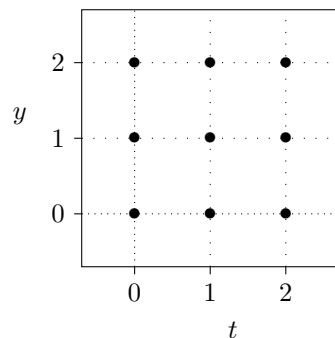
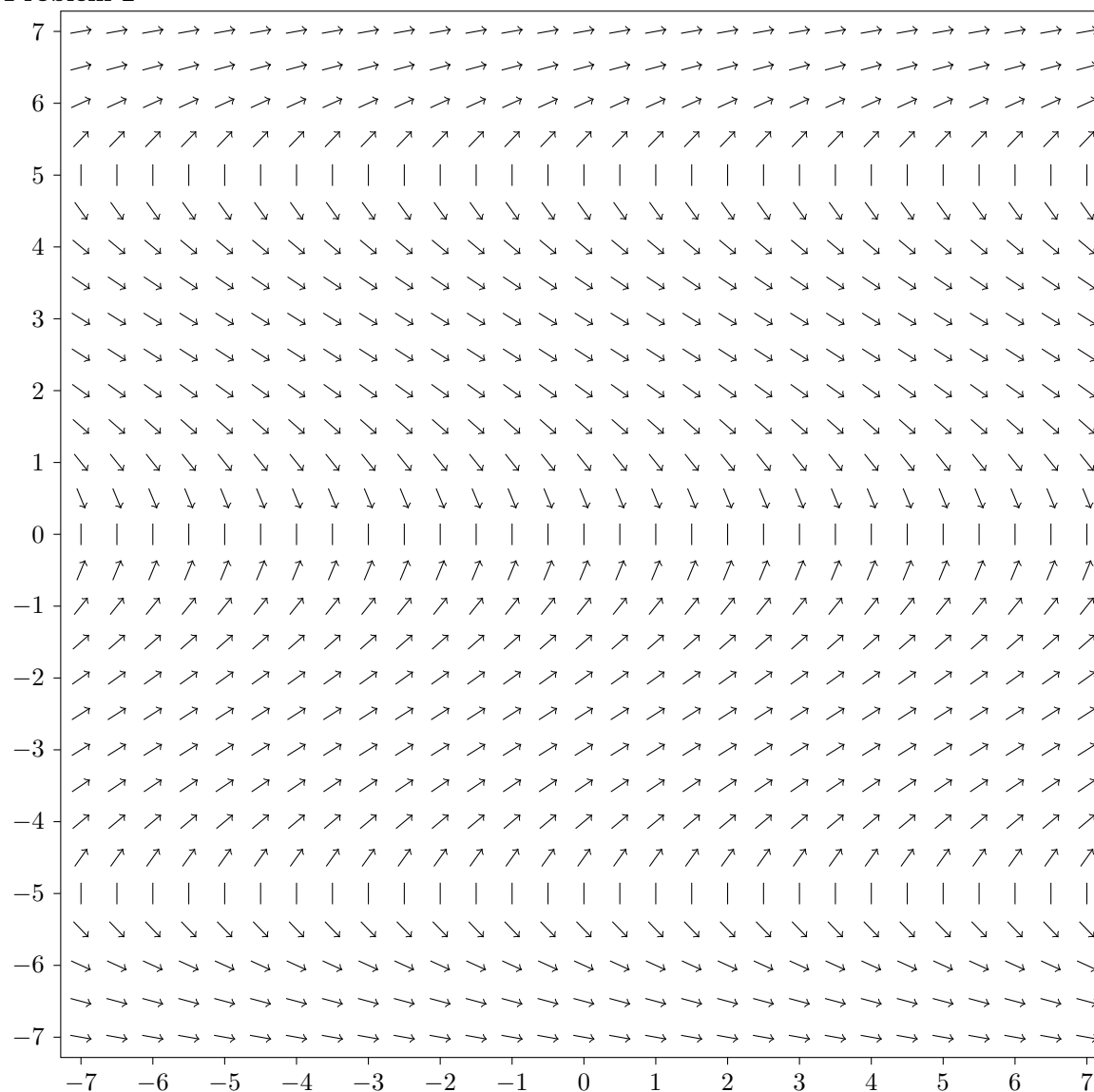


If you are taking the exam remotely, please print out this page for direction field questions. **You are only allowed** to take the exam remotely if you have prior permission from Professor Barton, have active COVID symptoms, or have been directed to self-isolate by a doctor or contact tracer.

Problem 1



Problem 2



Math 2584, Spring 2021

Exam 1 will occur:

- Thursday, February 11 at 9:30, in HILL 206
- Thursday, February 11 at 11:00, in SCEN 402
- Friday, February 12 at 8:35, in KIMP 102.

You are allowed a non-graphing calculator.

(AB 1) Is $y = e^t$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t} \frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 2) Is $y = e^{2t}$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t} \frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(AB 3) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 1$?

(AB 4) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 2$?

(AB 5) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 1$?

(AB 6) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 2$?

(AB 7) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

- $\frac{dy}{dt} + \arctan(t)y = e^t$, $y(3) = 7$.
- $\frac{1}{1+t^2} \frac{dy}{dt} - t^5y = \cos(6t)$, $y(2) = -1$, $y'(2) = 3$.
- $\frac{d^2y}{dt^2} - 5\sin(t)y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.
- $e^t \frac{d^2y}{dt^2} + 3(t-4) \frac{dy}{dt} + 4t^6y = 2$, $y(3) = 1$, $y'(3) = -1$.
- $\frac{d^2y}{dt^2} + \cos(t) \frac{dy}{dt} + 3\ln(1+t^2)y = 0$, $y(2) = 3$.
- $\frac{d^3y}{dt^3} - t^7 \frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)y = t^3$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$, $y'''(1) = 3$.
- $(1+t^2) \frac{d^3y}{dt^3} - 3 \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 2y = t^3$, $y(3) = 9$, $y'(3) = 7$, $y''(3) = 5$.
- $\frac{d^3y}{dt^3} - e^t \frac{d^2y}{dt^2} + e^{2t} \frac{dy}{dt} + e^{3t}y = e^{4t}$, $y(-1) = 1$, $y'(-1) = 3$.
- $(2 + \sin t) \frac{d^3y}{dt^3} + \cos t \frac{d^2y}{dt^2} + \frac{dy}{dt} + 5\sin(t)y = t^3$, $y(7) = 2$.

