0A. Complete Ordered Fields

[Definition: Field] A field is a set $\mathbf{F}$ along with operations $+$ and $\cdot$ with the following properties:

- If $a, b \in \mathbf{F}$, then $a+b \in \mathbf{F}$.
- If $a, b \in \mathbf{F}$, then $a \cdot b \in \mathbf{F}$. (We often write $ab = a \cdot b$.)
- Commutivity of addition: if $a, b \in \mathbf{F}$, then $a+b = b+a$.
- Associativity of addition: if $a, b, c \in \mathbf{F}$, then $(a+b)+c = a+(b+c)$.
- Additive identity: There is an element of $\mathbf{F}$, denoted 0 or $0_{\mathbf{F}}$, such that if $a \in \mathbf{F}$ then $a+0 = 0+a = a$.
- Additive inverses: For each $a \in \mathbf{F}$, there is a $b \in \mathbf{F}$ such that $a+b = b+a = 0$.
- Commutivity of multiplication: if $a, b \in \mathbf{F}$, then $ab = ba$.
- Associativity of multiplication: if $a, b, c \in \mathbf{F}$, then $(ab)c = a(bc)$.
- Multiplicative identity: There is an element of $\mathbf{F}$, denoted 1 or $1_{\mathbf{F}}$, such that if $a \in \mathbf{F}$ then $a1 = 1a = a$.
- Multiplicative inverses: For each $a \in \mathbf{F} \setminus \{0\}$, there is a $b \in \mathbf{F}$ such that $ab = ba = 1$.
- Distributivity: if $a, b, c \in \mathbf{F}$, then $a(b+c) = ab+ac$.
- Nontriviality: $1 \neq 0$.

(Problem 10) Show that if $a, b, c \in \mathbf{F}$ satisfy $a+b = a+c = 0$ then $b = c$.

(Problem 20) Show that if $a, b, c \in \mathbf{F}$ satisfy $ab = ac = 1$ then $b = c$.

(Problem 30) Suppose that $\mathbf{F}$ is a field. Show that if $a \in \mathbf{F}$ then $0a = 0$.

[Definition: Inverses] If $\mathbf{F}$ is a field and $a \in \mathbf{F}$, then:

- $(-a)$ denotes the additive inverse of a.
- If $a \neq 0$, then a^{-1} or $\frac{1}{a}$ denote the multiplicative inverse of a.
- $a-b$ denotes $a+(-b)$.
- $\frac{a}{b}$ denotes ab^{-1}.

(Problem 40) Show that $(-(-a)) = a$.

(Problem 50) Suppose that $\mathbf{F}$ is a field and that $a \in \mathbf{F}$. Show that $a(-1) = (-a)1 = (-a)$.

(Problem 60) Suppose that $\mathbf{F}$ is a field and that $a \in \mathbf{F}$. Show that $aa = (-a)(-a)$.

[Definition: Ordered field] An ordered field is a field $\mathbf{F}$ together with a subset $P \subset \mathbf{F}$, called the positive subset, such that:

- If $a, b \in P$, then $a+b \in P$ and $ab \in P$.
- $0 \notin P$.
- If $a \in \mathbf{F}$ with $a \neq 0$, then $a \in P$ if and only if $(-a) \notin P$.

(Problem 70) Let $\mathbf{F}$ be an ordered field with positive subset P. Show that $1 \in P$.

(Problem 80) Let $a \in \mathbf{F}$ with $a \neq 0$. Show that $a \in P$ if and only if $a^{-1} \in P$.

[Definition: Comparison] Let $\mathbf{F}$ be an ordered field with positive subset P. Let $a, b \in \mathbf{F}$.

- We say that $a < b$ (or $b > a$) if $b-a \in P$.
- We say that $a \leq b$ (or $b \geq a$) if $b-a \in (P \cup \{0\})$.

(Problem 90) Show that the relation $<$ is transitive. Show that the relation $\leq$ is reflexive. Is either relation symmetric?

(Problem 100) Let $a, b, c \in \mathbf{F}$. Suppose that $a \leq b$. Show that $a+c \leq b+c$.

(Problem 110) Let $\mathbf{F}$ be an ordered field. Let $a, b \in \mathbf{F}$. Show that exactly one of the following is true:

- $a < b$
- $a = b$
- $a > b$

[Definition: Inductive set] Let $\mathbf{F}$ be an ordered field. A subset $I \subseteq \mathbf{F}$ is inductive if:

- $1 \in I$.
- If $x \in I$ then $x + 1 \in I$.

(Problem 120) Let $\mathbf{F}$ be an ordered field. Give three examples of inductive subsets of $\mathbf{F}$.

(Problem 130) Let $\mathbf{F}$ be an ordered field. Suppose that $\mathcal{I}$ is a collection of inductive subsets of F (that is, if $J \in \mathcal{I}$ then $J \subseteq \mathbf{F}$ and J is inductive). Show that $\bigcap_{J\in\mathcal{I}} J$ is inductive.

(Problem 140) Let $\mathbf{F}$ be an ordered field. Let N be the intersection of all inductive subsets of $\mathbf{F}$. (Thus N is an inductive set.) Let $m \in N$. Show that $1 \le m$.

(Problem 150) Let $m, k \in N$. Show that $m + k \in N$.

(Problem 170) Let $k \in N$. Show that there does not exist a $m \in N$ with $k < m < k + 1$.

(Problem 160) Let $m, k \in N$ with $k > m$. Show that $k - m \in N$.

(Problem 180) Let $\mathbf{F}$ and $\tilde{\mathbf{F}}$ be two ordered fields. Let N, $\widetilde{N}$ be the intersection of all inductive subsets of $\mathbf{F}$ and $\tilde{\mathbf{F}}$, respectively. If $m \in N$, let $J_m = \{n \in N : n \le m\}$. Suppose that there are two numbers $n, k \in N$ (not necessarily distinct) and two functions φ_k, φ_n that satisfy

- $\varphi_k : J_k \to \widetilde{N}$
- $\varphi_n : J_n \to \widetilde{N}$
- $\varphi_k(1) = \tilde{1}$
- $\varphi_n(1) = \tilde{1}$
- If $m \in J_k$ and $m + 1 \in J_k$ then $\varphi_k(m + 1) = \varphi_k(m) + 1$.
- If $m \in J_n$ and $m + 1 \in J_n$ then $\varphi_n(m + 1) = \varphi_n(m) + 1$.

Show that $\varphi_n(i) = \varphi_k(i)$ for all $i \in J_k \cap J_n$.

(Problem 190) Show that if $m \in N$ then a function $\varphi_m : J_m \to \widetilde{N}$ as in Problem 180 must exist.

(Problem 200) Let $\mathbf{F}$ and $\tilde{\mathbf{F}}$ be two ordered fields, with $\mathbf{N}$, $\tilde{\mathbf{N}}$ the respective minimal inductive sets. Let $\varphi : \mathbf{N} \to \tilde{\mathbf{N}}$ be given by $\varphi(n) = \varphi_n(n)$ for all $n \in N$. Let $\widetilde{\varphi}$ be defined as φ with the roles of $\mathbf{R}$ and $\tilde{\mathbf{R}}$ reversed. Show that

- $\varphi(n + k) = \varphi(n) + \varphi(k)$ for all $n, k \in \mathbf{N}$.
- $\varphi(nk) = \varphi(n)\varphi(k)$ for all $n, k \in \mathbf{N}$.
- If $n > k$ then $\varphi(n) > \varphi(k)$.
- $\varphi \circ \widetilde{\varphi}(\tilde{n}) = \tilde{n}$ and $\widetilde{\varphi} \circ \varphi(n) = n$ for all $n \in N$, $\tilde{n} \in \widetilde{N}$.

(Problem 210) We will define the natural numbers (up to isomorphism) as a minimal inductive set of an ordered field. Let $\mathbf{F}$ be an ordered field. Define subsets Z and Q of $\mathbf{F}$ that we expect to be isomorphic to the integers $\mathbf{Z}$ and rational numbers $\mathbf{Q}$.

[Definition: Upper and lower bounds] Let $\mathbf{F}$ be an ordered field and let $A \subseteq \mathbf{F}$. We say that $\ell \in \mathbf{F}$ is a lower bound for A if $\ell \le a$ for all $a \in A$. We say that u is an upper bound for A if $u \ge a$ for all $a \in A$.

[Definition: Least upper bound] Let $\mathbf{F}$ be an ordered field and let $A \subseteq F$. We say that $b \in F$ is the least upper bound for A if

- b is an upper bound for A.
- If c is an upper bound for A then $b \le c$.

[Definition: Greatest lower bound] Let $\mathbf{F}$ be an ordered field and let $A \subseteq F$. We say that $b \in F$ is the greatest lower bound for A if

- b is a lower bound for A.
- If c is a lower bound for A then $b \ge c$.

(Problem 220) Let $\mathbf{F}$ be an ordered field and let $A \subseteq F$. Show that A has at most one least upper bound.

(Problem 230) Let $\mathbf{F}$ be an ordered field and let $A \subseteq F$. Suppose that $\alpha \in A$ is an upper bound for A. Show that α is a least upper bound.

(Problem 240) Let $A \subseteq N$ where N is as in Problem 140. Suppose that A is nonempty and that A has an upper bound in N. Show that A has a least upper bound.

(Problem 250) Let $A \subseteq N$ where N is as in Problem 140. Suppose that A has a least upper bound. Show that the least upper bound is an element of A.

[Definition: Completeness] An ordered field $\mathbf{F}$ is complete if every nonempty subset of $\mathbf{F}$ with an upper bound has a least upper bound.

(Problem 251) Let F be a complete ordered field and let N be as in Problem 140. Show that N has no upper bound.

(Problem 252) Let F be a complete ordered field, let $r \in F$, and let $s \in F$ with $s > 0$. Show that there is a $n \in N$ with $n > r$ and an $m \in N$ with $1/m < s$.

[Definition: The real numbers] The real numbers $\mathbf{R}$ are a complete ordered field. (We saw in MATH 4513 that a complete ordered field exists. You can review the argument in Section 0B.)

(Problem 260) Let $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ be two ordered fields, with $\mathbf{N}$, $\widetilde{\mathbf{N}}$ the respective minimal inductive sets. Let $\varphi : \mathbf{N} \to \widetilde{\mathbf{N}}$ be the isomorphism in Problem 200. Let $\widetilde{\varphi}$ be defined as φ with the roles of $\mathbf{F}$ and $\widetilde{\mathbf{F}}$ reversed. Show that

- $\varphi(n+k) = \varphi(n) + \varphi(k)$ for all $n, k \in \mathbf{N}$.
- $\varphi(nk) = \varphi(n)\varphi(k)$ for all $n, k \in \mathbf{N}$.
- If $n > k$ then $\varphi(n) > \varphi(k)$.
- $\varphi \circ \widetilde{\varphi}$ and $\widetilde{\varphi} \circ \varphi$ are the identity functions.

(Problem 270) Let $\mathbf{Z}$, $\widetilde{\mathbf{Z}}$ be as in Problem 210. Extend φ to a function $\varphi : \mathbf{Z} \to \widetilde{\mathbf{Z}}$.

(Problem 280) Show that

- $\varphi(-z) = -\varphi(z)$ for all $z \in \mathbf{Z}$.
- $\varphi(n+k) = \varphi(n) + \varphi(k)$ for all $n, k \in \mathbf{Z}$.
- $\varphi(nk) = \varphi(n)\varphi(k)$ for all $n, k \in \mathbf{Z}$.
- If $n > k$ then $\varphi(n) > \varphi(k)$.
- $\varphi \circ \widetilde{\varphi}$ and $\widetilde{\varphi} \circ \varphi$ are the identity functions.

(Problem 290) Let $\mathbf{Q}$, $\widetilde{\mathbf{Q}}$ be as in Problem 210. Extend φ to a function $\varphi : \mathbf{Q} \to \widetilde{\mathbf{Q}}$.

(Problem 300) Show that

- $\varphi(q^{-1}) = \varphi(q)^{-1}$ for all $q \in \mathbf{Q}$.
- $\varphi(-z) = -\varphi(z)$ for all $z \in \mathbf{Q}$.
- $\varphi(n+k) = \varphi(n) + \varphi(k)$ for all $n, k \in \mathbf{Q}$.
- $\varphi(nk) = \varphi(n)\varphi(k)$ for all $n, k \in \mathbf{Q}$.
- If $n > k$ then $\varphi(n) > \varphi(k)$.
- $\varphi \circ \widetilde{\varphi}$ and $\widetilde{\varphi} \circ \varphi$ are the identity functions.

(Problem 301) Let $\psi(r) = \sup\{\varphi(q) : q \in \mathbf{Q}, q < r\}$. Show that $\psi : F \to \widetilde{F}$ is well defined and that $\psi(q) = \varphi(q)$ for all $q \in Q$.

(Problem 310) Show that

- $A \subseteq F$ has an upper bound if and only if $\psi(A) \subseteq \widetilde{F}$ has an upper bound, and $\sup \psi(A) = \psi(\sup A)$.
- $\psi(q^{-1}) = \psi(q)^{-1}$ for all $q \in \mathbf{R}$.
- $\psi(-z) = -\psi(z)$ for all $z \in \mathbf{R}$.
- $\psi(n+k) = \psi(n) + \psi(k)$ for all $n, k \in \mathbf{R}$.
- $\psi(nk) = \psi(n)\psi(k)$ for all $n, k \in \mathbf{R}$.
- If $n > k$ then $\psi(n) > \psi(k)$.

