \frac{1}{45000}RI$, $S(0) = 8997$, $I(0) = 3$, $R(0) = 0$.

(b) Use the phase plane method to find an equation relating the number of susceptible and recovered people. Solve for the number of recovered people.
$\frac{dR}{dS} = \frac{dR/dt}{dS/dt} = \frac{R-5400}{5S}$, so $\ln|R - 5400| = \frac{1}{5}\ln S + C$. Using our initial conditions, we see that $\ln\frac{5400-R}{5400} = \frac{1}{5}\ln\frac{S}{8997}$, or $\boxed{R = 5400 - 5400\left(\frac{S}{8997}\right)^{1/5}.}$

(c) Write an equation relating the number of infected people to the number of susceptible people.
$I + R + S = 9000$, so $I = 9000 - R - S = 3600 + 5400\left(\frac{S}{8997}\right)^{1/5}$.

(d) Does the number of infected people ever approach zero? If so, how many susceptible people are left at that time?
No. Observe that if $S \geq 0$ then $I \geq 3600$.

(e) Does the number of susceptible people ever approach zero? If so, how many infected people and how many resistant people are there at that time?
As $S \to 0$, we see that $I \to 3600$ and $R \to 5400$.

(AB 85) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y - 15, \qquad \frac{dy}{dt} = -5x + 6y - 8, \qquad x(0) = 2, \quad y(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 85) If

$$\frac{dx}{dt} = -6x + 9y - 15, \qquad \frac{dy}{dt} = -5x + 6y - 8, \qquad x(0) = 2, \quad y(0) = 5$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 6\sin 3t \\ 4\sin 3t + 2\cos 3t \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

(AB 86) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 4, \quad y(0) = -2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 86) If

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 4, \quad y(0) = -2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} (38/45)t^{10} + (1/9)t^9 + 2 \\ -(29/45)t^{10} - 1 \end{pmatrix}.$$

(AB 87) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y + e^{2t}\cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 5, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 87) If

$$\frac{dx}{dt} = 5y + e^{2t}\cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 5, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} -t\sin t + (1/2)t\cos t + 5\cos t - (19/2)\sin t \\ -(1/2)t\sin t - 5\sin t \end{pmatrix}.$$

(AB 88) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 88) If

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 4 - 4\cos(e^t) \\ 4\cos(e^t) - 4 \end{pmatrix} + e^{-2t} \begin{pmatrix} 3\sin(e^t) - 3e^t\cos(e^t) - 3\sin 1 + 3\cos 1 \\ 4e^t\cos(e^t) - 4\sin(e^t) - 4\cos 1 + 4\sin 1 \end{pmatrix}.$$

(AB 89) Find the general solution to the equation $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3}\ln t$ on the interval $0 < t < \infty$.

(Answer 89) If $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3}\ln t$, then $y(t) = c_1 e^{t/3} + c_2 t\, e^{t/3} + \frac{1}{2}t^2 e^{t/3}\ln t - \frac{3}{4}t^2 e^{t/3}$ for all $t > 0$.

(AB 90) Find the general solution to the equation $4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 8e^{-t/2}\arctan t$.

(Answer 90) If $4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 8e^{-t/2}\arctan t$, then $y(t) = c_1 e^{-t/2} + c_2 t\, e^{-t/2} + e^{-t/2}\big(t^2\arctan t + t - \arctan t - t\ln(1+t^2)\big)$.

(AB 91) Find the general solution to the equation $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$.

(Answer 91) If $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$, then $y = c_1 e^{-t} + c_2 e^{-t/2} - 2e^{-t}\sin(e^{t/2})$.

(AB 92) Solve the initial-value problem $9\frac{d^2y}{dt^2} + y = \sec^2(t/3)$, $y(0) = 4$, $y'(0) = 2$, on the interval $-3\pi/2 < t < 3\pi/2$.

(Answer 92) If $9\frac{d^2y}{dt^2} + y = \sec^2(t/3)$, $y(0) = 4$, $y'(0) = 2$, then

$$y(t) = 5\cos(t/3) + 6\sin(t/3) + (\sin(t/3))\ln(\tan(t/3) + \sec(t/3)) - 1$$

for all $-3\pi/2 < t < 3\pi/2$.

(AB 93) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - 2x = 0$, $t > 0$, is $x(t) = C_1t^2 + C_2t^{-1}$. Solve the initial-value problem $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(Answer 93) If $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$, then $y(t) = -4\sqrt{t} + 3t^2 + \frac{2}{t}$ for all $0 < t < \infty$.

(AB 94) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - t\frac{dx}{dt} - 3x = 0$, $t > 0$, is $x(t) = C_1t^3 + C_2t^{-1}$. Find the general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

(Answer 94) If $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$, then $y(t) = -\frac{3}{2}t^{-1}\ln t + C_1t^3 + C_2t^{-1}$ for all $t > 0$.

(AB 95) Suppose that $\frac{d^2x}{dt^2} = 18x^3$, $x(0) = 1$, $x'(0) = 3$. Let $v = \frac{dx}{dt}$. Find a formula for v in terms of x. Then find a formula for x in terms of t.

(Answer 95) We have that $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ and also $\frac{dv}{dt} = \frac{d^2x}{dt^2}$. Thus $v\frac{dv}{dx} = 18x^3$, $v(1) = 3$. Solving, we see that $v = 3x^2$.

