# Math 2584, Spring 2021

Exam 2 will occur:

- Thursday, March 11 at 9:30, in HILL 206
- Thursday, March 11 at 11:00, in SCEN 402
- Friday, March 12 at 8:35, in KIMP 102.

You are allowed a non-graphing calculator.

**(AB 1)** (14 points) For each of the following differential equations, find all the equilibrium solutions or state that no such solutions exist. You do not need to find the nonequilibrium solutions.

(If this problem appears on Exam 2, its points will not count towards your Exam 2 total. Instead, if your score on this problem is better than your score on Problem 4 on Exam 1, it will replace your score on that problem and your Exam 1 score will be adjusted accordingly.)

(a) $\frac{dy}{dt} = y^3 - yt$

(b) $\frac{dy}{dt} = t^2 e^y$

(c) $\frac{dy}{dt} = ty + t^3$

(d) $\frac{dy}{dt} = (y^2 + 3y + 2)\sin(t)$

(e) $\frac{dy}{dt} = \ln(y^t)$

**(AB 2)** Consider the differential equation $\ln(3/4 + t^2)\frac{dy}{dt} = \sin(y)$.

(a) Find all pairs of numbers $T_0$ and $Y_0$ such that the initial value problem $\ln(3/4 + t^2)\frac{dy}{dt} = \sin(y)$, $y(T_0) = Y_0$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 1.2.1.)

(b) Find a pair of numbers $T_0$ and $Y_0$ such that the initial value problem $\ln(3/4 + t^2)\frac{dy}{dt} = \sin(y)$, $y(T_0) = Y_0$ has no solutions.

**(AB 3)** Consider the differential equation $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-1)y = 0$.

(a) Find all triples of numbers $T_0$, $Y_0$, and $Y_1$ such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-1)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 4.1.1.)

(b) Find a triple of numbers $T_0$, $Y_0$, and $Y_1$ such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-1)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ has no solutions.

**(AB 4)** Consider the differential equation $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-3)y = 0$.

(a) Find all triples of numbers $T_0$, $Y_0$, and $Y_1$ such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-3)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 4.1.1.)

(b) Find a triple of numbers $T_0$, $Y_0$, and $Y_1$ such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-3)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ has no solutions.

**(AB 5)** Consider the differential equation $\frac{dy}{dt} = y^{3/8}$.

(a) Find all pairs of numbers $T_0$ and $Y_0$ such that the initial value problem $\frac{dy}{dt} = y^{3/8}$, $y(T_0) = Y_0$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 1.2.1.)

(b) Find two solutions to the initial value problem $\frac{dy}{dt} = y^{3/8}$, $y(16) = 0$. Where are your solutions equal?

(c) Find two solutions to the initial value problem $\frac{dy}{dt} = y^{3/8}$, $y(16) = 1$. Where are your solutions equal?

**(AB 6)** Consider the differential equation $\frac{dy}{dt} = 3y(\ln y)^{2/3}$.

($a$) Find all pairs of numbers $T_0$ and $Y_0$ such that the initial value problem $\frac{dy}{dt} = 3y(\ln y)^{2/3}$, $y(T_0) = Y_0$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 1.2.1.)

($b$) Find two solutions to the initial value problem

$$\frac{dy}{dt} = 3y(\ln y)^{2/3}, \quad y(7) = 1.$$

. Where are your solutions equal?

($c$) Find two solutions to the initial value problem

$$\frac{dy}{dt} = 3y(\ln y)^{2/3}, \quad y(7) = e.$$

Where are your solutions equal?

**(AB 7)**

($a$) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0, \quad y(0) = 2, \quad y'(0) = 7$$

is guaranteed a unique solution?

($b$) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0, \quad y(2) = 2, \quad y'(2) = 7$$

is guaranteed a unique solution?

($c$) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0, \quad y(5) = 2, \quad y'(5) = 7$$

is guaranteed a unique solution?

($d$) How many solutions are there to the initial value problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2} + 2t\frac{dy}{dt} - 2y = 0, \quad y(3/2) = 2, \quad y'(3/2) = 7?$$

**(AB 8)** You are given that $g(t)$ and $q(t)$ are continuous functions. Which of the following functions are possible solutions to the initial value problem $(t-5)^3\frac{d^2y}{dt^2} - (3t-15)^2\frac{dy}{dt} + q(t)y = g(t)$, $y(0) = -5$, $y'(0) = -1$? If a function cannot be a solution, explain why not.