- $\psi \circ \widetilde{\psi}$ and $\widetilde{\psi} \circ \psi$ are the identity functions.

1A. Undergraduate analysis

[Definition: Open cover] Let $K \subseteq \mathbf{R}$. A cover of K is a collection $\mathcal{U}$ of subsets of $\mathbf{R}$ that satisfies $K \subseteq \bigcup_{V\in\mathcal{U}} V$. We say that $\mathcal{U}$ is an open cover if every element V of $\mathcal{U}$ is an open set in $\mathbf{R}$.

[Definition: Finite subcover] Let $K \subseteq \mathbf{R}$ and let $\mathcal{U}$ be a cover of K. A subcover of $\mathcal{U}$ is any subcollection $\mathcal{U}_1 \subseteq \mathcal{U}$ such that $K \subseteq \bigcup_{V\in\mathcal{U}_1} V$. A finite subcover is a subcover that is also a collection of finitely many sets.

[Definition: Compact set] A set $K \subseteq \mathbf{R}$ is compact if every open cover of K has a finite subcover.

(Problem 320) State the Heine-Borel theorem.

(Problem 330) Let $K \subseteq \mathbf{R}$ be nonempty, closed, and bounded. Show that $\inf K \in K$ and $\sup K \in K$.

(Problem 340) Let $K \subseteq \mathbf{R}$ be nonempty, closed, and bounded. Let $\mathcal{U}$ be an open cover of K. Let $K_r = \{x \in K : x \leq r\}$. Then $K_r \subseteq K$ so $\mathcal{U}$ is also an open cover of K_r. Let $\mathcal{S} = \{r \in K$:there is a finite subcover of $K_r\}$. Show that $\mathcal{S}$ is nonempty and bounded.

(Problem 350) Let $s = \sup \mathcal{S}$. Show that $s = \sup K$.

(Problem 360) (The Heine-Borel theorem.) Let $K \subset \mathbf{R}$. Suppose that K is both closed and bounded. Show that K is compact.

[Definition: Continuous function] A function $f : X \to \mathbf{R}$, where X is a metric space, is continuous if, for every $x \in X$ and every $\varepsilon > 0$, there is a $\delta > 0$ depending on x and ε such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. f is uniformly continuous if δ may be chosen independent of x.

(Problem 370) Let X be compact and let $f : X \to \mathbf{R}$ be continuous. Show that f is bounded.

(Problem 380) Show that f is uniformly continuous.

(Problem 420) Show that $\mathbf{Q} \cap [0,1]$ is countable. That is, show that there is a sequence $\{q_k\}_{k=1}^\infty$ that contains each rational number in $[0,1]$ exactly once and contains no other numbers.

1A. Review: the Riemann Integral

[Definition: Partition] Suppose a, $b \in \mathbf{R}$ with $a < b$. We say that P is a *partition* of $[a,b]$ if:

- $P \subseteq [a,b]$,
- P is finite,
- $a \in P$ and $b \in P$.

We will write $P = \{x_0, x_1, \dots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = b$.

[Definition: $\sup_A$, $\inf_A$] If A is a set and $f : A \to \mathbf{R}$, then $\sup_A f = \sup\{f(x) : x \in A\}$ and $\inf_A f = \inf\{f(x) : x \in A\}$.

[Definition: Lower and upper Darboux sums] Let $[a,b] \subseteq \mathbf{R}$, $f : [a,b] \to \mathbf{R}$, and let P be a partition of $[a,b]$. Suppose that f is bounded on $[a,b]$, so $-\infty < \inf_{[a,b]} f \leq \sup_{[a,b]} f < \infty$.

Then the upper and lower Darboux sums of f with respect to the partition P are

$$L(f,P) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{[x_{j-1},x_j]} f,$$

$$U(f,P) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{[x_{j-1},x_j]} f.$$

[Definition: Lower and upper Darboux integrals] Let $[a,b] \subseteq \mathbf{R}$ be a closed bounded interval and let $f : [a,b] \to \mathbf{R}$ be a bounded function. We define

$$L(f,[a,b]) = \sup_P L(f,P), \qquad U(f,[a,b]) = \sup_P U(f,P)$$

where the supremum and infimum are over all partitions P of $[a,b]$.

[Definition: Riemann integral] Let $[a,b] \subseteq \mathbf{R}$ be a closed bounded interval and let $f : [a,b] \to \mathbf{R}$ be a bounded function. If $U(f,[a,b]) = L(f,[a,b])$, then we say that f is *Riemann integrable* on $[a,b]$ and write

$$\int_a^b f = U(f,[a,b]) = L(f,[a,b]).$$

1B. The Riemann integral is not good enough

(Problem 390) Let $f(x) = \frac{1}{\sqrt[3]{x}}$. Let P be any partition of $[0,1]$. What is $U(f,P)$?

(Problem 400) How did you define $\int_0^1 \frac{1}{\sqrt[3]{x}} dx$ in calculus? In undergraduate analysis?

(Problem 410) Can you write down a generalization of the Riemann integral that can be used to define $\int_0^1 \frac{1}{\sqrt[3]{\sin \pi/x}}\, dx$?

(Problem 440) Let $f : [0,1] \to \mathbf{R}$. Suppose that f has a singularity at every rational number in $[0,1]$. Can we generalize the approach in Problem 410 to define $\int_0^1 f$?

(Problem 450) Let $\xi \in (0,1)$. Let $\{q_k\}_{k=1}^\infty$ be as in Problem 420. Write down a formula for a function h_k that is continuous, nonnegative, integrates to $2^{-k}\xi$, and satisfies $h_k(x) = 0$ if $|x - q_k| > 2^{-k}\xi$. Let $f_n(x) = \sum_{k=1}^n h_k(x)$. What is $\int_{-1}^2 f_n$? What can you say about $\int_0^1 f_n$?

(Problem 460) Consider the semi-metric[1] space (X,d), where X is the set of all bounded Riemann integrable functions on $[0,1]$, and where $d(f,g) = \int_0^1 |f-g|$. Show that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in X.

(Problem 470) Suppose (for the sake of contradiction) that $\{f_n\}_{n=1}^\infty$ is convergent, that is, that there is a Riemann integrable function f such that $f_n \to f$ (that is, $\int_0^1 |f_n - f| \to 0$ as $n \to \infty$). Provide an upper bound on $\int_0^1 f$.

(Problem 480) Let $(x_{j-1}, x_j) \subseteq [0,1]$ with $x_{j-1} < x_j$. Show that there is a point x in (x_{j-1}, x_j) such that $f(x) \geq 1/2$.

(Problem 490) Show that $U(f,P) \geq 1/2$ for any partition P of $[0,1]$. Have we derived a contradiction?

2A. Outer Measure on **R**

[Definition: Extended real number] An extended real number is either a real number, $\infty = +\infty$, or $-\infty$.

[Definition: Ordering on the extended real numbers] Let a, b be extended real numbers. We say that $a \leq b$ if:

- a, $b \in \mathbf{R}$ and $a \leq b$,
- $a = -\infty$, or
- $b = \infty$.

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[1] A semi-metric space satisfies all of the axioms of a metric space, except that $d(x,y) = 0$ is no longer required to imply that $x = y$.

(Problem 500) Let E be a set of extended real numbers. What is $\sup E$? What is $\inf E$?

(Problem 510) Write down a rigorous definition of the expression $a+b$ where a and b are extended real numbers. Are there any pairs of values we do not want to allow?

(Problem 520) Write down a rigorous definition of the expression $\sum_{k=0}^{\infty} a_k$ where the a_ks are nonnegative extended real numbers.

[Definition: Length of an open interval] Let $I \subseteq \mathbf{R}$ be an open interval. The *length of* I, or $\ell(I)$, is defined to be

$$\ell(I) = \begin{cases} b-a & \text{if } I=(a,b) \text{ with } -\infty < a < b < \infty, \\ 0 & \text{if } I = \varnothing, \\ \infty & \text{if } I \text{ is unbounded.} \end{cases}$$

[Definition: Outer measure] Let $A \subseteq \mathbf{R}$. The *outer measure* $|A|$ of A is defined to be

$$|A| = \inf\Big\{\sum_{k=1}^{\infty} \ell(I_k) : \text{Each } I_k \text{ is an open interval and } A \subseteq \bigcup_{k=1}^{\infty} I_k\Big\}.$$

(Problem 580) Let $A \subseteq \mathbf{R}$ be a countable set. Show that $|A| = 0$.

(Problem 530) Let $[a,b] \subseteq \mathbf{R}$ be a closed bounded interval. Show that $|[a,b]| = b-a$.

(Problem 540) Let f be a bounded function defined on the closed and bounded interval $[a,b]$. Show that

$$U(f,[a,b]) = \inf\Big\{\sum_{j=1}^{n} |I_j| \sup_{I_j} f : \text{each } I_j \text{ is a closed interval and } [a,b] = \bigcup_{j=1}^{n} I_j\Big\}.$$

(Problem 550) Show that if $A \subseteq B \subseteq \mathbf{R}$ then $|A| \leq |B|$.

(Problem 560) Let I be an open interval. Show that $\ell(I) = |I|$.

(Problem 570) Let $I = [a,b)$ or $I = (a,b]$ be a bounded half-open interval. Show that $|I| = b-a$.

(Problem 590) Use the previous result to show that $[a,b]$ is uncountable whenever $a < b$.

(Problem 600) Show that if $A \subseteq \mathbf{R}$ and $B \subseteq \mathbf{R}$ then $|A \cup B| \leq |A| + |B|$.

(Problem 610) Show that if $A_k \subseteq \mathbf{R}$ for all $k \geq 0$ then $\left|\bigcup_{k=1}^{\infty} A_k\right| \leq \sum_{k=1}^{\infty} |A_k|$.

(Problem 620) Show that if A and B are disjoint sets, and if $\sup A < \inf B$, then $|A \cup B| = |A| + |B|$.

(Problem 630) Show that if A and B are disjoint sets, and if $\sup A \leq \inf B$, then $|A \cup B| = |A| + |B|$.

(Problem 640) Let $A \subseteq \mathbf{R}$, $t \in \mathbf{R}$, and let $A + t = \{a + t : a \in A\}$. Show that $|A + t| = |A|$.

[Axiom of Choice] The axiom of choice states that, if $\mathcal{E}$ is a collection of sets, and if the elements of $\mathcal{E}$ are pairwise-disjoint nonempty sets, there is a set V such that V contains exactly one element of each set in $\mathcal{E}$ and no other elements.

(Problem 650) Define the relation $\sim$ by $a \sim b$ if $a - b \in \mathbf{Q}$. If $r \in [-1,1]$, let $E_r = \{s \in [-1,1] : r \sim s\}$. Show that if r, $s \in [-1,1]$, then either $E_r = E_s$ or $E_r \cap E_s = \varnothing$.

(Problem 660) Let $\mathcal{E} = \{E_r : r \in [-1,1]\}$. Let $V = V_0$ be the set given by the axiom of choice. Show that if v, $w \in V$ with $v \neq w$ then $v - w \notin \mathbf{Q}$.

(Problem 670) Let q be a rational number in $[-2,2]$. Let $V_q = \{v + q : v \in V\}$. Show that $|V_q| = |V|$.

(Problem 680) Show that if q, $p \in \mathbf{Q} \cap [-2,2]$ then either $q = p$ or $V_q \cap V_p = \varnothing$.

(Problem 690) Show that

$$[-1,1] \subseteq \bigcup_{q \in [-2,2] \cap \mathbf{Q}} V_q \subseteq [-3,3].$$

(Problem 700) Show that $|V| > 0$.

(Problem 710) Let $\{q_k\}_{k=1}^{\infty}$ be a sequence that contains each rational number in $[-2,2]$ exactly once and contains no other numbers. Show that $\sum_{k=1}^{\infty} |V_{q_k}| \neq \left|\bigcup_{k=1}^{\infty} V_{q_k}\right|$.

(Problem 720) Show that there exist two disjoint sets A and B such that $|A \cup B| \neq |A| + |B|$.

2B. Undergraduate analysis

(Problem 730) Show that if $V \subseteq \mathbf{R}$ and V is open then $V = \bigcup_{j=1}^{\infty} I_j$ for some sequence $\{I_j\}_{j=1}^{\infty}$ of bounded open intervals.

(Problem 731) Let (X,d) be a measure space and let $Y \subseteq X$. Recall that $G \subseteq Y$ is relatively open (or open in Y) if, for every $g \in G$, there is a $r_g > 0$ such that if $d(g,y) < r_g$ and $y \in Y$, then $y \in G$. Show that G is relatively open if and only if $G = U \cap Y$ for some $U \subseteq X$ that is open (in X).

[Definition: Inverse image] Let $f : X \to Y$ be a function and let $A \subseteq Y$. Then $f^{-1}(A) = \{x \in X : f(x) \in A\}$.

(Problem 740) Let $X \subseteq \mathbf{R}$ and let $f : X \to \mathbf{R}$. Show that f is continuous if and only if $f^{-1}(U)$ is open in X whenever U is open in $\mathbf{R}$.