But then $\frac{dx}{dt} = 3x^2$, $x(0) = 1$, and so $x = \frac{1}{1-3t}$.

(AB 96) Suppose that a rocket of mass $m = 1000$ kg is launched straight up from the surface of the planet Trantor with initial velocity 10 km/sec. The radius of Trantor is 3,000 km. When the rocket is r meters from the center of the earth, it experiences a force due to gravity of magnitude GMm/r^2, where $GM = 6 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the rocket's position.
The initial value problem is

$$1000\frac{d^2r}{dt^2} = -\frac{6 \times 10^{17}}{r^2}, \quad r(0) = 3{,}000{,}000, \quad r'(0) = 10000$$

where r denotes the distance to the center of Trantor in meters and t denotes time in seconds.

(b) Find the velocity of the rocket as a function of position.
Let v be the rocket's velocity in meters/second. We have that

$$1000v\frac{dv}{dr} = -\frac{6 \times 10^{17}}{r^2}, \quad v(3{,}000{,}000) = 10000$$

and so

$$500v^2 = \frac{6 \times 10^{17}}{r} - 1.5 \times 10^{11}.$$

(c) How far away from the earth is the rocket when it stops moving and starts to fall back?
$v = 0$ when $r = 4 \times 10^6$ meters.

(AB 97) A 5-kg toolbox is dropped (from rest) out of a spaceship at an altitude of 9,000 km above the surface of Trantor. The radius of Trantor is 3,000 km. When the toolbox is r meters from the center of the earth, it experiences a force due to gravity of magnitude $5GM/r^2$, where $GM = 6 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the toolbox's position.
The initial value problem is

$$5\frac{d^2r}{dt^2} = -\frac{3 \times 10^{15}}{r^2}, \quad r(0) = 12{,}000{,}000, \quad r'(0) = 0$$

where r denotes the distance to the center of Trantor in meters and t denotes time in seconds.

(b) Find the velocity of the toolbox as a function of position.
Let v be the toolbox's velocity in meters/second. We have that

$$5v\frac{dv}{dr} = -\frac{3 \times 10^{15}}{r^2}, \quad v(12{,}000{,}000) = 0$$

and so

$$\frac{5}{2}v^2 = \frac{3 \times 10^{15}}{r} - 2.5 \times 10^8.$$

(c) How fast is the toolbox moving when it strikes the surface of Trantor?
When $r = 3{,}000{,}000$, $v = -10000\sqrt{3}$ meters/second.

(AB 98) A particle of mass $m = 3$ kg a distance r from an infinitely long string experiences a force due to gravity of magnitude Gm/r, where $G = 2000$ meters^2/second^2, directed directly toward the string. Suppose that the particle is initially 1000 meters from the string and takes off with initial velocity 200 meters/second directly away from the string.

(a) Formulate the initial value problem for the particle's position.
The initial value problem is

$$3\frac{d^2r}{dt^2} = -\frac{6000}{r}, \quad r(0) = 1000, \quad r'(0) = 200$$

where r denotes the distance to the string in meters and t denotes time in seconds.

(b) Find the velocity of the particle as a function of position.
Let v be the particle's velocity in meters/second. We have that

$$3v\frac{dv}{dr} = -\frac{6000}{r}, \quad v(1000) = 200$$

and so

$$v^2 = -4000 \ln r + 4000 \ln 1000 + 40000.$$

(c) How far away from the string is the particle when it stops moving and starts to fall back?
$v = 0$ when $r = 1000e^{10}$ meters.

(AB 99) A charged particle of mass $m = 20$ g a distance r meters from an electric dipole experiences a force of $3/r^3$ newtons, directed directly toward the dipole. Suppose that the particle is initially 3 meters from the dipole and is set in motion with initial velocity 5 meters/second away from the dipole.

(a) Formulate the initial value problem for the particle's position.

The initial value problem is

$$0.02\frac{d^2r}{dt^2} = -\frac{3}{r^3}, \quad r(0) = 3, \quad r'(0) = 5$$

where r denotes the distance to the dipole in meters and t denotes time in seconds.

(b) Find the velocity of the particle as a function of position.

Let v be the particle's velocity in meters/second. We have that

$$0.02v\frac{dv}{dr} = -\frac{3}{r^3}, \quad v(3) = 5$$

and so

$$v^2 = \frac{150}{r^2} + \frac{25}{3}.$$

(c) What is the limiting velocity of the particle?

As $r \to \infty$, we see that v approaches $\frac{5}{\sqrt{3}}$ meters/second.

(AB 100) Suppose that a bob of mass 300g hangs from a pendulum of length 15cm. The pendulum is set in motion from its equilibrium point with an initial velocity of 3 meters/second.

If θ denotes the angle between the pendulum and the vertical, then the pendulum satisfies the equation of motion $m\frac{d^2\theta}{dt^2} = -\frac{mg}{\ell}\sin\theta$, where m is the mass of the pendulum bob, ℓ is the length of the pendulum and g is the acceleration of gravity (which you may take to be 9.8 meters/second^2).

Find a formula for $\omega = \frac{d\theta}{dt}$ in terms of θ.

(Answer 100) Let θ be the angle between the pendulum and a vertical line (in radians), and let t denote time in seconds. Then

$$0.3\frac{d^2\theta}{dt^2} = -\frac{9.8}{0.5}\sin\theta, \quad \theta(0) = 0, \quad \theta'(0) = 20.$$

Let $\omega = \frac{d\theta}{dt}$ be the pendulum's angular velocity. We have that $\frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}$ and also $\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$. Thus $\omega\frac{d\omega}{d\theta} = -\frac{196}{3}\sin\theta$ and $\omega(0) = 20$. Solving, we see that $\frac{1}{2}\omega^2 = \frac{196}{3}\cos\theta + \frac{404}{3}$.