($a$) $y(t) = \frac{25}{t-5}$

($b$) $y(t) = \frac{5}{t-1} + 4t$

**(AB 9)** You are given that $y_1 = t^3 + 4t^2 + 4t$ and $y_2 = t^2 + 4t + 4$ are both solutions to the differential equation $(t+2)^2\frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$ for all $t$.

What is the longest interval containing the number $t = -3$ on which the general solution to $(t+2)^2\frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$ may be written $y = C_1(t^3 + 4t^2 + 4t) + C_2(t^2 + 4t + 4)$?

**(AB 10)** Find a solution to the initial value problem
$(t+2)^2\frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$, $y(0) = 4$, $y'(0) = 4$ that is not $y(t) = t^2 + 4t + 4$.

**(AB 11)** Suppose $y_1(t)$ and $y_2(t)$ are both solutions to the initial value problem $(t+2)^2\frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$, $y(5) = 3$, $y'(5) = 7$ on $(-\infty,\infty)$. What is the longest interval on which you are guaranteed that $y_1 = y_2$?

**(AB 12)** The function $y_1(t) = e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2} - (1+2t)\frac{dy}{dt} + (t+1)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

**(AB 13)** The function $y_1(t) = t$ is a solution to the differential equation $t^2\frac{d^2y}{dt^2} - (t^2+2t)\frac{dy}{dt} + (t+2)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

**(AB 14)** The function $y_1(x) = x^3$ is a solution to the differential equation $x^2\frac{d^2y}{dx^2} + (x^2\tan x - 6x)\frac{dy}{dx} + (12 - 3\,x\tan x)y = 0$. Find the general solution to this differential equation on the interval $0 < x < \pi/2$.

**(AB 15)** Find the general solution to the following differential equations.
(a) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$.
(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.
(c) $\frac{d^4y}{dt^4} + 7\frac{d^2y}{dt^2} - 144y = 0$.
(d) $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0$.

**(AB 16)** Solve the following initial-value problems. Express your answers in terms of real functions.
(a) $9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 2y = 0$, $y(0) = 3$, $y'(0) = 2$.
(b) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$, $y(0) = 1$, $y'(0) = 4$.

**(AB 17)** Are the functions $f_1(x) = 0$, $f_2(x) = x$, $f_3(x) = x^2$ linearly independent or linearly dependent on the interval $(-\infty,\infty)$? If they are linearly dependent, find constants $c_1$, $c_2$, and $c_3$, not all zero, such that $c_1f_1(x) + c_2f_2(x) + c_3f_3(x) = 0$ for all $x \in (-\infty,\infty)$.

**(AB 18)** Are the functions $f_1(x) = 3$, $f_2(x) = \cos^2 x$, $f_3(x) = \sin^2 x$ linearly independent or linearly dependent on the interval $(-\infty,\infty)$? If they are linearly dependent, find constants $c_1$, $c_2$, and $c_3$, not all zero, such that $c_1f_1(x) + c_2f_2(x) + c_3f_3(x) = 0$ for all $x \in (-\infty,\infty)$.

**(AB 19)** Are the functions $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$ linearly independent or linearly dependent on the interval $(-\infty,\infty)$? If they are linearly dependent, find constants $c_1$, $c_2$, and $c_3$, not all zero, such that $c_1f_1(x) + c_2f_2(x) + c_3f_3(x) = 0$ for all $x \in (-\infty,\infty)$.

**(AB 20)** An object with mass 5 kg stretches a spring 4 cm. It is attached to a viscous damper with damping constant 16 newton $\cdot$ seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Write the differential equation and initial conditions that describe the position of the object. You may use 9.8 meters/second$^2$ for the acceleration of gravity.

**(AB 21)** A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant $\beta$. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20$ N $\cdot$ s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30$ N $\cdot$ s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

**(AB 22)** A 3-kg object is attached to a spring with constant $k$ and to a viscous damper with damping constant 42 N·sec/m. The object is set in motion from its equilibrium position with initial velocity 5 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object.