(AB 8) A lake initially has a population of 600 trout. The birth rate of trout is proportional to the number of trout living in the lake. Fishermen are allowed to harvest 30 trout/year from the lake. Write an initial value problem for the fish population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 9) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write an initial value problem for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 10) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen. Write an initial value problem for the amount of hydrogen in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 11) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C . Write an initial value problem for the temperature of the cup of coffee. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 12) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Write an initial value problem for the amount of salt in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 13) I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write an initial value problem for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 14) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write an initial value problem for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 15) According to the Stefan-Boltzmann law, a black body in a dark vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

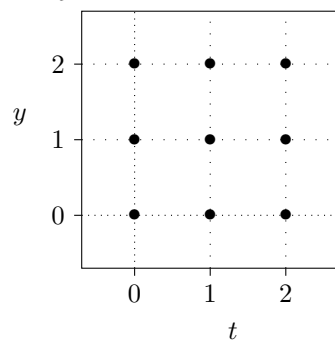
(AB 16) According to the Stefan-Boltzmann law, a black body in a vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). If there is some ambient light in the vacuum, then the black body absorbs energy at a rate proportional to the effective temperature of the light. The proportionality constant is the same in both cases; that is, if the object's temperature is equal to the effective temperature of the light, then its temperature will not change.

Suppose that a planet in space is currently at a temperature of 400 K. The effective temperature of its surroundings is 3 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 17) A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

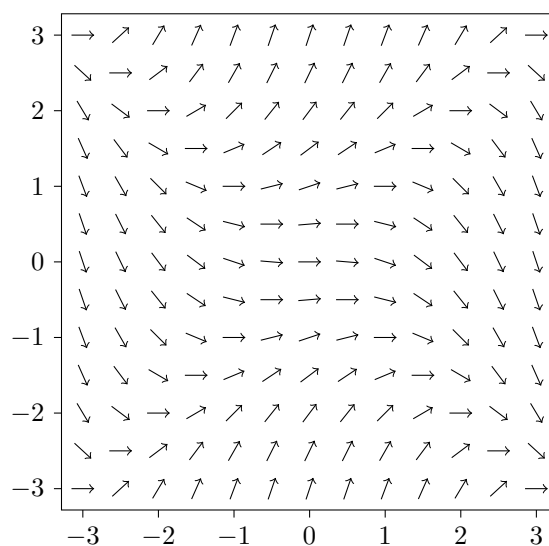
(AB 18) A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(AB 19) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.



(AB 20) Consider the differential equation $\frac{dy}{dx} = (y^2 - x^2)/3$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

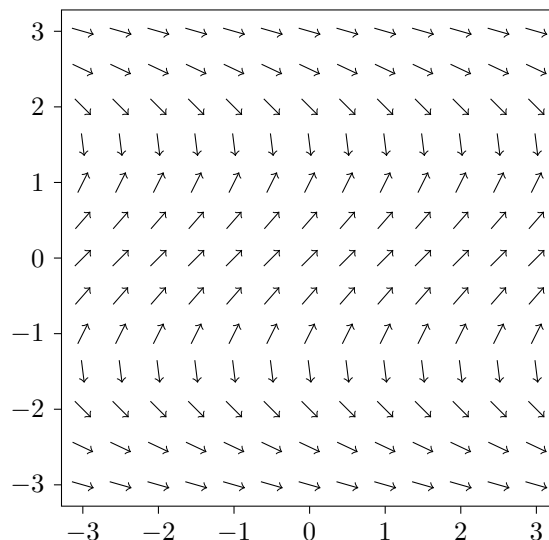
$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-2) = 1.$$



(AB 21) Consider the differential equation $\frac{dy}{dx} = \frac{2}{2-y^2}$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{2}{2-y^2}, \quad y(1) = 0.$$

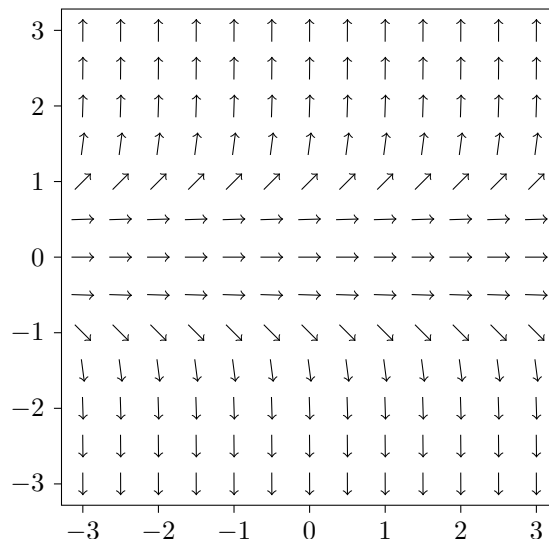
Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 22) Consider the differential equation $\frac{dy}{dx} = y^5$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = y^5, \quad y(1) = -1.$$

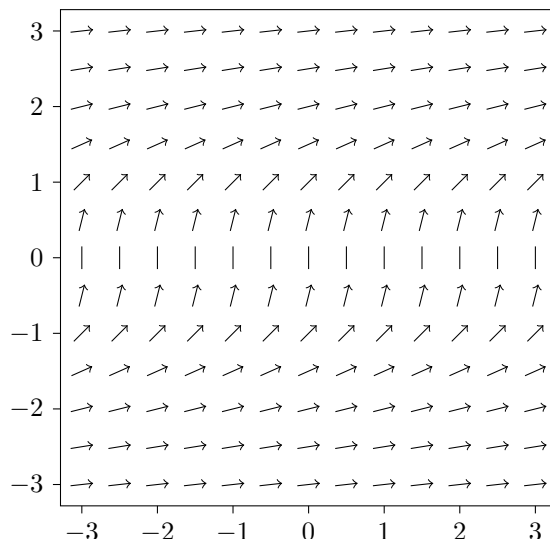
Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 23) Consider the differential equation $\frac{dy}{dx} = 1/y^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{1}{y^2}, \quad y(3) = 2.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?