(AB 101) Find the general solution to the following differential equations.

(a) $6\frac{d^2y}{dt^2}+5\frac{dy}{dt}+y=3e^{4t}$.

The general solution to $6\frac{d^2y}{dt^2}+5\frac{dy}{dt}+y=0$ is $y_g=C_1e^{-t/2}+C_2e^{-t/3}$. To solve $6\frac{d^2y}{dt^2}+5\frac{dy}{dt}+y=2e^{4t}$ we make the guess $y_p=Ae^{4t}$. The solution is $y=C_1e^{-t/2}+C_2e^{-t/3}+\frac{1}{39}e^{4t}$.

(b) $16\frac{d^2y}{dt^2}-y=e^{t/4}\sin t$.

The general solution to $16\frac{d^2y}{dt^2}-y=0$ is $y_g=C_1e^{t/4}+C_2e^{-t/4}$. To solve $16\frac{d^2y}{dt^2}-y=e^{t/4}\sin t$ we make the guess $y_p=Ae^{t/4}\sin t+Be^{t/4}\cos t$. The solution is $y=C_1e^{t/4}+C_2e^{-t/4}-(1/20)e^{t/4}\sin t-(1/40)e^{t/4}\cos t$.

(c) $\frac{d^2y}{dt^2}+49y=3t\sin 7t$.

The general solution to $\frac{d^2y}{dt^2}+49y=0$ is $y_g=C_1\sin 7t+C_2\cos 7t$. To solve $\frac{d^2y}{dt^2}+49y=3t\sin 7t$ we make the guess $y_p=At^2\sin 7t+Bt\sin 7t+Ct^2\cos 7t+Dt\cos 7t$. The solution is $y=C_1\sin 7t+C_2\cos 7t-(3/28)t^2\cos 7t+(3/4)t\sin 7t$.

(d) $\frac{d^2y}{dt^2}-4\frac{dy}{dt}=4e^{3t}$.

The general solution to $\frac{d^2y}{dt^2}-4\frac{dy}{dt}=0$ is $y_g=C_1+C_2e^{4t}$. To solve $\frac{d^2y}{dt^2}-4\frac{dy}{dt}=4e^{3t}$ we make the guess $y_p=Ae^{3t}$. The solution is $y=C_1+C_2e^{4t}-(4/3)e^{3t}$.

(e) $\frac{d^2y}{dt^2}+12\frac{dy}{dt}+85y=t\sin(3t)$.

The general solution to $\frac{d^2y}{dt^2}+12\frac{dy}{dt}+85y=0$ is $y_g=C_1e^{-6t}\cos(7t)+C_2e^{-6t}\sin(7t)$. To solve $\frac{d^2y}{dt^2}+12\frac{dy}{dt}+85y=t\sin(3t)$ we make the guess $y_p=At\sin(3t)+Bt\cos(3t)+C\sin(3t)+D\cos(3t)$. The solution is

$$y=C_1e^{-6t}\cos(7t)+C_2e^{-6t}\sin(7t)+\frac{19}{1768}t\sin(3t)-\frac{9}{1768}t\cos(3t)-\frac{1353}{781456}\sin(3t)+\frac{606}{781456}\cos(3t).$$

(f) $\frac{d^2y}{dt^2}+3\frac{dy}{dt}-10y=7e^{-5t}$.

The general solution to $\frac{d^2y}{dt^2}+3\frac{dy}{dt}-10y=0$ is $y_g=C_1e^{2t}+C_2e^{-5t}$. To solve $\frac{d^2y}{dt^2}+3\frac{dy}{dt}-10y=7e^{-5t}$ we make the guess $y_p=Ate^{-5t}$. The solution is $y=C_1e^{2t}+C_2e^{-5t}-(1/7)te^{-5t}$.

(g) $16\frac{d^2y}{dt^2}-24\frac{dy}{dt}+9y=6t^2+\cos(2t)$.

The general solution to $16\frac{d^2y}{dt^2}-24\frac{dy}{dt}+9y=0$ is $y_g=C_1e^{(3/4)t}+C_2te^{(3/4)t}$. To solve $16\frac{d^2y}{dt^2}-24\frac{dy}{dt}+9y=6t^2+\cos(2t)$ we make the guess $y_p=At^2+Bt+C+D\cos(2t)+E\sin(2t)$. The solution is $y=C_1e^{(3/4)t}+C_2te^{(3/4)t}+(2/3)t^2+(32/9)t+64/9-(48/5329)\cos(2t)-(55/5329)\sin(2t)$.

(h) $\frac{d^2y}{dt^2}-6\frac{dy}{dt}+25y=t^2e^{3t}$.

The general solution to $\frac{d^2y}{dt^2}-6\frac{dy}{dt}+25y=0$ is $y_g=C_1e^{3t}\cos(4t)+C_2e^{3t}\sin(4t)$. To solve $\frac{d^2y}{dt^2}-6\frac{dy}{dt}+25y=t^2e^{3t}$ we make the guess $y_p=At^2e^{3t}+Bte^{3t}+Ce^{3t}$. The solution is $y=C_1e^{3t}\cos(4t)+C_2e^{3t}\sin(4t)+(1/16)t^2e^{3t}-(1/128)e^{3t}$.