If $k = 100$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $k = 200$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

**(AB 23)** A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant $\beta$. The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

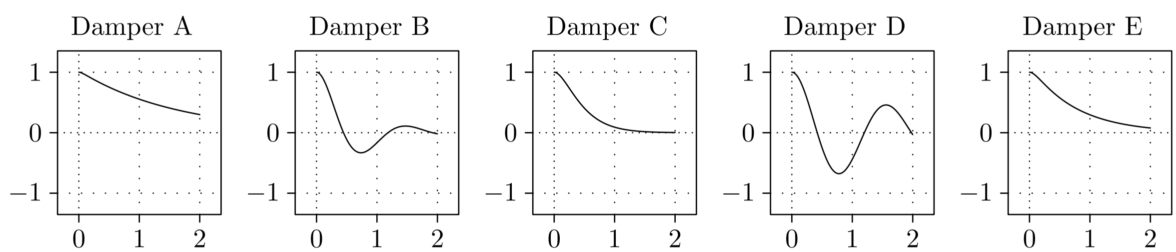
Write the differential equation and initial conditions that describe the position of the object. Then find the value of $\beta$ for which the system is critically damped. Be sure to include units for $\beta$.

**(AB 24)** An object of mass $m$ is attached to a spring with constant 80 N/m and to a viscous damper with damping constant 20 N·s/m. The object is pulled down to 5cm below its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of $m$ for which the system is critically damped. Be sure to include units for $m$.

**(AB 25)** Five objects, each with mass 3 kg, are attached to five springs, each with constant 48 N/m. Five dampers with unknown constants are attached to the objects. In each case, the object is pulled down to a distance 1 cm below the equilibrium position and released from rest. You are given that the system is critically damped in exactly one of the five cases.

Here are the graphs of the objects' positions with respect to time:




(a) For which damper is the system critically damped?
(b) For which dampers is the system overdamped?
(c) For which dampers is the system underdamped?
(d) Which damper has the highest damping constant? Which damper has the lowest damping constant?

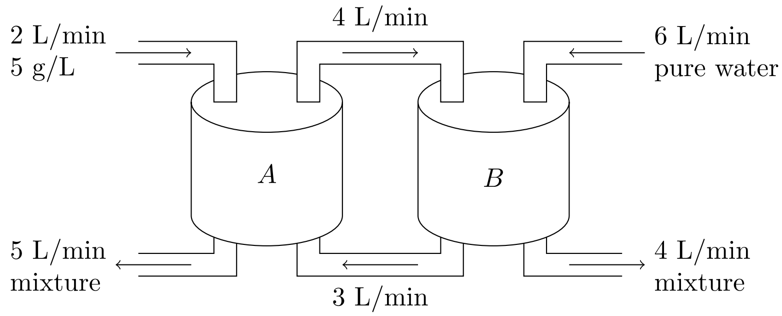
**(AB 26)** An object with mass 5 kg stretches a spring 4 cm. It is attached to a viscous damper with damping constant 16 newton · seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Using only first derivatives, write the differential equations and initial conditions for the object's position. That is, write a first-order system involving the object's position. You may use 9.8 meters/second$^2$ for the acceleration of gravity.

**(AB 27)** A forest is inhabited by rabbits and foxes. Every year, each rabbit gives birth to 3 babies and each fox gives birth to 5 babies on average. Foxes try to catch and eat rabbits; each month, each fox has a 0.02% chance of catching each of the rabbits present in the forest. Each month, every fox that does not eat a rabbit starves to death.

Write the differential equations for the number of foxes and rabbits in the forest.

**(AB 28)** Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.




Write the differential equations and initial conditions that describe the amount of salt in each tank.

**(AB 29)** Write the system

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y$$

in matrix form.

**(AB 30)** Write the system

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = 4y - x$$

in matrix form.

# Answer key

**(AB 1)** (14 points) For each of the following differential equations, find all the equilibrium solutions or state that no such solutions exist. You do not need to find the nonequilibrium solutions.

(a) $\frac{dy}{dt} = y^3 - yt$
$y = 0$.

(b) $\frac{dy}{dt} = t^2 e^y$
There are no equilibrium solutions.

(c) $\frac{dy}{dt} = ty + t^3$
There are no equilibrium solutions.

(d) $\frac{dy}{dt} = (y^2 + 3y + 2)\sin(t)$
$y = -1$ and $y = -2$.

(e) $\frac{dy}{dt} = \ln(y^t)$
$y = 1$.