(Problem 750) Let $f : X \to Y$ be a function and let $A \subseteq Y$. Show $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$.

(Problem 760) Let $\mathcal{B} \subseteq 2^Y$ be a collection of subsets of Y. Show that $f^{-1}\left(\bigcup_{A\in\mathcal{B}} A\right) = \bigcup_{A\in\mathcal{B}} f^{-1}(A)$.

(Problem 770) Let $\mathcal{B} \subseteq 2^Y$ be a collection of subsets of Y. Show that $f^{-1}\left(\bigcap_{A\in\mathcal{B}} A\right) = \bigcap_{A\in\mathcal{B}} f^{-1}(A)$.

(Problem 780) Let $f : X \to Y$ and $g : Y \to Z$. Let $A \subseteq Z$. Show that $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))$.

[Definition: Increasing function] Let $X \subseteq \mathbf{R}$ and let $f : X \to \mathbf{R}$.

- f is increasing if $f(x) \leq f(y)$ for all x, $y \in X$ with $x < y$.
- f is strictly increasing if $f(x) < f(y)$ for all x, $y \in X$ with $x < y$.

2B. Measurable spaces and functions

(Problem 790) What properties of the outer measure $|\cdot|$ did we use in the proof of Problems 650–720?

[Definition: σ-algebra; measure] Let X be a set. Let $\mathcal{S}$ be a collection of subsets of $\mathcal{S}$. Let $\mu : \mathcal{S} \to [0,\infty]$. We say that $(X,\mathcal{S})$ is a measurable space and μ is a measure on $(X,\mathcal{S})$ if:

- $\mu(\varnothing) = 0$.
- If $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$ is a sequence of pairwise-disjoint subsets of X then $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$.
- $\mathcal{S}$ is a σ-algebra on X, that is,
  - $\varnothing \in \mathcal{S}$,
  - If $E \in \mathcal{S}$, then $X \setminus E \in \mathcal{S}$,
  - If $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$ then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$.

We call the elements of $\mathcal{S}$ measurable sets.

(Problem 800) Is $\mu(E) = |E|$ a measure on $(\mathbf{R}, 2^{\mathbf{R}})$, where $2^{\mathbf{R}}$ is the collection of all subsets of $\mathbf{R}$?

(Problem 810) Does there exist a measure μ on $(\mathbf{R}, 2^{\mathbf{R}})$ that satisfies

- $\mu((a,b)) = b - a$ for every $-\infty \leq a < b \leq \infty$,
- $\mu(t + E) = \mu(E)$ for all $E \subseteq \mathbf{R}$ and all $t \in \mathbf{R}$?

(Problem 820) Let X be a nonempty set. Show that $\mathcal{S} = \{\emptyset, X\}$ is a σ-algebra on X.

(Problem 830) Let X be a nonempty set. Show that $\mathcal{S} = 2^X$ is a σ-algebra on X.

(Problem 840) Let X be a nonempty set. Let $C = \{E \subseteq X : E \text{ is countable}\}$. Let $\mathcal{S} = C \cup \{E \subseteq X : X \setminus E \in C\}$. Show that $\mathcal{S}$ is a σ-algebra on X.

(Problem 850) Let $(X, \mathcal{S})$ be a measurable space (that is, let X be a set and let $\mathcal{S}$ be a σ-algebra on X). Show that $X \in \mathcal{S}$.

(Problem 860) Let $(X, \mathcal{S})$ be a measurable space. Show that if $D, E \in \mathcal{S}$ then $D \cup E \in \mathcal{S}$.

(Problem 870) Show $D \cap E \in \mathcal{S}$.

(Problem 880) Show $D \setminus E \in \mathcal{S}$.

(Problem 890) Let $(X, \mathcal{S})$ be a measurable space. Let $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$. Show that $\bigcap_{k=1}^{\infty} E_k \in \mathcal{S}$.

[Exercise 2B.11] Let $(Y, \mathcal{T})$ be a measurable space. Let $X \in \mathcal{T}$ and let $\mathcal{S} = \{E \in \mathcal{T} : E \subseteq X\}$. Then $(X, \mathcal{S})$ is a measurable space.

(Problem 900) Let X be a set. Let $\mathcal{T}$ be a collection of σ-algebras on X. Show that $\bigcap_{\mathcal{S} \in \mathcal{T}} \mathcal{S}$ is also a σ-algebra on X.

[Definition: Smallest σ-algebra] Let X be a set and let $\mathcal{A} \subseteq 2^X$ be a collection of subsets of $\mathcal{S}$. The intersection of all σ-algebras containing $\mathcal{A}$ is the smallest σ-algebra containing $\mathcal{A}$.

(Problem 910) Let $X = \mathbf{R}$ and let $\mathcal{A} = \{\{3\}, \{5\}\}$. What is the smallest σ-algebra containing $\mathcal{A}$?

(Problem 920) Let $X = \mathbf{R}$ and let $\mathcal{A} = \{\{x\} : x \in \mathbf{R}\}$. What is the smallest σ-algebra containing $\mathcal{A}$?

[Definition: Borel set] A set $E \subseteq \mathbf{R}$ is called a Borel set if E is in the smallest σ-algebra on $\mathbf{R}$ that contains all the open subsets of $\mathbf{R}$.

(Problem 930) Show that all closed subsets of $\mathbf{R}$ are Borel sets.

(Problem 940) Show that all countable subsets of $\mathbf{R}$ are Borel sets.

(Problem 950) Let $-\infty < a < b < \infty$. Show that $[a, b)$ is a Borel set.

[Definition: Measurable function] Let $(X, \mathcal{S})$ be a measurable space. Let $f : X \to \mathbf{R}$. We say that f is $\mathcal{S}$-measurable if $f^{-1}(B) \in \mathcal{S}$ for every Borel set B.

(Problem 960) Show that any function $f : X \to \mathbf{R}$ is 2^X-measurable.

(Problem 980) Let $\mathcal{S} = \{\emptyset, X\}$. Suppose that $f : X \to \mathbf{R}$ is $\mathcal{S}$-measurable. Show that f is constant.

(Problem 970) Let $(X, \mathcal{S})$ be a measurable space and let $f : X \to \mathbf{R}$ be a constant function. Show that f is $\mathcal{S}$-measurable.

(Problem 981) Let X be a set and let $\mathcal{S}$, $\mathcal{T}$ be two σ-algebras on X. Suppose that $\mathcal{T} \subseteq \mathcal{S}$ and that $f : X \to \mathbf{R}$ is $\mathcal{T}$-measureable. Show that f is $\mathcal{S}$-measurable.

(Problem 982) Let Y be a set, let $\mathcal{T}$ be a σ-algebra on Y, and let $f : Y \to \mathbf{R}$ be $\mathcal{T}$-measurable. Let $X \in \mathcal{T}$ and let $\mathcal{S}$ be as in Exercise 2B.11. Show that $f\big|_X$ is $\mathcal{S}$-measurable.

[Definition: Characteristic function] Let $E \subseteq X$. Then $\chi_E : X \to \mathbf{R}$ is the piecewise defined function given by

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}$$

(Problem 1000) Show that χ_E is $\mathcal{S}$-measurable if and only if $E \in \mathcal{S}$.

(Problem 1010) Let $(X, \mathcal{S})$ be a measurable space and let $f : X \to \mathbf{R}$. Suppose that $f^{-1}((a, \infty)) \in \mathcal{S}$ for all $a \in \mathbf{R}$. Let $\mathcal{T} = \{A \subseteq \mathbf{R} : f^{-1}(A) \in \mathcal{S}\}$. Show that $\mathcal{T}$ is a σ-algebra on $\mathbf{R}$.

(Problem 1020) Let $-\infty < a < b < \infty$. Show that $(a, b) \in \mathcal{T}$.

(Problem 1030) Let $U \subseteq \mathbf{R}$ be open. Show that $U \in \mathcal{T}$.

(Problem 1040) Is f measurable?

(Problem 1041) Let $(X, \mathcal{S})$ be a measurable space. Let $f : X \to \mathbf{R}$ be $\mathcal{S}$-measurable. Let $a \in \mathbf{R}$. Show that af is also $\mathcal{S}$-measurable.

[Definition: Borel measurable] If $X \subseteq \mathbf{R}$, then $f : X \to \mathbf{R}$ is Borel measurable if $f^{-1}(B)$ is a Borel set for every Borel set $B \subseteq \mathbf{R}$. That is, f is Borel measurable if f is $\mathcal{B}$-measurable where $\mathcal{B}$ is the set of all Borel subsets of $\mathbf{R}$.

(Problem 1042) Let $f : X \to \mathbf{R}$ be Borel measurable. Show that X is Borel.

(Problem 1050) Let $f : \mathbf{R} \to \mathbf{R}$ be continuous. Show that f is Borel measurable.

(Problem 1060) Let $X \subseteq \mathbf{R}$ be a Borel set and let $f : X \to \mathbf{R}$ be continuous. Show that f is Borel measurable.

(Problem 1070) Let $X \subseteq \mathbf{R}$ be a Borel set and let $f : X \to \mathbf{R}$ be increasing. Show that f is Borel measurable.

(Problem 1080) Let $(X, \mathcal{S})$ be a measurable space. Let $Y \subseteq \mathbf{R}$. Let $f : X \to Y$ be $\mathcal{S}$-measurable and let $g : Y \to \mathbf{R}$ be Borel measurable. Show that $g \circ f : X \to \mathbf{R}$ is $\mathcal{S}$-measurable.

(Problem 1081) Let B be a Borel set. Let $a \in \mathbf{R}$. Show that $aB = \{ab : b \in B\}$ is Borel.

(Problem 1090) Let $(X, \mathcal{S})$ be a measurable space. Let f, $g : X \to \mathbf{R}$ be $\mathcal{S}$-measurable functions. Show that $f + g$ is $\mathcal{S}$-measurable.

(Problem 1100) Show that fg is $\mathcal{S}$-measurable.

(Problem 1110) If $g(x) \neq 0$ for all $x \in X$, show that f/g is $\mathcal{S}$-measurable.

(Problem 1120) Let $(X, \mathcal{S})$ be a measurable space and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $\mathcal{S}$-measurable functions $f_n : X \to \mathbf{R}$. Suppose that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in X$. Show that f is $\mathcal{S}$-measurable.

[Definition: Borel subsets of the extended real numbers] Let $B \subseteq [-\infty, \infty]$. We say that B is a Borel set if $B \cap \mathbf{R}$ is a Borel set. If $(X, \mathcal{S})$ is a measurable space and $f : X \to [-\infty, \infty]$, we say that f is $\mathcal{S}$-measureable if $f^{-1}(B) \in \mathcal{S}$ for every Borel set $B \subseteq [-\infty, \infty]$.

(Problem 1121) Let $(X, \mathcal{S})$ be a measure space and let $f : X \to [-\infty, \infty]$. Let $\widetilde{X} = f^{-1}(\mathbf{R})$ and let $\widetilde{\mathcal{S}} = \{E \in \mathcal{S} : E \subseteq X\}$. Show that f is $\mathcal{S}$-measurable if and only if

- $f^{-1}(\{\infty\}) \in \mathcal{S}$,
- $f^{-1}(\{-\infty\}) \in \mathcal{S}$,
- $(\widetilde{X}, \widetilde{\mathcal{S}})$ is a measurable space, and
- $f\big|_{\widetilde{X}} : \widetilde{X} \to \mathbf{R}$ is $\widetilde{\mathcal{S}}$-measurable.

(Problem 1130) Suppose $(X, \mathcal{S})$ is a measurable space and $f : X \to [-\infty, \infty]$ satisfies $f^{-1}((a, \infty]) \in \mathcal{S}$ for all $a \in \mathbf{R}$. Show that f is $\mathcal{S}$-measurable.

(Problem 1140) Let $(X, \mathcal{S})$ be a measurable space. Let $f_n : X \to [-\infty, \infty]$ be $\mathcal{S}$-measurable for each n. Show that $g(x) = \sup\{f_n(x) : n \in \mathbf{N}\}$ is $\mathcal{S}$-measurable.

(Problem 1150) Show that $h(x) = \inf\{f_n(x) : n \in \mathbf{N}\}$ is $\mathcal{S}$-measurable.

(Problem 1151) If $\{f_i\}_{i \in \mathcal{I}}$ is a collection of measurable functions, and $\mathcal{I}$ is an uncountable index set, must $g(x) = \sup\{f_i(x) : i \in \mathcal{I}\}$ be $\mathcal{S}$-measurable?

2C. Measures and their properties.

[Definition: measure] Let X be a set. Let $\mathcal{S}$ be a σ-algebra on X. We say that μ is a measure on $(X, \mathcal{S})$ if:

- $\mu : \mathcal{S} \to [0, \infty]$
- $\mu(\varnothing) = 0$.
- If $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$ is a sequence of pairwise-disjoint subsets of X then $\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \sum_{k=1}^{\infty} \mu(E_k)$.

We call $(X, \mathcal{S}, \mu)$ a measure space.

(Problem 1160) Let X be a set. Let μ satisfy $\mu(\{x\}) = 1$ for every $x \in X$. Can we extend μ to a measure on $(X, 2^X)$?