(i) $\frac{d^2y}{dt^2}+10\frac{dy}{dt}+25y=3e^{-5t}$.

The general solution to $\frac{d^2y}{dt^2}+10\frac{dy}{dt}+25y=0$ is $y_g=C_1e^{-5t}+C_2te^{-5t}$. To solve $\frac{d^2y}{dt^2}+10\frac{dy}{dt}+25y=3e^{-5t}$ we make the guess $y_p=At^2e^{-5t}$. The solution is $y=C_1e^{-5t}+C_2te^{-5t}+(3/2)t^2e^{-5t}$.

(j) $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=3\cos(2t)$.

The general solution to $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=0$ is $y_g=C_1e^{-2t}+C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=3\cos(2t)$, we make the guess $y_p=A\cos(2t)+B\sin(2t)$. The solution is $y=C_1e^{-2t}+C_2e^{-3t}+\frac{15}{52}\sin(2t)+\frac{3}{52}\cos(2t)$.

(k) $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=5\sin(4t)$.

The general solution to $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=0$ is $y_g=C_1e^{-3t}+C_2te^{-3t}$. To solve $\frac{d^2y}{dt^2}+6\frac{dy}{dt}+9y=5\sin(4t)$, we make the guess $y_p=A\cos(4t)+B\sin(4t)$.

(l) $\frac{d^2y}{dt^2}+9y=5\sin(3t)$.

The general solution to $\frac{d^2y}{dt^2}+9y=0$ is $y_g=C_1\cos(3t)+C_2\sin(3t)$. To solve $\frac{d^2y}{dt^2}+9y=5\sin(3t)$, we make the guess $y_p=C_1t\cos(3t)+C_2t\sin(3t)$. The solution is $y=C_1\cos(3t)+C_2\sin(3t)-\frac{5}{6}t\cos(3t)$.

(m) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$.

The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ is $y_g = C_1e^{-t} + C_2te^{-t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2t^2$, we make the guess $y_p = At^2 + Bt + C$. The solution is $y = C_1e^{-t} + C_2te^{-t} + 2t^2 - 8t + 12$.

(n) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$.

The general solution to $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 0$ is $y_g = C_1 + C_2e^{-2t}$. To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} = 3t$, we make the guess $y_p = At^2 + Bt$. The solution is $y = C_1 + C_2e^{-2t} + \frac{3}{4}t^2 - \frac{3}{4}t$.

(o) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$.

The general solution to $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 0$ is $y_g = C_1e^{3t} + C_2e^{4t}$. To solve $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 5t + \cos(2t)$, we make the guess $y_p = At + B + C\cos(2t) + D\sin(2t)$. The solution is $y = C_1e^{3t} + C_2e^{4t} + \frac{5}{12}t + \frac{35}{144} + \frac{16}{177}\cos 2t - \frac{28}{177}\sin 2t$.

(p) $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$.

The general solution to $\frac{d^2y}{dt^2} - 9y = 0$ is $y_g = C_1e^{3t} + C_2e^{-3t}$. To solve $\frac{d^2y}{dt^2} - 9y = 2e^t + e^{-t} + 5t + 2$, we make the guess $y_p = Ae^t + Be^{-t} + Ct + D$.

(q) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$.

The general solution to $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$ is $y_g = C_1e^{2t} + C_2te^{2t}$. To solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 3e^{2t} + 5\cos t$, we make the guess $y_p = At^2e^{2t} + B\cos t + C\sin t$.

(AB 102) Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $16\frac{d^2y}{dt^2} - y = 3e^t$, $y(0) = 1$, $y'(0) = 0$.

If $16\frac{d^2y}{dt^2} - y = 3e^t$, $y(0) = 1$, $y'(0) = 0$, then $y(t) = \frac{1}{5}e^t + \frac{4}{5}e^{-t/4}$.

(b) $\frac{d^2y}{dt^2} + 49y = \sin 7t$, $y(\pi) = 3$, $y'(\pi) = 4$.

If $\frac{d^2y}{dt^2} + 49y = \sin 7t$, $y(\pi) = 3$, $y'(\pi) = 4$, then $y(t) = -\frac{1}{14}t\cos(7t) + \frac{\pi - 42}{14}\cos(7t) - \frac{55}{98}\sin(7t)$.

(c) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 25t^2$, $y(2) = 0$, $y'(2) = 3$.

If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y = 25t^2$, $y(2) = 0$, $y'(2) = 0$, then $y(t) = -\frac{5}{2}t^2 - \frac{3}{2}t + \frac{19}{20} + \frac{11}{60}e^{2t-4} - \frac{356}{30}e^{-5t+10}$.

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$.

If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} = 4e^{3t}$, $y(0) = 1$, $y'(0) = 3$, then $y(t) = -\frac{4}{3}e^{3t} + \frac{7}{4}e^{4t} + \frac{7}{12}$.

(AB 103) A 3-kg mass stretches a spring 5 cm. It is attached to a viscous damper with damping constant 27 N·s/m and is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(20t)$ N upwards.

Write the differential equation and initial conditions that describe the position of the object.

(Answer 103) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in meters).