**(AB 2)** Consider the differential equation $\ln(3/4 + t^2)\frac{dy}{dt} = \sin(y)$.

(a) Find all pairs of numbers $T_0$ and $Y_0$ such that the initial value problem $\ln(3/4 + t^2)\frac{dy}{dt} = \sin(y)$, $y(T_0) = Y_0$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 1.2.1.)

$(T_0, Y_0)$ fails to satisfy the conditions if $T_0 = \pm 1/2$.

(b) Find a pair of numbers $T_0$ and $Y_0$ such that the initial value problem $\ln(3/4 + t^2)\frac{dy}{dt} = \sin(y)$, $y(T_0) = Y_0$ has no solutions.
Here are four possible answers:

- $(T_0, Y_0) = (1/2, 3\pi/2)$,
- $(T_0, Y_0) = (1/2, 0)$,
- $(T_0, Y_0) = (-1/2, \pi/2)$,
- $(T_0, Y_0) = (-1/2, 8)$.

In general, $T_0 = \pm 1/2$, and $Y_0$ may be any real number you like except for $0$, $\pm\pi$, $\pm 2\pi \ldots$

**(AB 3)** Consider the differential equation $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-1)y = 0$.

(a) Find all triples of numbers $T_0$, $Y_0$, and $Y_1$ such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-1)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 4.1.1.)
$(T_0, Y_0, Y_1)$ fails to satisfy the conditions if $T_0 = 3$.

(b) Find a triple of numbers $T_0$, $Y_0$, and $Y_1$ such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-1)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ has no solutions.
Here are four possible answers:

- $(T_0, Y_0, Y_1) = (3, 2, 0)$,
- $(T_0, Y_0, Y_1) = (3, 0, 1)$,
- $(T_0, Y_0, Y_1) = (3, -2, 2)$,
- $(T_0, Y_0, Y_1) = (3, 4, -5)$.

In general, $T_0 = 3$. $Y_0$ and $Y_1$ may be any numbers you like as long as $Y_1 + 2Y_0 \neq 0$.

**(AB 4)** Consider the differential equation $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-3)y = 0$.

(a) Find all triples of numbers $T_0$, $Y_0$, and $Y_1$ such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-3)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 4.1.1.)

$(T_0, Y_0, Y_1)$ fails to satisfy the conditions if $T_0 = 3$.

(b) Find a triple of numbers $T_0$, $Y_0$, and $Y_1$ such that the initial value problem $(t-3)\frac{d^2y}{dt^2} + (t-2)\frac{dy}{dt} + (t-3)y = 0$, $y(T_0) = Y_0$, $y'(T_0) = Y_1$ has no solutions.

Here are four possible answers:

- $(T_0, Y_0, Y_1) = (3, 2, 5)$,
- $(T_0, Y_0, Y_1) = (3, 0, 1)$,
- $(T_0, Y_0, Y_1) = (3, -2, 2)$,
- $(T_0, Y_0, Y_1) = (3, 0, -5)$.

In general, $T_0 = 3$. $Y_0$ and $Y_1$ may be any numbers you like as long as $Y_1 \neq 0$.

**(AB 5)** Consider the differential equation $\frac{dy}{dt} = y^{3/8}$.

(a) Find all pairs of numbers $T_0$ and $Y_0$ such that the initial value problem $\frac{dy}{dt} = y^{3/8}$, $y(T_0) = Y_0$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 1.2.1.)

$(T_0, Y_0)$ fails to satisfy the conditions if $Y_0 = 0$.

(b) Find two solutions to the initial value problem $\frac{dy}{dt} = y^{3/8}$, $y(16) = 0$. Where are your solutions equal?

$y(t) = 0$ and $y(t) = (\frac{5}{8}t - 10)^{8/5}$. The solutions are equal only at $t = 16$.

(c) Find two solutions to the initial value problem $\frac{dy}{dt} = y^{3/8}$, $y(16) = 1$. Where are your solutions equal?

$y(t) = (\frac{5}{8}t - 9)^{8/5}$ and $y(t) = \begin{cases} 0, & t \leq 14.4, \\ (\frac{5}{8}t - 9)^{8/5}, & t \geq 14.4. \end{cases}$ The solutions are equal for all $t \geq 14.4$.