(Problem 1170) Let X be a set, $\mathcal{S}$ a σ-algebra on X, and $w : X \to [0, \infty]$ a function. Show that

$$\mu(E) = \sum_{x \in E} w(x) = \sup\Big\{\sum_{x \in D} w(x) : D \subseteq E, D \text{ finite}\Big\}$$

is a measure.

(Problem 1180) Let $\mu(E) = 0$ if E is countable and $\mu(E) = 3$ if E is uncountable. Is μ a measure on $(\mathbf{R}, 2^{\mathbf{R}})$?

(Problem 1190) Let μ be as in Problem 1180. Is there a σ-algebra $\mathcal{S}$ on $\mathbf{R}$ such that $\mu\big|_{\mathcal{S}}$ is a measure on $\mathcal{S}$?

(Problem 1360) Let $(X, \mathcal{S})$ be a measurable space. Let μ satisfy

- $\mu : \mathcal{S} \to [0, \infty]$
- If $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$ then $\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leq \sum_{k=1}^{\infty} \mu(E_k)$.
- If A, $B \in \mathcal{S}$ with $A \cap B = \varnothing$ then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Show that $(X, \mathcal{S}, \mu)$ is a measure space.

(Problem 1200) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let D, $E \in \mathcal{S}$ with $D \subseteq E$. Show that $\mu(D) \leq \mu(E)$.

(Problem 1210) Show that if $\mu(D) < \infty$ then $\mu(E \setminus D) = \mu(E) - \mu(D)$.

(Problem 1220) Give an example of a measure space $(X, \mathcal{S}, \mu)$ and sets D, $E \in \mathcal{S}$ with $D \subsetneq E$, $\mu(D) = \mu(E) = \infty$, and such that $\mu(E \setminus D) = 0$.

(Problem 1230) Give an example where $\mu(E \setminus D) = \infty$.

(Problem 1240) Give an example where $\mu(E \setminus D) = 7$.

(Problem 1250) Let $(X, \mathcal{S}, \mu)$ be a measure space and let $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$. We do not require that the E_ks be pairwise-disjoint. Show that $\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leq \sum_{k=1}^{\infty} \mu(E_k)$.

(Problem 1260) Let $(X, \mathcal{S}, \mu)$ be a measure space and let $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$. We require that $E_k \subseteq E_{k+1}$ for all k. Show that $\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \lim_{k \to \infty} \mu(E_k)$.

(Problem 1270) Let $(X, \mathcal{S}, \mu)$ be a measure space and let $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$. We require that $E_k \supseteq E_{k+1}$ for all k and that $\mu(E_k) < \infty$ for at least one k. Show that $\mu\Big(\bigcap_{k=1}^{\infty} E_k\Big) = \lim_{k \to \infty} \mu(E_k)$.

[Exercise 2C.10] Give an example of a measure space $(X, \mathcal{S}, \mu)$ and a sequence of sets $\{E_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$ such that $\mu\Big(\bigcap_{k=1}^{\infty} E_k\Big) \neq \lim_{k \to \infty} \mu(E_k)$.

(Problem 1280) Let $(X, \mathcal{S}, \mu)$ be a measure space and let D, $E \in \mathcal{S}$. Show that $\mu(D \cup E) + \mu(D \cap E) = \mu(D) + \mu(E)$.

2D. Undergraduate analysis

(Problem 1290) Let $G \subseteq \mathbf{R}$ be open. Show that G is the union of countably many pairwise-disjoint open intervals.

(Problem 1300) Let $X \subseteq \mathbf{R}$ and let $f : X \to \mathbf{R}$ be strictly increasing. Show that f is one-to-one.

(Problem 1310) Let $X \subseteq \mathbf{R}$ and let $f : X \to \mathbf{R}$ be increasing. Let $E = \{y \in \mathbf{R} : f^{-1}(y)$ contains more than one element$\}$. Show that E is countable.

(Problem 1311) Let $f : \mathbf{R} \to \mathbf{R}$ be increasing. Let $E = \{x \in \mathbf{R} : f$ is not continuous at $x\}$. Show that E is countable.

(Bonus Problem 1312) Let $f : \mathbf{R} \to \mathbf{R}$ be increasing. Let $E = \{y \in \mathbf{R} : y \notin f(\mathbf{R})\}$. Show that E is the union of countably many disjoint intervals.

(Problem 1320) Let $X \subseteq \mathbf{R}$ and let $f_n : X \to \mathbf{R}$ be increasing for each n. Suppose that there is a function $f : X \to \mathbf{R}$ such that $f_n(x) \to f(x)$ for each $x \in X$. Show that f is also increasing.

(Problem 1330) Let X, Y be two metric spaces. Let $f : X \to Y$ and let $f_n : X \to Y$ for each $n \in \mathbf{N}$. Suppose that each f_n is continuous and that $f_n \to f$ uniformly on X. Show that f is continuous.

(Problem 1331) Let X, Y be two metric spaces. Let $f : X \to Y$ and let $f_n : X \to Y$ for each $n \in \mathbf{N}$. Suppose that each f_n is uniformly continuous and that $f_n \to f$ uniformly on X. Show that f is uniformly continuous.

(Problem 1340) A sequence of functions $f_n : X \to Y$ is uniformly Cauchy if, for every $\varepsilon > 0$, there is a $K \in \mathbf{N}$ such that if $m, n \in \mathbf{N}$ with $m \geq n \geq K$, then $d(f_n(x), f_m(x)) < \varepsilon$ for all $x \in X$. Suppose that $\{f_n\}_{n=1}^\infty$ is uniformly Cauchy and that Y is complete. Show that $f_n \to f$ uniformly for some function $f : X \to Y$.

2D. Lebesgue measure.

(Problem 1350) Let $\mu(E) = |E|$, where $|E|$ is the outer measure of Section 2A. Is μ a measure on $(\mathbf{R}, 2^{\mathbf{R}})$?

(Problem 1370) Let $A, Z \subseteq \mathbf{R}$. Suppose that $|Z| = 0$. Show that $|A \cup Z| = |A| + |Z|$.

(Problem 1371) Let $A, Z \subseteq \mathbf{R}$. Suppose that $|Z| = \infty$. Show that $|A \cup Z| = |A| + |Z|$.

(Problem 1380) Let $A \subseteq \mathbf{R}$ and $b < c$, $b, c \in \mathbf{R}$. Suppose that $A \cap (b, c) = \varnothing$. Show that $|A \cup (b, c)| = |A| + |(b, c)|$.

(Problem 1390) Let $G \subseteq \mathbf{R}$ be open. Suppose $G = \cup_{k=1}^\infty I_k$ where each I_k is a (possibly empty or infinite) open interval and the I_ks are pairwise-disjoint. Show that $|G| = \sum_{k=1}^\infty \ell(I_k)$.

(Problem 1400) Let $A, G \subseteq \mathbf{R}$. Suppose that $A \cap G = \varnothing$ and that G is open. Show that $|A \cup G| = |A| + |G|$.

(Problem 1410) Let $A, F \subseteq \mathbf{R}$. Suppose that $A \cap F = \varnothing$ and that F is closed. Show that $|A \cup F| = |A| + |F|$.

(Problem 1420) Let $\mathcal{L} = \{D \subseteq \mathbf{R}$: If $\varepsilon > 0$, then there is a closed set F with $F \subseteq D$ and with $|D \setminus F| < \varepsilon\}$. Let $F \subseteq \mathbf{R}$ be closed. Show that $F \in \mathcal{L}$.

(Problem 1421) Show that $D \in \mathcal{L}$ if and only if, for every $\varepsilon > 0$, there is an open set G with $D \cup G = \mathbf{R}$ and with $|G \cap D| < \varepsilon$.

(Problem 1440) Show that if $\{D_k\}_{k=1}^\infty \subseteq \mathcal{L}$ then $\bigcap_{k=1}^\infty D_k \in \mathcal{L}$.

(Problem 1440) Let $F \subseteq \mathbf{R}$ be closed. Suppose that $|F| < \infty$. Show that $\mathbf{R} \setminus F \in \mathcal{L}$.

(Problem 1450) Let $F \subseteq \mathbf{R}$ be closed. Show that $\mathbf{R} \setminus F \in \mathcal{L}$ even if $|F| = \infty$.

(Problem 1460) Let $D \in \mathcal{L}$. Show that $\mathbf{R} \setminus D \in \mathcal{L}$.

(Problem 1461) Show that if $D \in \mathcal{L}$ then for all $\varepsilon > 0$, there is an open set G with $D \subseteq G$ and with $|G \setminus D| < \varepsilon$.

(Problem 1462) Show that if for all $\varepsilon > 0$, there is an open set G with $D \subsetneq G$ and with $|G \setminus D| < \varepsilon$, then $D \in \mathcal{L}$.

(Problem 1470) Show that if $\{D_k\}_{k=1}^{\infty} \subseteq \mathcal{L}$ then $\bigcup_{k=1}^{\infty} D_k \in \mathcal{L}$.

(Problem 1530) Show that the set of all Lebesgue measurable subsets of $\mathbf{R}$ is a σ-algebra.

(Problem 1480) Show that if B is a Borel set and $\varepsilon > 0$, then there exists a closed set F with $F \subsetneq B$ and with $|B \setminus F| < \varepsilon$.

(Problem 1490) Let A, $B \subsetneq \mathbf{R}$. Suppose that B is a Borel set and that $A \cap B = \emptyset$. Show that $|A \cup B| = |A| + |B|$.

[Exercise 2D.10] Let A, $B \subsetneq \mathbf{R}$. Suppose that $B \in \mathcal{L}$, where $\mathcal{L}$ is as in Problem 1420, and that $A \cap B = \emptyset$. Show that $|A \cup B| = |A| + |B|$.

(Problem 1500) Show that there exists an $A \subsetneq \mathbf{R}$ that is not a Borel set.

(Problem 1510) Let $\mathcal{B}$ be the set of all Borel subsets of $\mathbf{R}$. Let $\mu(E) = |E|$ for all $E \in \mathcal{B}$. Show that $(\mathbf{R}, \mathcal{B}, \mu)$ is a measure space.

(Problem 1520) Let $A \subsetneq \mathbf{R}$. Show that the following statements are equivalent:

(a) If $\varepsilon > 0$, then there is a closed set F with $F \subsetneq A$ and with $|A \setminus F| < \varepsilon$.
(b) There exists a sequence $\{F_k\}_{k=1}^{\infty}$ such that $F_k \subsetneq A$ and F_k is closed for each $k \in \mathbf{N}$, and such that $|A \setminus \bigcup_{k=1}^{\infty} F_k| = 0$.
(c) There exists a Borel set B with $B \subsetneq A$ and with $|A \setminus B| = 0$.
(d) If $\varepsilon > 0$, then there is an open set G with $G \supsetneq A$ and with $|G \setminus A| < \varepsilon$.
(e) There exists a sequence $\{G_k\}_{k=1}^{\infty}$ such that $G_k \supsetneq A$ and G_k is open for each $k \in \mathbf{N}$, and such that $|\bigcap_{k=1}^{\infty} G_k \setminus A| = 0$.
(f) There exists a Borel set B with $B \supsetneq A$ and with $|B \setminus A| = 0$.

[Definition: Lebesgue measureable] A set $A \subsetneq \mathbf{R}$ is Lebesgue measurable if there is a Borel set $B \subsetneq A$ such that $|A \setminus B| = 0$.

(Problem 1590) Let $\mathcal{L}$ be the set of all Lebesgue measureable subsets of $\mathbf{R}$. Let $\mu(E) = |E|$ for all $E \in \mathcal{L}$. Show that $(\mathbf{R}, \mathcal{L}, \mu)$ is a measure space. (We refer to outer measure restricted to $\mathcal{L}$ as *Lebesgue measure*.)

(Problem 1600) Show that there exists an $A \subsetneq [0, 1]$ that is not Lebesgue measurable.

[Exercise 2D.12] Suppose that $A \subsetneq \mathbf{R}$ is a bounded set. Let b, $c \in \mathbf{R}$ be such that $A \subsetneq [b, c]$. Suppose that

$$|A \cup ([b, c] \setminus A)| = |A| + |[b, c] \setminus A|.$$

Show that A is Lebesgue measurable.

[Exercise 2D.13] Suppose that $A \subsetneq \mathbf{R}$. Suppose that

$$|(A \cap [-n, n]) \cup ([-n, n] \setminus A)| = |A \cap [-n, n]| + |[-n, n] \setminus A|$$

for all $n \in \mathbf{N}$. Show that A is Lebesgue measurable.

(Problem 1610) Suppose that $(\mathbf{R}, \mathcal{S}, \mu)$ is a measure space, where $\mu(E) = |E|$ for all $E \in \mathcal{S}$. Suppose further that $\mathcal{S}$ contains all closed bounded intervals. Show that $\mathcal{S} \subsetneq \mathcal{L}$, where $\mathcal{L}$ is the set of all Lebesgue measurable sets.

* * *

[Definition: Lebesgue measurable] Let $A \subsetneq \mathbf{R}$ and let $f : A \to \mathbf{R}$. We say that f is Lebesgue measurable if, whenever $B \subsetneq \mathbf{R}$ is Borel, we have that $f^{-1}(B)$ is a Lebesgue measurable set (that is, satisfies one of the six equivalent conditions of Problem 1520).

