Math 2584, Spring 2021

Exam 3 will occur:
- Thursday, April 8 at 9:30, in HILL 206
- Thursday, April 8 at 11:00, in SCEN 402
- Friday, April 9 at 8:35, in KIMP 102.

You are allowed a non-graphing calculator.

Please check your final exam schedule. If you have 3 or more final exams scheduled for the same day, and you would like to reschedule the final exam for this class, please let me know by email by the day of the third exam.

(AB 1) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 2) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 3) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 4) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 5) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 6) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 7) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 8) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det\begin{pmatrix} 4-m & 1 & -1 \\ 1 & 4-m & -1 \\ -1 & -1 & 4-m \end{pmatrix} = -(m-3)^2(m-6).$

(AB 9) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, \qquad x(0) = 3, \quad y(0) = 2, \quad z(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det\begin{pmatrix} 3-m & 1 & 0 \\ 2 & 3-m & -2 \\ 0 & 1 & 3-m \end{pmatrix} = -(m-3)^3.$

(AB 10) Find the solution to the initial-value problem

$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, x(0) = 4, \quad y(0) = 3, \quad z(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det\begin{pmatrix} 1-m & 1 & 1 \\ 1 & 3-m & -1 \\ 0 & 2 & 2-m \end{pmatrix} = -(m-2)^3.$

(AB 11) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

(*a*) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.
(*b*) Use the phase plane method to find an equation relating the number of susceptible and infected people.
(*c*) Use the phase plane method to find an equation relating the number of resistant and susceptible people.
(*d*) As $t \to \infty$, $I \to 0$ and the number of people who never contract the disease approaches a limit. Let S_∞ be the limiting number of people who never contract the disease. Find an (algebraic, not differential) equation involving S_∞.
(*e*) What is the maximum number of people that are infected at any one time?

(AB 12) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(AB 13) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

(a) Set up the system of differential equations that describes the number of susceptible, infected, and resistant people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(b) Assume that there are initially no recovered or vaccinated people and 7 infected people. Use the phase plane method to find an equation relating S and I.

(c) Find an algebraic equation for the maximum number of people that are infected with the virus at any one time.

(AB 14) A small town has a population of 16,000 people. It is expected that soon, one of them will be infected with a contagious disease.

Epidemiologists observe the town and expect each infected person to encounter 10 people per day, who are distributed among susceptible, vaccinated, recovered and infected people according to their proportion of the total population. Each time an infected person encounters a susceptible person, there is a 6% chance that the susceptible person becomes infected. Each infected person has a 25% chance per day of recovering. A recovered person can never contract the disease again.

A (not very effective) vaccine can be distributed before the epidemic starts. Each time an infected person encounters a vaccinated person, there is a 2% chance they become infected. (Once a vaccinated person becomes infected, they are identical to an infected person who was never vaccinated.)

The mayor of the town would like to be sure that no more than 2,000 people are ever infected.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

(b) Use the phase plane method to find an equation relating the number of susceptible and recovered people.

(c) Use the phase plane method to find an equation relating the number of recovered people to the number of vaccinated (and never-infected) people.

(d) Find an equation relating the number of infected people to the number of recovered people.

(e) How many people must be vaccinated in order for there to be at most 2000 previously-infected (and thus resistant) people when the epidemic ends?

(AB 15) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers.

Each infected person encounters an average of 20 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. If an infected person encounters a recovered person, the recovered person has a 1% chance of contracting the disease again. Each infected person has a 12% chance of recovering from the disease on any given day.

(*a*) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.
(*b*) Use the phase plane method to find an equation relating the number of susceptible and recovered people. Solve for the number of recovered people.
(*c*) Write an equation relating the number of infected people to the number of susceptible people.
(*d*) Does the number of infected people ever approach zero? If so, how many susceptible people are left at that time?
(*e*) Does the number of susceptible people ever approach zero? If so, how many infected people and how many resistant people are there at that time?

(AB 16) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y - 15, \qquad \frac{dy}{dt} = -5x + 6y - 8, \qquad x(0) = 2, \quad y(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 17) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 4, \quad y(0) = -2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 18) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y + e^{2t}\cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 5, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 19) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(AB 20) Find the general solution to the equation $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3}\ln t$ on the interval $0 < t < \infty$.