(AB 24) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y - 2)(y + 1)^2$. By hand, sketch some typical solutions.

(AB 25) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 26) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = y^2(y - 2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(AB 27) I owe my bank a debt of B dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

- Formulate a differential equation for the amount of money I owe.
- Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

(AB 28) A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

- Write the differential equation for the amount of salt in the tank.
- Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

(AB 29) A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where v is her velocity in meters per second.

- Write a differential equation for her velocity. Assume her velocity is always downwards.
- Find the (negative) critical points of this differential equation and classify them as to stability. Be sure to include units.
- What is the real-world meaning of these critical points?

(AB 30) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen.

- Write a differential equation for the amount of hydrogen left in the tank.
- Find the critical points of this differential equation and classify them as to stability. Be sure to include units.
- What is the real-world meaning of these critical points?

(AB 31) Young oak trees grow in a large field with area 1 square kilometer. Initially, there are 30 trees in the field. Each tree shades 20 square meters. The trees produce acorns. Squirrels scatter the acorns across the field at random. Squirrels eat most of the acorns; on average, each tree produces two acorns per year that are not eaten by squirrels. Every acorn not eaten by squirrels and planted in sunlight sprouts.

- Write a differential equation for the number of trees in the field.
- Find the critical points of this differential equation and classify them as to stability.
- What is the real-world meaning of these critical points?

(AB 32) For each of the following differential equations, determine whether it is linear, separable, exact, homogeneous, a Bernoulli equation, or of the form $\frac{dy}{dt} = f(At + By + C)$. Then solve the differential equation.

- $\frac{dy}{dt} = \frac{t + \cos t}{\sin y - y}$
- $\ln y + y + t + \left(\frac{t}{y} + t\right) \frac{dy}{dt} = 0$
- $(t^2 + 1) \frac{dy}{dt} = ty - t^2 - 1$
- $t^2 \frac{dy}{dt} = y^2 + t^2 - ty$
- $4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5) \frac{dy}{dt} = 0$
- $\frac{dy}{dt} = -y \tan 2t - y^3 \cos 2t$
- $(t + y) \frac{dy}{dt} = 5y - 3t$
- $\frac{dy}{dt} = \frac{1}{3t + 2y + 7}$
- $\frac{dy}{dt} = -y^3 \cos(2t)$
- $4ty \frac{dy}{dt} = 3y^2 - 2t^2$
- $t \frac{dy}{dt} = 3y - \frac{t^2}{y^5}$
- $\frac{dy}{dt} = \csc^2(y - t)$
- $\frac{dy}{dt} = 8y - y^8$
- $t \frac{dy}{dt} = -\cos t - 3y$
- $\frac{dy}{dt} = \cot(y/t) + y/t$
- $\frac{dy}{dt} = ty + t^2 \sqrt{[3]y}$

(AB 33) For each of the following differential equations, determine whether it is linear, separable, exact, homogeneous, Bernoulli, or a function of a linear term. Then solve the given initial-value problem.

- $\cos(t + y^3) + 2t + 3y^2 \cos(t + y^3) \frac{dy}{dt} = 0, y(\pi/2) = 0$
- $\frac{dy}{dt} = \sin t \cos y, y(\pi) = \pi/2$
- $2ty \frac{dy}{dt} = 4t^2 - y^2, y(1) = 3.$
- $t \frac{dy}{dt} = -1 - y^2, y(1) = 1$
- $\frac{dy}{dt} = (2y + 2t - 5)^2, y(0) = 3.$
- $\frac{dy}{dt} = 2y - \frac{6}{y^2}, y(0) = 7.$
- $\frac{dy}{dt} = -3y - \sin t e^{-3t}, y(0) = 2$
- $\frac{dy}{dt} = (y^2 - 5y + 4)te^{t^2}, y(0) = 4$

(AB 34) Solve the initial-value problem $\frac{dy}{dt} = \frac{t-5}{y}, y(0) = 3$ and determine the range of t -values in which the solution is valid.

(AB 35) Solve the initial-value problem $\frac{dy}{dt} = y^2$, $y(0) = 1/4$ and determine the range of t -values in which the solution is valid.

Answer key

(AB 1) Is $y = e^t$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t} \frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(Answer 1) No, $y = e^t$ is not a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t} \frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(AB 2) Is $y = e^{2t}$ a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t} \frac{dy}{dt} - \frac{4}{1-2t}y = 0$?

(Answer 2) Yes, $y = e^{2t}$ is a solution to the differential equation $\frac{d^2y}{dt^2} + \frac{4t}{1-2t} \frac{dy}{dt} - \frac{4}{1-2t}y = 0$.

(AB 3) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 1$?

(Answer 3) No.

(AB 4) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 2$?

(Answer 4) No.

(AB 5) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 3y$, $y(0) = 1$?

(Answer 5) Yes.

(AB 6) Is $y = e^{3t}$ a solution to the initial value problem $\frac{dy}{dt} = 2y$, $y(0) = 2$?

(Answer 6) No.

(AB 7) For each of the following initial-value problems, tell me whether we expect to have an infinite family of solutions, no solutions, or a unique solution. Do not find the solution to the differential equation.

(a) $\frac{dy}{dt} + \arctan(t)y = e^t$, $y(3) = 7$.