(Problem 1620) Let $f : \mathbf{R} \to \mathbf{R}$ be increasing and continuous. Let $\mathcal{S} = \{E \subsetneq \mathbf{R} : f(E) \text{ is Borel}\}$. Show that $\mathcal{S}$ contains all intervals.

(Bonus Problem 1621) Show that the previous problem is valid if f is merely increasing (not necessarily continuous).

(Problem 1630) Show that $\mathcal{S}$ is a σ-algebra.

(Problem 1640) Show that if $f:\mathbf{R}\to\mathbf{R}$ is increasing, and if $B\subseteq\mathbf{R}$ is a Borel set, then $f(B)$ is also a Borel set.

* * *

(Problem 1641) Let $F_0=F_{1,0}=[0,1]$. If n, $k\in\mathbf{N}$ and $F_{k,n-1}=[a_{k,n-1},b_{k,n-1}]$ exists, let $F_{2k-1,n}=[a_{k,n-1},\frac{2}{3}a_{k,n-1}+\frac{1}{3}b_{k,n-1}]$, and let $F_{2k,n}=[\frac{1}{3}a_{k,n-1}+\frac{2}{3}b_{k,n-1},b_{k,n-1}]$. Show that

- $F_{k,n}$ exists for all $n\in\mathbf{N}$ and for $1\leq k\leq 2^n$.
- $F_{k,n}$ is a closed interval of length 3^{-n}.
- If $1\leq k<j\leq 2^n$, $x\in F_{k,n}$ and $y\in F_{j,n}$, then $x<y$. In particular, $F_{k,n}\cap F_{j,n}=\emptyset$ if $j\neq k$.

Let $F_n=\bigcup_{k=1}^{2^n}F_{k,n}$.

(Problem 1642) If $n\in\mathbf{N}$ and $1<k<2^{n-1}$, let $G_{k,n}=F_{k,n-1}\setminus(F_{2k-1,n}\cup F_{2k,n})$. Show that

- $G_{k,n}$ is an open interval of length 3^{-n}.
- If $1\leq k<j\leq 2^{n-1}$, $x\in G_{k,n}$ and $y\in G_{j,n}$, then $x<y$. In particular, $G_{k,n}\cap G_{j,n}=\emptyset$ if $j\neq k$.
- If $n\neq m$ then $G_{k,n}\cap G_{j,m}=\emptyset$.

[Definition: The Cantor set] The Cantor set $C=\bigcap_{k=0}^{\infty}F_k=[0,1]\setminus\bigcup_{k=1}^{\infty}G_k$.

(Problem 1650) Show that C is closed.

(Problem 1660) Show that $|C|=0$.

(Problem 1670) Let $I\subseteq\mathbf{R}$ be an interval. Show that if $I\subseteq C$ then I contains at most one point.

(Problem 1680) Let $\Lambda_k(x)=\frac{|F_k\cap(-\infty,x)|}{|F_k|}$. Show that Λ_k is continuous.

(Problem 1690) Sketch the graphs of Λ_0, Λ_1, and Λ_2.

(Problem 1700) Show that if $n\geq m$, then $\Lambda_n(x)=\Lambda_m(x)$ for all $x\in G_m$.

(Problem 1710) Show that $\{\Lambda_k\}_{k=1}^{\infty}$ is uniformly Cauchy.

[Definition: The Cantor function] Let $\Lambda(x)=\lim_{k\to\infty}\Lambda_k(x)$.

(Problem 1720) Show that Λ exists and is continuous, increasing, and surjective $\Lambda:[0,1]\to[0,1]$.

(Problem 1730) Show that $\Lambda([0,1]\setminus C)$ is countable.

(Problem 1740) Show that $\Lambda(C)=[0,1]$.

(Problem 1750) Show that C is uncountable.

(Problem 1760) Let $A\subsetneq[0,1]$ be a set that is not a Borel set. Let $E=C\cap\Lambda^{-1}(A)$. Show that E is Lebesgue measurable but that $\Lambda(E)$ is not Lebesgue measurable.

2E. Undergraduate analysis

[Definition: Pointwise convergence] Let X be a set, let $f:X\to\mathbf{R}$, and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of functions from X to R. We say that the sequence $\{f_k\}_{k=1}^{\infty}$ converges to f pointwise if $\lim_{k\to\infty}f_k(x)=f(x)$ for each $x\in X$. That is, if for every $\varepsilon>0$ and every $x\in X$ there is a $N\in\mathbf{N}$ such that $|f_k(x)-f(x)|<\varepsilon$ for all $k\geq N$.

[Definition: Uniform convergence] Let X be a set, let $f:X\to\mathbf{R}$, and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of functions from X to R. We say that the sequence $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly if for every $\varepsilon>0$ there is a $N\in\mathbf{N}$ such that $|f_k(x)-f(x)|<\varepsilon$ for all $k\geq N$ and all $x\in X$.

(Problem 1770) Give an example of a sequence of functions that converges pointwise but not uniformly.

(Problem 1771) Let X be a metric space and let $Y\subseteq X$ be a subspace. Show that $G\subseteq Y$ is open in Y (relatively open) if and only if there is a $U\subseteq X$ that is open in X and satisfies $G=Y\cap U$.

(Problem 1772) Let X be a metric space and let $Y \subseteq X$ be a subspace. Show that $F \subseteq Y$ is closed in Y (relatively closed) if and only if there is a $D \subseteq X$ that is closed in X and satisfies $F = Y \cap D$.

(Problem 1773) Let $X \subseteq \mathbf{R}$ and let $f : X \to \mathbf{R}$. Show that f is continuous everywhere on X if and only if, for every $U \subseteq \mathbf{R}$ open, the set $f^{-1}(U)$ is relatively open.

(Problem 1780) Show that if the sequence $\{f_k\}_{k=1}^{\infty}$ converges uniformly to f, then the sequence $\{f_k\}_{k=1}^{\infty}$ converges pointwise to f.

(Problem 1790) Let $X \subseteq \mathbf{R}$, let $x_0 \in X$, let $f : X \to \mathbf{R}$, and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of functions from X to R. Suppose that each f_k is continuous at x_0 and that the sequence $\{f_k\}_{k=1}^{\infty}$ converges to f uniformly on X. Show that f is also continuous at x_0.

(Problem 1800) Give an example of a sequence of continuous functions that converge pointwise to a discontinuous function.

(Problem 1810) Let $F \subseteq \mathbf{R}$ be a closed set. Let $g : F \to \mathbf{R}$ be continuous. Show that there exists a function $h : \mathbf{R} \to \mathbf{R}$ such that h is continuous and such that $h(x) = g(x)$ for all $x \in F$.

2E. Convergence of Measurable Functions.

(Problem 1820) Let X be a set. Let $f : X \to \mathbf{R}$ and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of functions from X to $\mathbf{R}$. Let $Y = \{x \in X : \lim_{k\to\infty} f_k(x) = f(x)\}$ and let $A_{n,m,k} = \left\{x \in X : |f_k(x) - f(x)| < \frac{1}{n}\right\}$. Write Y in terms of unions and intersections of the sets $A_{n,m,k}$.

(Problem 1830) Let $\{m_n\}_{n=1}^{\infty}$ be a sequence of natural numbers. Show that f_k converges uniformly to f on the set

$$\bigcap_{n=1}^{\infty} \bigcap_{k=m_n}^{\infty} \left\{x \in X : |f_k(x) - f(x)| < \tfrac{1}{n}\right\}.$$

(Problem 1840) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $f : X \to \mathbf{R}$ be $\mathcal{S}$-measurable and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$. Suppose that f_k converges to f pointwise on X. Let

$$A_{n,m} = \bigcap_{k=m}^{\infty} \left\{x \in X : |f_k(x) - f(x)| < \tfrac{1}{n}\right\}.$$

Show that $\lim_{m\to\infty} \mu(A_{n,m}) = \mu(X)$.

(Problem 1850) Suppose further that $\mu(X) < \infty$. Choose some $\varepsilon > 0$. For each $n \in \mathbf{N}$, let m_n be such that $\mu(A_{n,m_n}) > \mu(X) - \frac{\varepsilon}{2^n}$. Show that $\mu(X \setminus \bigcap_{n=1}^{\infty} A_{n,m_n}) < \varepsilon$.

(Problem 1860) [Egorov's Theorem] Let $(X, \mathcal{S}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $f : X \to \mathbf{R}$ be $\mathcal{S}$-measurable and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$. Suppose that f_k converges to f pointwise on X. Show that for every $\varepsilon > 0$ there is a set $E \subseteq X$ with $\mu(X \setminus E) < \varepsilon$ such that f_k converges to f uniformly on E.

[Exercise 2E.5] Give an example of a Borel measurable function $f : \mathbf{R} \to \mathbf{R}$ and a sequence of Borel measurable functions $f_k : \mathbf{R} \to \mathbf{R}$ such that $f_k \to f$ pointwise but such that f_k does not converge uniformly on any set of infinite measure.

(Problem 1861) What is the best analogue to Egorov's Theorem available in $\mathbf{R}$? In an arbitrary measure space of infinite measure?

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[Definition: Simple function] A function is called simple if it takes on only finitely many values.

(Problem 1870) Let $(X, \mathcal{S})$ be a measurable space and let $f : X \to \mathbf{R}$ be a simple function. Let $f(X) = \{c_k : 1 \leq k \leq n\}$. Let $E_k = f^{-1}(\{c_k\})$. Show that if $x \in X$ then $x \in E_k$ for exactly one value of k (so $X = \bigcup_{k=1}^{n} E_k$ and $E_j \cap E_k = \emptyset$ if $j \neq k$).

(Problem 1871) Show that $f = \sum_{k=1}^{n} c_k \chi_{E_k}$.

(Problem 1880) Show that f is $\mathcal{S}$-measurable if and only if $\{E_k : 1 \leq k \leq n\} \subseteq \mathcal{S}$.

(Problem 1890) Let X be a set, let $k \in \mathbf{N}$, and let $f : X \to \mathbf{R}$. Define f_k as

$$f_k(x) = \operatorname{sgn}(f(x)) \min\left(k, \frac{\lfloor 2^k |f(x)| \rfloor}{2^k}\right)$$

where $\lfloor y \rfloor = \inf\{n \in \mathbf{Z} : n \leq y\}$. Find $f(X)$ and show that f_k is simple.

(Problem 1900) Show that $\{f_k(x)\}_{k=1}^{\infty}$ is increasing if $f(x) \geq 0$ and decreasing if $f(x) \leq 0$.

(Problem 1910) Let $X_m = \{x \in X : |f(x)| < m\}$. Show that $f_k \to f$ uniformly on X_m for each $m \in \mathbf{N}$.

(Problem 1920) Show that $\lim_{k\to\infty} f_k(x) = f(x)$ for all $x \in X$ and that, if f is bounded, then f_k converges to f uniformly on X.

(Problem 1930) Let $\mathcal{S}$ be a σ-algebra on X. Suppose in addition that f is $\mathcal{S}$-measurable. Show that f_k is $\mathcal{S}$-measurable for each $k \in \mathbf{N}$.

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(Problem 1940) Give an example of a Borel measurable function $f : \mathbf{R} \to \mathbf{R}$ such that f is not continuous at any point $x \in \mathbf{R}$.

(Problem 1950) Let B be a Lebesgue measurable set. Show that for all $\varepsilon > 0$, there is an open set U with $|U| < \varepsilon$ such that $B \setminus U$ is closed and such that $(\mathbf{R} \setminus B) \setminus U$ is also closed.

(Problem 1960) Show that $g = \chi_B\big|_{\mathbf{R}\setminus U}$ is continuous everywhere on $\mathbf{R} \setminus U$.

(Problem 1970) Show that if $s : \mathbf{R} \to \mathbf{R}$ is simple and Lebesgue measurable and $\varepsilon > 0$, then there is an open set U with $|U| < \varepsilon$ such that $s\big|_{\mathbf{R}\setminus U}$ is continuous everywhere on $\mathbf{R} \setminus U$.

(Problem 1980) Let $f : \mathbf{R} \to \mathbf{R}$ be Lebesgue measurable. Let $B \subseteq \mathbf{R}$ be Lebesgue measurable with $|B| < \infty$. For all $\varepsilon > 0$, show that there is an open set G such that $|G| < \varepsilon$ and such that f is bounded on $B \setminus G$.

(Problem 1990) [Luzin's theorem] Let $f : \mathbf{R} \to \mathbf{R}$ be Lebesgue measurable. For all $\varepsilon > 0$, show that there is an open set U and a continuous function $h : \mathbf{R} \to \mathbf{R}$ such that $|U| < \varepsilon$ and such that $f(x) = h(x)$ for all $x \notin U$.

(Problem 2000) Need f be continuous on $\mathbf{R}$?

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(Problem 2010) Let $s : \mathbf{R} \to \mathbf{R}$ be Lebesgue measurable and simple. Show that there exists a function $g : \mathbf{R} \to \mathbf{R}$ that is Borel measurable and such that the set $\{x \in \mathbf{R} : f(x) \neq g(x)\}$ is Borel and satisfies $|\{x \in \mathbf{R} : s(x) \neq g(x)\}| = 0$.