**(AB 6)** Consider the differential equation $\frac{dy}{dt} = 3y(\ln y)^{2/3}$.

(a) Find all pairs of numbers $T_0$ and $Y_0$ such that the initial value problem $\frac{dy}{dt} = 3y(\ln y)^{2/3}$, $y(T_0) = Y_0$ does not satisfy the conditions of the Picard-Lindelöf Theorem (that is, of Theorem 1.2.1.)

$(T_0, Y_0)$ fails to satisfy the conditions if $Y_0 = 1$.

(b) Find two solutions to the initial value problem

$$\frac{dy}{dt} = 3y(\ln y)^{2/3}, \quad y(7) = 1.$$

. Where are your solutions equal?

$y(t) = 1$ and $y(t) = e^{(t-7)^3}$. The solutions are equal only at $t = 7$.

(c) Find two solutions to the initial value problem

$$\frac{dy}{dt} = 3y(\ln y)^{2/3}, \quad y(7) = e.$$

Where are your solutions equal?

$y_1 = e^{(t-6)^3}$ and $y_2 = \begin{cases} e^{(t-6)^3}, & t \geq 6, \\ 1, & t \leq 6. \end{cases}$ The two solutions are equal on the interval $[6, \infty)$.

**(AB 7)**

(a) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0,\quad y(0)=2,\quad y'(0)=7$$

is guaranteed a unique solution?

$(-\infty,3/2)$.

(b) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0,\quad y(2)=2,\quad y'(2)=7$$

is guaranteed a unique solution?

$(3/2,3)$.

(c) What is the largest interval on which the problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0,\quad y(5)=2,\quad y'(5)=7$$

is guaranteed a unique solution?

$(3,\infty)$.

(d) How many solutions are there to the initial value problem

$$(t-3)(2t-3)\frac{d^2y}{dt^2}+2t\frac{dy}{dt}-2y=0,\quad y(3/2)=2,\quad y'(3/2)=7?$$

This initial value problem has no solutions.

**(AB 8)** You are given that $g(t)$ and $q(t)$ are continuous functions. Which of the following functions are possible solutions to the initial value problem $(t-5)^3\frac{d^2y}{dt^2}-(3t-15)^2\frac{dy}{dt}+q(t)y=g(t)$, $y(0)=-5$, $y'(0)=-1$? If a function cannot be a solution, explain why not.

(a) $y(t)=\frac{25}{t-5}$

This is a possible solution.

(b) $y(t)=\frac{5}{t-1}+4t$

This is not a possible solution; the solution is guaranteed to exist (and be differentiable) for all $t<5$, including $t=1$.

**(AB 9)** You are given that $y_1=t^3+4t^2+4t$ and $y_2=t^2+4t+4$ are both solutions to the differential equation $(t+2)^2\frac{d^2y}{dt^2}-(4t+8)\frac{dy}{dt}+6y=0$ for all $t$.

What is the longest interval containing the number $t=-3$ on which the general solution to $(t+2)^2\frac{d^2y}{dt^2}-(4t+8)\frac{dy}{dt}+6y=0$ may be written $y=C_1(t^3+4t^2+4t)+C_2(t^2+4t+4)$?

**(Answer 9)** $(-\infty,-2)$.

**(AB 10)** Find a solution to the initial value problem
$(t+2)^2\frac{d^2y}{dt^2}-(4t+8)\frac{dy}{dt}+6y=0$, $y(0)=4$, $y'(0)=4$ that is not $y(t)=t^2+4t+4$.

**(Answer 10)** There are many possible answers. They include:

$$y(t)=\begin{cases}t^3+4t^2+4t, & t\le -2,\\ t^2+4t+4, & t\ge -2,\end{cases}\qquad y(t)=\begin{cases}-2t^2-8t-8, & t\le -2,\\ t^2+4t+4, & t\ge -2,\end{cases}\qquad y(t)=\begin{cases}0, & t\le -2,\\ t^2+4t+4, & t\ge -2.\end{cases}$$

**(AB 11)** Suppose $y_1(t)$ and $y_2(t)$ are both solutions to the initial value problem $(t+2)^2\frac{d^2y}{dt^2} - (4t+8)\frac{dy}{dt} + 6y = 0$, $y(5) = 3$, $y'(5) = 7$ on $(-\infty, \infty)$. What is the longest interval on which you are guaranteed that $y_1 = y_2$?