(AB 21) Find the general solution to the equation $4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 8e^{-t/2}\arctan t$.

(AB 22) Find the general solution to the equation $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$.

(AB 23) Solve the initial-value problem $9\frac{d^2y}{dt^2} + y = \sec^2(t/3)$, $y(0) = 4$, $y'(0) = 2$, on the interval $-3\pi/2 < t < 3\pi/2$.

(AB 24) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - 2x = 0$, $t > 0$, is $x(t) = C_1t^2 + C_2t^{-1}$. Solve the initial-value problem $t^2\frac{d^2y}{dt^2} - 2y = 9\sqrt{t}$, $y(1) = 1$, $y'(1) = 2$ on the interval $0 < t < \infty$.

(AB 25) The general solution to the differential equation $t^2\frac{d^2x}{dt^2} - t\frac{dx}{dt} - 3x = 0$, $t > 0$, is $x(t) = C_1t^3 + C_2t^{-1}$. Find the general solution to the differential equation $t^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} - 3y = 6t^{-1}$.

(AB 26) Suppose that $\frac{d^2x}{dt^2} = 18x^3$, $x(0) = 1$, $x'(0) = 3$. Let $v = \frac{dx}{dt}$. Find a formula for v in terms of x. Then find a formula for x in terms of t.

(AB 27) Suppose that a rocket of mass $m = 1000$ kg is launched straight up from the surface of the planet Trantor with initial velocity 10 km/sec. The radius of Trantor is 3,000 km. When the rocket is r meters from the center of the earth, it experiences a force due to gravity of magnitude GMm/r^2, where $GM = 6 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the rocket's position.
(b) Find the velocity of the rocket as a function of position.
(c) How far away from the earth is the rocket when it stops moving and starts to fall back?

(AB 28) A 5-kg toolbox is dropped (from rest) out of a spaceship at an altitude of 9,000 km above the surface of Trantor. The radius of Trantor is 3,000 km. When the toolbox is r meters from the center of the earth, it experiences a force due to gravity of magnitude $5GM/r^2$, where $GM = 6 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the toolbox's position.
(b) Find the velocity of the toolbox as a function of position.
(c) How fast is the toolbox moving when it strikes the surface of Trantor?

(AB 29) A particle of mass $m = 3$ kg a distance r from an infinitely long string experiences a force due to gravity of magnitude Gm/r, where $G = 2000$ meters^2/second^2, directed directly toward the string. Suppose that the particle is initially 1000 meters from the string and takes off with initial velocity 200 meters/second directly away from the string.

(a) Formulate the initial value problem for the particle's position.
(b) Find the velocity of the particle as a function of position.
(c) How far away from the string is the particle when it stops moving and starts to fall back?

(AB 30) A charged particle of mass $m = 20$ g a distance r meters from an electric dipole experiences a force of $3/r^3$ newtons, directed directly toward the dipole. Suppose that the particle is initially 3 meters from the dipole and is set in motion with initial velocity 5 meters/second away from the dipole.

(a) Formulate the initial value problem for the particle's position.
(b) Find the velocity of the particle as a function of position.
(c) What is the limiting velocity of the particle?

(AB 31) Suppose that a bob of mass 300g hangs from a pendulum of length 15cm. The pendulum is set in motion from its equilibrium point with an initial velocity of 3 meters/second.

If θ denotes the angle between the pendulum and the vertical, then the pendulum satisfies the equation of motion $m\frac{d^2\theta}{dt^2} = -\frac{mg}{\ell}\sin\theta$, where m is the mass of the pendulum bob, ℓ is the length of the pendulum and g is the acceleration of gravity (which you may take to be 9.8 meters/second^2).

Find a formula for $\omega = \frac{d\theta}{dt}$ in terms of θ.