We expect a unique solution.

(b) $\frac{1}{1+t^2} \frac{dy}{dt} - t^5y = \cos(6t)$, $y(2) = -1$, $y'(2) = 3$.

We do not expect any solutions.

(c) $\frac{d^2y}{dt^2} - 5 \sin(t)y = t$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$.

We do not expect any solutions.

(d) $e^t \frac{d^2y}{dt^2} + 3(t-4) \frac{dy}{dt} + 4t^6y = 2$, $y(3) = 1$, $y'(3) = -1$.

We expect a unique solution.

(e) $\frac{d^2y}{dt^2} + \cos(t) \frac{dy}{dt} + 3 \ln(1+t^2)y = 0$, $y(2) = 3$.

We expect an infinite family of solutions.

(f) $\frac{d^3y}{dt^3} - t^7 \frac{d^2y}{dt^2} + \frac{dy}{dt} + 5 \sin(t)y = t^3$, $y(1) = 2$, $y'(1) = 5$, $y''(1) = 0$, $y'''(1) = 3$.

We do not expect any solutions.

(g) $(1+t^2) \frac{d^3y}{dt^3} - 3 \frac{d^2y}{dt^2} + 5 \frac{dy}{dt} + 2y = t^3$, $y(3) = 9$, $y'(3) = 7$, $y''(3) = 5$.

We expect a unique solution.

(h) $\frac{d^3y}{dt^3} - e^t \frac{d^2y}{dt^2} + e^{2t} \frac{dy}{dt} + e^{3t}y = e^{4t}$, $y(-1) = 1$, $y'(-1) = 3$.

We expect an infinite family of solutions.

(i) $(2 + \sin t) \frac{d^3y}{dt^3} + \cos t \frac{d^2y}{dt^2} + \frac{dy}{dt} + 5 \sin(t)y = t^3$, $y(7) = 2$.

We expect an infinite family of solutions.

(AB 8) A lake initially has a population of 600 trout. The birth rate of trout is proportional to the number of trout living in the lake. Fishermen are allowed to harvest 30 trout/year from the lake. Write an initial value problem for the fish population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 8) Independent variable: $t = \text{time (in years)}$.

Dependent variable: $P = \text{Number of trout in the lake}$

Initial condition: $P(0) = 600$.

Parameters: $\alpha = \text{birth rate (in 1/years)}$.

Differential equation: $\frac{dP}{dt} = \alpha P - 30$.

(AB 9) Five songbirds are blown off course onto an island with no birds on it. The birth rate of songbirds on the island is then proportional to the number of birds living on the island, and the death rate is proportional to the square of the number of birds living on the island. Write an initial value problem for the bird population. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 9) Independent variable: $t = \text{time (in years)}$.

Dependent variable: $P = \text{Number of birds on the island}$.

Parameters: $\alpha = \text{birth rate parameter (in 1/years)}$; $\beta = \text{death rate parameter (in 1/(bird-years))}$.

Initial condition: $P(0) = 5$.

Differential equation: $\frac{dP}{dt} = \alpha P - \beta P^2$.

(AB 10) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen. Write an initial value problem for the amount of hydrogen in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 10) Independent variable: $t = \text{time (in minutes)}$.

Dependent variable: $H = \text{Amount of hydrogen in the tank (in grams)}$.

Parameters: $\alpha = \text{reaction rate parameter (in 1/second-grams)}$.

Initial condition: $H(0) = 50$.

Differential equation: $\frac{dH}{dt} = -\alpha H(126.9H - 3345)$.

(AB 11) According to Newton's law of cooling, the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that a cup of coffee is initially at a temperature of 95°C and is placed in a room at a temperature of 25°C . Write an initial value problem for the temperature of the cup of coffee. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 11) Independent variable: $t = \text{time (in seconds)}$.

Dependent variable: $T = \text{Temperature of the cup (in degrees Celsius)}$

Initial condition: $T(0) = 95$.

Differential equation: $\frac{dT}{dt} = -\alpha(T - 25)$, where α is a positive parameter (constant of proportionality) with units of 1/seconds.

(AB 12) A large tank initially contains 600 liters of water in which 3 kilograms of salt have been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 3 liters of the well-mixed solution flows out. Write an initial value problem for the amount of salt in the tank. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 12) Independent variable: $t = \text{time (in minutes)}$.

Dependent variable: $Q = \text{amount of dissolved salt (in kilograms)}$.

Initial condition: $Q(0) = 3$.

Differential equation: $\frac{dQ}{dt} = 0.01 - \frac{3Q}{600-t}$.

(AB 13) I want to buy a house. I borrow \$300,000 and can spend \$1600 per month on mortgage payments. My lender charges 4% interest annually, compounded continuously. Suppose that my payments are also made continuously. Write an initial value problem for the amount of money I owe. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 13) Independent variable: t = time (in years).

Dependent variable: B = balance of my loan (in dollars).

Initial condition: $B(0) = 300,000$.

Differential equation: $\frac{dB}{dt} = 0.04B - 12 \cdot 1600$

(AB 14) A hole in the ground is in the shape of a cone with radius 1 meter and depth 20 cm. Initially, it is empty. Water leaks into the hole at a rate of 1 cm^3 per minute. Water evaporates from the hole at a rate proportional to the area of the exposed surface. Write an initial value problem for the volume of water in the hole. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 14) Independent variable: t = time (in minutes).

Dependent variables:

h = depth of water in the hole (in centimeters)

V = volume of water in the hole (in cubic centimeters); notice that $V = \frac{1}{3}\pi(5h)^2h$

Initial condition: $V(0) = 0$.

Differential equation: $\frac{dV}{dt} = 1 - \alpha\pi(5h)^2 = 1 - 25\alpha\pi(3V/25\pi)^{2/3}$, where α is a proportionality constant with units of cm/s.