Then

$$3\frac{d^2x}{dt^2} + 27\frac{dx}{dt} + 588x = 3\cos(20t), \qquad u(0) = 0, \quad u'(0) = 0.$$

(AB 104) An object weighing 4 lbs stretches a spring 2 inches. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $3\cos(\omega t)$ pounds, directed upwards. There is no damping.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of ω for which resonance occurs. Be sure to include units for ω.

(Answer 104) Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Then

$$\frac{4}{32}\frac{d^2x}{dt^2} + 24x = 3\cos(\omega t), \qquad x(0) = 0, \quad x'(0) = 0.$$

Resonance occurs when $\omega = 8\sqrt{3}\ \mathrm{s}^{-1}$.

(AB 105) A 4-kg object is suspended from a spring. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $7\sin(\omega t)$ newtons, directed upwards.

(a) Write the differential equation and initial conditions that describe the position of the object.
Let t denote time (in seconds) and let x denote the object's displacement above equilibrium (in feet). Let k denote the constant of the spring (in N·s/m). Then

$$4\frac{d^2x}{dt^2} + kx = 7\sin(\omega t), \qquad x(0) = 0, \quad x'(0) = 0.$$

(b) It is observed that resonance occurs when $\omega = 20$ rad/sec. What is the constant of the spring? Be sure to include units.
The spring constant is $k = 1600$ N·s/m.

(AB 106) A 1-kg object is suspended from a spring with constant 225 N/m. There is no damping. It is initially at rest at equilibrium. At time $t = 0$, an external force begins to act on the object; at time t seconds, the force is $15\cos(\omega t)$ newtons, directed upwards. Illustrated are the object's position for three different values of ω. You are given that the three values are $\omega = 15$ radians/second, $\omega = 16$ radians/second, and $\omega = 17$ radians/second. Determine the value of ω that will produce each image.

(Answer 106) $\omega = 15$ radians/second in Picture B. $\omega = 16$ radians/second in Picture A. $\omega = 17$ radians/second in Picture C.

(AB 107) Using the definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ (**not** the table in your notes, on Blackboard, or in your book), find the Laplace transforms of the following functions.

(a) $f(t) = e^{-11t}$

(b) $f(t) = t$

(c) $f(t) = \begin{cases} 3e^t, & 0 < t < 4, \\ 0, & 4 \leq t. \end{cases}$

(Answer 107)

(a) $\mathcal{L}\{t\} = \frac{1}{s^2}$.

(b) $\mathcal{L}\{e^{-11t}\} = \frac{1}{s+11}$.

(c) $\mathcal{L}\{f(t)\} = \frac{3-3e^{4-4s}}{s-1}$.

(AB 108) Find the Laplace transforms of the following functions. You may use the table in your notes, on Blackboard, or in your book.

(a) $f(t) = t^4 + 5t^2 + 4$
$\mathcal{L}\{t^4 + 5t^2 + 4\} = \frac{96}{s^5} + \frac{10}{s^3} + \frac{4}{s}$.

(b) $f(t) = (t+2)^3$
$\mathcal{L}\{(t+2)^3\} = \mathcal{L}\{t^3 + 6t^2 + 12t + 8\} = \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{8}{s}$.

(c) $f(t) = 9e^{4t+7}$
$\mathcal{L}\{9e^{4t+7}\} = \mathcal{L}\{9e^7 e^{4t}\} = \frac{9e^7}{s-4}$.

(d) $f(t) = -e^{3(t-2)}$
$\mathcal{L}\{-e^{3(t-2)}\} = \mathcal{L}\{-e^{-6}e^{3t}\} = -\frac{e^{-6}}{s-3}$.

(e) $f(t) = (e^t + 1)^2$
$\mathcal{L}\{(e^t+1)^2\} = \mathcal{L}\{e^{2t} + 2e^t + 1\} = \frac{1}{s-2} + \frac{2}{s-1} + \frac{1}{s}$.

(f) $f(t) = 8\sin(3t) - 4\cos(3t)$
$\mathcal{L}\{8\sin(3t) - 4\cos(3t)\} = \frac{24}{s^2+9} - \frac{4s}{s^2+9}$.

(g) $f(t) = t^2 e^{5t}$
$\mathcal{L}\{t^2 e^{5t}\} = \frac{2}{(s-5)^3}$.

(h) $f(t) = 7e^{3t}\cos 4t$
$\mathcal{L}\{7e^{3t}\cos 4t\} = \frac{7s-21}{(s-3)^2+16}$.

(i) $f(t) = 4e^{-t}\sin 5t$
$\mathcal{L}\{4e^{-t}\sin 5t\} = \frac{20}{(s+1)^2+25}$.

(j) $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$
If $f(t) = \begin{cases} 0, & t < 3, \\ e^t, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-3s}e^3}{s-1}$.

(k) $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$
If $f(t) = \begin{cases} 0, & t < 1, \\ t^2 - 2t + 2, & t \geq 1, \end{cases}$ then $\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{1}{s} + \frac{2}{s^3}\right)$.

(l) $f(t) = \begin{cases} 0, & t < 1, \\ t-2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$
If $f(t) = \begin{cases} 0, & t < 1, \\ t-2, & 1 \leq t < 2, \\ 0, & t \geq 2 \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2}$.

(m) $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$
If $f(t) = \begin{cases} 5e^{2t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{5}{s-2} - \frac{5e^6 e^{-3s}}{s-2}$.