Answer key

(AB 1) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 1) If

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = -5, \quad y(0) = 3$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 4t - 5 \\ -2t + 3 \end{pmatrix}.$$

(AB 2) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 2) If

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = -3, \quad y(0) = 2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} -3\cos t + 16\sin t \\ 2\cos t + 7\sin t \end{pmatrix}.$$

(AB 3) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 3) If

$$\frac{dx}{dt} = 7x - \frac{9}{2}y, \qquad \frac{dy}{dt} = -18x - 17y, \qquad x(0) = 0, \quad y(0) = 1$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-20t} \begin{pmatrix} 3/20 \\ 9/10 \end{pmatrix} + e^{10t} \begin{pmatrix} -3/20 \\ 1/10 \end{pmatrix}.$$

(AB 4) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 4) If

$$\frac{dx}{dt} = 13x - 39y, \qquad \frac{dy}{dt} = 12x - 23y, \qquad x(0) = 5, \quad y(0) = 2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 5\cos 12t + \sin 12t \\ 2\cos 12t + 2\sin 12t \end{pmatrix}.$$

(AB 5) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 5) If

$$\frac{dx}{dt} = -6x + 9y, \qquad \frac{dy}{dt} = -5x + 6y, \qquad x(0) = 1, \quad y(0) = -3$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos 3t - 11\sin 3t \\ -3\cos 3t - (23/3)\sin 3t \end{pmatrix}.$$

(AB 6) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 6) If

$$\frac{dx}{dt} = -6x + 4y, \qquad \frac{dy}{dt} = -\frac{1}{4}x - 4y, \qquad x(0) = 1, \quad y(0) = 4$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-5t} \begin{pmatrix} 15t + 1 \\ (15/4)t + 4 \end{pmatrix}.$$

(AB 7) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 7) If

$$\frac{dx}{dt} = -2x + 2y, \qquad \frac{dy}{dt} = 2x - 5y, \qquad x(0) = 1, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 4/5 \\ 2/5 \end{pmatrix} + e^{-6t} \begin{pmatrix} 1/5 \\ -2/5 \end{pmatrix}.$$

(AB 8) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 4-m & 1 & -1 \\ 1 & 4-m & -1 \\ -1 & -1 & 4-m \end{pmatrix} = -(m-3)^2(m-6).$

(Answer 8) If

$$\frac{dx}{dt} = 4x + y - z, \qquad \frac{dy}{dt} = x + 4y - z, \qquad \frac{dz}{dt} = 4z - x - y, \quad x(0) = 3, \quad y(0) = 9, \quad z(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{3t} \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix} + e^{6t} \begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}.$$

(AB 9) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, \qquad x(0) = 3, \quad y(0) = 2, \quad z(0) = 1.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 3-m & 1 & 0 \\ 2 & 3-m & -2 \\ 0 & 1 & 3-m \end{pmatrix} = -(m-3)^3.$

(Answer 9) If

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = 2x + 3y - 2z, \qquad \frac{dz}{dt} = y + 3z, \qquad x(0) = 3, \quad y(0) = 2, \quad z(0) = 1$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{3t} \begin{pmatrix} 2t^2 + 2t + 3 \\ 4t + 2 \\ 2t^2 + 2t + 1 \end{pmatrix}.$$

(AB 10) Find the solution to the initial-value problem

$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, x(0) = 4, \quad y(0) = 3, \quad z(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

Hint: $\det \begin{pmatrix} 1-m & 1 & 1 \\ 1 & 3-m & -1 \\ 0 & 2 & 2-m \end{pmatrix} = -(m-2)^3.$

(Answer 10) If

$$\frac{dx}{dt} = x + y + z, \qquad \frac{dy}{dt} = x + 3y - z, \qquad \frac{dz}{dt} = 2y + 2z, \qquad x(0) = 4, \quad y(0) = 3, \quad z(0) = 5$$

then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 4 + 4t + 2t^2 \\ 2t + 3 \\ 5 + 6t + 2t^2 \end{pmatrix}.$$

(AB 11) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers and which cannot be caught twice. Each infected person encounters an average of 8 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. Each infected person has a 17% chance of recovering from the disease on any given day.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.
Let t denote time (in days), let S denote the number of susceptible people (who have never had the disease), let I denote the number of infected people (who currently have the disease), and let R denote the number of recovered people (who now are resistant, that is, cannot get the disease again). Then
$\boxed{\frac{dS}{dt} = -\frac{1}{22500} SI,\ \frac{dI}{dt} = \frac{1}{22500} SI - 0.17I,\ \frac{dR}{dt} = 0.17I,\ S(0) = 8997,\ I(0) = 3,\ R(0) = 0.}$