(AB 15) According to the Stefan-Boltzmann law, a black body in a dark vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). Suppose that a planet in space is currently at a temperature of 400 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 15) Independent variable: t = time (in seconds).

Dependent variable: T = object's temperature (in kelvins)

Parameter: σ = proportionality constant (in $1/(\text{seconds} \cdot \text{kelvin}^3)$)

Initial condition: $T(0) = 400$.

Differential equation: $\frac{dT}{dt} = -\sigma T^4$.

(AB 16) According to the Stefan-Boltzmann law, a black body in a vacuum radiates energy (in the form of light) at a rate proportional to the fourth power of its temperature (in kelvins). If there is some ambient light in the vacuum, then the black body absorbs energy at a rate proportional to the effective temperature of the light. The proportionality constant is the same in both cases; that is, if the object's temperature is equal to the effective temperature of the light, then its temperature will not change.

Suppose that a planet in space is currently at a temperature of 400 K. The effective temperature of its surroundings is 3 K. Write an initial value problem for the planet's temperature. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 16) Independent variable: t = time (in seconds).

Dependent variable: T = object's temperature (in kelvins)

Parameter: σ = proportionality constant (in $1/(\text{seconds} \cdot \text{kelvin}^3)$)

Initial condition: $T(0) = 400$.

Differential equation: $\frac{dT}{dt} = -\sigma T^4 + 81\sigma$.

(AB 17) A ball is thrown upwards with an initial velocity of 10 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 17) Independent variable: $t = \text{time (in seconds)}$.

Dependent variable: $v = \text{velocity of the ball (in meters/second, where a positive velocity denotes upward motion)}$.

Parameters: $\alpha = \text{proportionality constant of the drag force (in newton-seconds/meter)}$

$m = \text{mass of the ball (in kilograms)}$

Initial condition: $v(0) = 10$.

Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/m)v$.

(AB 18) A ball of mass 300 g is thrown upwards with an initial velocity of 20 meters per second. The ball experiences a downwards force due to the Earth's gravity and a drag force proportional to the square of its velocity. Write an initial value problem for the ball's velocity. Specify your independent and dependent variables, any unknown parameters, and your initial condition.

(Answer 18) Independent variable: $t = \text{time (in seconds)}$.

Dependent variable: $v = \text{velocity of the ball (in meters/second, where a positive velocity denotes upward motion)}$.

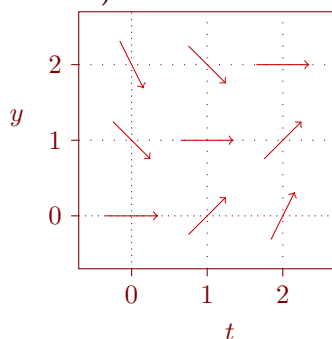
Parameter: $\alpha = \text{proportionality constant of the drag force (in newton-seconds/meter)}$.

Initial condition: $v(0) = 20$.

Differential equation: $\frac{dv}{dt} = -9.8 - (\alpha/0.3)v|v|$, or $\frac{dv}{dt} = \begin{cases} -9.8 - (\alpha/0.3)v^2, & v \geq 0, \\ -9.8 + (\alpha/0.3)v^2, & v < 0. \end{cases}$

(AB 19) Here is a grid. Draw a small direction field (with nine slanted segments) for the differential equation $\frac{dy}{dt} = t - y$.

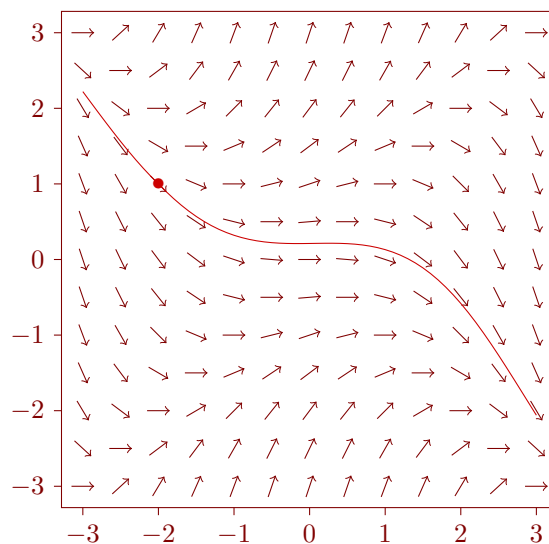
(Answer 19) Here is the direction field for the differential equation $\frac{dy}{dt} = t - y$.



(AB 20) Consider the differential equation $\frac{dy}{dx} = (y^2 - x^2)/3$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = (y^2 - x^2)/3, \quad y(-2) = 1.$$

(Answer 20)

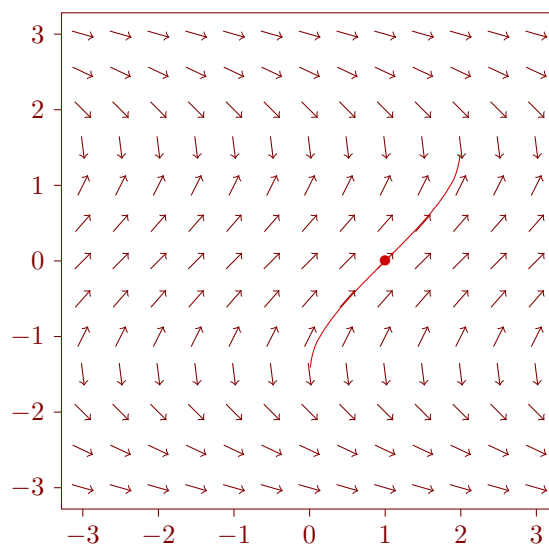


(AB 21) Consider the differential equation $\frac{dy}{dx} = \frac{2}{2-y^2}$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{2}{2-y^2}, \quad y(1) = 0.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?