(n) $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3 \end{cases}$
If $f(t) = \begin{cases} 7t^2 e^{-t}, & t < 3, \\ 0, & t \geq 3, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{14}{(s+1)^3} - \frac{14e^{-3s-3}}{(s+1)^3} - \frac{42e^{-3s-3}}{(s+1)^2} - \frac{63e^{-3s-3}}{s+1}$.

(o) $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi \end{cases}$
If $f(t) = \begin{cases} \cos 3t, & t < \pi, \\ \sin 3t, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = \frac{s}{s^2+9} + e^{-\pi s}\frac{s}{s^2+9} - e^{-\pi s}\frac{3}{s^2+9}$.

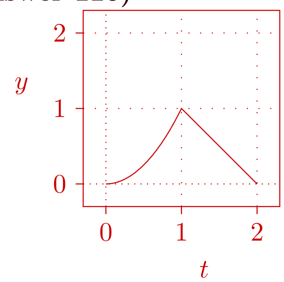
(p) $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi \end{cases}$
If $f(t) = \begin{cases} 0, & t < \pi/2, \\ \cos t, & \pi/2 \leq t < \pi, \\ 0, & t \geq \pi, \end{cases}$ then $\mathcal{L}\{f(t)\} = -e^{-\pi s/2}\frac{1}{s^2+1} + e^{-\pi s}\frac{s}{s^2+1}$.

(q) $f(t) = t\,e^t \sin t$
$\mathcal{L}\{t\,e^t \sin t\} = \frac{2s-2}{(s^2-2s+2)^2}$.
(r) $f(t) = t^2 \sin 5t$
$\mathcal{L}\{t^2 \sin 5t\} = \frac{40s^2-10(s^2+25)}{(s^2+25)^3}$.
(s) You are given that $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$. Find $\mathcal{L}\{t\,J_0(t)\}$. (The function J_0 is called a Bessel function and is important in the theory of partial differential equations in polar coordinates.)
$\mathcal{L}\{t\,J_0(t)\} = \frac{s}{(s^2+1)^{3/2}}$.
(t) You are given that $\mathcal{L}\{\frac{1}{\sqrt{t}}e^{-1/t}\} = \frac{\sqrt{\pi}}{\sqrt{s}}e^{-2\sqrt{s}}$. Find $\mathcal{L}\{\sqrt{t}\,e^{-1/t}\}$.
$\mathcal{L}\{\sqrt{t}\,e^{-1/t}\} = \frac{\sqrt{\pi}(1+2\sqrt{s})}{2s\sqrt{s}}e^{-2\sqrt{s}}$

(AB 109) For each of the following problems, find y.
(a) $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$
If $\mathcal{L}\{y\} = \frac{2s+8}{s^2+2s+5}$, then $y = 2e^{-t}\cos(2t) + 3e^{-t}\sin(2t)$.
(b) $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$
If $\mathcal{L}\{y\} = \frac{5s-7}{s^4}$, then $y = \frac{5}{2}t^2 - \frac{7}{6}t^3$.
(c) $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$
If $\mathcal{L}\{y\} = \frac{s+2}{(s+1)^4}$, then $y = \frac{1}{2}t^2e^{-t} + \frac{1}{6}t^3e^{-t}$.
(d) $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$
If $\mathcal{L}\{y\} = \frac{2s-3}{s^2-4}$, then $y = (1/4)e^{2t} + (7/4)e^{-2t}$.
(e) $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$
If $\mathcal{L}\{y\} = \frac{1}{s^2(s-3)^2}$, then $y = \frac{2}{27} + \frac{1}{9}t - \frac{2}{27}e^{3t} + \frac{1}{9}t\,e^{3t}$.
(f) $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$
If $\mathcal{L}\{y\} = \frac{s+2}{s(s^2+4)}$, then $y = \frac{1}{2} - \frac{1}{2}\cos(2t) + \frac{1}{2}\sin(2t)$.
(g) $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$
If $\mathcal{L}\{y\} = \frac{s}{(s^2+1)(s^2+9)}$, then $y = \frac{1}{8}\cos t - \frac{1}{8}\cos 3t$.
(h) $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$
If $\mathcal{L}\{y\} = \frac{(2s-1)e^{-2s}}{s^2-2s+2}$, then $y = 2\,\mathcal{U}(t-2)\,e^{t-2}\,\cos(t-2) + \mathcal{U}(t-2)\,e^{t-2}\,\sin(t-2)$.
(i) $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$
If $\mathcal{L}\{y\} = \frac{(s-2)e^{-s}}{s^2-4s+3}$, then $y = \frac{1}{2}\,\mathcal{U}(t-1)\,e^{3(t-1)} + \frac{1}{2}\,\mathcal{U}(t-1)\,e^{t-1}$.
(j) $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$
If $\mathcal{L}\{y\} = \frac{s}{(s^2+9)^2}$, then $y = \int_0^t \frac{1}{3}\sin 3r\cos(3t-3r)\,dr = \frac{1}{6}t\sin(3t)$.
(k) $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$
If $\mathcal{L}\{y\} = \frac{4}{(s^2+4s+8)^2}$, then $y = e^{-2t}\int_0^t \sin(2r)\sin(2t-2r)\,dr = \frac{1}{4}e^{-2t}\sin(2t) - \frac{1}{2}e^{-2t}t\cos(2t)$.
(l) $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$
If $\mathcal{L}\{y\} = \frac{s}{s^2-9}\,\mathcal{L}\{\sqrt{t}\}$, then $y = \int_0^t \frac{e^{3r}+e^{-3r}}{2}\sqrt{t-r}\,dr$.