(Problem 2020) Let $f : \mathbf{R} \to \mathbf{R}$ be Lebesgue measurable. Show that there exists a function $g : \mathbf{R} \to \mathbf{R}$ that is Borel measurable and satisfies $|\{x \in \mathbf{R} : f(x) \neq g(x)\}| = 0$.

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(Problem 2021) Let $\{x_n\}_{n=1}^{\infty} \subseteq [-\infty, \infty]$. Suppose that $x_n \leq x_{n+1}$ for all n. Show that $\lim_{n\to\infty} x_n$ exists and satisfies $\lim_{n\to\infty} x_n = \sup_{n\in\mathbf{N}} x_n$.

3A. Integration with Respect to a Measure.

[Definition: $\mathcal{S}$-partition] Let $\mathcal{S}$ be a σ-algebra on a set X. An $\mathcal{S}$-partition of X is a finite collection of disjoint sets in S whose union is all of X. (So $P = \{A_j : 1 \leq j \leq m\}$, $m < \infty$, $A_j \cap A_k = \varnothing$ if $j \neq k$, and $X = \cup_{j=1}^m A_j$.)

[Definition: Lower Lebesgue sum] Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $f : X \to [0, \infty]$ be $\mathcal{S}$-measurable. Let $P = \{A_j : 1 \leq j \leq m\}$ be an $\mathcal{S}$-partition of X. The lower Lebesgue sum $\mathcal{L}(f, P)$ is

$$\mathcal{L}(f, P) = \sum_{j=1}^{m} \mu(A_j) \inf_{A_j} f.$$

(If either $\mu(A_j) = 0$ or $\inf_{A_j} f = 0$, then we take $\mu(A_j) \inf_{A_j} f = 0$ even if the other quantity is ∞.)

[Definition: Integral of a nonnegative function] Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $f : X \to [0, \infty]$ be $\mathcal{S}$-measurable. Then

$$\int f \, d\mu = \sup\{\mathcal{L}(f, P) : P \text{ is an } \mathcal{S}\text{-partition of } X\}.$$

(Problem 2100) Suppose that $(X, \mathcal{S}, \mu)$ is a measure space and $E \in \mathcal{S}$. What is $\int \chi_E \, d\mu$?

(Problem 2110) What is $\int \chi_{\mathbf{Q}} \, d\lambda$? What is $\int \chi_{[0,1]\setminus\mathbf{Q}} \, d\lambda$?

[Exercise 3A.5] Let $\{b_k\}_{k=1}^\infty$ be a sequence of nonnegative real numbers. Define the function $b : \mathbf{N} \to \mathbf{R}$ by $b(k) = b_k$. Let $\mu(E) = \#E$ denote the counting measure. Then $\int b \, d\mu = \sum_{k=1}^\infty b_k$.

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(Bonus Problem 2030) Let $X = [a, b]$ for some a, $b \in \mathbf{R}$, $a < b$. Let $a = x_0 < x_1 < \cdots < x_m = b$. Let $A_0 = \{x_j : 0 \leq l \leq m\}$ and let $A_k = (x_{k-1}, x_k)$ for $1 \leq k \leq m$. Show that $P = \{A_j : 0 \leq j \leq m\}$ is a $\mathcal{L}$-partition and a $\mathcal{B}$-partition, where $\mathcal{L}$ and $\mathcal{B}$ are the σ-algebras of Borel and Lebesgue measurable sets, respectively.

(Bonus Problem 2040) Let P be the partition in Problem 2030. Let $f : [a, b] \to \mathbf{R}$ be Lebesgue measurable. How does $\mathcal{L}(f, P)$ relate to the upper and lower Darboux sums $U(f, A_0)$ and $L(f, A_0)$ of f over $A_0 = \{x_0, \ldots, x_m\}$?

(Bonus Problem 2050) Suppose that $X = [a, b]$, $\mathcal{S}$ denotes the Borel (or Lebesgue) sets, λ denotes Lebesgue measure, and $f : X \to [0, \infty)$ is bounded and Borel measurable. How does $\int f \, d\lambda$ compare to the lower Riemann integral $L(f, [a, b])$?

(Bonus Problem 2060) Let $(X, \mathcal{S}, \mu)$ be a measurable space and $f : X \to [0, \infty)$ be bounded. How would you define an upper Lebesgue sum $\mathcal{U}(f, P)$? How would you define an integral in terms of the upper Lebesgue sum?

(Bonus Problem 2061) Let P, Q be two partitions of X. What can you say about $\mathcal{L}(f, P)$ and $\mathcal{U}(f, Q)$?

[Exercise 3B.4a] If $\mu(X) < \infty$ and $f : X \to [0, \infty)$ is a bounded $\mathcal{S}$-measurable function, then the "upper" and "lower" Lebesgue integrals are equal.

[Exercise 3B.4b]

[Exercise 3B.4c]

(Bonus Problem 2080) Why did we use the "lower" Lebesgue integral (instead of the "upper" Lebesgue integral) as the definition of Lebesgue integral?

(Bonus Problem 2090) Suppose that $X = [a, b]$, $\mathcal{S}$ denotes the Borel (or Lebesgue) subsets of X, λ denotes Lebesgue measure, and $f : X \to [0, \infty)$ is bounded and Borel measurable. How does the (upper) Lebesgue integral compare to the upper Riemann integral $U(f, [a, b])$?

(Bonus Problem 2091) Suppose that $X = [a, b]$, $\mathcal{S}$ denotes the Borel (or Lebesgue) subsets of X, λ denotes Lebesgue measure, and $f : X \to [0, \infty)$ is bounded. Show that if f is Riemann integrable, then f is Lebesgue measurable, and moreover the Riemann and Lebesgue integrals of f coincide.

(Bonus Problem 2092) Give an example of a function that is Lebesgue integrable (measurable) but not Riemann integrable.

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(Problem 2120) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let f be an $\mathcal{S}$-measurable simple function, so $f = \sum_{k=1}^{m} c_k \chi_{E_k}$ for some partition $\{E_k\}$ and distinct values c_k. What is $\int f \, d\mu$ in terms of c_k and E_k?

(Problem 2121) Suppose that $f = \sum_{k=1}^{m} c_k \chi_{E_k}$, where the E_ks are pairwise-disjoint, but possibly empty or not having union X, and where the numbers c_k need not be distinct. What is $\int f \, d\mu$?

(Bonus Problem 2122) What is $\int f \, d\mu$ if we relax the requirement that the E_ks be pairwise disjoint?

(Problem 2130) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $f : X \to [0, \infty]$ be $\mathcal{S}$-measurable. Show that

$$\int f \, d\mu = \sup\left\{\int s \, d\mu : s \text{ is simple and } 0 \leq s(x) \leq f(x) \text{ for all } x \in X\right\}.$$

(Problem 2140) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let f, $g : X \to [0, \infty]$ be $\mathcal{S}$-measurable. Suppose that $f(x) \leq g(x)$ for all $x \in X$. Show that $\int f \, d\mu \leq \int g \, d\mu$.

(Problem 2150) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $f_k : X \to [0, \infty]$ be $\mathcal{S}$-measurable. Suppose that $f_k(x) \leq f_{k+1}(x)$ for all $x \in X$ and all $k \in \mathbf{N}$. Show that $\lim_{k\to\infty} \int f_k \, d\mu$ exists.

(Problem 2160) Let $f(x) = \lim_{k\to\infty} f_k(x)$. By Problem 1120, f is $\mathcal{S}$-measurable. Show that $\lim_{k\to\infty} \int f_k \, d\mu \leq \int f \, d\mu$.

(Problem 2170) Let $s \leq f$ be a $\mathcal{S}$-measurable simple function. Let $0 < t < 1$. Let

$$s_k(x) = \begin{cases} t\,s(x), & \text{if } t\,s(x) \leq f_k(x), \\ 0, & \text{otherwise.} \end{cases}$$

Show that s_k is simple and $\mathcal{S}$-measurable.

(Problem 2171) What can you say about $\int s_k \, d\mu$ and $\int f_k \, d\mu$?

(Problem 2180) What is $\lim_{k\to\infty} \int s_k \, d\mu$?

(Problem 2190) Show that $\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu$.

[Exercise 3A.8] There exists a sequence of simple Borel measurable functions from $\mathbf{R}$ to $[0, \infty)$ such that $\lim_{k\to\infty} f_k(x) = 0$ for all $x \in \mathbf{R}$ but $\lim_{k\to\infty} \int f_k \, d\lambda = 1$.

(Problem 2240) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let f be a nonnegative $\mathcal{S}$-measurable function. Let $c \geq 0$. Show that $\int cf \, d\mu = c \int f \, d\mu$.

(Problem 2210) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let f, g be two nonnegative $\mathcal{S}$-measurable simple functions. Show that $f + g$ is also simple.

(Problem 2220) Show that $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$ if f and g are nonnegative and simple.

(Problem 2230) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let f, g be two nonnegative $\mathcal{S}$-measurable functions. By Problem 1090, $f + g$ is also $\mathcal{S}$-measurable. Show that $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$.

[Definition: Positive and negative parts] Let $f : X \to [-\infty, \infty]$ be a function. We define

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) > 0, \\ 0 & \text{if } f(x) \leq 0, \end{cases} \qquad f^-(x) = \begin{cases} -f(x) & \text{if } f(x) < 0, \\ 0 & \text{if } f(x) \geq 0. \end{cases}$$

(Problem 2250) Show that $f(x) = f^+(x) - f^-(x)$ and that $f^\pm : X \to [0, \infty]$.

(Problem 2260) Let $(X, \mathcal{S})$ be a measurable space and $f : X \to [-\infty, \infty]$. Suppose f is $\mathcal{S}$-measurable. Show that f^+ and f^- are $\mathcal{S}$-measurable.

[Definition: Integral of a real-valued function] Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $f : X \to [-\infty, \infty]$ be $\mathcal{S}$-measurable. If $\int f^+ \, d\mu < \infty$ or $\int f^- \, d\mu < \infty$ (or both), we define

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

(Problem 2290) What are the two ways that $\int f \, d\mu$ can fail to exist?

(Problem 2270) Show that the definition of the integral of a nonnegative function given above coincides with this new definition in the case where f is nonnegative.

(Problem 2280) Show that $\int |f| \, d\mu < \infty$ if and only if $\int f^+ \, d\mu < \infty$ and $\int f^- \, d\mu < \infty$.

(Problem 2330) Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $f : X \to [-\infty, \infty]$, and $\int f \, d\mu$ exists. Show that $\left|\int f \, d\mu\right| \leq \int |f| \, d\mu$.

(Problem 2300) Suppose $(X, \mathcal{S}, \mu)$ is a measure space, $f : X \to [-\infty, \infty]$, and $\int f \, d\mu$ exists. Let $c \in \mathbf{R}$. Show that $\int cf \, d\mu = c \int f \, d\mu$.

(Problem 2310) Suppose $(X, \mathcal{S}, \mu)$ is a measure space, f, $g : X \to [-\infty, \infty]$, and $\int |f| \, d\mu < \infty$, $\int |g| \, d\mu < \infty$. Show that $\int f + g \, d\mu = \int f \, d\mu + \int g \, d\mu$.

(Problem 2320) Suppose $(X, \mathcal{S}, \mu)$ is a measure space, f, $g : X \to [-\infty, \infty]$, $f(x) \leq g(x)$ for all $x \in X$, and $\int f \, d\mu$, $\int g \, d\mu$ exist. Show that $\int f \, d\mu \leq \int g \, d\mu$.

3B. Limits of integrals and integrals of limits

[Definition: Integration on a subset] Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $E \in \mathcal{S}$. If $f : X \to [-\infty, infty]$ is a $\mathcal{S}$-measurable function, then we let

$$\int_E f \, d\mu = \int \chi_E f \, d\mu$$

provided the right-hand side exists.

(Problem 2350) Let A, $B \in \mathcal{S}$ be disjoint. Show that $\int_A f \, d\mu + \int_B f \, d\mu = \int_{A \cup B} f \, d\mu$.

(Problem 2340) Show that $\left|\int_E f \, d\mu\right| \leq \mu(E) \sup_E |f|$.

(Problem 2360) [The Bounded Convergence Theorem] Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of $\mathcal{S}$-measurable functions from X to $\mathbf{R}$ that converges pointwise on X to a function $f : X \to \mathbf{R}$. Suppose that $\mu(X) < \infty$ and that $\sup_{x \in X} \sup_{k \in \mathbf{N}} |f_k(x)| < \infty$. Show that $\lim_{k\to\infty} \int f_k \, d\mu = \int f \, d\mu$.

(Problem 2380) Suppose that $\mu(E) = 0$. Show that $\int_E f \, d\mu = 0$.

(Problem 2370) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let f, $g : X \to [-\infty, \infty]$ be two $\mathcal{S}$-measurable functions. Suppose that $\mu\{x \in X : f(x) \neq g(x)\} = 0$. Show that $\int f \, d\mu$ exists if and only if $\int g \, d\mu$ exists, and that $\int f \, d\mu = \int g \, d\mu$ if they exist.

[Definition: Almost everywhere] Let $(X, \mathcal{S}, \mu)$ be a measure space. If $E \in \mathcal{S}$, then E contains μ-almost every element in X if $\mu(X \setminus E) = 0$. If μ is clear from context, we say E contains almost every element in X.