**(Answer 11)** $(-2, \infty)$.

**(AB 12)** The function $y_1(t) = e^t$ is a solution to the differential equation $t\frac{d^2y}{dt^2} - (1+2t)\frac{dy}{dt} + (t+1)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

**(Answer 12)** $y(t) = C_1\, e^t + C_2\, t^2 e^t$.

**(AB 13)** The function $y_1(t) = t$ is a solution to the differential equation $t^2\frac{d^2y}{dt^2} - (t^2+2t)\frac{dy}{dt} + (t+2)y = 0$. Find the general solution to this differential equation on the interval $t > 0$.

**(Answer 13)** $y(t) = C_1\, t + C_2\, te^t$.

**(AB 14)** The function $y_1(x) = x^3$ is a solution to the differential equation $x^2\frac{d^2y}{dx^2} + (x^2\tan x - 6x)\frac{dy}{dx} + (12 - 3\,x\tan x)y = 0$. Find the general solution to this differential equation on the interval $0 < x < \pi/2$.

**(Answer 14)** $y(x) = C_1\, x^3 + C_2\, x^3 \sin x$.

**(AB 15)** Find the general solution to the following differential equations.

(a) $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$.
If $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 85y = 0$, then $y = C_1e^{-6t}\cos(7t) + C_2e^{-6t}\sin(7t)$.

(b) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$.
If $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = 0$, then $y = C_1e^{(-2+\sqrt{2})t} + C_2e^{(-2-\sqrt{2})t}$.

(c) $\frac{d^4y}{dt^4} + 7\frac{d^2y}{dt^2} - 144y = 0$.
If $\frac{d^4y}{dt^4} + 7\frac{d^2y}{dt^2} - 144y = 0$, then $y = C_1e^{3t} + C_2e^{-3t} + C_3\cos 4t + C_4\sin 4t$.

(d) $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0$.
If $\frac{d^4y}{dt^4} - 8\frac{d^2y}{dt^2} + 16y = 0$, then $y = C_1e^{2t} + C_2te^{2t} + C_3e^{-2t} + C_4te^{-2t}$.

**(AB 16)** Solve the following initial-value problems. Express your answers in terms of real functions.

(a) $9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 2y = 0$, $y(0) = 3$, $y'(0) = 2$.
If $9\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 2y = 0$, $y(0) = 3$, $y'(0) = 2$, then $y = 3e^{-t/3}\cos(t/3) + 9e^{-t/3}\sin(t/3)$.

(b) $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$, $y(0) = 1$, $y'(0) = 4$.
If $\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + 25y = 0$, $y(0) = 1$, $y'(0) = 4$, then $y = e^{-5t} + 9te^{-5t}$.

**(AB 17)** Are the functions $f_1(x) = 0$, $f_2(x) = x$, $f_3(x) = x^2$ linearly independent or linearly dependent on the interval $(-\infty, \infty)$? If they are linearly dependent, find constants $c_1$, $c_2$, and $c_3$, not all zero, such that $c_1f_1(x) + c_2f_2(x) + c_3f_3(x) = 0$ for all $x \in (-\infty, \infty)$.

**(Answer 17)** They are linearly dependent. $c_1 = 1$ (or any other nonzero number), $c_2 = 0$, $c_3 = 0$.

**(AB 18)** Are the functions $f_1(x) = 3$, $f_2(x) = \cos^2 x$, $f_3(x) = \sin^2 x$ linearly independent or linearly dependent on the interval $(-\infty, \infty)$? If they are linearly dependent, find constants $c_1$, $c_2$, and $c_3$, not all zero, such that $c_1f_1(x) + c_2f_2(x) + c_3f_3(x) = 0$ for all $x \in (-\infty, \infty)$.

**(Answer 18)** They are linearly dependent. $c_1 = 1$, $c_2 = -3$, $c_3 = -3$ is a valid answer (although not the only possible answer).

**(AB 19)** Are the functions $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$ linearly independent or linearly dependent on the interval $(-\infty, \infty)$? If they are linearly dependent, find constants $c_1$, $c_2$, and $c_3$, not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$ for all $x \in (-\infty, \infty)$.

**(Answer 19)** They are linearly independent.

**(AB 20)** An object with mass 5 kg stretches a spring 4 cm. It is attached to a viscous damper with damping constant 16 newton · seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Write the differential equation and initial conditions that describe the position of the object. You may use 9.8 meters/second$^2$ for the acceleration of gravity.