(AB 110) Sketch the graph of $y = t^2 - t^2\,\mathcal{U}(t-1) + (2-t)\,\mathcal{U}(t-1)$.

(Answer 110)



(AB 111) Solve the following initial-value problems using the Laplace transform. Do you expect the graphs of y, $\frac{dy}{dt}$, or $\frac{d^2y}{dt^2}$ to show any corners or jump discontinuities? If so, at what values of t?

(a) $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$

If $\frac{dy}{dt} - 9y = \sin 3t$, $y(0) = 1$, then $y(t) = -\frac{1}{30}\sin 3t - \frac{1}{90}\cos 3t + \frac{31}{30}e^{9t}$.

(b) $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$

If $\frac{dy}{dt} - 2y = 3e^{2t}$, $y(0) = 2$, then $y = 3te^{2t} + 2e^{2t}$.

(c) $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$

If $\frac{dy}{dt} + 5y = t^3$, $y(0) = 3$, then $y = \frac{1}{5}t^3 - \frac{3}{25}t^2 + \frac{6}{125}t - \frac{6}{625} + \frac{1881}{625}e^{-5t}$.

(d) $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$, $y(0) = 1$, $y'(0) = 1$, then $y(t) = e^{2t} - te^{2t}$.

(e) $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$, then $y(t) = \frac{1}{5}(e^{-t} - e^t\cos t + 7e^t\sin t)$.

(f) $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$

If $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = e^{3t}$, $y(0) = 2$, $y'(0) = 3$, then $y(t) = 4e^{3t} - 2e^{4t} - te^{3t}$.

(g) $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$

If $\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 8y = t^2e^{2t}$, $y(0) = 3$, $y'(0) = 2$, then $y(t) = -\frac{15}{8}e^{4t} + \frac{39}{8}e^{2t} - \frac{1}{4}te^{2t} - \frac{1}{2}t^2e^{2t} - t^3e^{2t}$.

(h) $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} - 4y = e^t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{40}e^{2t} - \frac{1}{72}e^{-2t} - \frac{1}{90}e^t\cos(3t) - \frac{1}{45}e^t\sin(3t)$.

(i) $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$

If $\frac{d^2y}{dt^2} + 9y = \cos(2t)$, $y(0) = 1$, $y'(0) = 5$, then $y(t) = \frac{1}{5}\cos 2t + \cos 3t + \frac{5}{3}\sin 3t$.

(j) $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \le t < 4, \\ 0, & 4 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 0$

If $\frac{dy}{dt} + 3y = \begin{cases} 2, & 0 \le t < 4, \\ 0, & 4 \le t \end{cases}$, $y(0) = 2$, then

$$y(t) = \frac{2}{3} + \frac{4}{3}e^{-3t} - \frac{2}{3}\mathcal{U}(t-4) + \frac{2}{3}\mathcal{U}(t-4)e^{12-3t}.$$

The graph of y has a corner at $t = 4$. The graph of $\frac{dy}{dt}$ has a jump at $t = 4$.

(k) $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 4y = \begin{cases} \sin t, & 0 \le t < 2\pi, \\ 0, & 2\pi \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = (1/6)(1 - \mathcal{U}(t-2\pi))(2\sin t - \sin 2t).$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2\pi$.

(l) $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \begin{cases} 1, & 0 \le t < 10, \\ 0, & 10 \le t \end{cases}$, $y(0) = 0$, $y'(0) = 0$, then

$$y(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t} - \mathcal{U}(t-10)\left[\frac{1}{2} + \frac{1}{2}e^{-2(t-10)} - e^{-(t-10)}\right].$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 10$. The graph of $\frac{d^2y}{dt^2}$ has a jump discontinuity at $t = 10$.

(m) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \le t < 2, \\ 4, & 2 \le t \end{cases}$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = 2$.

If $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \begin{cases} 0, & 0 \le t < 2, \\ 4, & 2 \le t \end{cases}$, $y(0) = 3$, $y'(0) = 2$, $y''(0) = 1$, then

$$y(t) = 3e^{-t} + 5te^{-t} + 4t^2e^{-t} + \mathcal{U}(t-2)(4 - 4e^{2-t} - 4te^{2-t} - 2(t-2)^2e^{2-t}).$$

The graph of $\frac{d^2y}{dt^2}$ has a corner at $t = 2$. The graph of $\frac{d^3y}{dt^3}$ has a jump discontinuity at $t = 2$.

(n) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \begin{cases} 0, & 0 \le t < 2, \\ 3, & 2 \le t \end{cases}$, $y(0) = 2$, $y'(0) = 1$, then

$$y(t) = 2e^{-2t} + 5te^{-2t} + \frac{3}{4}u_2(t) - \frac{3}{4}e^{-2t+4}u_2(t) - \frac{3}{2}(t-2)e^{-2t+4}u_2(t).$$

The graph of $\frac{dy}{dt}$ has a corner at $t = 2$. The graph of $\frac{d^2y}{dt^2}$ has a jump at $t = 2$.

(o) $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 9y = t\sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t r\sin 3r \sin(3t - 3r)\,dr$.

(p) $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 9y = \sin(3t)$, $y(0) = 0$, $y'(0) = 0$, then $y(t) = \frac{1}{3}\int_0^t \sin 3r \sin(3t - 3r)\,dr = \frac{1}{6}\sin 3t - \frac{1}{2}t\cos 3t$.