(Problem 2390) Show that the Bounded Convergence Theorem is still true if we relax the requirement that the f_ks are uniformly bounded to the requirement that there is some $c \in \mathbf{R}$ such that $|f_k(x)| \leq c$ for μ-almost every $x \in X$.

(Problem 2400) Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $g : X \to [0, \infty]$ be $\mathcal{S}$-measurable. Suppose that g is bounded. Show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that, if $B \in \mathcal{S}$ and $\mu(B) < \delta$, then $\int_B g \, d\mu < \varepsilon$.

(Problem 2410) Show that the preceding problem is true provided $\int g\,d\mu < \infty$ even if g is not bounded.

[Exercise 3A.1] Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $g : X \to [0, \infty]$ be $\mathcal{S}$-measurable. Suppose that $\int g\,d\mu < \infty$. Show that if $E \in \mathcal{S}$ and $\mu(E) = \infty$ then $\inf_E g = 0$.

(Problem 2420) Show that for every $\varepsilon > 0$ there is an $E \in \mathcal{S}$ with $\mu(E) < \infty$ and $\int_{X\setminus E} g\,d\mu < \varepsilon$.

(Problem 2430) [The Dominated Convergence Theorem] Let $(X, \mathcal{S}, \mu)$ be a measure space, and let $f, f_k : X \to [-\infty, \infty]$, $g : X \to [0, \infty]$ be $\mathcal{S}$-measurable. Suppose that

- $\mu(X) < \infty$,
- $\lim_{k\to\infty} f_k(x) = f(x)$ for almost every $x \in X$.
- $\int g\,d\mu < \infty$
- $\sup_{k\in\mathbf{N}} |f_k(x)| \leq g(x)$ for almost every $x \in X$.

Show that $\lim_{k\to\infty} \int f_k\,d\mu = \int f\,d\mu$.

(Problem 2440) [The Dominated Convergence Theorem] Show that the previous problem is true even if $\mu(X) = \infty$.

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[Definition: The Lebesgue space] Let $(X, \mathcal{S}, \mu)$ be a measure space. Let $f : X \to [-\infty, \infty]$ be $\mathcal{S}$-measurable. Then the L^1-norm of f is defined by

$$\|f\|_1 = \int |f|\,d\mu.$$

The Lebesgue space $L^1(\mu)$ is defined by

$$L^1(\mu) = \{f : f \text{ is an } \mathcal{S}\text{-measurable function } f : X \to \mathbf{R} \text{ and } \|f\|_1 < \infty\}.$$

(Problem 2450) Let $f, g \in L^1(\mu)$. Show that

- $\|f\|_1 \geq 0$.
- $\|f\|_1 = 0$ if and only if $f = 0$ almost everywhere.
- $\|cf\|_1 = |c|\,\|f\|_1$ for all $c \in \mathbf{R}$.
- $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$.

(Problem 2451) Is $\|\cdot\|_1$ a norm on the vector space $L^1(\mu)$?

(Problem 2460) Let μ denote the counting measure on $\mathbf{N}$. What is $L^1(\mu)$? (This space is often called ℓ^1.)

(Problem 2470) Let $f \in L^1(\mu)$ and let $\varepsilon > 0$. Show that there is a simple function $s \in L^1(\mu)$ that satisfies $\|f - s\|_1 < \varepsilon$.

[Definition: Lebesgue space on the real numbers] $L^1(\mathbf{R}) = L^1(\lambda)$ where λ denotes Lebesgue measure. (The underlying σ-algebra is the Lebesgue or Borel measurable sets.)

[Definition: Step function] We say that $g : \mathbf{R} \to \mathbf{R}$ is a step function if $g = \sum_{j=1}^m a_j \chi_{I_j}$, where each I_j is an interval and each $a_j \in \mathbf{R}$.

(Problem 2480) Let g be a step function. Show that we may require the intervals I_j to be pairwise disjoint.

(Problem 2500) Let $f \in L^1(\mathbf{R})$ and let $\varepsilon > 0$. Show that there is a step function $s \in L^1(\mathbf{R})$ that satisfies $\|f - s\|_1 < \varepsilon$.

(Problem 2510) Let $f \in L^1(\mathbf{R})$ and let $\varepsilon > 0$. Show that there is a continuous function $g \in L^1(\mathbf{R})$ that satisfies $\|f - g\|_1 < \varepsilon$.

(Bonus Problem 2511) Can we require that g have *compact support*, that is, that $g(x) = 0$ outside of some bounded set?

(Bonus Problem 2512) Can we require that g be continuously differentiable? Twice differentiable? Smooth (differentiable to order m for all $m \in \mathbf{N}$)?

4A. Hardy-Littlewood maximal function

(Problem 2520) (Markov's inequality) Let $(X, \mathcal{S}, \mu)$ be a measure space, let $h \in L^1(\mu)$, and let $c \in (0, \infty)$. Show that

$$\mu\{x \in X : |h(x)| \geq c\} \leq \frac{1}{c}\|h\|_1.$$

[Definition: $3I$] Let I be a bounded nonempty open interval in $\mathbf{R}$. Then $3I$ is the open interval with the same center as I and three times its length.

(Problem 2530) Show that if I, J are bounded nonempty non-disjoint open intervals, and $\ell(I) \geq \ell(J)$, then $J \subseteq 3I$.

(Bonus Problem 2540) Let $I_1 = (0, 10), I_2 = (9, 15), I_3 = (14, 22), I_4 = (21, 31)$. What subsets of $\{I_1, I_2, I_3, I_4\}$ are pairwise disjoint?

(Bonus Problem 2550) Find $I_1 \cup I_2 \cup I_3 \cup I_4$.

(Bonus Problem 2560) Find $\bigcup_{I \in \mathcal{J}} 3I$ for each of the sets $\mathcal{J}$ you found in Problem 2540.

(Problem 2570) [The Vitali covering lemma] Let $\{I_k\}_{k=1}^n$ be a list of finitely many bounded nonempty open intervals in $\mathbf{R}$. Show that there exists a sublist $\{I_{k_j}\}_{j=1}^m$ such that the I_{k_j}s are pairwise disjoint and $\bigcup_{k=1}^n I_k \subseteq \bigcup_{j=1}^m 3I_{k_j}$.

[Definition: Hardy-Littlewood maximal function] Let $h : \mathbf{R} \to \mathbf{R}$ be measurable. Then

$$h^*(b) = \sup_{t>0} \frac{1}{2t} \int_{b-t}^{b+t} |h|.$$

(Problem 2571) Show that if h is bounded and $b \in \mathbf{R}$ then $h^*(b) \leq \sup_{\mathbf{R}} |h|$.

(Bonus Problem 2580) Let $h = \chi_{[0,1]}$. Find h^*.

[Exercise 4A.9] Let $h : \mathbf{R} \to \mathbf{R}$ be measurable and let $c > 0$. Then $\{b \in \mathbf{R} : h^*(b) > c\}$ is open.

(Problem 2590) Suppose $h \in L^1(\mathbf{R})$ and $c \in (0, \infty)$. Show that

$$|\{b \in \mathbf{R} : h^*(b) > c\}| < \frac{3}{c}\|h\|_1.$$

4B. Undergraduate analysis

[Definition: Derivative] Let $I \subseteq \mathbf{R}$ be an open interval, $b \in I$ and $g : I \to \mathbf{R}$. Then $g'(b) = \lim_{t \to 0} \frac{g(b+t)-g(b)}{t}$ if the limit exists, in which case g is differentiable at b.

(Problem 2600) Let $f : \mathbf{R} \to \mathbf{R}$ be continuous. Show that if $b \in \mathbf{R}$ then

$$\lim_{t \to 0^+} \frac{1}{2t} \int_{b-t}^{b+t} |f - f(b)| = 0.$$

(Problem 2620) Let $f \in L^1(\mathbf{R})$ be continuous. Define $g(x) = \int_{-\infty}^{x} f$. Let $b \in \mathbf{R}$. Show that g is differentiable at b and that $g'(b) = f(b)$.

4B. Derivatives of integrals

(Problem 2610) Let $f \in L^1(\mathbf{R})$. Show that

$$\lim_{t\to 0^+} \frac{1}{2t}\int_{b-t}^{b+t} |f - f(b)| = 0$$

for almost every $b \in \mathbf{R}$.

(Problem 2660) Let $f \in L^1(\mathbf{R})$. Show that $f(b) = \lim_{t\to 0^+} \frac{1}{2t}\int_{b-t}^{b+t} f$ for almost every $b \in \mathbf{R}$.

(Problem 2630) Let $f \in L^1(\mathbf{R})$. Define $g(x) = \int_{-\infty}^{x} f$. Let $b \in \mathbf{R}$. Show that g is differentiable at b with $g'(b) = f(b)$ for almost every $b \in \mathbf{R}$.

(Problem 2640) Let $E \subseteq [0,1]$. Suppose that E has the property that $|E \cap [0,b]| = \frac{b}{2}$ for all $b \in [0,1]$. Show that E is not Lebesgue measurable.

(Bonus Problem 2650) Can you show that such a set E exists?

[Definition: Density] Let $E \subseteq \mathbf{R}$ and $b \in \mathbf{R}$. The density of E at b is $\lim_{t\to 0^+} \frac{|E\cap(b-t,b+t)|}{2t}$ provided the limit exists.

(Problem 2670) Let $E \subseteq \mathbf{R}$ be a Lebesgue measurable set with $|E| < \infty$. Show that the density of E is 1 at almost every $b \in E$ and that the density of E is 0 at almost every $b \notin E$.

(Problem 2680) Show that the above result is still true even if $|E| = \infty$.

(Problem 2690) Let $G \subseteq \mathbf{R}$ be open and nonempty. Show that there exist two closed sets F, $\widehat{F} \subseteq G \setminus \mathbf{Q}$ with $F \cap \widehat{F} = \varnothing$ and $|F| > 0$, $|\widehat{F}| > 0$.

(Problem 2700) Let S be the set of all nonempty bounded open intervals in $\mathbf{R}$ with rational endpoints. Is S countable or uncountable?

(Problem 2710) If $G \subseteq \mathbf{R}$ is open, is there an $I \in S$ with $I \subseteq G$?

(Problem 2720) Show that there exists a Borel set E such that $0 < |E \cap I| < |I|$ for all nonempty bounded open intervals I.

(Problem 2730) Why don't Problems 2680 and 2720 contradict each other?

5A. Undergraduate analysis

[Definition: Cartesian product] Let A and B be sets. The Cartesian product $A \times B$ of A and B is the set of all ordered pairs (a,b) with $a \in A$ and $b \in B$.

5A. Products of measure spaces

[Definition: Rectangle] If X and Y are sets, then $R \subseteq X \times Y$ is called a rectangle if there exist sets $A \subseteq X$, $B \subseteq Y$ with $R = A \times B$.

[Definition: Measurable rectangle] Let $(X,\mathcal{S})$ and $(Y,\mathcal{T})$ be measurable spaces. A set $R \subseteq X \times Y$ is called a measurable rectangle if there exist sets $A \in \mathcal{S}$, $B \in \mathcal{T}$ with $R = A \times B$.

[Definition: Product σ-algebra] Let $(X,\mathcal{S})$ and $(Y,\mathcal{T})$ be measurable spaces. Then $\mathcal{S} \otimes \mathcal{T}$ is the smallest σ-algebra on $X \times Y$ that contains all measurable rectangles in $X \times Y$.

(Problem 2740) How is $\mathcal{S} \otimes \mathcal{T}$ different from $\mathcal{S} \times \mathcal{T}$?

[Definition: Cross section of a set] Let X and Y be sets and let $E \subseteq X \times Y$. If $a \in X$ or $b \in Y$, then the cross sections $[E]_a$ and $[E]^b$ are defined by

$$[E]_a = \{y \in Y : (a,y) \in E\}, \quad [E]^b = \{x \in X : (x,b) \in E\}.$$

(Problem 2750) Let $X = Y = \mathbf{R}$ and let $E = \{(x,y) : x^2 + y^2 < 25\}$. Draw $[E]_3$ and $[E]^4$ in $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$.

(Problem 2760) Suppose $a \in X$, $A \subseteq X$ and $b \in Y$, $B \subseteq Y$. What are $[A \times B]_a$ and $[A \times B]^b$?

(Problem 2770) Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces. Let

$$\mathcal{E} = \{E \subseteq X \times Y : [E]_a \in \mathcal{T},\ [E]^b \in \mathcal{S} \text{ for all } a \in X,\ b \in Y\}.$$

Show that $\mathcal{E}$ contains all measurable rectangles in $X \times Y$.

(Problem 2780) Show that $\mathcal{E}$ is a σ-algebra.

(Problem 2790) What can you conclude about $\mathcal{E}$ and $\mathcal{S} \otimes \mathcal{T}$?

(Problem 2791) Is $\mathcal{E} \subseteq \mathcal{S} \otimes \mathcal{T}$?

[Definition: Cross sections of functions] Let $f : X \times Y \to \mathbf{R}$ be a function. If $a \in X$ and $b \in Y$, we let

$$[f]_a(y) = f(a, y), \quad [f]^b(x) = f(x, b)$$

for all $y \in Y$, $x \in X$.

(Problem 2800) Let $A \subseteq X$, $B \subseteq Y$, and $f = \chi_{A \times B}$. If $a \in X$, what is $[f]_a$? If $b \in B$, what is $[f]^b$?