(b) Use the phase plane method to find an equation relating the number of susceptible and infected people.
$\frac{dI}{dS} = \frac{3825}{S} - 1$, so $\boxed{I = 9000 - S - 3825 \ln \frac{8997}{S}.}$

(c) Use the phase plane method to find an equation relating the number of resistant and susceptible people.
$\frac{dR}{dS} = -\frac{3825}{S}$, so $\boxed{R = 3825 \ln \frac{8997}{S}.}$

(d) As $t \to \infty$, $I \to 0$ and the number of people who never contract the disease approaches a limit. Let S_∞ be the limiting number of people who never contract the disease. Find an (algebraic, not differential) equation involving S_∞.
$\boxed{9000 - S_\infty - 3825 \ln \frac{8997}{S_\infty} = 0.}$ On the exam you will not be expected to solve this equation.

(e) What is the maximum number of people that are infected at any one time?
When I is maximized, $0 = \frac{dI}{dt} = \frac{1}{22500} SI - 0.17I$, and so $S = 3825$. Thus, the maximum value of I is
$\boxed{I = 9000 - 3825 - 3825 \ln \frac{62979}{33750}.}$

(AB 12) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000} SI$, $\frac{dI}{dt} = \frac{0.2}{5000} SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 susceptible people are vaccinated each day (and thus become resistant without being infected first).

Set up the system of differential equations that describes the number of susceptible, infected, and recovered people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

(Answer 12) $\frac{dS}{dt} = -\frac{0.2}{5000} SI - 15$, $\frac{dI}{dt} = \frac{0.2}{5000} SI - 0.1I$, $\frac{dR}{dt} = 0.1I + 15$.

(AB 13) Suppose that a disease is spreading through a town of 5000 people, and that its transmission in the absence of vaccination is given by the Kermack-McKendrick SIR model $\frac{dS}{dt} = -\frac{0.2}{5000}SI$, $\frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, $\frac{dR}{dt} = 0.1I$ where t denotes time (in days), S denotes the number of susceptible people, I denotes the number of infected people, and R denotes the number of recovered, disease-resistant people.

Suppose we modify the model by assuming that 15 people are vaccinated each day (and thus become resistant without being infected first). No testing is available, and so vaccines are distributed among susceptible, resistant, and infected people according to their proportion of the total population. A vaccine administered to an infected or resistant person has no effect.

(a) Set up the system of differential equations that describes the number of susceptible, infected, and resistant people. You may use the same variable names as before. (You may let R denote all disease-resistant people, both vaccinated and recovered.)

$\boxed{\frac{dS}{dt} = -\frac{0.2}{5000}SI - 15\frac{S}{5000},\ \frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I,\ \frac{dR}{dt} = 0.1I + 15\frac{S}{5000}.}$

(b) Assume that there are initially no recovered or vaccinated people and 7 infected people. Use the phase plane method to find an equation relating S and I.

We compute that $\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = \frac{0.2SI - 500I}{-0.2SI - 15S} = \frac{0.2S - 500}{S}\frac{I}{-0.2I - 15}$. This is a separable differential equation, which we solve to see that $-0.2I - 15\ln I = 0.2S - 500\ln S + C$. Applying the initial conditions $I(0) = 7$, $S(0) = 4993$, we see that $\boxed{-0.2(I-7) - 15\ln(I/7) = 0.2(S-4993) - 500\ln(S/4993).}$

(c) Find an algebraic equation for the maximum number of people that are infected with the virus at any one time.

The maximum I value occurs when $0 = \frac{dI}{dt} = \frac{0.2}{5000}SI - 0.1I$, or $S = 2500$. The maximum I value then satisfies

$\boxed{-0.2(I-7) - 15\ln(I/7) = 0.2(2500-4993) - 500\ln(2500/4993).}$

(AB 14) A small town has a population of 16,000 people. It is expected that soon, one of them will be infected with a contagious disease.