(Answer 21)



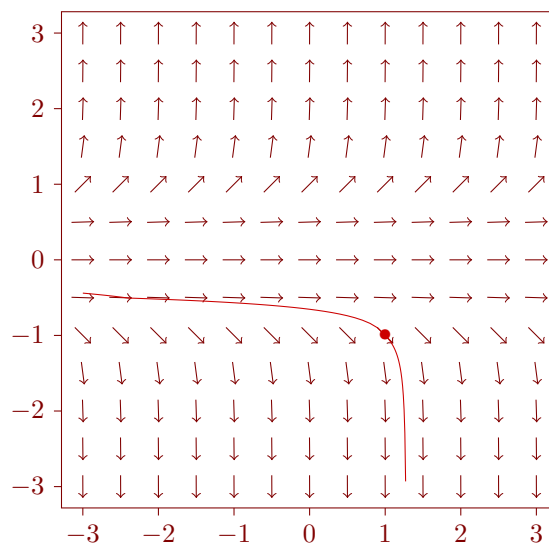
The domain of definition of the solution appears to be $0 < t < 2$.

(AB 22) Consider the differential equation $\frac{dy}{dx} = y^5$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = y^5, \quad y(1) = -1.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?

(Answer 22)



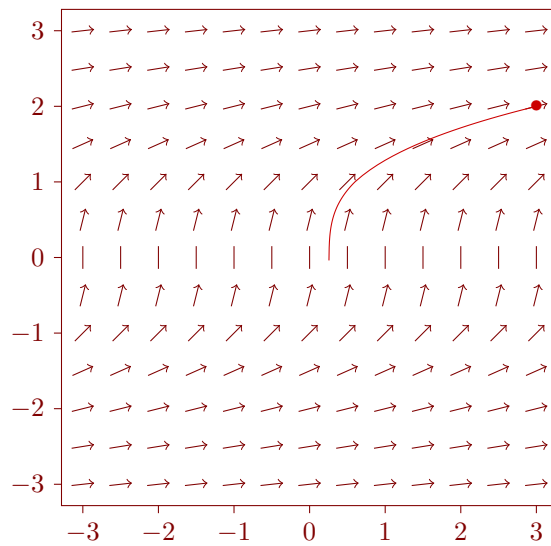
The domain of definition of the solution appears to be approximately $t < 1.3$.

(AB 23) Consider the differential equation $\frac{dy}{dx} = 1/y^2$. Here is the direction field for this differential equation. Sketch, approximately, the solution to

$$\frac{dy}{dx} = \frac{1}{y^2}, \quad y(3) = 2.$$

Based on your sketch, what is the (approximate) domain of definition of the solution to this differential equation?

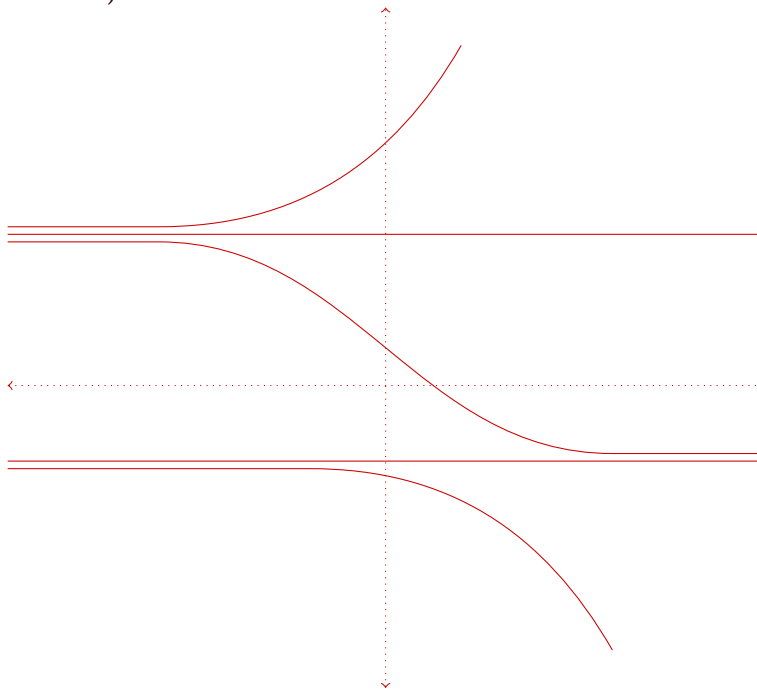
(Answer 23)



The domain of definition of the solution appears to be $\frac{1}{4} < t$.

(AB 24) Consider the autonomous first-order differential equation $\frac{dy}{dx} = (y - 2)(y + 1)^2$. By hand, sketch some typical solutions.

(Answer 24)



(AB 25) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = \sin y$. Classify each critical point as asymptotically stable, unstable, or semistable.

(Answer 25)

Critical points: $y = k\pi$ for any integer k .



If k is even then $y = k\pi$ is unstable.

If k is odd then $y = k\pi$ is stable.

(AB 26) Find the critical points and draw the phase portrait of the differential equation $\frac{dy}{dx} = y^2(y - 2)$. Classify each critical point as asymptotically stable, unstable, or semistable.

(Answer 26)

Critical points: $y = 0$ and $y = 2$.



$y = 0$ is semistable. $y = 2$ is unstable.

(AB 27) I owe my bank a debt of B dollars. The bank assesses an interest rate of 5% per year, compounded continuously. I pay the debt off continuously at a rate of 1600 dollars per month.

- (a) Formulate a differential equation for the amount of money I owe.

$$\frac{dB}{dt} = 0.05B - 19200, \text{ where } t \text{ denotes time in years.}$$

- (b) Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

The critical point is $B = \$384,000$. It is unstable. If my initial debt is less than \$384,000, then I will eventually pay it off (have a debt of zero dollars), but if my initial debt is greater than \$384,000, my debt will grow exponentially. The critical point $B = 384,000$ corresponds to the balance that will allow me to make interest-only payments on my debt.

(AB 28) A large tank contains 600 liters of water in which salt has been dissolved. Every minute, 2 liters of a salt solution with a concentration of 5 grams per liter flows in, and 2 liters of the well-mixed solution flows out.