**(Answer 20)** Let $t$ denote time (in seconds) and let $x$ denote the object's displacement above equilibrium (in meters). Then

$$5\frac{d^2x}{dt^2} + 16\frac{dx}{dt} + 1225x = 0, \qquad x(0) = -0.04, \quad x'(0) = 3.$$

**(AB 21)** A 2-kg object is attached to a spring with constant 80 N/m and to a viscous damper with damping constant $\beta$. The object is pulled down to 10cm below its equilibrium position and released with no initial velocity.

Write the differential equation and initial conditions that describe the position of the object.

If $\beta = 20$ N · s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $\beta = 30$ N · s/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

**(Answer 21)** Let $t$ denote time (in seconds) and let $x$ denote the object's displacement above equilibrium (in meters). Then

$$2\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.1, \quad x'(0) = 0.$$

If $\beta = 20$ N · s/m, then the system is underdamped, and we do expect to see decaying oscillations.

If $\beta = 30$ N · s/m, then the system overdamped, and we do not expect to see decaying oscillations.

**(AB 22)** A 3-kg object is attached to a spring with constant $k$ and to a viscous damper with damping constant 42 N·sec/m. The object is set in motion from its equilibrium position with initial velocity 5 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object.

If $k = 100$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

If $k = 200$ N/m, is the system overdamped, underdamped, or critically damped? Do you expect to see decaying oscillations in the solutions?

**(Answer 22)** Let $t$ denote time (in seconds) and let $x$ denote the object's displacement above equilibrium (in meters). Then

$$3\frac{d^2x}{dt^2} + 42\frac{dx}{dt} + kx = 0, \qquad x(0) = 0, \quad x'(0) = -5.$$

If $k = 100$ N/m, then the system is overdamped, and we do not expect to see decaying oscillations.

If $k = 200$ N/m, then the system underdamped, and we do expect to see decaying oscillations.

**(AB 23)** A 4-kg object is attached to a spring with constant 70 N/m and to a viscous damper with damping constant $\beta$. The object is pushed up to 5cm above its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of $\beta$ for which the system is critically damped. Be sure to include units for $\beta$.

**(Answer 23)** Let $t$ denote time (in seconds) and let $x$ denote the object's displacement above equilibrium (in meters). Then

$$4\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + 70x = 0, \qquad x(0) = 0.05, \quad x'(0) = -3.$$

Critical damping occurs when $\beta = 4\sqrt{70}$ N $\cdot$ s/m.

**(AB 24)** An object of mass $m$ is attached to a spring with constant 80 N/m and to a viscous damper with damping constant 20 N·s/m. The object is pulled down to 5cm below its equilibrium position and released with initial velocity 3 m/s downwards.

Write the differential equation and initial conditions that describe the position of the object. Then find the value of $m$ for which the system is critically damped. Be sure to include units for $m$.

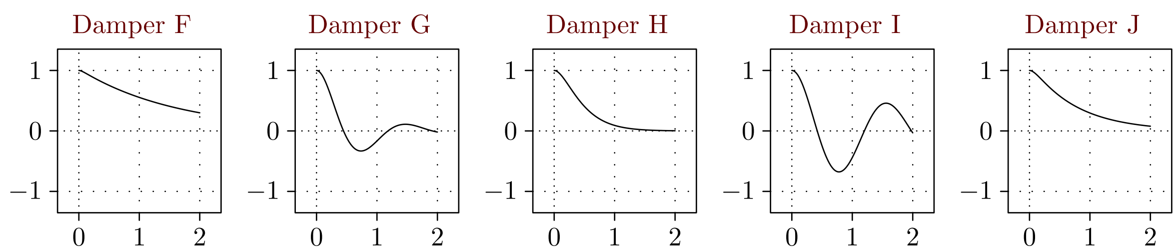
**(Answer 24)** Let $t$ denote time (in seconds) and let $x$ denote the object's displacement above equilibrium (in meters). Then

$$m\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 80x = 0, \qquad x(0) = -0.05, \quad x'(0) = -3.$$

Critical damping occurs when $m = \frac{5}{4}$ kg.

**(AB 25)** Five objects, each with mass 3 kg, are attached to five springs, each with constant 48 N/m. Five dampers with unknown constants are attached to the objects. In each case, the object is pulled down to a distance 1 cm below the equilibrium position and released from rest. You are given that the system is critically damped in exactly one of the five cases.