Epidemiologists observe the town and expect each infected person to encounter 10 people per day, who are distributed among susceptible, vaccinated, recovered and infected people according to their proportion of the total population. Each time an infected person encounters a susceptible person, there is a 6% chance that the susceptible person becomes infected. Each infected person has a 25% chance per day of recovering. A recovered person can never contract the disease again.

A (not very effective) vaccine can be distributed before the epidemic starts. Each time an infected person encounters a vaccinated person, there is a 2% chance they become infected. (Once a vaccinated person becomes infected, they are identical to an infected person who was never vaccinated.)

The mayor of the town would like to be sure that no more than 2,000 people are ever infected.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.
Let t denote time (in days). Let V denote the number of never-infected vaccinated people, S denote the number of never-infected susceptible people, I denote the number of infected people, and R denote the number of recovered, disease-resistant people. We have a parameter, v, for the number of people vaccinated at the start of the epidemic. Then

$$\frac{dS}{dt} = -\frac{0.6}{16000}IS, \quad \frac{dV}{dt} = -\frac{0.2}{16000}IV, \quad \frac{dI}{dt} = \frac{0.6}{16000}IS + \frac{0.2}{16000}IV - 0.25I, \quad \frac{dR}{dt} = 0.25I$$

and

$$S(0) = 15999 - v, \quad V(0) = v, \quad I(0) = 1, \quad R(0) = 0.$$

(b) Use the phase plane method to find an equation relating the number of susceptible and recovered people.
We compute that $\frac{dS}{dR} = \frac{dS/dt}{dR/dt} = \frac{-\frac{0.6}{16000}IS}{0.25I} = \frac{-6}{40000}S$. This is a separable differential equation, which we solve to see that $S = Ce^{-6R/40000}$. Applying the initial conditions $S(0) = 15999 - v$, $R(0) = 0$, we see that $\boxed{S = (15999 - v)e^{-6R/40000}.}$

(c) Use the phase plane method to find an equation relating the number of recovered people to the number of vaccinated (and never-infected) people.
We compute that $\frac{dV}{dR} = \frac{dV/dt}{dR/dt} = \frac{-\frac{0.2}{16000}IV}{0.25I} = \frac{-2}{40000}V$. This is a separable differential equation, which we solve to see that $V = Ce^{-2R/40000}$. Applying the initial conditions $V(0) = v$, $R(0) = 0$, we see that $\boxed{V = ve^{-2R/40000}.}$

(d) Find an equation relating the number of infected people to the number of recovered people.
$I + S + V + R = 16{,}000$, so $I = 16000 - R - S - V = 16000 - R - ve^{-2R/40000} - (15999 - v)e^{-6R/40000}$.

(e) How many people must be vaccinated in order for there to be at most 2000 previously-infected (and thus resistant) people when the epidemic ends?
If $I = 0$ and $R = 2000$, then $0 = 14000 - ve^{-1/10} - (15999 - v)e^{-3/10}$. Solving, we see that $\boxed{v = \frac{14000-(15999)e^{-3/10}}{e^{-1/10}+e^{-3/10}}.}$ Thus, at least this many people must be vaccinated.

(AB 15) An isolated town has a population of 9000 people. Three of them are simultaneously infected with a contagious disease to which no member of the town has previously been exposed, but from which one eventually recovers.

Each infected person encounters an average of 20 other townspeople per day. Encounters are distributed among susceptible, infected, and recovered people according to their proportion of the total population. If an infected person encounters a susceptible person, the susceptible person has a 5% chance of contracting the disease. If an infected person encounters a recovered person, the recovered person has a 1% chance of contracting the disease again. Each infected person has a 12% chance of recovering from the disease on any given day.

(a) Set up the initial value problem that describes the number of susceptible, infected, and recovered people.

Let t denote time (in days), let S denote the number of susceptible people (who have never had the disease), let I denote the number of infected people (who currently have the disease), and let R denote the number of recovered people (who now are resistant, that is, are less likely to get the disease again). Then $\frac{dS}{dt} = -\frac{1}{9000}SI$, $\frac{dI}{dt} = \frac{1}{9000}SI + \frac{1}{45000}RI - 0.12I$, $\frac{dR}{dt} = 0.12I - \frac{1}{45000}RI$, $S(0) = 8997$, $I(0) = 3$, $R(0) = 0$.