(Problem 2801) If $a \in X$ and $E \subseteq X \times Y$, what can you say about $[\chi_E]_a$ and $\chi_{[E]_a}$?

(Problem 2810) Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces. Let $a \in X$ and $b \in Y$. Let $f : X \times Y \to \mathbf{R}$ be a $\mathcal{S} \otimes \mathcal{T}$-measurable function. Show that $[f]_a$ is $\mathcal{T}$-measurable and $[f]^b$ is $\mathcal{S}$-measurable.

* * *

[Definition: Algebra] Let W be a set and let $\mathcal{A}$ be a set of subsets of W. We say that $\mathcal{A}$ is an algebra on W if:

- $\varnothing \in \mathcal{A}$
- If $E \in \mathcal{A}$ then $W \setminus E \in \mathcal{A}$
- If $E, F \in \mathcal{A}$, then $E \cup F \in \mathcal{A}$.

(Problem 2840) Show that all algebras are closed under finite intersections.

(Problem 2850) Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be measurable spaces. Let $\mathcal{A}$ be the set of all finite unions of measurable rectangles in $\mathcal{S} \otimes \mathcal{T}$. Show that $\mathcal{A}$ is an algebra on $X \times Y$.

(Problem 2851) Let $A \times B$ and $C \times D$ be two measurable rectangles in $X \times Y$. Show that $(A \times B) \cup (C \times D) = (A \times B) \cup H \cup J$, where H and J are two measurable rectangles that satisfy $(A \times B) \cap H = (A \times B) \cap J = H \cap J = \varnothing$.

(Problem 2860) Let E be a finite union of measurable rectangles. Show that E is a finite union of disjoint measurable rectangles.

[Definition: Monotone class] Let W be a set and let $\mathcal{M}$ be a set of subsets of W. We say that $\mathcal{M}$ is a monotone class on W if:

- If $E_1 \subseteq E_2 \subseteq \dots$ is an increasing sequence of sets in $\mathcal{M}$ then $\bigcup_{k=1}^\infty E_k \in \mathcal{M}$.
- If $E_1 \supseteq E_2 \supseteq \dots$ is a decreasing sequence of sets in $\mathcal{M}$ then $\bigcap_{k=1}^\infty E_k \in \mathcal{M}$.

(Problem 2870) Show that the set of all intervals in $\mathbf{R}$ is a monotone class.

(Problem 2880) Show that the set of all intervals in $\mathbf{R}$ is not an algebra.

(Problem 2890) Let $\mathcal{C}$ be a collection of monotone classes on a set W. Show that $\bigcap_{\mathcal{M} \in \mathcal{C}} \mathcal{M}$ is a monotone class on W.

[Definition: Smallest monotone class] Let $\mathcal{B}$ be a set of subsets of a set W. Let $\mathcal{C} = \{\mathcal{M} : \mathcal{B} \subseteq \mathcal{M} \subseteq 2^W, \mathcal{M} \text{ is a monotone class}\}$. We call $\bigcap_{\mathcal{M} \in \mathcal{C}} \mathcal{M}$ the smallest monotone class containing $\mathcal{B}$.

(Problem 2900) Let $\mathcal{B}$ be a set of subsets of a set W. Show that the smallest σ-algebra containing $\mathcal{B}$ contains the smallest monotone class containing $\mathcal{B}$.

(Problem 2910) Let $\mathcal{A}$ be an algebra on a set W. Let $\mathcal{M}$ be the smallest monotone class containing $\mathcal{A}$. Show that $\mathcal{M}$ is a σ-algebra.

(Problem 2911) Show that $\mathcal{M}$ is the smallest σ-algebra containing $\mathcal{A}$.

* * *

[Definition: Finite measure] A measure μ on a measurable space $(X, \mathcal{S})$ is finite if $\mu(X) < \infty$.

[Definition: σ-finite] A measure μ on a measurable space $(X, \mathcal{S})$ is σ-finite if there exists a sequence $\{X_k\}_{k=1}^{\infty} \subseteq \mathcal{S}$ with $X = \bigcup_{k=1}^{\infty} X_k$ and $\mu(X_k) < \infty$ for each $k \in \mathbf{N}$.

(Problem 2912) Show that we may also require $X_k \subseteq X_{k+1}$ for all $k \in \mathbf{N}$.

(Problem 2920) Give an example of a finite measure.

(Problem 2930) Show that Lebesgue measure on $\mathbf{R}$ is σ-finite but not finite.

(Problem 2940) Show that the counting measure on $\mathbf{N}$ is σ-finite but not finite.

(Problem 2950) Show that the counting measure on $\mathbf{R}$ is not σ-finite.

(Problem 2960) Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be measure spaces. Suppose $\mu(X) < \infty$. If $E \in \mathcal{S} \otimes \mathcal{T}$, show that $y \mapsto \mu([E]^y)$ is a $\mathcal{T}$-measurable function on Y.

(Problem 2970) Show that the result is still true if $(X, \mathcal{S}, \mu)$ is a σ-finite measure space.

[Definition: Integral notation] If $(X, \mathcal{S}, \mu)$ is a measure space and $g : X \to [-\infty, \infty]$ is a $\mathcal{S}$-measurable function then

$$\int_X g(x)\, d\mu(x) = \int g\, d\mu$$

where $d\mu(x)$ indicates that variables other than x should be treated as constants.

[Definition: Product of measures] Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \nu)$ be σ-finite measure spaces. Let $E \in \mathcal{S} \otimes \mathcal{T}$. We define

$$(\mu \times \nu)(E) = \int_Y \mu([E]^y)\, d\nu(y).$$

(Problem 2980) Let $A \in \mathcal{S}$ and $B \in \mathcal{T}$. Show that $(\mu \times \nu)(A \times B) = \mu(A) \cdot \nu(B)$.

(Problem 2981) Show that if $E \in \mathcal{S} \otimes \mathcal{T}$ then $(\mu \times \nu)(E)$ is well defined and nonnegative.

(Problem 2990) Show that $\mu \times \nu$ is a measure on $(X \times Y, \mathcal{S} \otimes \mathcal{T})$.

(Problem 2991) Let $E_1 \subseteq E_2 \subseteq \dots$ be an increasing sequence of sets in $\mathcal{S} \otimes \mathcal{T}$. Show that

$$\int_Y \mu([\bigcup_{k=1}^{\infty} E_k]^y)\, d\nu(y) = \lim_{k\to\infty} \int_Y \mu([E_k]^y)\, d\nu(y).$$

(Problem 2992) Let $E_1 \supseteq E_2 \supseteq \dots$ be a decreasing sequence of sets in $\mathcal{S} \otimes \mathcal{T}$. If $\mu(X) < \infty$, show that

$$\int_Y \mu([\bigcap_{k=1}^{\infty} E_k]^y)\, d\nu(y) = \lim_{k\to\infty} \int_Y \mu([E_k]^y)\, d\nu(y).$$

(Problem 3000) Suppose $\mu(X) < \infty$ and $\mu(Y) < \infty$. Show that

$$\int_Y \mu([E]^y)\, d\nu(y) = \int_X \nu([E]_x)\, d\mu(x)$$

for all $E \in \mathcal{S} \otimes \mathcal{T}$.

(Problem 3010) Show that

$$\int_Y \mu([E]^y)\, d\nu(y) = \int_X \nu([E]_x)\, d\mu(x)$$

for all $E \in \mathcal{S} \otimes \mathcal{T}$ if X and Y are σ-finite (and not necessarily finite).

5B. Iterated integrals

(Problem 3020) Let $(X,\mathcal{S},\mu)$ and $(Y,\mathcal{T},\nu)$ be σ-finite measure spaces. Let $f: X\times Y\to[0,\infty]$ be a $\mathcal{S}\otimes\mathcal{T}$-measurable function. Let $g(x)=\int_Y f(x,y)\,d\nu(y)=\int_Y [f]_x(y)\,d\nu(y)$. Show that $g:X\to\mathbf{R}$ is $\mathcal{S}$-measurable.

[Definition: Iterated integral] Let $(X,\mathcal{S},\mu)$ and $(Y,\mathcal{T},\nu)$ be measure spaces. Let $f: X\times Y\to[-\infty,\infty]$ be a $\mathcal{S}\otimes\mathcal{T}$-measurable function. We define

$$\begin{aligned}\int_X\int_Y f(x,y)\,d\nu(y)\,d\mu(x) &= \int_X\Big(\int_Y f(x,y)\,d\nu(y)\Big)d\mu(x)\\ &= \int_X\Big(\int_Y [f]_x\,d\nu(y)\Big)d\mu.\end{aligned}$$

(Problem 3030) [Tonelli's Theorem] Let $(X,\mathcal{S},\mu)$ and $(Y,\mathcal{T},\nu)$ be σ-finite measure spaces. Let $f: X\times Y\to[0,\infty]$ be a $\mathcal{S}\otimes\mathcal{T}$-measurable function. Show that

$$\int_{X\times Y} f\,d(\mu\times\nu)=\int_X\int_Y f(x,y)\,d\nu(y)\,d\mu(x)=\int_Y\int_X f(x,y)\,d\mu(x)\,d\nu(y).$$

[Exercise 5B.1] Tonelli's theorem can fail for measurable functions $f: X\times Y\to\mathbf{R}$.

(Problem 3040) Let $\mathcal{B}$ be the σ-algebra of Borel subsets of $[0,1]$. Let μ denote the counting measure. Is $([0,1],\mathcal{B},\mu)$ σ-finite?

(Problem 3050) Let λ denote Lebesgue measure on $\mathcal{B}$. Let $D=\{(x,x):x\in[0,1]\}$. Find

$$\int_{[0,1]}\int_{[0,1]}\chi_D(x,y)\,d\mu(y)\,d\lambda(x).$$

(Problem 3060) Find

$$\int_{[0,1]}\int_{[0,1]}\chi_D(x,y)\,d\lambda(x)\,d\mu(y).$$

(Problem 3070) Show that if $x_{j,k}\geq 0$ for all $j,k\in\mathbf{N}$, then $\sum_{j=1}^\infty\sum_{k=1}^\infty x_{j,k}=\sum_{k=1}^\infty\sum_{j=1}^\infty x_{j,k}$.

(Problem 3080) Let $(X,\mathcal{S},\mu)$ and $(Y,\mathcal{T},\nu)$ be σ-finite measure spaces. Let $f: X\times Y\to[-\infty,\infty]$ be a $\mathcal{S}\otimes\mathcal{T}$-measurable function. Suppose that $\int_{X\times Y}|f|d(\mu\times\nu)<\infty$. Show that $\int_X|f(x,y)|\,d\mu(x)<\infty$ for ν-almost every $y\in Y$ and that $\int_Y|f(x,y)|\,d\nu(y)<\infty$ for μ-almost every $x\in X$.

(Problem 3090) What can you say about the set $\{x\in X:\int_Y f(x,y)\,d\nu(y)$ does not exist$\}$?

(Problem 3100) Show that $x\mapsto\int_Y f(x,y)\,d\nu(y)$ is a $\mathcal{S}$-measurable function and that $y\mapsto\int_X f(x,y)\,d\mu(x)$ is a $\mathcal{T}$-measurable function (up to a set of measure zero).

(Problem 3110) [Fubini's theorem] Show that if $\int_{X\times Y}|f|d(\mu\times\nu)<\infty$ then

$$\int_{X\times Y} f\,d(\mu\times\nu)=\int_X\int_Y f(x,y)\,d\nu(y)\,d\mu(x)=\int_Y\int_X f(x,y)\,d\mu(x)\,d\nu(y).$$

* * *

[Definition: Region under a graph] Let X be a set and let $f:X\to[0,\infty]$. The region under the graph of f is

$$U_f=\{(x,t):x\in X, 0<t<f(x)\}.$$

(Problem 3120) Let $E_{m,k}=f^{-1}([\frac{m}{k},\frac{m+1}{k}])\times(0,\frac{m}{k})$. Show that $U_f=\bigcup_{k=1}^\infty\bigcup_{m=0}^\infty E_{m,k}$.

(Problem 3130) Let $(X,\mathcal{S},\mu)$ be a σ-finite measure space. Let $f:X\to[0,\infty]$ be a $\mathcal{S}$-measurable function. Let $\mathcal{B}$ be the Borel sets and let λ denote Lebesgue measure on $\mathbf{R}$. Show that $U_f\in\mathcal{S}\otimes\mathcal{B}$.

(Problem 3140) Show that $(\mu \times \lambda)(U_f) = \int_X f\,d\mu$.

(Problem 3150) Show that $\int_X f\,d\mu = \int_{(0,\infty)} \mu(\{x \in X : t < f(x)\})\,d\lambda(t)$.

(Problem 3160) Show that if $\int_X f\,d\mu < \infty$ then $\int_{(0,\infty)} \mu(\{x \in X : t < f(x)\})\,d\lambda(t) = \int_0^\infty \mu(\{x \in X : t < f(x)\})\,dt$, where the right hand integral is an improper Riemann integral.

(Problem 3170) Use the above result to give another proof of Markov's inequality (that is, $\mu(\{x \in X : t < f(x)\}) \leq \frac{1}{t} \int_X f\,d\mu$).

(Problem 3171) Can you do the converse, that is, use Markov's inequality to get the above result?