- (a) Write the differential equation for the amount of salt in the tank.

$$\frac{dQ}{dt} = 10 - Q/300, \text{ where } Q \text{ denotes the amount of salt in grams and } t \text{ denotes time in minutes.}$$

- (b) Find the critical points of this differential equation and classify them as to stability. What is the real-world meaning of the critical points?

The critical point is $Q = 3000$. It is stable. No matter the initial amount of salt in the tank, as time passes the amount of salt will approach 3000 g or 3 kg.

(AB 29) A skydiver with a mass of 70 kg falls from an airplane. She deploys a parachute, which produces a drag force of magnitude $2v^2$ newtons, where v is her velocity in meters per second.

- (a) Write a differential equation for her velocity. Assume her velocity is always downwards.

$$\text{If } v \leq 0 \text{ then } 70 \frac{dv}{dt} = -70 * 9.8 + 2v^2. \text{ (If } v > 0 \text{ then } 70 \frac{dv}{dt} = -70 * 9.8 - 2v^2.)$$

- (b) Find the (negative) critical points of this differential equation and classify them as to stability. Be sure to include units.

$v = -\sqrt{343}$ meters/second is a stable critical point.

- (c) What is the real-world meaning of these critical points?

As $t \rightarrow \infty$, her velocity will approach the stable critical point of $v = -\sqrt{343}$ meters/second.

(AB 30) A tank contains hydrogen gas and iodine gas. Initially, there are 50 grams of hydrogen and 3000 grams of iodine. These two gases react to form hydrogen iodide at a rate proportional to the product of the amount of iodine remaining and the amount of hydrogen remaining. The reaction consumes 126.9 grams of iodine for every gram of hydrogen.

- (a) Write a differential equation for the amount of hydrogen left in the tank.
 $\frac{dH}{dt} = -\alpha H(126.9H - 3345)$, where H is the amount of hydrogen remaining (in grams), t denotes time in minutes, and α is a positive parameter.
- (b) Find the critical points of this differential equation and classify them as to stability. Be sure to include units.
 $H = 0$ grams (unstable) and $H = 26.36$ grams (stable).
- (c) What is the real-world meaning of these critical points?
 As $t \rightarrow \infty$, the amount of hydrogen in the tank will tend towards 26.36 grams. The $H = 0$ critical point is not physically meaningful, as if $H = 0$ grams then the model predicts -3345 grams of iodine in the tank.

(AB 31) Young oak trees grow in a large field with area 1 square kilometer. Initially, there are 30 trees in the field. Each tree shades 20 square meters. The trees produce acorns. Squirrels scatter the acorns across the field at random. Squirrels eat most of the acorns; on average, each tree produces two acorns per year that are not eaten by squirrels. Every acorn not eaten by squirrels and planted in sunlight sprouts.

- (a) Write a differential equation for the number of trees in the field.
 $\frac{dP}{dt} = 2P(1 - 20P/1,000,000)$, where P is the number of trees in the field and t denotes time in years.
- (b) Find the critical points of this differential equation and classify them as to stability.
 $P = 0$ (unstable) and $P = 50,000$ trees (stable).
- (c) What is the real-world meaning of these critical points?
 If there are initially no trees in the field, then the number of trees will remain at the $P = 0$ equilibrium. However, if there are initially any trees in the field, then as $t \rightarrow \infty$, the number of trees will approach 50,000.

(AB 32) For each of the following differential equations, determine whether it is linear, separable, exact, homogeneous, a Bernoulli equation, or of the form $\frac{dy}{dt} = f(At + By + C)$. Then solve the differential equation.