(b) Use the phase plane method to find an equation relating the number of susceptible and recovered people. Solve for the number of recovered people.

$\frac{dR}{dS} = \frac{dR/dt}{dS/dt} = \frac{R-5400}{5S}$, so $\ln|R - 5400| = \frac{1}{5}\ln S + C$. Using our initial conditions, we see that $\ln\frac{5400-R}{5400} = \frac{1}{5}\ln\frac{S}{8997}$, or $\boxed{R = 5400 - 5400\left(\frac{S}{8997}\right)^{1/5}.}$

(c) Write an equation relating the number of infected people to the number of susceptible people.

$I + R + S = 9000$, so $I = 9000 - R - S = 3600 + 5400\left(\frac{S}{8997}\right)^{1/5}$.

(d) Does the number of infected people ever approach zero? If so, how many susceptible people are left at that time?

No. Observe that if $S \geq 0$ then $I \geq 3600$.

(e) Does the number of susceptible people ever approach zero? If so, how many infected people and how many resistant people are there at that time?

As $S \to 0$, we see that $I \to 3600$ and $R \to 5400$.

(AB 16) Find the solution to the initial-value problem

$$\frac{dx}{dt} = -6x + 9y - 15, \qquad \frac{dy}{dt} = -5x + 6y - 8, \qquad x(0) = 2, \quad y(0) = 5.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 16) If

$$\frac{dx}{dt} = -6x + 9y - 15, \qquad \frac{dy}{dt} = -5x + 6y - 8, \qquad x(0) = 2, \quad y(0) = 5$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 6\sin 3t \\ 4\sin 3t + 2\cos 3t \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

(AB 17) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 4, \quad y(0) = -2.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 17) If

$$\frac{dx}{dt} = 6x + 8y + t^8 e^{2t}, \qquad \frac{dy}{dt} = -2x - 2y, \qquad x(0) = 4, \quad y(0) = -2$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} (38/45)t^{10} + (1/9)t^9 + 2 \\ -(29/45)t^{10} - 1 \end{pmatrix}.$$

(AB 18) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 5y + e^{2t} \cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 5, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 18) If

$$\frac{dx}{dt} = 5y + e^{2t} \cos t, \qquad \frac{dy}{dt} = -x + 4y, \qquad x(0) = 5, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} -t \sin t + (1/2)t \cos t + 5 \cos t - (19/2) \sin t \\ -(1/2)t \sin t - 5 \sin t \end{pmatrix}.$$

(AB 19) Find the solution to the initial-value problem

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0.$$

Express your final answer in terms of real functions (no complex numbers or complex exponentials).

(Answer 19) If

$$\frac{dx}{dt} = 2x + 3y + \sin(e^t), \qquad \frac{dy}{dt} = -4x - 5y, \qquad x(0) = 0, \quad y(0) = 0$$

then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 4 - 4\cos(e^t) \\ 4\cos(e^t) - 4 \end{pmatrix} + e^{-2t} \begin{pmatrix} 3\sin(e^t) - 3e^t \cos(e^t) - 3\sin 1 + 3\cos 1 \\ 4e^t \cos(e^t) - 4\sin(e^t) - 4\cos 1 + 4\sin 1 \end{pmatrix}.$$

(AB 20) Find the general solution to the equation $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3} \ln t$ on the interval $0 < t < \infty$.

(Answer 20) If $9\frac{d^2y}{dt^2} - 6\frac{dy}{dt} + y = 9e^{t/3} \ln t$, then $y(t) = c_1 e^{t/3} + c_2 t e^{t/3} + \frac{1}{2}t^2 e^{t/3} \ln t - \frac{3}{4}t^2 e^{t/3}$ for all $t > 0$.

(AB 21) Find the general solution to the equation $4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 8e^{-t/2} \arctan t$.

(Answer 21) If $4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 8e^{-t/2} \arctan t$, then $y(t) = c_1 e^{-t/2} + c_2 t e^{-t/2} + e^{-t/2}\big(t^2 \arctan t + t - \arctan t - t \ln(1 + t^2)\big)$.

(AB 22) Find the general solution to the equation $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$.