Here are the graphs of the objects' positions with respect to time:




(*a*) For which damper is the system critically damped?
The system is critically damped for Damper H.

(*b*) For which dampers is the system overdamped?
The system overdamped for Dampers J and F.

(*c*) For which dampers is the system underdamped?
The system underdamped for Dampers G and I.

(*d*) Which damper has the highest damping constant? Which damper has the lowest damping constant?
Damper F has the highest damping constant. Damper I has the lowest damping constant.

**(AB 26)** An object with mass 5 kg stretches a spring 4 cm. It is attached to a viscous damper with damping constant 16 newton $\cdot$ seconds/meter. The object is pulled down an additional 2 cm and is released with initial velocity 3 meters/second upwards.

Using only first derivatives, write the differential equations and initial conditions for the object's position. That is, write a first-order system involving the object's position. You may use 9.8 meters/second$^2$ for the acceleration of gravity.

**(Answer 26)** Let $t$ denote time (in seconds), let $x$ denote the object's displacement above equilibrium (in meters), and let $v$ denote the object's velocity (in meters per second). Then

$$5\frac{dv}{dt} + 16v + 1225x = 0, \qquad \frac{dx}{dt} = v, \qquad x(0) = -0.04, \quad v(0) = 3.$$

**(AB 27)** A forest is inhabited by rabbits and foxes. Every year, each rabbit gives birth to 3 babies and each fox gives birth to 5 babies on average. Foxes try to catch and eat rabbits; each month, each fox has a 0.02% chance of catching each of the rabbits present in the forest. Each month, every fox that does not eat a rabbit starves to death.

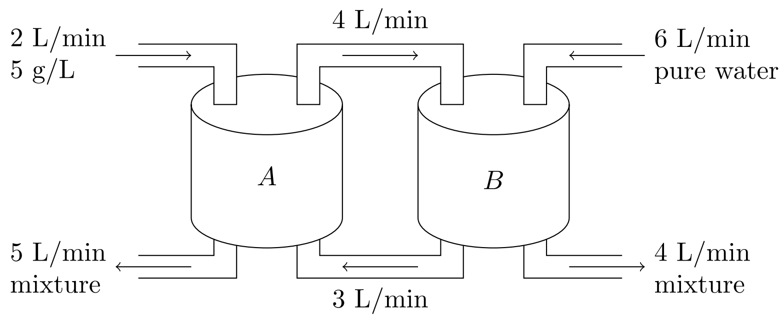
Write the differential equations for the number of foxes and rabbits in the forest.

**(Answer 27)** Let $t$ denote time (in months), let $R$ denote the number of rabbits, and let $F$ denote the number of foxes.

Then

$$\frac{dR}{dt} = \frac{3}{12}R - 0.0002RF, \quad \frac{dF}{dt} = \frac{5}{12}F - (F - 0.0002RF).$$

**(AB 28)** Consider the following system of tanks. Tank A initially contains 200 L of water in which 3 kg of salt have been dissolved, and Tank B initially contains 300 L of water in which 2 kg of salt have been dissolved. Salt water flows into each tank at the rates shown, and the well-stirred solution flows between the two tanks and is drained away through the pipes shown at the indicated rates.




Write the differential equations and initial conditions that describe the amount of salt in each tank.

**(Answer 28)** Let $t$ denote time (in minutes).

Let $x$ denote the amount of salt (in grams) in tank A.

Let $y$ denote the amount of salt (in grams) in tank B.

Then $x(0) = 3000$ and $y(0) = 2000$.

If $t < 50$, then

$$\frac{dx}{dt} = 10 - \frac{9x}{200 - 4t} + \frac{3y}{300 + 3t}, \qquad \frac{dy}{dt} = \frac{4x}{200 - 4t} - \frac{7y}{300 + 3t}.$$

**(AB 29)** Write the system

$$\frac{dx}{dt} = 6x + 8y, \qquad \frac{dy}{dt} = -2x - 2y$$

in matrix form.

**(Answer 29)**

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

**(AB 30)** Write the system

$$\frac{dx}{dt} = 5y, \qquad \frac{dy}{dt} = 4y - x$$

in matrix form.

**(Answer 30)**

$$\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} 0 & 5 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$