(q) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$

If $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = \sqrt{t+1}$, $y(0) = 2$, $y'(0) = 1$, then $y(t) = 7e^{-2t} - 5e^{-3t} + \int_0^t (e^{-2r} - e^{-3r})\sqrt{t - r + 1}\,dr$.

(r) $y(t) + \int_0^t r\,y(t-r)\,dr = t$.

If $y(t) + \int_0^t y(r)(t-r)\,dr = t$, then $y(t) = \sin t$.

(s) $y(t) = te^t + \int_0^t (t-r)\,y(r)\,dr$.

If $y(t) = te^t + \int_0^t (t-r)\,y(r)\,dr$, then $y(t) = -\frac{1}{8}e^{-t} + \frac{1}{8}e^t + \frac{3}{4}te^t + \frac{1}{4}t^2e^t$.

(t) $\frac{dy}{dt} = 1 - \sin t - \int_0^t y(r)\,dr$, $y(0) = 0$.

If $\frac{dy}{dt} = 1 - \sin t - \int_0^t y(r)\,dr$, $y(0) = 0$, then $y(t) = \sin t - \frac{1}{2}t\sin t$.

(u) $\frac{dy}{dt} + 2y + 10\int_0^t e^{4r}y(t-r)\,dr = 0$, $y(0) = 7$.

If $\frac{dy}{dt} + 2y + 10\int_0^t e^{4r}y(t-r)\,dr = 0$, $y(0) = 7$, then $y = 7e^t\cos t - 21e^t\sin t$.

(v) $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$.

If $6\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + y = 4\mathcal{U}(t-2)$, $y(0) = 0$, $y'(0) = 1$, then

$$y = 6e^{-t/3} - 6e^{-t/2} + 4\mathcal{U}(t-2) - 12\mathcal{U}(t-2)e^{-(t-2)/3} + 8\mathcal{U}(t-2)e^{-(t-2)/2}.$$

The graph of $y'(t)$ has a corner at $t = 2$, and the graph of $y''(t)$ has a jump at $t = 2$.

(w) $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$.

If $\frac{dy}{dt} + 9y = 7\delta(t-2)$, $y(0) = 3$, then $y(t) = 3e^{-9t} + 7\mathcal{U}(t-2)e^{-9t+18}$. The graph of $y(t)$ has a jump at $t = 2$.

(x) $\frac{d^2y}{dt^2} + 4y = -2\delta(t-4\pi)$, $y(0) = 1/2$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 4y = \delta(t-4\pi)$, $y(0) = 1/2$, $y'(0) = 0$, then

$$y = \frac{1}{2}\cos(2t) - \mathcal{U}(t-4\pi)\sin(2t).$$

The graph of $y(t)$ has a corner at $t = 4\pi$, and graph of $y'(t)$ has a jump at $t = 4\pi$.

(y) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + \mathcal{U}(t-2)$, $y(0) = 1$, $y'(0) = 0$

If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 2\delta(t-1) + \mathcal{U}(t-2)$, $y(0) = 1$, $y'(0) = 0$, then

$$y = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} + \frac{1}{2}\mathcal{U}(t-1)e^{-t+1} - \frac{1}{2}\mathcal{U}(t-1)e^{-3t+3} + \frac{1}{3}\mathcal{U}(t-2) - \frac{1}{2}e^{-t+2}\mathcal{U}(t-2) + \frac{1}{6}\mathcal{U}(t-2)e^{-3t+6}.$$

The graph of $y(t)$ has a corner at $t=1$. The graph of $y'(t)$ has a corner at $t=2$, and a jump at $t=1$. $y''(t)$ has an impulse at $t=1$, and a jump at $t=2$.

(z) $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 5\delta(t-4)$, $y(0)=1$, $y'(0)=0$, $y''(0)=2$.
If $\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - \frac{dy}{dt} + 2y = 5\delta(t-4)$, $y(0)=3$, $y'(0)=0$, $y''(0)=0$, then

$$y = \frac{3}{5}e^{2t} + \frac{4}{5}\cos t - \frac{2}{5}\sin t + \mathcal{U}(t-4)(e^{2t} - \cos t - 2\sin t)$$

The graph of $\frac{dy}{dt}$ has a corner at $t=4$, and the graph of $\frac{d^2y}{dt^2}$ has a jump at $t=4$.

(AB 112) At time $t=0$, a group of 7 birds, all 1 year old, are blown onto an isolated island. A given bird has a probability of e^{-3t} of being alive after t years. A bird that is t years old and is still alive on average produces te^{-t} chicks per year. Let $B(t)$ be the birth rate at time t years. Find an integrodifferential equation (and, if necessary, an initial condition) for the birth rate.

(Answer 112) $B(t) = 7(t+1)e^{-4(t+1)} + \int_0^t B(t-r)\,re^{-4r}\,dr$.

(AB 113) Let I be the number of people with a rare infections disease. Suppose that 300 people simultaneously contract the disease at time $t=0$. Each infected person infects three new people per day on average. A person who was infected r days ago has a probability of $4r^2e^{-2r}$ of recovering on day r. Write an integrodifferential equation (and, if necessary, an initial condition) for the number of infected people.

(Answer 113)

$$\frac{dI}{dt} = 3I - \int_0^t 12r^2e^{-2r}\,I(t-r)\,dr - 1200t^2e^{-2t}, \qquad I(0) = 300.$$