(Answer 22) If $2\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \sin(e^{t/2})$, then $y = c_1 e^{-t} + c_2 e^{-2t} - 2e^{-t} \sin(e^{t/2})$.

(AB 23) Solve the initial-value problem $9\frac{d^2y}{dt^2}+y=\sec^2(t/3)$, $y(0)=4$, $y'(0)=2$, on the interval $-3\pi/2<t<3\pi/2$.

(Answer 23) If $9\frac{d^2y}{dt^2}+y=\sec^2(t/3)$, $y(0)=4$, $y'(0)=2$, then

$$y(t)=5\cos(t/3)+6\sin(t/3)+(\sin(t/3))\ln(\tan(t/3)+\sec(t/3))-1$$

for all $-3\pi/2<t<3\pi/2$.

(AB 24) The general solution to the differential equation $t^2\frac{d^2x}{dt^2}-2x=0$, $t>0$, is $x(t)=C_1t^2+C_2t^{-1}$. Solve the initial-value problem $t^2\frac{d^2y}{dt^2}-2y=9\sqrt{t}$, $y(1)=1$, $y'(1)=2$ on the interval $0<t<\infty$.

(Answer 24) If $t^2\frac{d^2y}{dt^2}-2y=9\sqrt{t}$, $y(1)=1$, $y'(1)=2$, then $y(t)=-4\sqrt{t}+3t^2+\frac{2}{t}$ for all $0<t<\infty$.

(AB 25) The general solution to the differential equation $t^2\frac{d^2x}{dt^2}-t\frac{dx}{dt}-3x=0$, $t>0$, is $x(t)=C_1t^3+C_2t^{-1}$. Find the general solution to the differential equation $t^2\frac{d^2y}{dt^2}-t\frac{dy}{dt}-3y=6t^{-1}$.

(Answer 25) If $t^2\frac{d^2y}{dt^2}-t\frac{dy}{dt}-3y=6t^{-1}$, then $y(t)=-\frac{3}{2}t^{-1}\ln t+C_1t^3+C_2t^{-1}$ for all $t>0$.

(AB 26) Suppose that $\frac{d^2x}{dt^2}=18x^3$, $x(0)=1$, $x'(0)=3$. Let $v=\frac{dx}{dt}$. Find a formula for v in terms of x. Then find a formula for x in terms of t.

(Answer 26) We have that $\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=v\frac{dv}{dx}$ and also $\frac{dv}{dt}=\frac{d^2x}{dt^2}$. Thus $v\frac{dv}{dx}=18x^3$, $v(1)=3$. Solving, we see that $v=3x^2$.

But then $\frac{dx}{dt}=3x^2$, $x(0)=1$, and so $x=\frac{1}{1-3t}$.

(AB 27) Suppose that a rocket of mass $m=1000$ kg is launched straight up from the surface of the planet Trantor with initial velocity 10 km/sec. The radius of Trantor is 3,000 km. When the rocket is r meters from the center of the earth, it experiences a force due to gravity of magnitude GMm/r^2, where $GM=6\times10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the rocket's position.
The initial value problem is

$$1000\frac{d^2r}{dt^2}=-\frac{6\times10^{17}}{r^2},\quad r(0)=3{,}000{,}000,\quad r'(0)=10000$$

where r denotes the distance to the center of Trantor in meters and t denotes time in seconds.

(b) Find the velocity of the rocket as a function of position.
Let v be the rocket's velocity in meters/second. We have that

$$1000v\frac{dv}{dr}=-\frac{6\times10^{17}}{r^2},\quad v(3{,}000{,}000)=10000$$

and so

$$500v^2=\frac{6\times10^{17}}{r}-1.5\times10^{11}.$$

(c) How far away from the earth is the rocket when it stops moving and starts to fall back?
$v=0$ when $r=4\times10^6$ meters.

(AB 28) A 5-kg toolbox is dropped (from rest) out of a spaceship at an altitude of 9,000 km above the surface of Trantor. The radius of Trantor is 3,000 km. When the toolbox is r meters from the center of the earth, it experiences a force due to gravity of magnitude $5GM/r^2$, where $GM = 6 \times 10^{14}$ meters^3/second^2.