- (a) $\frac{dy}{dt} = \frac{t + \cos t}{\sin y - y}$
 $\frac{dy}{dt} = \frac{t + \cos t}{\sin y - y}$ is separable and has solution $\frac{1}{2}y^2 + \cos y = -\frac{1}{2}t^2 - \sin t + C$.
- (b) $\ln y + y + t + \left(\frac{t}{y} + t\right)\frac{dy}{dt} = 0$
 $\ln y + y + t + \left(\frac{t}{y} + t\right)\frac{dy}{dt} = 0$ is exact and has solution $t \ln y + ty + \frac{1}{2}t^2 = C$.
- (c) $(t^2 + 1)\frac{dy}{dt} = ty - t^2 - 1$
 $(t^2 + 1)\frac{dy}{dt} = ty - t^2 - 1$ is linear and has solution $y = -\sqrt{t^2 + 1} \ln(t + \sqrt{t^2 + 1}) + C\sqrt{t^2 + 1}$.
- (d) $t^2\frac{dy}{dt} = y^2 + t^2 - ty$
 $t^2\frac{dy}{dt} = y^2 + t^2 - ty$ is homogeneous and has solution $y = \frac{t}{C - \ln|t|} + t$.
- (e) $4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5)\frac{dy}{dt} = 0$
 $4ty^3 + 6t^3 + (3 + 6t^2y^2 + y^5)\frac{dy}{dt} = 0$ is exact and has solution $2t^2y^3 + \frac{3}{2}t^4 + 3y + \frac{1}{6}y^6 = C$.
- (f) $\frac{dy}{dt} = -y \tan 2t - y^3 \cos 2t$
 $\frac{dy}{dt} = -y \tan 2t - y^3 \cos 2t$ is Bernoulli. Let $v = y^{-2}$. Then $\frac{dv}{dt} = 2v \tan 2t + 2 \cos 2t$, so $v = \frac{1}{2} \sin(2t) + t \sec(2t) + C \sec(2t)$ and $y = \frac{1}{\sqrt{\frac{1}{2} \sin(2t) + t \sec(2t) + C \sec(2t)}}$.
- (g) $(t + y)\frac{dy}{dt} = 5y - 3t$
 $(t + y)\frac{dy}{dt} = 5y - 3t$ is homogeneous and has solution $(y - 3t)^2 = Ct^2(y - t)$.
- (h) $\frac{dy}{dt} = \frac{1}{3t + 2y + 7}$
If $\frac{dy}{dt} = \frac{1}{3t + 2y + 7}$, then $\frac{3t + 2y + 7}{3} - \frac{2}{9} \ln|3t + 2y + 7 + 2/3| = \ln|t| + C$.
- (i) $\frac{dy}{dt} = -y^3 \cos(2t)$
 $\frac{dy}{dt} = -y^3 \cos(2t)$ is separable and has solution $y = \pm \frac{1}{\sqrt{\sin(2t) + C}}$ or $y = 0$.
- (j) $4ty\frac{dy}{dt} = 3y^2 - 2t^2$
 $4ty\frac{dy}{dt} = 3y^2 - 2t^2$ is homogeneous (and also Bernoulli) and has solution $2 \ln(y^2/t^2 + 2) = -\ln|t| + C$.
- (k) $t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$
 $t\frac{dy}{dt} = 3y - \frac{t^2}{y^5}$ is a Bernoulli equation. Let $v = y^6$. Then $t\frac{dv}{dt} = 18v - 6t^2$, so $v = Ct^{18} + \frac{3}{8}t^2$ and $y = \sqrt[6]{Ct^{18} + \frac{3}{8}t^2}$.
- (l) $\frac{dy}{dt} = \csc^2(y - t)$
If $\frac{dy}{dt} = \csc^2(y - t)$, then $\tan(y - t) - y = C$.
- (m) $\frac{dy}{dt} = 8y - y^8$
 $\frac{dy}{dt} = 8y - y^8$ is Bernoulli (and also separable, but separating variables results in an impossible integral). Make the substitution $v = y^{-7}$. Then $\frac{dv}{dt} = -56v + 7$, so $v = Ce^{-56t} + \frac{1}{8}$ and $y = \frac{1}{\sqrt[7]{Ce^{-56t} + 1/8}}$.
- (n) $t\frac{dy}{dt} = -\cos t - 3y$
 $t\frac{dy}{dt} = -\cos t - 3y$ is linear and has solution $y = \frac{1}{t} \sin t + \frac{2}{t^2} \cos t - \frac{2}{t^3} \sin t + \frac{C}{t^3}$.
- (o) $\frac{dy}{dt} = \cot(y/t) + y/t$
 $\frac{dy}{dt} = \cot(y/t) + y/t$ is homogeneous and has solution $\sec(y/t) = Ct$.
- (p) $\frac{dy}{dt} = ty + t^2\sqrt{[3]y}$
 $\frac{dy}{dt} = ty + t^2\sqrt{[3]y}$ is Bernoulli. Make the substitution $v = y^{2/3}$. Then $v = -t - \frac{3}{2} + Ce^{t^2/3}$ and so $y = \sqrt{(-t - \frac{3}{2} + Ce^{t^2/3})^3}$.

(AB 33) For each of the following differential equations, determine whether it is linear, separable, exact, homogeneous, Bernoulli, or a function of a linear term. Then solve the given initial-value problem.

(a) $\cos(t + y^3) + 2t + 3y^2 \cos(t + y^3) \frac{dy}{dt} = 0$, $y(\pi/2) = 0$

If $\cos(t + y^3) + 2t + 3y^2 \cos(t + y^3) \frac{dy}{dt} = 0$, $y(\pi/2) = 0$, then $\sin(t + y^3) + t^2 = 1 + \pi^2/4$.

(b) $\frac{dy}{dt} = \sin t \cos y$, $y(\pi) = \pi/2$

If $\frac{dy}{dt} = \sin t \cos y$, $y(\pi) = \pi/2$, then $y = \pi/2$ for all t .

(c) $2ty \frac{dy}{dt} = 4t^2 - y^2$, $y(1) = 3$.

If $2ty \frac{dy}{dt} = 4t^2 - y^2$, $y(1) = 3$, then $y = \sqrt{\frac{4}{3}t^2 + \frac{23}{3t}}$.

(d) $t \frac{dy}{dt} = -1 - y^2$, $y(1) = 1$

If $t \frac{dy}{dt} = -1 - y^2$, $y(1) = 1$, then $y = \tan(\pi/4 - \ln t)$.

(e) $\frac{dy}{dt} = (2y + 2t - 5)^2$, $y(0) = 3$.

If $\frac{dy}{dt} = (2y + 2t - 5)^2$, $y(0) = 3$, then $y = \frac{1}{2} \tan(2t + \pi/4) + \frac{5}{2} - t$.

(f) $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$.

If $\frac{dy}{dt} = 2y - \frac{6}{y^2}$, $y(0) = 7$, then $y = \sqrt[3]{340e^{6t} + 3}$.

(g) $\frac{dy}{dt} = -3y - \sin t e^{-3t}$, $y(0) = 2$

If $\frac{dy}{dt} = -3y - \sin t e^{-3t}$, $y(0) = 2$, then $y = e^{-3t} \cos t + e^{-3t}$.

(h) $\frac{dy}{dt} = (y^2 - 5y + 4)te^{t^2}$, $y(0) = 4$

If $\frac{dy}{dt} = (y^2 - 5y + 4)te^{t^2}$, $y(0) = 4$, then $y = 4$ for all t .

(AB 34) Solve the initial-value problem $\frac{dy}{dt} = \frac{t-5}{y}$, $y(0) = 3$ and determine the range of t -values in which the solution is valid.

(Answer 34) $y = \sqrt{(t-5)^2 - 16} = \sqrt{(t-1)(t-9)} = \sqrt{t^2 - 10t + 9}$. The solution is valid for all $t < 1$.

(AB 35) Solve the initial-value problem $\frac{dy}{dt} = y^2$, $y(0) = 1/4$ and determine the range of t -values in which the solution is valid.

(Answer 35) $y = \frac{1}{4-t}$. The solution is valid for all $t < 4$.