(a) Formulate the initial value problem for the toolbox's position.
The initial value problem is

$$5\frac{d^2r}{dt^2} = -\frac{3 \times 10^{15}}{r^2}, \quad r(0) = 12{,}000{,}000, \quad r'(0) = 0$$

where r denotes the distance to the center of Trantor in meters and t denotes time in seconds.

(b) Find the velocity of the toolbox as a function of position.
Let v be the toolbox's velocity in meters/second. We have that

$$5v\frac{dv}{dr} = -\frac{3 \times 10^{15}}{r^2}, \quad v(12{,}000{,}000) = 0$$

and so

$$\frac{5}{2}v^2 = \frac{3 \times 10^{15}}{r} - 2.5 \times 10^8.$$

(c) How fast is the toolbox moving when it strikes the surface of Trantor?
When $r = 3{,}000{,}000$, $v = -10000\sqrt{3}$ meters/second.

(AB 29) A particle of mass $m = 3$ kg a distance r from an infinitely long string experiences a force due to gravity of magnitude Gm/r, where $G = 2000$ meters^2/second^2, directed directly toward the string. Suppose that the particle is initially 1000 meters from the string and takes off with initial velocity 200 meters/second directly away from the string.

(a) Formulate the initial value problem for the particle's position.
The initial value problem is

$$3\frac{d^2r}{dt^2} = -\frac{6000}{r}, \quad r(0) = 1000, \quad r'(0) = 200$$

where r denotes the distance to the string in meters and t denotes time in seconds.

(b) Find the velocity of the particle as a function of position.
Let v be the particle's velocity in meters/second. We have that

$$3v\frac{dv}{dr} = -\frac{6000}{r}, \quad v(1000) = 200$$

and so

$$v^2 = -4000 \ln r + 4000 \ln 1000 + 40000.$$

(c) How far away from the string is the particle when it stops moving and starts to fall back?
$v = 0$ when $r = 1000e^{10}$ meters.

(AB 30) A charged particle of mass $m = 20$ g a distance r meters from an electric dipole experiences a force of $3/r^3$ newtons, directed directly toward the dipole. Suppose that the particle is initially 3 meters from the dipole and is set in motion with initial velocity 5 meters/second away from the dipole.

(a) Formulate the initial value problem for the particle's position.

The initial value problem is

$$0.02\frac{d^2r}{dt^2} = -\frac{3}{r^3}, \quad r(0) = 3, \quad r'(0) = 5$$

where r denotes the distance to the dipole in meters and t denotes time in seconds.

(b) Find the velocity of the particle as a function of position.

Let v be the particle's velocity in meters/second. We have that

$$0.02v\frac{dv}{dr} = -\frac{3}{r^3}, \quad v(3) = 5$$

and so

$$v^2 = \frac{150}{r^2} + \frac{25}{3}.$$

(c) What is the limiting velocity of the particle?

As $r \to \infty$, we see that v approaches $\frac{5}{\sqrt{3}}$ meters/second.

(AB 31) Suppose that a bob of mass 300g hangs from a pendulum of length 15cm. The pendulum is set in motion from its equilibrium point with an initial velocity of 3 meters/second.

If θ denotes the angle between the pendulum and the vertical, then the pendulum satisfies the equation of motion $m\frac{d^2\theta}{dt^2} = -\frac{mg}{\ell}\sin\theta$, where m is the mass of the pendulum bob, ℓ is the length of the pendulum and g is the acceleration of gravity (which you may take to be 9.8 meters/second^2).

Find a formula for $\omega = \frac{d\theta}{dt}$ in terms of θ.

(Answer 31) Let θ be the angle between the pendulum and a vertical line (in radians), and let t denote time in seconds. Then

$$0.3\frac{d^2\theta}{dt^2} = -\frac{9.8}{0.5}\sin\theta, \quad \theta(0) = 0, \quad \theta'(0) = 20.$$

Let $\omega = \frac{d\theta}{dt}$ be the pendulum's angular velocity. We have that $\frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}$ and also $\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$. Thus $\omega\frac{d\omega}{d\theta} = -\frac{196}{3}\sin\theta$ and $\omega(0) = 20$. Solving, we see that $\frac{1}{2}\omega^2 = \frac{196}{3}\cos\theta + \frac{404}{3}$.